JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 26, Number 2, Summer 2014

APPROXIMATIONS OF SOLUTIONS TO A RETARDED TYPE FRACTIONAL DIFFERENTIAL EQUATION WITH A DEVIATED ARGUMENT

PRADEEP KUMAR, D.N. PANDEY AND D. BAHUGUNA

Communicated by Neville Ford

ABSTRACT. In the present work, we are concerned with approximations of solutions to a retarded type fractional differential equation with a deviated argument in a separable Hilbert space H. We consider an integral equation associated with a given problem and then consider a sequence of approximate integral equations. We prove the existence, uniqueness and convergence to each of the approximate integral equations by using analytic semigroup theory and the fixed point method. We also prove that the limiting function satisfies the associated integral equation. Finally, we consider Faedo-Galerkin approximations of solutions and prove some convergence results.

1. Introduction. In this article, we consider the following retarded type fractional differential equation with a deviated argument in a separable Hilbert space $(H, \|\cdot\|, (\cdot,\cdot))$:

(1.1)

$$\left.\begin{array}{ll} {}^cD_t^{\eta}[u(t)+g(t,u(t))]+Au(t)=f(t,u(t),u[h(u(t),t)]), & \eta\in[0,1), \\ u(0)=u_0, & 0<t\le T<\infty \end{array}\right\}$$

where ${}^cD_t^{\eta}$ is the Caputo fractional derivative of order η and $A : D(A) \subset H \to H$ is a closed, densely defined, positive definite, self-adjoint linear operator which satisfies assumption (H1), stated later. Functions f, g

2010 AMS *Mathematics subject classification.* Primary 34G10, 34G20, 34K30, 35K90, 47N20.

Keywords and phrases. Analytic semigroup, retarded type fractional differential equation, deviated argument, Banach fixed point theorem, Faedo-Galerkin approximation.

The first author is the corresponding author.

The third author would like to acknowledge financial aid from the Department of Science and Technology, New Delhi, under its research project SR/S4/MS:796/12.

Received by the editors on February 21, 2013, and in revised form on July 18, 2013.

DOI:10.1216/JIE-2014-26-2-215 Copyright ©2014 Rocky Mountain Mathematics Consortium

and h are suitably defined and satisfy certain conditions to be stated later.

In the present work, we are interested in Faedo-Galerkin approximations of solutions to problem (1.1). In **[20]**, Milleta has discussed the Faedo-Galerkin approximations of solutions to the particular case of (1.1) in the cases when $\eta = 1$, $h \equiv 0$ and $f(t,u) = M(u)$. For a nice introduction and related study of various problems in this direction, we refer to the reader to **[1, 2, 3, 4, 20, 22, 23]** and the references cited therein.

In **[23]**, Muslim et al. have established the existence, uniqueness and convergence of approximations of solutions in a separable Hilbert space and convergence of the Faedo-Galerkin approximations of solutions to the following problem:

$$\begin{aligned} u(t) &= u_0 + \frac{1}{\Gamma\beta}\int_0^t (t-s)^{\beta-1}(-Au(s))ds \\ &\quad + \frac{1}{\Gamma\beta}\int_0^t (t-s)^{\beta-1} f(s,u(s),u(a(s)))\, ds, \end{aligned}$$

where $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ on a Banach space $(H, \|\cdot\|, (\cdot,\cdot))$, the functions $f : [T] \times H \times H \to H$ and $a : [0,T] \to [0,T]$ are suitable functions.

In **[24]**, Ntouyas et al. proved existence results for semilinear neutral functional differential inclusions with finite or infinite delay in Banach spaces to the following problem

(1.2)
$$\frac{d}{dt}[y(t) - f(t,y_t)] = Ay(t) + F(t,y_t), \text{ almost everywhere } t \in J := [0,T],$$

(1.3)
$$y(t) = \Phi(t), \quad t \in J_0 := [-r,0],$$

where $f : J \times D \to E$, $F : J \times D \to \mathcal{P}(E)$ is a multivalued map, $\mathcal{D} = \{\Psi : [-r,0] \to E : \Psi \text{ is continuous}\}$, $\Phi \in \mathcal{D}$, $0 < r < \infty$, E is a real separable Banach space with norm $\|\cdot\|$ and $\mathcal{P}(E)$ is the family of all nonempty subsets of E.

For earlier work on the existence and uniqueness of solutions to differential equations of fractional order, we refer to **[5, 12, 14, 15,**

16, 17, 18, 19, 21, 27, 28, 31, 33] and the references cited therein.

The book [**6**] by El'sgol'ts and Norkin provides a comprehensive study of differential equations with deviated arguments. The existence, uniqueness, almost automorphic solutions and asymptotic behaviors of differential equations with deviating arguments has been studied by many authors like Grimm [**8**], Obreg [**25**], Driver [**10**], Gal [**7**] (see also [**9, 11, 13, 29, 32**]) and the references cited therein.

The rest of the paper is organized as follows. In Section 2, we put some notations, notions and results that are required for proving the main results. In Section 3, we consider an integral equation associated with problem (1.1) and then consider a sequence of approximate integral equations and establish the existence and uniqueness of solutions to each of the approximate integral equations. We also prove the convergence of solutions to each of the approximate integral equations in Section 4 and then prove that the limiting function satisfies the associated integral equation. In Section 5, we consider the Faedo-Galerkin approximations of solutions and prove some convergence results for such approximations. Finally, we give an example to demonstrate the applications of abstract results obtained in the earlier sections.

2. Preliminaries and assumptions. In this section, we present some assumptions, preliminaries and lemmas required for proving the main results. The details of the material presented here can be found in [**26**]. We shall use the following assumption on operator A:

(H1) Let A be a closed, positive definite, self-adjoint linear operator from the domain $D(A) \subset H$ into H with $D(A)$ dense in H. We also assume that A has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots,$$

where $\lambda_m \to \infty$ as $m \to \infty$ and a corresponding complete orthonormal system of eigenfunctions $\{\phi_i\}$, i.e.,

$$A\phi_i = \lambda_i \phi_i \quad \text{and} \quad \langle \phi_i, \phi_j \rangle = \delta_{ij},$$

where $\delta_{ij} = 1$ for $i = j$, zero otherwise.

Assumption (H1) implies that $-A$ generates an analytic semigroup of bounded linear operators $S(t)$, $t \geq 0$. Then there exist constants $\widetilde{M} \geq 1$

and $\omega \geq 0$ such that

$$\|S(t)\| \leq \widetilde{M}e^{\omega t}, \quad t \geq 0.$$

We also note that **[26,** Lemma 4.2, page 52]

$$\left\| \frac{d^i}{dt^i} S(t) \right\| \leq M_i, \quad t > t_0$$

for some positive constant M_i.

Without loss of generality, we may assume that $\|S(t)\|$ is uniformly bounded by M, i.e., $\|S(t)\| \leq M$ for $t \geq 0$, and that $0 \in \rho(-A)$, i.e., $-A$ is invertible. This allows us to define the positive fractional power A^α for $0 \leq \alpha \leq 1$ as closed linear operator with domain $D(A^\alpha) \subseteq H$. Furthermore, $D(A^\alpha)$ is dense in H endowed with the norm

$$\|x\|_\alpha = \|A^\alpha x\|.$$

Henceforth, we denote the space $D(A^\alpha)$ by H_α endowed with the norm $\|\cdot\|_\alpha$. Also, for each $\alpha > 0$, we define $H_{-\alpha} = (H_\alpha)^*$, the dual space of H_α endowed with the norm $\|x\|_{-\alpha} = \|A^{-\alpha} x\|$.

Lemma 2.1 (**[26,** pages 72, 74, 195–196]). *Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$ with $\|S(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$. Then*

(i) *H_α is a Hilbert space for $0 \leq \alpha \leq 1$;*
(ii) *for any $0 < \delta \leq \alpha$ implies $D(A^\alpha) \subset D(A^\delta)$, the embedding $H_\alpha \hookrightarrow H_\delta$ is continuous;*
(iii) *the operator $A^\alpha S(t)$ is bounded for every $t > 0$ and*

$$\|A^\alpha S(t)\| \leq C_\alpha t^{-\alpha}.$$

We denote the space of all H_α-valued continuous functions on $[0, t]$ by $\mathcal{C}_t^\alpha = C([0, t]; H_\alpha)$, for all $t \in (0, T]$. Then $\mathcal{C}_t^\alpha$ is a Banach space endowed with the norm,

$$\|\psi\|_{t,\alpha} := \sup_{0 \leq r \leq t} \|\psi(r)\|_\alpha, \quad \psi \in \mathcal{C}_t^\alpha.$$

For $0 \leq \alpha < 1$, define

$$\mathcal{C}_T^{\alpha-1} = \{y \in \mathcal{C}_T^\alpha : \|y(t) - y(s)\|_{\alpha-1} \leq L|t - s|, \quad \forall t, s \in [0, T]\},$$

where L is a suitable positive constant to be specified later.

We assume the following conditions:

(H2) Let $U_1 \subset \mathrm{Dom}\,(f)$ be an open subset of $\mathbf{R}_+ \times H_\alpha \times H_{\alpha-1}$ and, for each $(t,u,v) \in U_1$, there is a neighborhood $V_1 \subset U_1$ of (t,u,v). The nonlinear map $f : \mathbf{R}_+ \times H_\alpha \times H_{\alpha-1} \to H$ satisfies the following condition:

$$\|f(t,x,\psi) - f(s,y,\widetilde{\psi})\| \le L_f[|t-s|^{\theta_1} + \|x-y\|_\alpha + \|\psi - \widetilde{\psi}\|_{\alpha-1}],$$

where $0 < \theta_1 \le 1$, $0 \le \alpha < 1$, $L_f > 0$ is a constant, $(t,x,\psi) \in V_1$ and $(s,y,\widetilde{\psi}) \in V_1$.

(H3) Let $U_2 \subset \mathrm{Dom}\,(h)$ be an open subset of $H_\alpha \times \mathbf{R}_+$ and, for each $(x,t) \in U_2$, there is a neighborhood $V_2 \subset U_2$ of (x,t). The map $h : H_\alpha \times [0,T] \to [0,T]$ satisfies the following condition:

$$|h(x,t) - h(y,s)| \le L_h[\|x-y\|_\alpha + |t-s|^{\theta_2}],$$

where $0 < \theta_2 \le 1$, $0 \le \alpha < 1$, $L_h > 0$ is a constant, $(x,t),(y,s) \in V_2$ and $h(\cdot,0) = 0$.

(H4) Let $U_3 \subset \mathrm{Dom}\,(g)$ be an open subset of $[0,T] \times H_{\alpha-1}$ and, for each $(t,x) \in U_3$, there is a neighborhood $V_3 \subset U_3$ of (x,t). There exists a positive constant β, $0 < \alpha < \beta < 1$, such that the function $A^\beta g$ is continuous for $(t,u) \in [0,T_0] \times H_{\alpha-1}$ such that

$$\|A^\beta g(t,x) - A^\beta g(s,y)\| \le L_g\{|t-s| + \|x-y\|_{\alpha-1}\},$$

and

$$L_g\|A^{\alpha-\beta-1}\| \le \delta < 1$$

where $L_g, \delta > 0$ is a positive constant and $(x,t),(y,s) \in V_3$.

3. Approximate integral equations. The existence of a solution to (1.1) is closely related to the following integral equation (3.9).

Definition 1 ([**31,** Definition 1.2]). Let $f \in L^1((0,T),H)$ and $\alpha \ge 0$. Then the expression

$$I_t^\alpha f(t) = (f * \Theta_\alpha)(t) = \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1} f(s)\,ds,$$

$$t > 0, \quad \alpha > 0, \tag{3.1}$$

where $I_t^0 f(t) = f(t)$ and

$$\Theta_\alpha(t) = \begin{cases} \dfrac{1}{\Gamma\alpha} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and $\Theta_0(t) = 0$, is called the Riemann-Liouville integral of order α of f.

Definition 2 (**[31,** Definition 1.3]**).** Let $f \in C^{m-1}((0,T),H)$, $(\Theta_{m-\alpha} * f) \in W^{m,1}((0,T),H)$ $(m \in \mathbf{N},\ 0 \leq m-1 < \alpha < m)$. Then the expression

$$(3.2) \qquad {}^cD_t^\alpha f(t) = D_t^m I_t^{m-\alpha}\bigg(f(t) - \sum_{0}^{m-1} f^i(0)\Theta_{i+1}(t)\bigg),$$

where $D_t^m = d^m/(dt^m)$, is called the Caputo fractional derivative of order α of f.

Then, by definitions (1) and (3.2), we can rewrite (1.1) as

(3.3)

$$\begin{aligned} u(t) &= (u_0 + g(0,u_0)) - g(t,u(t)) \\ &\quad - \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}[Au(s) + g(s,u(s))]\,ds \\ &\quad + \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1} f(s,u(s),u(h(u(s),s)))\,ds, \quad t \in [0,T]. \end{aligned}$$

For a fixed $R > 0$, we choose $0 < T_0 = T_0(\alpha,\beta,u_0) \leq T$ sufficiently small, such that

$$(3.4) \quad C_{\alpha+1-\beta} L_g \|A^{-1}\| \frac{T_0^{\eta(\beta-\alpha)}}{\beta-\alpha} + C_\alpha L_f[2 + LL_h]\frac{T_0^{\eta(1-\alpha)}}{1-\alpha} \leq 1-\delta,$$

where $\delta = L_g\|A^{\alpha-\beta-1}\| < 1$ and $T_0 < \min(d_1,d_2)$ with

$$(3.5) \qquad d_1 = \left(\frac{R}{4}(\beta-\alpha)(C_{1+\alpha-\beta}L_g)^{-1}\right)^{1/[\eta(\beta-\alpha)]},$$

$$(3.6) \qquad d_2 = \left(\frac{R}{4}(1-\alpha)(C_\alpha[2+LL_h]L_f)^{-1}\right)^{1/[\eta(1-\alpha)]}$$

and satisfying the following

$$\|(S(t^\eta\theta)-I)A^\alpha[u_0+g_n(0,u_0)]\|+\|A^{\alpha-\beta}\|L_g[T_0+\|A^{-1}\|R]\leq\frac{R}{2}, \tag{3.7}$$

for all $t\in[0,T_0]$ and

$$C_{\alpha+1-\beta}N_1\frac{T_0^{\eta(\beta-\alpha)}}{\beta-\alpha}+C_\alpha N\frac{T_0^{\eta(1-\alpha)}}{1-\alpha}\leq\frac{R}{2}. \tag{3.8}$$

For more details of choosing such a T_0, we refer to [**7,** Theorem 2.2].

We set

$$\mathcal{W}=\{u\in\mathcal{C}_{T_0}^{\alpha}\cap\mathcal{C}_{T_0}^{\alpha-1}:u(0)=u_0,\quad \|u-u_0\|_{T_0,\alpha}\leq R\}.$$

Clearly, $\mathcal{W}$ is a closed, bounded subset of $\mathcal{C}_{T_0}^{\alpha-1}$ and complete.

Definition 3 ([**5,** page 434])**.** By a solution of problem (1.1), we mean a function $u:[0,T]\to H_\alpha$ satisfying the following three conditions:

(i) $u(\cdot)+g(\cdot,u(\cdot))\in\mathcal{C}_T^{\alpha-1}\cap C([0,T],H)$.
(ii) $u(t)+g(t,u(t))\in D(A)$ and $(t,u(t),u[h(u(t),t)])\in U_1$ for all $t\in[0,T]$.
(iii) $d^\eta/dt^\eta[u(t)+g(t,u(t))]+Au(t)=f(t,u(t),u[h(u(t),t)])$ for all $t\in(0,T]$.
(iv) $u(0)=u_0$.

Definition 4 ([**30,** Definition 2.7])**.** By the mild solution of Cauchy problem (1.1), we mean a continuous function $u:(0,T_0]\to H$ which satisfies the following integral equation associated with (1.1):

$$\begin{aligned}u(t)=&\int_0^\infty\theta\xi_\eta(\theta)S(t^\eta\theta)[u(0)+g(0,u_0)]\,d\theta-g(t,u(t))\\&+\eta\int_0^t\int_0^\infty\theta\xi_\eta(\theta)(t-s)^{\eta-1}AS((t-s)^\eta\theta)g(s,u(s))\,d\theta\,ds\\&+\eta\int_0^t\int_0^\infty\theta\xi_\eta(\theta)(t-s)^{\eta-1}S((t-s)^\eta\theta)\\&\times f(s,u(s),u[h(u(s),s)])\,d\theta\,ds,\quad t\in(0,T_0].\end{aligned} \tag{3.9}$$

where

$$\xi_\eta(\theta) = \frac{1}{\eta}\theta^{-1-(1/\eta)}\rho_\eta(\theta^{-1/\eta}) \geq 0,$$

$$\rho_\eta(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\eta-1}\frac{\Gamma(n\eta+1)}{n!}\sin(n\pi\eta), \quad \theta \in (0,\infty),$$

ξ_η is a probability density function defined on $(0,\infty)$, that is,

$$\int_0^\infty \xi_\eta(\theta)d\theta = 1.$$

Also, we have

$$\int_0^\infty \theta^\gamma \xi_\eta(\theta) = \int_0^\infty \frac{1}{\theta^{\gamma\beta}}\rho_\eta(\theta) = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma\eta)}, \quad \text{for any } \gamma \in [0,1].$$

For more details, we refer to **[5, 31, 33]**.

Let $H_n \subseteq H$ denote the finite dimensional subspace spanned by $\{u_0, u_1, \cdots, u_n\}$, and let $P^n : H \to H_n$ be the corresponding projection operator for $n = 0, 1, 2, \ldots$. We define

$$g_n : \mathbf{R}_+ \times H \longrightarrow H \quad \text{as} \quad g_n(t,u(t)) = g(t, P^n u(t)) \quad \text{and} \tag{3.10}$$

$$f_n : \mathbf{R}_+ \times H \times H \longrightarrow H \quad \text{given by}$$

$$f_n(s,u(s),u[h(u(s),s)]) = f(s, P^n u(s), P^n u[h(u(s),s)]). \tag{3.11}$$

For $n = 0, 1, \ldots$, we define a map $\mathcal{F}_n : \mathcal{W} \to \mathcal{W}$ given by

$$\begin{aligned}(\mathcal{F}_n u)(t) = &\int_0^\infty \theta\xi_\eta(\theta)S(t^\eta\theta)[u(0)+g_n(0,u_0)]\,d\theta - g_n(t,u(t)) \\ &+ \eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}AS((t-s)^\eta\theta)g_n(s,u(s))\,d\theta\,ds \\ &+ \eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}S((t-s)^\eta\theta) \\ &\times f_n(s,u(s),u[h(u(s),s)])\,d\theta\,ds, \quad t \in (0,T_0].\end{aligned} \tag{3.12}$$

Theorem 3.1. *Let assumptions* (H1)–(H4) *hold and also let* $u_0 \in H_\alpha$ *for* $0 \leq \alpha < 1$. *Then there exists a unique* $u_n \in \mathcal{C}_{T_0}^{\alpha-1} \cap \mathcal{C}_{T_0}^{\alpha}$ *such that* $\mathcal{F}_n u_n = u_n$ *for each* $n = 0, 1, 2, \ldots$, u_n *satisfies the following* *approximate integral equation corresponding to the integral equation* (3.9),

$$
\begin{aligned}
u_n(t) = &\int_0^\infty \theta\xi_\eta(\theta)S(t^\eta\theta)[u(0) + g_n(0, u_0)]\, d\theta - g_n(t, u_n(t)) \\
&+ \eta \int_0^t \int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1} AS((t-s)^\eta\theta) g_n(s, u_n(s))\, d\theta\, ds \\
&+ \eta \int_0^t \int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1} S((t-s)^\eta\theta) \\
&\times f_n(s, u_n(s), u[h(u_n(s), s)])\, d\theta\, ds, \quad t \in (0, T_0].
\end{aligned} \tag{3.13}
$$

Proof. In order to prove this theorem, first we need to show that $\mathcal{F}_n u \in \mathcal{C}_{T_0}^{\alpha-1}$ for any $u \in \mathcal{C}_{T_0}^{\alpha-1}$. Clearly, $\mathcal{F}_n : \mathcal{C}_{T_0}^{\alpha} \to \mathcal{C}_{T_0}^{\alpha}$.

If $u \in \mathcal{C}_{T_0}^{\alpha-1}$, $T_0 > t_2 > t_1 > 0$, and $0 \le \alpha < 1$, then we get

(3.14)

$$
\begin{aligned}
&\|(\mathcal{F}_n u)(t_2) - (\mathcal{F}_n u)(t_1)\|_{\alpha-1} \\
&\quad \le \int_0^\infty \theta\xi_\eta(\theta)\|(S(t_2^\eta\theta) - S(t_1^\eta\theta))(u_0 + g_n(0, u_0))\|_{\alpha-1}\, d\theta \\
&\qquad + \|A^{\alpha-1-\beta}\|\|A^\beta g_n(t_2, u(t_2)) - A^\beta g_n(t_1, u(t_1))\| \\
&\qquad + \int_0^{t_1}\int_0^\infty \xi_\eta(\theta) \\
&\qquad \left\|\Big(\eta\theta(t_2-s)^{\eta-1} AS((t_2-s)^\eta\theta) - \eta\theta(t_1-s)^{\eta-1} AS((t_1-s)^\eta\theta)\Big)\right\| \\
&\quad \times \|A^{\alpha-1} g_n(s, u(s))\|\, d\theta\, ds \\
&\qquad + \int_{t_1}^{t_2}\int_0^\infty \xi_\eta(\theta)\|\eta\theta(t_2-s)^{\eta-1} AS((t_2-s)^\eta\theta)\| \\
&\qquad \|A^{\alpha-1} g_n(s, u(s))\|\, d\theta\, ds \\
&\qquad + \int_0^{t_1}\int_0^\infty \xi_\eta(\theta)\|(\eta\theta(t_2-s)^{\eta-1} S((t_2-s)^\eta\theta) - \eta\theta(t_1-s)^{\eta-1} \\
&\qquad S((t_1-s)^\eta\theta))A^{\alpha-1}\|\|f_n(s, u(s), u[h(u(s), s)])\|\, d\theta\, ds \\
&\qquad + \int_{t_1}^{t_2} \int_0^\infty \xi_\eta(\theta)\|\eta\theta(t_2-s)^{\eta-1} S((t_2-s)^\eta\theta)A^{\alpha-1}\|
\end{aligned}
$$

$$\|f_n(s,u(s),u[h(u(s),s)])\|\,d\theta\,ds$$

For the first part of the right hand side of (3.14), we have

$$\begin{aligned}
&\int_0^\infty \xi_\eta(\theta)\|(S(t_2^\eta\theta)-S(t_1^\eta\theta))(u_0+g_n(0,u_0))\|_{\alpha-1}\,d\theta \\
&\quad\le \int_0^\infty \xi_\eta(\theta)\left[\int_{t_1}^{t_2}\frac{d}{dt}S((t^\eta\theta))\,dt\right]\|A^{\alpha-1}(u_0+g_n(0,u_0))\|\,d\theta \\
&\quad\le \int_0^\infty \xi_\eta(\theta)[M_1(t_2-t_1)]\|A^{\alpha-1}(u_0+g_n(0,u_0))\|\,d\theta \\
&\quad\le C_1(t_2-t_1), \qquad (3.15)
\end{aligned}$$

where $C_1=[\|u_0\|_{\alpha-1}+\|A^{\alpha-\beta-1}\|\|g_n(0,u_0)\|_\beta]M$.

For the second part of the right hand side of (3.14), we can see that

$$\begin{aligned}
&\|A^{\alpha-\beta-1}\|\|A^\beta g_n(t_2,u(t_2))-A^\beta g_n(t_1,u(t_1))\| \\
&\qquad\le \|A^{\alpha-\beta-1}\|L_g[(t_2-t_1)+\|u(t_2)-u(t_1)\|_{\alpha-1}] \\
&\qquad\le C_2(t_2-t_1), \qquad (3.16)
\end{aligned}$$

where $C_2=\|A^{\alpha-\beta-1}\|[L_g(1+L)]$. To handle the third and fifth parts of the right hand side of (3.14), observe that

$$\begin{aligned}
&\int_0^{t_1}\int_0^\infty \xi_\eta(\theta)\|[\eta\theta(t_2-s)^{\eta-1}AS((t_2-s)^\eta\theta)-\eta\theta(t_1-s)^{\eta-1} \\
&\quad AS((t_1-s)^\eta\theta)]\| \\
&\quad\times\|A^{\alpha-2}f_n(s,u(s),u[h(u(s),s)])\|\,d\theta\,ds \\
&\quad\le \int_0^{t_1}\int_0^\infty \xi_\eta(\theta)\left\|\left[\frac{d}{dt}S((t-s)^\eta\theta)\Big|_{t=t_2}-\frac{d}{dt}S((t-s)^\eta\theta)\Big|_{t=t_1}\right]\right\| \\
&\quad\times\|A^{\alpha-2}\|N\,d\theta\,ds \\
&\quad\le \int_0^{t_1}\int_0^\infty \xi_\eta(\theta)\left[\int_{t_1}^{t_2}\left|\frac{d^2}{dt^2}S((t-s)^\eta\theta)\right|dt\right]\|A^{\alpha-2}\|N\,d\theta\,ds \\
&\quad\le \int_0^{t_1}\int_0^\infty \xi_\eta(\theta)[M_2(t_2-t_1)]\|A^{\alpha-2}\|N\,d\theta\,ds \\
&\quad\le C_3(t_2-t_1), \qquad (3.17)
\end{aligned}$$

where $C_3 = NM_2\|A^{\alpha-2}\|T_0$. Similarly, for the third part of (3.14), we have

$$\begin{aligned}\int_0^{t_1}\int_0^{\infty} &\xi_\eta(\theta)\|[\eta\theta(t_2-s)^{\eta-1}S((t_2-s)^\eta\theta)\\ &-\eta\theta(t_1-s)^{\eta-1}S((t_1-s)^\eta\theta)]A^{\alpha-\beta}\|\\ &\times\|A^\beta g_n(s,u(s))\|\,d\theta\,ds\\ &\le C_4(t_2-t_1) \qquad (3.18)\end{aligned}$$

where $C_4 = N_1M_2\|A^{\alpha-\beta-1}\|T_0$. For the sixth part of (3.14), we have

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^{\infty} &\xi_\eta(\theta)\|\eta\theta(t_2-s)^{\eta-1}AS((t_2-s)^\eta\theta)\|\\ &\|A^{\alpha-2}f_n(s,u(s),u[h(u(s),s)])\|\,d\theta\,ds\\ &\le \int_{t_1}^{t_2}\int_0^{\infty}\xi_\eta(\theta)\left\|\frac{d}{dt}S((t-s)^\eta\theta)\Big|_{t=t_2}\right\|\|A^{\alpha-2}\|N\,d\theta\,ds\\ &\le C_5(t_2-t_1), \qquad (3.19)\end{aligned}$$

where $C_5 = \|A^{\alpha-2}\|M_1N$. Finally, for the fourth part of (3.14), we have

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^{\infty} &\xi_\eta(\theta)\|\eta\theta(t_2-s)^{\eta-1}AS((t_2-s)^\eta\theta)A^{\alpha-\beta-1}\|\\ &\|A^\beta g_n(s,u(s))\|\,d\theta\,ds\\ &\le C_6(t_2-t_1), \qquad (3.20)\end{aligned}$$

where $C_6 = \|A^{\alpha-\beta-1}\|M_1N_1$.

We use (3.15), (3.16) and (3.17)-(3.20) in (3.14) to get the following inequality:

$$\|(\mathcal{F}_nu)(t_2)-(\mathcal{F}_nu)(t_1)\|_{\alpha-1} \le L|t_2-t_1|, \qquad (3.21)$$

where $L = \max\{C_i,\ i=1,2,\dots 6\}$. Hence, $\mathcal{F}_n : \mathcal{C}_{T_0}^{\alpha-1} \to \mathcal{C}_{T_0}^{\alpha-1}$ follows.

Our next task is to show that $\mathcal{F}_n : \mathcal{W} \to \mathcal{W}$. Now, for $t \in [0,T_0]$ and $u \in \mathcal{W}$, we have

$$\|(\mathcal{F}_nu)(t)-u_0\|_\alpha$$

$$\le \int_0^\infty \theta\xi_\eta(\theta)\|(S(t^\eta\theta) - I)A^\alpha[u_0 + g_n(0, u_0)]\|\, d\theta$$
$$+ \|A^{\alpha-\beta}\|\|A^\beta g_n(s, u(s))) - A^\beta g_n(0, u(0))\|$$
$$+ \eta \int_0^t \int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}\|S((t-s)^\eta\theta)A^{1+\alpha-\beta}\|$$
$$\|A^\beta g_n(s, u(s)))])\|\, d\theta\, ds$$
$$+ \eta \int_0^t \int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}\|S((t-s)^\eta\theta)A^\alpha\|$$
$$\|f_n(s, u(s), u(h(u(s), s))])\|\, d\theta\, ds$$
$$\le \|(S(t^\eta\theta) - I)A^\alpha[u_0 + g_n(0, u_0)]\|$$
$$+ \|A^{\alpha-\beta}\|L_g[T_0 + \|A^{-1}\|R]$$
$$+ C_{1+\alpha-\beta}N_1\frac{T_0^{\eta(\beta-\alpha)}}{\beta-\alpha} + C_\alpha N\frac{T_0^{\eta(1-\alpha)}}{1-\alpha}.$$

Hence, from (3.7) and (3.8), we get

$$\|\mathcal{F}_n u - u_0\|_{T_0,\alpha} \le R.$$

Therefore, $\mathcal{F}_n : \mathcal{W} \to \mathcal{W}$.

Now, if $t \in [0, T_0]$ and $u, v \in \mathcal{W}$, then

$$\|(\mathcal{F}_n u)(t) - (\mathcal{F}_n v)(t)\|_\alpha$$
$$\le \|A^{\alpha-\beta}\|\|A^\beta g_n(t, u(s)) - A^\beta g_n(t, v(s))\|$$
$$+ \eta \int_0^t \int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}\|S((t-s)^\eta\theta)A^{1+\alpha-\beta}\|$$
$$\times \|A^\beta g_n(s, u(s))) - A^\beta g_n(s, v(s))\|\, d\theta\, ds$$
$$+ \eta \int_0^t \int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}\|S((t-s)^\eta\theta)A^\alpha\|$$
$$\|f_n(s, u(s), u(h[u(s), s]))$$
$$- f_n(s, v(s), v(h[v(s), s])))\|\, d\theta\, ds. \tag{3.22}$$

We have the following inequalities:

$$\|A^\beta g_n(s, u(s))) - A^\beta g_n(t, v(t))\| \le L_g\|A^{-1}\|\|u - v\|_{T_0,\alpha}, \tag{3.23}$$

$$\|f_n(s,u(s),u[h(u(s),s)]) - f_n(s,v(s),v[h(v(s),s)])\| \leq L_f(2+LL_h)\|u-v\|_{T_0,\alpha}. \tag{3.24}$$

We use the inequalities (3.23) and (3.24) in (3.22) and get

$$\|(\mathcal{F}_n u)(t) - (\mathcal{F}_n v)(t)\|_\alpha \leq \Bigg[L_g\|A^{\alpha-\beta-1}\| + C_{1+\alpha-\beta}L_g\|A^{-1}\|\frac{T_0^{\eta(\beta-\alpha)}}{\beta-\alpha} + C_\alpha L_f(2+LL_h)\frac{T_0^{\eta(1-\alpha)}}{1-\alpha}\Bigg]\|u-v\|_{T_0,\alpha}. \tag{3.25}$$

Hence, from inequality (3.4), we get the following inequality:

$$\|\mathcal{F}_n u - \mathcal{F}_n v\|_{T_0,\alpha} < \|u-v\|_{T_0,\alpha},$$

i.e., the map $\mathcal{F}_n$ is a contraction on $\mathcal{W}$. Therefore, the map $\mathcal{F}_n$ has a unique fixed point $u_n \in \mathcal{W}$, given by

$$\begin{aligned} u_n(t) = &\int_0^\infty \theta\xi_\eta(\theta)S(t)[u_0+g_n(0,u_0)]\,d\theta - g_n(t,u_n(t)) \\ &+ \eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}AS((t-s)^\eta\theta)g_n(s,u_n(s)))\,d\theta\,ds, \\ &+ \eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}S((t-s)^\eta\theta) \\ &\quad f_n(s,u_n(s),u_n(h(u_n(s),s)))\,d\theta\,ds, \quad t\in(0,T_0]. \end{aligned} \tag{3.26}$$

This completes the proof of Theorem 3.1. □

Lemma 3.2. *Assume that assumptions* (H1)–(H3) *are satisfied. We have the following results*

(i) *If* $u_0 \in D(A^\alpha)$, *then* $u_n(t) \in D(A^\vartheta)$ *for all* $t \in (0,T_0]$,
(ii) *If* $u_0 \in D(A)$, *then* $u_n(t) \in D(A^\vartheta)$ *for all* $t \in (0,T_0]$,

for $0<\vartheta<\beta<1$.

Proof. Since we have proved that $u_n \in \mathcal{W} \subseteq \mathcal{C}_{T_0}^{\alpha-1}$, then u_n must be Hölder continuous on $[0,T_0]$. Furthermore, the inequalities (H2)–(H4) imply the Hölder continuity of $f(t,u_n(t),u_n(h(u_n(t),t)))$ and $g(t,u_n(t))$

on $[0, T_0]$. We also note that [**26,** Theorem 3.2, page 111]

$$\eta \int_0^t \int_0^\infty \theta \xi_\eta(\theta)(t-s)^{\eta-1} AS((t-s)^\eta \theta) g_n(s, u_n(s))\, d\theta\, ds \in D(A).$$

Hence, we can easily prove that $u_n(t) \in D(A)$. For more details, we refer to [**5,** Theorem 2.2]. Part (i) follows from (ii) and the fact that $D(A) \subset D(A^\vartheta)$, $0 < \vartheta \le 1$ (see Lemma 2.1 (ii)). □

Lemma 3.3. *Suppose that assumptions* (H1)–(H4) *are satisfied. We have the following inequalities*:

(i) *If* $u_0 \in D(A^\alpha)$, *then for any* $t_0 \in (0, T_0]$

$$\|u_n(t)\|_\vartheta \le U_{t_0}, \quad t \in [t_0, T_0],\ n = 1, 2, \ldots,$$

for some constant U_{t_0}, *independent of* n.

(ii) *If* $u_0 \in D(A)$, *then there exists a constant* U_0 *such that*

$$\|u_n(t)\|_\vartheta \le U_0, \quad t \in [0, T_0],\ n = 1, 2, \ldots .$$

Proof. Let $u_0 \in D(A^\alpha)$. Applying A^ϑ on both sides of (3.26), for $t \in [t_0, T_0]$ and $\alpha < \vartheta < \beta$, we have

$$\begin{aligned} \|u_n(t)\|_\vartheta \le & \int_0^\infty \xi_\eta(\theta) \|A^\vartheta S(t^\eta \theta)(u_0 + g_n(0, u_0)\|\, d\theta \\ & + \|A^{\vartheta-\beta}\|\, \|A^\beta g_n(t, u_n(t))\| \\ & + \eta \int_0^t \int_0^\infty \theta \xi_\eta(\theta)(t-s)^{\eta-1} \|A^{1+\vartheta-\beta} S((t-s)^\eta \theta)\| \\ & \times \|A^\beta g_n(s, u_n(s))\|\, d\theta\, ds \\ & + \eta \int_0^t \int_0^\infty \theta \xi_\eta(\theta)(t-s)^{\eta-1} \\ & \times \|S((t-s)^\eta \theta)) A^\vartheta\| \\ & \|f_n(s, u_n(s), u_n(h(u_n(s), s)))\|\, d\theta\, ds \\ & \le C_\vartheta t_0^{-\eta\vartheta} (\|u_0\| + \|g_n(0, u_0)\| \\ & + \|A^{\vartheta-\beta}\| N_1 \\ & + C_{1+\vartheta-\beta} N_1 \frac{T^{\eta(\beta-\vartheta)}}{\beta-\vartheta} + C_\vartheta N \frac{T^{\eta(1-\vartheta)}}{1-\vartheta} \le U_{t_0}. \end{aligned} \tag{3.27}$$

Similarly, we can find the estimate

$$\begin{aligned}\|u_n(t)\|_\vartheta &\le M(\|A^\vartheta u_0\| + \|g_n(0,\widetilde{u_0}\|_\vartheta) + \|A^{\vartheta-\beta}\|N_1 \\ &+ C_{1+\vartheta-\beta}N_1\frac{T^{\eta(\beta-\vartheta)}}{\beta-\vartheta} + C_\vartheta N\frac{T^{\eta(1-\vartheta)}}{1-\vartheta} \le U_0, \qquad (3.28)\end{aligned}$$

for given $u_0 \in D(A)$ and $t \in (0, T_0]$. □

4. Convergence of solutions. In this section we establish the convergence of the solution $u_n \in H_\alpha(T_0)$ of each approximate integral equation to a unique solution u of (3.9).

Theorem 4.1. *Let us assume that conditions* (H1)–(H3) *are satisfied. If* $u_0 \in D(A^\alpha)$, *then for* $t_0 \in (0, T_0]$

$$\|u_n - u_m\|_{T_0,\alpha} \longrightarrow 0, \quad \text{as } m,n \to \infty,$$

i.e., u_n *is a Cauchy sequence in* $\mathcal{W}$ *on* $[t_0, T_0]$.

Proof. Let $0 < \alpha < \vartheta < \beta$. For $n \ge m$, we have

$$\begin{aligned}&\|f_n(t,u_n(t),u_n[h(u_n(t),t)]) - f_m(t,u_m(t),u_m[h(u_m(t),t)])\| \\ &\quad \le \|f_n(t,u_n(t),u_n[h(u_n(t),t)]) \\ &\qquad - f_n(t,u_m(t),u_m[h(u_m(t),t)])\| \\ &\qquad + \|f_n(t,u_m(t),u_m[h(u_m(t),t)]) \\ &\qquad - f_m(t,u_m(t),u_m[h(u_m(t),t)])\| \\ &\quad \le L_f(2+LL_h)\|u_n(t)-u_m(t)\|_\alpha \\ &\qquad + L_f[\|(P^n-P^m)u_m(t)\|_\alpha \\ &\qquad + \|A^{-1}\|\|(P^n-P^m)u_m(h(u_m(t),t))\|_\alpha].\end{aligned}$$

Also,

$$\begin{aligned}\|(P^n-P^m)u_m(t)\|_\alpha &\le \|A^{\alpha-\vartheta}(P^n-P^m)A^\vartheta u_m(t)\| \\ &\le \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m(t)\|. \qquad (4.1)\end{aligned}$$

Thus, we have

$$\|f_n(t,u_n(t),u_n[h(u_n(t),t)]) - f_m(t,u_m(t),u_m[h(u_m(t),t)])\|$$

$$\leq L_f(2+LL_h)\|u_n(t)-u_m(t)\|_\alpha + L_f\left[\frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m(t)\| \right.$$
$$\left. + \frac{\|A^{-1}\|}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m(h(u_m(t),t))\|\right].$$

Similarly,

$$\begin{aligned}
&\|A^\beta g_n(t,u_n(t)) - A^\beta g_m(t,u_m(t))\| \\
&\quad \leq \|A^\beta g_n(t,u_n(t)) - A^\beta g_n(t,u_m(t))\| \\
&\qquad + \|A^\beta g_n(t,u_m(t)) - A^\beta g_m(t,u_m(t))\| \\
&\quad \leq L_g\|A^{-1}\|\left[\|u_n(t)-u_m(t)\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m(t)\|\right].
\end{aligned}$$

Now, for $0 < t_0' < t_0$, we may write

$$\begin{aligned}
\|u_n(t)-u_m(t)\|_\alpha \leq & \int_0^\infty \xi_\eta(\theta)\|S(t^\eta\theta)A^\alpha(g_n(0,u_0)-g_m(0,u_0))\|\,d\theta \\
& + \|A^{\alpha-\beta}\|\,\|A^\beta g_n(t,u_n(t)) - A^\beta g_m(t,u_m(t))\| \\
& + \eta\left(\int_0^{t_0'} + \int_{t_0'}^t\right)\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1} \\
& \|A^{1+\alpha-\beta}S((t-s)^\eta\theta))\| \\
& \times \|A^\beta g_n(s,u_n(s)) - A^\beta g_m(s,u_m(s))\|\,d\theta\,ds \\
& + \eta\left(\int_0^{t_0'} + \int_{t_0'}^t\right)\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1} \\
& \|A^\alpha S((t-s)^\eta\theta)\| \\
& \times \|f_n(s,u_n(s),u_n(h(u_n(s),s))) \\
& - f_m(s,u_m(s),u_m(h(u_m(s),s)))\| d\theta ds.
\end{aligned}$$

We estimate the first term as

$$\begin{aligned}
&\int_0^\infty \xi_\eta(\theta)\|S(t^\eta\theta)A^\alpha(g_n(0,u_0)-g_m(0,u_0))\|\,d\theta \\
&\quad \leq \int_0^\infty \xi_\eta(\theta)M\|A^{\alpha-\beta}\|\,\|A^\beta g(0,P^n u_0) - A^\beta g(0,P^m u_0)\|\,d\theta \\
&\quad \leq M\|A^{\alpha-\beta-1}\|L_g\|(P^n-P^m)A^\alpha u_0\|\int_0^\infty \xi_\eta(\theta)\,d\theta
\end{aligned}$$

$$\leq M\|A^{\alpha-\beta-1}\|L_g\|(P^n - P^m)A^{\alpha}u_0\|$$
$$\leq \frac{1}{\lambda_m^{\vartheta-\alpha}} M\|A^{\alpha-\beta-1}\|L_g\| \, \|A^{\vartheta}u_0\|.$$

The first and third integrals are estimated as

$$\eta \int_0^{t_0'} \int_0^{\infty} \theta\xi_\eta(\theta)(t-s)^{\eta-1}\|A^{1+\alpha-\beta}S((t-s)^\eta\theta)\|$$
$$\|A^\beta g_n(s,u_n(s)) - A^\beta g_m(s,u_m(s))\| \, d\theta\, ds$$
$$\leq \frac{2C_{1+\alpha-\beta}N_1}{\beta-\alpha}(t^{\eta(\beta-\alpha)} - (t-t_0')^{\eta(\beta-\alpha)})$$
$$\leq \frac{2C_{1+\alpha-\beta}N_1}{\beta-\alpha}(t-\delta_1 t_0')^{\eta(\beta-\alpha)-1}t_0', \quad 0<\delta_1<1,$$
$$\leq \frac{2C_{1+\alpha-\beta}N_1}{\beta-\alpha}(t_0-t_0')^{\eta(\beta-\alpha)-1}t'_0. \tag{4.2}$$

$$\eta \int_0^{t_0'} \int_0^{\infty} \xi_\eta(\theta)(t-s)^{\eta-1}\|A^{\alpha}S((t-s)^\eta\theta)\|$$
$$\|f_n(s,u_n(s),u_n(h(u_n(s),s)))$$
$$- f_m(s,u_m(s),u_m(h(u_m(s),s)))\| \, d\theta\, ds$$
$$\leq \frac{2C_\alpha N}{1-\alpha}(t^{\eta(1-\alpha)} - (t-t_0')^{\eta(1-\alpha)})$$
$$\leq \frac{2C_\alpha N}{1-\alpha}(t-\delta_2 t_0')^{\eta(1-\alpha)-1}t_0', \quad 0<\delta_2<1,$$
$$\leq \frac{2C_\alpha N}{1-\alpha}(t_0-t_0')^{\eta(1-\alpha)-1}t'_0. \tag{4.3}$$

For the second and fourth integrals, we have

$$\eta \int_{t_0'}^{t} \int_0^{\infty} \theta\xi_\eta(\theta)(t-s)^{\eta-1}\|A^{1+\alpha-\beta}S((t-s)^\eta\theta)\|$$
$$\|A^\beta g_n(s,u_n(s)) - A^\beta g_m(s,u_m(s))\| \, d\theta\, ds$$
$$\leq \eta C_{1+\alpha-\beta}L_g\|A^{-1}\| \bigg(\frac{U_{t_0'}{T_0}^{\eta(\beta-\alpha)}}{\lambda_m^{\vartheta-\alpha}\eta(\beta-\alpha)} \tag{4.4}$$
$$+ \int_{t_0'}^{t} (t-s)^{\eta(\beta-\alpha)-1}\|u_n(s)-u_m(s)\|_\alpha ds \bigg)$$

$$\begin{aligned}
&\eta \int_{t_0'}^{t} \int_0^\infty \theta \xi_\eta(\theta)(t-s)^{\eta-1} \|A^\alpha S((t-s)^\eta \theta)\| \\
&\qquad \times f_n(s, u_n(s), u_n[h(u_n(s), s)]) \\
&\qquad\quad - f_m(s, u_m(s), u_m[h(u_m(s), s)])\, d\theta\, ds \\
&\qquad \leq \eta C_\alpha L_f \bigg((1+\|A^{-1}\|) \frac{U_{t_0'} {T_0}^{\eta(1-\alpha)}}{\lambda_m^{\vartheta-\alpha} \eta(1-\alpha)} \\
&\qquad\quad + (2+LL_h) \int_{t_0'}^{t} (t-s)^{\eta(1-\alpha)-1} \|u_n(s)-u_m(s)\|_\alpha ds \bigg). \qquad (4.5)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|u_n(t)-u_m(t)\|_\alpha \leq{}& \frac{1}{\lambda_m^{\vartheta-\alpha}} M \|A^{\alpha-\beta-1}\| L_g \| \|A^\vartheta u_0\| \\
&+ \|A^{\alpha-\beta-1}\| L_g \bigg(\|u_n(t)-u_m(t)\|_\alpha + \frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}} \bigg) \\
&+ 2 \bigg(\frac{C_{1+\alpha-\beta} N_1}{(t_0-t_0')^{1-\eta(\beta-\alpha)}} + \frac{C_\alpha N}{(t_0-t_0')^{1-\eta(1-\alpha)}} \bigg) t_0' \\
&+ C_{\alpha,\beta} \frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}} \\
&+ \int_{t_0'}^{t} \bigg(\frac{C_\alpha L_f (2+LL_h)}{(t-s)^{\eta(\alpha-1)+1}} + \frac{C_{1+\alpha-\beta} L_g \|A^{-1}\|}{(t-s)^{\eta(\alpha-\beta)+1}} \bigg) \\
&\|u_n(s)-u_m(s)\|_\alpha ds,
\end{aligned}$$

where

$$C_{\alpha,\beta} = (1+\|A^{-1}\|) C_\alpha L_f \frac{{T_0}^{\eta(1-\alpha)}}{1-\alpha} + C_{1+\alpha-\beta} L_g \|A^{-1}\| \frac{{T_0}^{\eta(\beta-\alpha)}}{\beta-\alpha}.$$

Since $\|A^{\alpha-\beta-1}\| L_g < 1$, we have

$$\begin{aligned}
\|u_n(t)-u_m(t)\|_\alpha \leq{}& \frac{1}{(1-\|A^{\alpha-\beta-1}\| L_g)} \\
&\times \bigg\{ \|A^{\alpha-\beta-1}\| L_g \bigg(M \frac{1}{\lambda_m^{\vartheta-\alpha}} \|A^\vartheta u_0\| + \frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}} \bigg) \\
&\quad + 2 \bigg(\frac{C_{1+\alpha-\beta} N_1}{(t_0-t_0')^{1-\eta(\beta-\alpha)}}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{C_\alpha N}{(t_0 - t_0')^{1-\eta(1-\alpha)}}\bigg) t_0' + C_{\alpha,\beta} \frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}} \bigg\} \\
&+ \int_{t_0'}^{t} \bigg(\frac{C_\alpha L_f (2 + L L_h)}{(t-s)^{\eta(\alpha-1)+1}} \\
&+ C_{1+\alpha-\beta} L_g \|A^{-1}\| (t-s)^{\eta(\alpha-\beta)+1} \bigg) \\
&\|u_n(s) - u_m(s)\|_\alpha ds.
\end{aligned}
$$

Applying Gronwall's inequality and estimating $t - s$ by T_0, we get the following:

$$
\begin{aligned}
\|u_n(t) - u_m(t)\|_\alpha \leq & \frac{1}{(1 - \|A^{\alpha-\beta-1}\| L_g)} \\
& \bigg\{ \bigg(\|A^{\alpha-\beta-1}\| L_g M \|A^\vartheta u_0\| \\
& + \|A^{\alpha-\beta-1}\| L_g U_{t_0'} + C_{\alpha,\beta} U_{t_0'} \bigg) \frac{1}{\lambda_m^{\vartheta-\alpha}} \\
& + 2 \bigg(\frac{C_{1+\alpha-\beta} N_1}{(t_0 - t_0')^{1-\eta(\beta-\alpha)}} + \frac{C_\alpha N}{(t_0 - t_0')^{1-\eta(1-\alpha)}} \bigg) t_0' \bigg\} C.
\end{aligned}
$$

Letting $m \to \infty$ and taking the supremum over $[t_0, T_0]$, we obtain

$$
\begin{aligned}
& \|u_n - u_m\|_{T_0,\alpha} \\
& \leq \frac{2}{(1 - \|A^{\alpha-\beta-1}\| L_g)} \bigg(\frac{C_{1+\alpha-\beta} N_1}{(t_0 - t_0')^{1-\eta(\beta-\alpha)}} + \frac{C_\alpha N}{(t_0 - t_0')^{1-\eta(1-\alpha)}} \bigg) t_0' C.
\end{aligned}
$$

As t_0' is arbitrary, the right hand side may be made as small as desired by taking t_0' sufficiently small. This completes the proof of Theorem 4.1. □

Similarly, we can prove the following corollary.

Corollary 4.2. *If $u_0 \in D(A)$, then*

$$\|u_n - u_m\|_{T_0,\alpha} \longrightarrow 0, \quad \text{as } m, n \to \infty,$$

i.e., u_n is a Cauchy sequence in $\mathcal{W}$ on $(0, T_0]$.

With the help of Theorems 3.1 and 4.1, we have the following result for the convergence of solutions to each of the approximate integral equations.

Theorem 4.3. *Let us suppose that assumptions* (H1)–(H4) *are satisfied, and let* $u_0 \in D(A^\alpha)$ *or* $D(A)$*. Then there exists a unique function* $u_n \in \mathcal{W}$*,*

$$\begin{aligned} u_n(t) = &\int_0^\infty \xi_\eta(\theta) S(t^\eta \theta)[u(0) + g_n(0, u_0)]\, d\theta - g_n(t, u_n(t)) \\ &+ \eta \int_0^t \int_0^\infty \theta \xi_\eta(\theta)(t-s)^{\eta-1} A S((t-s)^\eta \theta) g_n(s, u_n(s))\, d\theta\, ds \\ &+ \eta \int_0^t \int_0^\infty \theta \xi_\eta(\theta)(t-s)^{\eta-1} S((t-s)^\eta \theta) \\ &\times f_n(s, u_n(s), u_n(h_n(u_n(s), s)))\, d\theta\, ds, \quad t \in (0, T_0] \end{aligned}$$

and $u \in \mathcal{W}$*,*

$$\begin{aligned} u(t) = &\int_0^\infty \xi_\eta(\theta) S(t^\eta \theta)[u(0) + g(0, u_0)]\, d\theta - g(t, u(t)) \\ &+ \eta \int_0^t \int_0^\infty \theta \xi_\eta(\theta)(t-s)^{\eta-1} A S((t-s)^\eta \theta) g(s, u(s))\, d\theta\, ds \\ &+ \eta \int_0^t \int_0^\infty \theta \xi_\eta(\theta)(t-s)^{\eta-1} S((t-s)^\eta \theta) \\ &\times f(s, u(s), u(h(u(s), s)))\, d\theta\, ds, \quad t \in (0, T_0], \end{aligned}$$

such that $u_n \to u$ *as* $n \to \infty$ *in* $\mathcal{W}$ *and* u *satisfies* (3.9) *on* $(0, T_0]$.

5. Faedo-Galerkin approximations. In this section, we will study the Faedo-Galerkin approximation solution of (1.1) and prove the convergence result for such an approximation.

We have proved a unique solution $u \in \mathcal{W}$ of the integral equation:

$$\begin{aligned} u(t) = &\int_0^\infty \xi_\eta(\theta) S(t^\eta \theta)[u(0) + g(0, u_0)]\, d\theta - g_n(t, u(t)) \\ &+ \eta \int_0^t \int_0^\infty \theta \xi_\eta(\theta)(t-s)^{\eta-1} A S((t-s)^\eta \theta) g(s, u(s))\, d\theta\, ds \end{aligned}$$

$$+\eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}S((t-s)^\eta\theta)$$

$$f(s,u(s),u(h(u(s),s)))\,d\theta\,ds,\quad t\in[0,T_0]. \tag{5.1}$$

Then it has the representation

$$u(t)=\sum_{i=0}^{\infty}\alpha_i(t)\phi_i,\qquad \alpha_i(t)=(u(t),\phi_i),\quad i=0,1,\dots; \tag{5.2}$$

where the ϕ_i's are defined in (H1).

Also, we have a unique solution $u_n\in\mathcal{W}$ of the approximate integral equation

$$u_n(t)=\int_0^\infty \xi_\eta(\theta)S(t^\eta\theta)[u(0)+g_n(0,u_0)]\,d\theta-g_n(t,u_n(t))$$

$$+\eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}AS((t-s)^\eta\theta)g_n(s,u_n(s))\,d\theta\,ds$$

$$+\eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}S((t-s)^\eta\theta)$$

$$f_n(s,u_n(s),u_n(h(u_n(s),s)))\,d\theta\,ds,\quad t\in[0,T_0]. \tag{5.3}$$

Let $P^n u_n(t)=\widehat{u}_n(t)$ be the orthogonal projection of (5.3) on the first n elements of $\{\phi_i\}$ satisfying the following equation:

$$\widehat{u}_n(t)=\int_0^\infty \xi_\eta(\theta)S(t^\eta\theta)P^n[u(0)+g_n(0,u_0)]\,d\theta-P^n g_n(t,u_n(t))$$

$$+\eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}AS((t-s)^\eta\theta)P^n g_n(s,u_n(s))\,d\theta\,ds$$

$$+\eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}S((t-s)^\eta\theta)P^n$$

$$f_n(s,u_n(s),u_n(h(u_n(s),s)))\,d\theta\,ds,\quad t\in[0,T_0]. \tag{5.4}$$

Using (3.10) and (3.11) in (5.4), we get

$$\widehat{u}_n(t)=\int_0^\infty \xi_\eta(\theta)S(t^\eta\theta)P^n[u(0)+g_n(0,u_0)]\,d\theta-P^n g(t,\widehat{u}_n(t))$$

$$+\eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}AS((t-s)^\eta\theta)P^n g(s,\widehat{u}_n(s))\,d\theta\,ds$$
$$+\eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}S((t-s)^\eta\theta)P^n$$

(5.5)

$$f(s,\widehat{u}_n(s),\hat{u}_n(h(\widehat{u}_n(s),s)))\,d\theta\,ds,\quad t\in[0,T_0].$$

The solution $\widehat{u}_n$ of (5.5) has the following representation

$$\widehat{u}_n(t)=\sum_{i=0}^n\alpha_i^n(t)\phi_i,\qquad \alpha_i^n(t)=(\hat{u}_n(t),\phi_i),\quad i=0,1,\ldots; \tag{5.6}$$

Then we get a system of equations from (5.4) and (5.6)

$$\frac{d^\beta}{dt^\beta}[\alpha_i^n(t)+H_i^n(t,\alpha_0^n,\alpha_1^n,\cdots,\alpha_n^n)]+\lambda_i\alpha_i^n(t)$$
$$=F_i^n(t,\alpha_0^n,\alpha_1^n,\cdots,\alpha_n^n,\tau_0^n,\tau_1^n,\cdots,\tau_n^n)$$
$$\alpha_i^n(0)=u_i, \tag{5.7}$$

where

$$F_i^n=\left(f(t,\sum_{i=0}^n\alpha_i^n\phi_i,\sum_{i=0}^n\tau_i^n\phi_i),\phi_i\right),$$
$$H_i^n=\left(g(t,\sum_{i=0}^n\alpha_i^n\phi_i),\phi_i\right),$$
$$\tau_i^n=\alpha_i^n(h(\alpha_0^n,\alpha_1^n,\ldots,\alpha_n^n,t))$$

and $u_i=(u_0,\phi_i)$ for $i=1,2,\ldots,n$. Convergence of $\alpha_i^n(t)\to\alpha_i(t)$ follows from the following theorem and the fact that

$$A^\alpha[u(t)-\widehat{u}_n(t)]=A^\alpha\left[\sum_{i=0}^\infty(\alpha_i(t)-\alpha_i^n(t))\phi_i\right]$$
$$=\sum_{i=0}^\infty\lambda_i^\alpha(\alpha_i(t)-\alpha_i^n(t))\phi_i.$$

Thus, we have

$$\|A^{\alpha}[u(t) - \widehat{u}_n(t)]\|^2 \geq \sum_{i=0}^{n} \lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2.$$

Theorem 5.1. *Let us suppose that propositions* (H1)–(H4) *are satisfied. Then we have the following*:

(a) *If* $u_0 \in D(A^{\alpha})$, *then for any* $0 < t_0 \leq T_0$,

$$\sup_{t_0 \leq t \leq T_0} \left[\sum_{i=0}^{n} \lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2 \right] \longrightarrow 0, \quad \text{as } n \to \infty.$$

(b) *If* $u_0 \in D(A)$, *then*

$$\sup_{0 \leq t \leq T_0} \left[\sum_{i=0}^{n} \lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2 \right] \longrightarrow 0, \quad \text{as } n \to \infty.$$

As a consequence of Theorems 3.1 and 4.1, we have the following result.

Proposition 5.2. *Let us suppose that assumptions* (H1)–(H4) *are satisfied. Then we have the following*:

(a) *If* $u_0 \in D(A^{\alpha})$, *then for any* $0 < t_0 \leq T_0$,

$$\|\widehat{u}_n - \widehat{u}_m\|_{T,\alpha} \longrightarrow 0, \quad \text{as } m, n \to \infty,$$

i.e., $\widehat{u}_n$ *is a cauchy sequence in* $\mathcal{W}$ *on* $[t_0, T_0]$.

(b) *If* $u_0 \in D(A)$, *then*

$$\|\widehat{u}_n - \widehat{u}_m\|_{T,\alpha} \longrightarrow 0, \quad \text{as } m, n \to \infty,$$

i.e., $\widehat{u}_n$ *is a cauchy sequence in* $\mathcal{W}$ *on* $[0, T_0]$.

Proof. Letting $n \geq m$ and $0 \leq \alpha < \vartheta$, we have

$$\begin{aligned} \|\widehat{u}_n(t) - \widehat{u}_m(t)\|_{\alpha} &= \|P^n u_n(t) - P^m u_m(t)\|_{\alpha} \\ &\leq \|P^n[u_n(t) - u_m(t)]\|_{\alpha} + \|(P^n - P^m)u_m\|_{\alpha} \\ &\leq \|u_n(t) - u_m(t)\|_{\alpha} + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^{\vartheta} u_m\|. \end{aligned}$$

If $u_0 \in D(A^\alpha)$, then the result in (a) follows from Theorem 4.1. If $u_0 \in D(A)$, (b) follows from Corollary 4.2. □

For the convergence of $\widehat{u}_n(t) \to u(t)$, we have the following theorem.

Theorem 5.3. *Let assumptions* (H1)–(H4) *be satisfied, and let* $u_0 \in D(A^\alpha)$ *or* $D(A)$. *Then there exists a unique function* $\widehat{u}_n \in \mathcal{W}$ *satisfying*

$$\begin{aligned}\widehat{u}_n(t) = &\int_0^\infty \xi_\eta(\theta)S(t^\eta\theta)[u(0)+g_n(0,u_0)]\,d\theta - g_n(t,\hat{u}_n(t))\\ &+\eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}AS((t-s)^\eta\theta)g_n(s,\hat{u}_n(s))\,d\theta\,ds\\ &+\eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}S((t-s)^\eta\theta)\\ &f_n(s,\widehat{u}_n(s),\hat{u}_n(h_n(\widehat{u}_n(s),s)))\,d\theta\,ds, \quad t\in[0,T_0],\end{aligned}$$

and $u \in \mathcal{W}$

$$\begin{aligned}u(t) = &\int_0^\infty \xi_\eta(\theta)S(t^\eta\theta)[u(0)+g(0,u_0)]d\theta - g_n(t,\widehat{u}(t))\\ &+\eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}AS((t-s)^\eta\theta)g(s,u(s))\,d\theta\,ds\\ &+\eta\int_0^t\int_0^\infty \theta\xi_\eta(\theta)(t-s)^{\eta-1}S((t-s)^\eta\theta)\\ &f(s,u(s),u(h(u(s),s)))\,d\theta\,ds, \quad t\in[0,T_0],\end{aligned}$$

such that $\widehat{u}_n \to u$ *as* $n\to\infty$ *in* $\mathcal{W}$ *and* u *satisfies* (3.9) *on* $[0,T_0]$.

6. Example. We consider the following fractional order partial differential equation with a deviated argument:

(6.1)

$$\begin{cases}\partial_t^\eta[w(t,x)+\partial_x f_1(t,w(t,x))]-\partial_x^2[w(t,x)]\\ \quad = f_2(x,w(t,x))+f_3(t,x,w(t,x)), \quad x\in(0,1),\ t>0,\eta\in[0,1),\\ \quad w(t,0)=w(t,1)=0,\\ \quad w(0,x)=u_0, \quad x\in(0,1),\end{cases}$$

where

$$f_2(x,w(t,x)) = \int_0^x K(x,s)w(s,k(t)|w(s,t)|)\,ds.$$

The function $f_3 : \mathbf{R}_+ \times [0,1] \times \mathbf{R} \to \mathbf{R}$ is measurable in its second variable x, locally Hölder continuous in its first variable t, locally Lipschitz continuous in its third variable w and uniformly in x. Further, we assume that $k : \mathbf{R}_+ \to \mathbf{R}_+$ is locally Hölder continuous in t with $k(0) = 0$ and $K(\cdot,\cdot) \in C^1([0,1] \times [0,1]; \mathbf{R})$.

Let $X = L^2((0,1); \mathbf{R})$, $Au = d^2u/dx^2$, $D(A) = H^2(0,1) \cap H_0^1(0,1)$, $X_{1/2} = D((A)^{1/2}) = H_0^1(0,1)$ and $X_{-1/2} = (H_0^1(0,1))^* = H^{-1}(0,1) \equiv H^1(0,1)$.

For $x \in (0,1)$, we define the function $f : \mathbf{R}_+ \times X_{1/2} \times X_{-1/2} \to X$ by

$$f(t,u,\xi)(x) = f_2(x,\xi) + f_3(t,x,u), \tag{6.2}$$

where $f_2 : [0,1] \times X \to H_0^1(0,1)$ is given by

$$f_2(t,\xi) = \int_0^x K(x,y)\xi(y)\,dy, \tag{6.3}$$

and $f_3 : \mathbf{R} \times [0,1] \times H^2(0,1) \to H_0^1(0,1)$ satisfies the following

$$\|f_3(t,x,u)\| \le Q(x,t)(1 + \|u\|_{H^2(0,1)}), \tag{6.4}$$

with $Q(\cdot,t) \in X$ and Q is continuous in its second argument. Next we assume that the function $h : H_0^1(0,1) \times \mathbf{R}_+ \to \mathbf{R}_+$ is defined by

$$h(u(x,t),t) = k(t)|u(x,t)|. \tag{6.5}$$

For $u \in D(A)$ and $\lambda \in \mathbf{R}$ with $-Au = \lambda u$, we have

$$\langle -Au, u\rangle = \langle \lambda u, u\rangle$$

$$\|u'\|_{L^2} = \lambda\|u\|_{L^2}, \tag{6.6}$$

so we have $\lambda > 0$. The solution u of $-Au = \lambda u$ is

$$u(x) = D_1 \cos(\sqrt{\lambda}x) + D_2 \sin(\sqrt{\lambda}x). \tag{6.7}$$

Using the boundary condition, we get $D_1 = 0$ and $\lambda = \lambda_n = n^2\pi^2$ for $n \in \mathbf{N}$. Thus, for $n \in \mathbf{N}$, we have

$$u_n(x) = D_2 \sin(\sqrt{\lambda_n}x).$$

Also $\langle u_n, u_m \rangle = 0, m \neq n$ and $\langle u_n, u_n \rangle = 1$. So, for $u \in D(A)$, there exists a sequence α_n of real numbers such that $u(x) = \sum_{n\in\mathbf{N}} \alpha_n u_n(x)$ with $\sum_{n\in\mathbf{N}}(\alpha_n)^2 < \infty$ and $\sum_{n\in\mathbf{N}}(\alpha_n)^2(\lambda_n)^2 < \infty$.

The semigroup is given by

$$S(t)u = \sum_{n\in\mathbf{N}} \exp(-n^2 t)\langle u, u_m\rangle u_m.$$

The abstract formulation of (6.1) can be written as the following:

$$\frac{d^\eta}{dt^\eta}[u(t) + g(t, u(t))] + Au(t) = f(t, u(t), u[h(u(t), t)]), \quad t > 0,\ \eta[0, 1),$$

$$u(0) = u_0, \tag{6.8}$$

where $u(t) = w(t, \cdot)$ that is $u(t)(x) = w(t, x)$, $x \in (0, 1)$. The function $g : \mathbf{R}_+ \times X_{1/2} \to X$, such that $g(t, u(t))(x) = \partial_x f_1(t, w(t, x))$.

It's not difficult to prove that all the assumptions (H1)–(H4) are satisfied. For more details, see [**7**].

Acknowledgments. We highly appreciate the valuable comments and suggestions of the referees on our manuscript which helped to considerably improve the quality of the manuscript.

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Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, India, Pin- 208016
Email address: **prdipk@gmail.com**

Department of Mathematics, Indian Institute of Technology Roorkee, India, Pin- 247667
Email address: **dwij.iitk@gmail.com, dwijpfma@iitr.ernet.in**

Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, India, Pin- 208016
Email address: **dhiren@iitk.ac.in**