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AN INTEGRAL BOUNDARY VALUE PROBLEM FOR NONLINEAR DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER ON AN UNBOUNDED DOMAIN

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ABSTRACT. This paper investigates the existence of solutions for nonlinear fractional differential equations with integral boundary conditions on an unbounded domain. An example illustrating how the theory can be applied in practice is also included.

1. Introduction. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. These characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integer-order models. For more details and applications, we refer the reader to the books [**13, 26, 27, 34, 35**]. For some recent developments on the topic, see [**4, 8, 12, 16, 21, 24, 25, 36, 40, 42**] and the references therein.

Boundary value problems (BVPs) on infinite intervals arise naturally in the study of radially symmetric solutions of nonlinear elliptic equations and various physical phenomena [**2, 18**]. For BVPs of integer order on infinite intervals, there is a lot of work, see [**3, 15, 22, 28, 31**] and the references therein. For BVPs of fractional order on infinite intervals, some excellent results dealing with nonlinear fractional differential equations are shown in [**1, 11, 19, 29, 30, 38, 41, 43**]. In [**11**], by employing Schauder's fixed point theorem combined with the diagonalization method, Arara, Benchohra, Hamidi and Nieto successfully discussed the existence of the bounded solution of the following

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problem on infinite intervals:

$$^{C}D_{0+}^{\alpha}y(t) = f\big(t, y(t)\big) \quad \text{for } t \in [0, +\infty), \text{ with } y(0) = y_0,$$

where $1 < \alpha \leq 2$ and y is bounded on J, for a known $f \in C(J \times \mathbf{R}, \mathbf{R})$ and $y_0 \in \mathbf{R}$. Here, $^{C}D_{0+}^{\alpha}$ is a Caputo fractional derivative of order α. Zhao and Ge [**43**] used the Leray Schauder nonlinear alternative theorem to show the existence of positive solutions to the following fractional order differential equation:

$$\begin{gathered} D_{0+}^{\alpha}u(t) + f\big(t, u(t)\big) = 0 \quad \text{for } t \in [0, +\infty), \\ \text{with } u(0) = 0 \text{ and } \lim_{t \to +\infty} D_{0+}^{\alpha-1}u(t) = \beta u(\xi), \end{gathered}$$

where $1 < \alpha \leq 2$, $f \in C(J \times \mathbf{R}, [0, +\infty))$, $0 \leq \xi, \eta < \infty$ and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative.

Inspired by the papers mentioned above, we will discuss the existence of the solutions for nonlinear fractional differential equations with integral boundary conditions on an unbounded domain. Precisely, we consider the following problem

$$\begin{gathered} D^{\alpha}u(t) + f\big(t, u(t)\big) = 0 \quad \text{for } t \in J, \\ \text{with } u(0) = 0 \text{ and } D^{\alpha-1}u(+\infty) = \lambda \int_0^{\tau} u(t)dt, \end{gathered} \tag{1.1}$$

where $1 < \alpha \leq 2$, $f \in C(J \times \mathbf{R}, \mathbf{R})$, $0 \leq \lambda$, $\tau < \infty$ and D^{α} denotes the Riemann-Liouville fractional derivative of order α. Our results assume the following conditions.

(H_1) $\Gamma(\alpha + 1) > \lambda\tau^{\alpha}$.

(H_2) There exists a nonnegative measurable function b defined on $J := [0, +\infty)$ such that

$$|f(t, x) - f(t, y)| \leq b(t)\,|x - y| \quad \text{for } t \in J,\ x, y \in \mathbf{R},$$

and

$$\alpha \int_0^{+\infty} (1 + t^{\alpha-1})\, b(t)\, dt \leq \Gamma(\alpha + 1) - \lambda\tau^{\alpha}.$$

(H_3) The function $a(t) := |f(t, 0)|$ $(t \in J)$ satisfies

$$\int_0^{+\infty} a(t)\, dt < +\infty.$$

Integral boundary value problems are an important and significant branch of the theory of BVPs. Recently, integral boundary value problems of fractional differential equations on *finite* intervals have been gaining more importance and attention; see, for instance, [**5–7, 9, 10, 14, 17, 20, 23, 32, 37**]. To the authors' knowledge, no one has studied the integral boundary value problem of fractional differential equations (1.1) on *infinite* intervals, and we will fill this gap in the literature. The purpose of this paper is to improve and generalize the results mentioned to some degree.

Remark 1.1. Problem (1.1) can be considered as a fractional oscillation model describing damped oscillations with viscoelastic intrinsic damping of the oscillator,

$$D^{\alpha}u(t) + b(t)u(t) = g(t),$$

with boundary conditions

$$u(0) = 0 \quad \text{and} \quad D^{\alpha-1}u(+\infty) = \lambda \int_0^{\tau} u(t)\,dt,$$

where the integral boundary condition can be regraded as a continuous distribution of the values of the unknown function on an arbitrary finite segment of the interval.

For the forthcoming analysis, we define the space

$$X = \Big\{u \in C(J) : \sup_{t\in J} |u(t)|/(1+t^{\alpha-1}) < +\infty\Big\},$$

equipped with the norm

$$\|u\|_X = \sup_{t\in J} \frac{|u(t)|}{1+t^{\alpha-1}}.$$

Obviously, X is a Banach space.

2. Preliminaries. For the convenience of the reader, in this section we first present some useful definitions and lemmas.

Definition 2.1 [**26**]. The Riemann-Liouville fractional derivative of order δ for a continuous function f is defined by

$$D^\delta f(t) = \frac{1}{\Gamma(n-\delta)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\delta-1} f(s)\, ds, \quad n = [\delta]+1,$$

provided the right hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 [**26**]. The Riemann-Liouville fractional integral of order δ for a function f is defined as

$$I^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s)\, ds, \quad \delta > 0,$$

provided that this integral exists.

Lemma 2.1 [**33**]. *Let $U \subset X$ be a bounded set. Then U is relatively compact in X if the following conditions hold*:

(i) *for any $u \in U$ the function $u(t)/(1+t^{\alpha-1})$ is equicontinuous on any compact subinterval of J.*

(ii) *For any $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 0$ such that*

$$\left| \frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}} \right| < \varepsilon \quad \textit{for any } t_1, t_2 \geq T \textit{ and } u \in U.$$

Lemma 2.2 [**26**]. *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) - C_1 t^{\alpha-1} - C_2 t^{\alpha-2} - \cdots - C_N t^{\alpha-N},$$

for some $C_1, C_2, \ldots, C_N \in \mathbf{R}$ with $N = [\alpha]+1$.

Remark 2.1. For $\alpha \geq 0$ and $\beta > -1$, we have

$$D^\alpha \left[\frac{t^\beta}{\Gamma(1+\beta)}\right] = \frac{t^{(\beta-\alpha)}}{\Gamma(1+\beta-\alpha)} \quad \text{and} \quad I^\alpha \left[\frac{t^\beta}{\Gamma(1+\beta)}\right] = \frac{t^{(\beta+\alpha)}}{\Gamma(1+\beta+\alpha)}.$$

First, we calculate an explicit expression for the Green's function related to the associated linear problem.

Lemma 2.3. *Let* $h \in C(J)$ *with* $\int_0^\infty h(s)\,ds < \infty$*. If* $1 < \alpha \leq 2$ *and* $\Gamma(\alpha+1) \neq \lambda\tau^\alpha$*, then the fractional boundary value problem*

$$D^\alpha u(t) + h(t) = 0 \quad \text{for } t \in J, \tag{2.1}$$

with $u(0) = 0$ *and*

$$D^{\alpha-1}u(+\infty) = \lambda \int_0^\tau u(t)\,dt$$

has a unique solution

$$u(t) = \int_0^{+\infty} G(t,s)h(s)\,ds,$$

where

$$G(t,s) = \begin{cases} \frac{-[\Gamma(\alpha+1)-\lambda\tau^\alpha](t-s)^{\alpha-1}+[\Gamma(\alpha+1)-\lambda(\tau-s)^\alpha]t^{\alpha-1}}{\Gamma(\alpha)[\Gamma(\alpha+1)-\lambda\tau^\alpha]} & s \leq t,\ s \leq \tau, \\ \frac{[\Gamma(\alpha+1)-\lambda(\tau-s)^\alpha]t^{\alpha-1}}{\Gamma(\alpha)[\Gamma(\alpha+1)-\lambda\tau^\alpha]} & 0 \leq t \leq s \leq \tau, \\ \frac{-[\Gamma(\alpha+1)-\lambda\tau^\alpha](t-s)^{\alpha-1}+\Gamma(\alpha+1)t^{\alpha-1}}{\Gamma(\alpha)[\Gamma(\alpha+1)-\lambda\tau^\alpha]} & 0 \leq \tau \leq s \leq t, \\ \frac{\alpha t^{\alpha-1}}{\Gamma(\alpha+1)-\lambda\tau^\alpha} & s \geq t,\ s \geq \tau. \end{cases} \tag{2.2}$$

Proof. By Lemma 2.2, we can reduce (2.1) to the integral equation

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds + c_1t^{\alpha-1} + c_2t^{\alpha-2}, \tag{2.3}$$

with constants c_1, $c_2 \in \mathbf{R}$.

Firstly, the condition $u(0) = 0$ implies that $c_2 = 0$.

Secondly, from (2.3) and the condition $D^{\alpha-1}u(+\infty) = \lambda\int_0^\tau u(t)\,dt$, we have

$$\begin{aligned} D^{\alpha-1}u(\infty) &= -\int_0^\infty h(s)\,ds + c_1\Gamma(\alpha) = \lambda\int_0^\tau u(t)\,dt \\ &= -\frac{\lambda}{\Gamma(\alpha+1)}\int_0^\tau (\tau-s)^\alpha h(s)\,ds + \frac{\lambda c_1\tau^\alpha}{\alpha}. \end{aligned} \tag{2.4}$$

As a consequence, we deduce that

$$c_1 = \frac{\alpha}{\Gamma(\alpha+1)-\lambda\tau^\alpha}\bigg(\int_0^\infty h(s)\,ds - \frac{\lambda}{\Gamma(\alpha+1)}\int_0^\tau (\tau-s)^\alpha h(s)\,ds\bigg).$$

Substituting c_1 and c_2 into (2.3), we have

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds \\ &\quad + \frac{\alpha t^{\alpha-1}}{\Gamma(\alpha+1)-\lambda\tau^\alpha}\int_0^\infty h(s)\,ds \\ &\quad - \frac{\alpha\lambda t^{\alpha-1}}{\Gamma(\alpha+1)-\lambda\tau^\alpha}\int_0^\tau \frac{(\tau-s)^\alpha}{\Gamma(\alpha+1)}h(s)\,ds \\ &= \int_0^\infty G(t,s)h(s)\,ds, \end{aligned} \tag{2.5}$$

where $G(t,s)$ is defined by (2.2). □

Lemma 2.4. *If* (H_1) *holds, then the Green's function* $G(t,s)$ *satisfies* $G(t,s)\ge 0$ *and*

$$\frac{G(t,s)}{1+t^{\alpha-1}} \le \frac{\alpha}{\Gamma(\alpha+1)-\lambda\tau^\alpha}. \tag{2.6}$$

Proof. We see using (H_1) that $G(t,s)\ge 0$.

If $s\le t$, $s\le\tau$ and $0\le\tau\le s\le t$, then

$$\frac{G(t,s)}{1+t^{\alpha-1}} \le \frac{\alpha\Gamma(\alpha)t^{\alpha-1}}{\Gamma(\alpha)(\Gamma(\alpha+1)-\lambda\tau^\alpha)(1+t^{\alpha-1})} \le \frac{\alpha}{\Gamma(\alpha+1)-\lambda\tau^\alpha}.$$

For $s\ge t$, $s\ge\tau$ and $0\le t\le s\le\tau$, the conclusion is obvious. □

3. Main results. This section is devoted to proving our existence result for the problem (1.1).

Theorem 3.1. *If* (H_1), (H_2) *and* (H_3) *hold, then the problem* (1.1) *has at least one solution.*

Proof. First, notice that from condition (H_2) we have that

$$|f(t,x)| \le a(t) + b(t)\,|x|, \quad t \in J,\ x \in \mathbf{R}.$$

Now define the operator A by

$$(Au)(t) = \int_0^\infty G(t,s) f(s,u(s))\,ds, \tag{3.1}$$

so that the problem (1.1) has a solution u if and only if u solves the operator equation $u = Au$. Therefore, in what follows, we will set out to verify that the operator A has a fixed point.

To this end, we divide the proof into different steps:

Step 1. Choose a constant R such that

$$R \ge \frac{\alpha \int_0^\infty a(s)\,ds}{\Gamma(\alpha+1) - \lambda\tau^\alpha - \alpha\int_0^\infty b(s)(1+s^{\alpha-1})\,ds}$$

and let $U = \{u \in X : \|u\|_X \le R\}$. It is obvious that, if u is a continuous function on J, then $Au \in C(J)$, too. To show that $A(U) \subset U$, let $u \in U$ and $t \in J$. We have

$$\begin{aligned} \frac{|Au(t)|}{1+t^{\alpha-1}} &= \frac{1}{1+t^{\alpha-1}}\left|\int_0^\infty G(t,s) f(s,u(s))\,ds\right| \\ &\le \frac{\alpha}{\Gamma(\alpha+1)-\lambda\tau^\alpha}\int_0^\infty \big[a(s) + b(s)|u(s)|\big]\,ds \\ &\le \frac{\alpha}{\Gamma(\alpha+1)-\lambda\tau^\alpha}\int_0^\infty \big[a(s) + b(s)(1+s^{\alpha-1})\|u\|_X\big]\,ds \\ &\le R. \end{aligned} \tag{3.2}$$

Hence, $\|Au\|_X \le R$, and thus $A(U) \subset U$.

Step 2. $A : U \to U$ is continuous. If $u_n,\ u \in U$ $(n = 1, 2, \dots)$ such that $\|u_n - u\|_X \to 0$ as $n \to +\infty$, then, for all $t \in [0, \infty)$,

$$
\begin{aligned}
(3.3)\quad & \left|\frac{Au_n(t)}{1+t^{\alpha-1}} - \frac{Au(t)}{1+t^{\alpha-1}}\right| \\
&= \frac{1}{1+t^{\alpha-1}} \int_0^\infty G(t,s)\big|f(s,u_n(s)) - f(s,u(s))\big|\, ds \\
&\le \frac{\alpha}{\Gamma(\alpha+1) - \lambda\tau^\alpha} \int_0^\infty b(s)\,|u_n(s) - u(s)|\, ds \\
&\le \frac{\alpha}{\Gamma(\alpha+1) - \lambda\tau^\alpha} \int_0^\infty b(s)\,(1+s^{\alpha-1})\,\|u_n - u\|_X\, ds \\
&\le \|u_n - u\|_X,
\end{aligned}
$$

so we conclude that $\|Au_n - Au\|_X \to 0$ as $n \to +\infty$. Hence, A is continuous.

Step 3. Let V be a subset of U. Following Lemma 2.1, we verify that $A(V)$ is a relatively compact set in two steps.

First, let $I \subset J$ be a compact interval, t_1, $t_2 \in I$ and $t_1 < t_2$. Then for any $u \in V$, we have

(3.4)

$$
\begin{aligned}
\left|\frac{Au(t_2)}{1+t_2^{\alpha-1}} - \frac{Au(t_1)}{1+t_1^{\alpha-1}}\right| &= \left|\int_0^\infty \left(\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right) f(s,u(s))\, ds\right| \\
&\le \int_0^\infty \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right| \\
&\quad \times \big[a(s) + b(s)(1+s^{\alpha-1})\|u\|_X\big]\, ds.
\end{aligned}
$$

Since it is continuous on $J \times J$, we have that $G(t,s)/(1+t^{\alpha-1})$ is a uniformly continuous function on the compact set $I \times I$. Moreover, for $s \ge t$, we have that this function depends only on t, so consequently it is uniformly continuous on $I \times (J \backslash I)$. Therefore, we have that, for all $s \in J$ and $t_1, t_2 \in I$, the following property holds:

For all $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that, if $|t_1 - t_2| < \delta$, then

$$
\left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right| < \varepsilon.
$$

This property, together with (3.4) and the fact that

$$\int_0^\infty \big[a(s) + b(s)(1 + s^{\alpha-1})\, R\big]\, ds < \infty, \tag{3.5}$$

imply that $Au(t)/(1 + t^{\alpha-1})$ is equicontinuous on I.

Second, to verify condition (ii) in Lemma 2.1, we use the following property:

$$\lim_{t\to\infty} \frac{G(t,s)}{1+t^{\alpha-1}} = \frac{1}{\Gamma(\alpha+1) - \lambda\tau^\alpha} \times \begin{cases} \frac{\lambda(\tau^\alpha - (\tau-s)^\alpha)}{\Gamma(\alpha)} & 0 \le s \le \tau, \\ \frac{\lambda\tau^\alpha}{\Gamma(\alpha)} & \tau \le s. \end{cases} \tag{3.6}$$

From this property, it is not difficult to verify that, for any $\varepsilon > 0$ given, there exists a constant $T = T(\varepsilon) > 0$ such that

$$\left| \frac{G(t_1,s)}{1+t_1^{\alpha-1}} - \frac{G(t_2,s)}{1+t_2^{\alpha-1}} \right| < \varepsilon \quad \text{for any } t_1, t_2 \ge T \text{ and } s \in J.$$

Now, from (3.4) and (3.5), we have that the same property holds for $Au(t)/(1 + t^{\alpha-1})$, uniformly for $u \in V$. Hence, $A(V)$ is equiconvergent at ∞.

Consequently, Lemma 2.1 yields that $A(V)$ is relatively compact. Therefore, by Schauder's fixed point theorem, the operator A has a fixed point in U, that is, the problem (1.1) has at least one solution in X. $\square$

4. Example. Consider the following boundary value problem on an unbounded domain:

$$\begin{gathered} D^{3/2}u(t) + \frac{\sqrt[4]{t}\,\ln(1+|u(t)|)}{(1+\sqrt{t})(3+t)^2} + \left(\frac{\pi}{2} - \arctan t\right) e^{-t} = 0 \text{ for } t \in [0,+\infty), \\ u(0) = 0, \qquad D^{1/2}u(+\infty) = \lambda \int_0^\tau u(t)\, dt, \end{gathered} \tag{4.1}$$

where $\alpha = 3/2$, $\lambda > 0$ and $\tau > 0$. By a simple computation, we have

$$|f(t,x) - f(t,y)| = \left| \frac{\sqrt[4]{t}\,[\ln(1+|x|) - \ln(1+|y|)]}{(1+\sqrt{t})(3+t)^2} \right| \le \frac{\sqrt[4]{t}\,|x-y|}{(1+t^{1/2})(3+t)^2}, \tag{4.2}$$

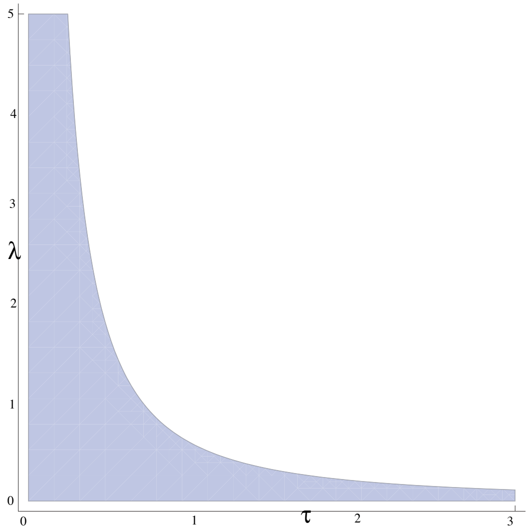

FIGURE 1. Set of $(\lambda, \tau) \in \mathbf{R}^+ \times \mathbf{R}^+$ satisfying (4.3).

for $t \in J$, $x, y \in \mathbf{R}$. Noting that

$$a(t) := |f(t,0)| = \left(\frac{\pi}{2} - \arctan t\right) e^{-t}, \qquad b(t) = \frac{\sqrt[4]{t}}{(1+\sqrt{t})(3+t)^2},$$

we have

$$\Gamma(\alpha+1) = \frac{3\sqrt{\pi}}{4},$$
$$\int_0^{+\infty} a(t)\,dt \le \int_0^{+\infty} \pi e^{-t}\,dt = \pi,$$
$$\alpha \int_0^{+\infty} (1+t^{\alpha-1}) b(t)\,dt = \frac{\sqrt[4]{3}\,\pi}{4\sqrt{2}}.$$

Therefore, if

$$\lambda\sqrt{\tau^3} < \frac{3\sqrt{\pi}}{4} - \frac{\sqrt[4]{3}\,\pi}{4\sqrt{2}} \approx 0.598445 \tag{4.3}$$

(for instance, if $\lambda = 1/2$ and $\tau = 1$, see Figure 1) then all conditions of Theorem 3.1 are satisfied. By Theorem 3.1, the BVP (4.1) has at least one solution.

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