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NONLOCAL INITIAL BOUNDARY VALUE PROBLEM FOR A FRACTIONAL INTEGRODIFFERENTIAL EQUATION IN A BANACH SPACE

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ABSTRACT. In this paper, we study the existence and uniqueness of solutions for fractional integrodifferential equations with nonlocal initial condition in a Banach space. The results are established by the application of the contraction mapping principle and the Krasnoselkii fixed point theorem. An application is also given.

1. Introduction. In this paper, we consider an initial boundary value problem (IBVP for short) for a fractional integrodifferential equation with a nonlocal initial condition, of the form

$$\begin{cases} {}^cD^q x(t) = \int_0^t k(t,s,x(s))\,ds \quad t \in I = [0,1], \\ x(0) = \int_0^1 g(s)x(s)\,ds, \end{cases} \tag{1.1}$$

where ${}^cD^q$ is the standard Caputo fractional derivative of order $0 < q < 1$, and $x : I \to E$ for a Banach space, E. We assume that $g \in L^1([0,1], R_+)$ with $g(t) \in [0,1)$, and k is a given E-valued function satisfying some conditions that will be specified later.

Fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, engineering, etc. In fact, fractional differential equations are considered as providing alternative models to nonlinear differential equations [**4**]. For more details on the geometric and physical interpretation of fractional derivatives of the Caputo type, see [**5**].

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In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Kilbas et al. [**9**], Lakshmikantham et al. [**10**], Miller and Ross [**13**], Samko et al. [**18**], Podlubny [**17**] and the papers in [**1–3, 8, 9, 11, 12, 14–16, 19**] and the references therein. In [**1**], Ahmad and Nieto obtain results for a nonlinear boundary value problem of fractional integro differential equations with integral boundary conditions. In [**2, 3**], Anguraj et al. proved the existence of solutions of a Cauchy problem for a semilinear integrodifferential equation with a nonlocal initial condition. In [**6**], the authors have discussed ω periodic solutions to fractional integrodifferential equations with infinite delay.

Recently, N'Guerekata [**15, 16**] studied the existence of solutions of fractional abstract differential equations with nonlocal initial conditions. In [**8**], Jaradat et al. discussed the mild solution for fractional semilinear initial value problems. In [**14**], Mophou et al. investigated existence results for some fractional differential equations with nonlocal initial conditions. In [**19**], Tidke studied global solutions to nonlinear mixed Volterra-Fredholm integrodifferential equations with nonlocal initial conditions. Lakshmikantham and Vatsala [**11**] initiated the basic theory of fractional differential equations. Lv et al. [**12**] proved the existence of solutions to fractional differential equations with nonlocal initial conditions in Banach spaces. Motivated by [**7, 20**], we study in this paper the existence of solutions to fractional integrodifferential equations with nonlocal initial conditions in Banach spaces by using fractional calculus and fixed point theorems.

2. Preliminaries. In this section, we introduce definitions and preliminary facts which are used throughout this paper. Let E be a real Banach space with zero element θ. Denote by $C = C([0,1],E)$ the Banach space of all continuous functions $x : [0,1] \to E$ with norm $\|x\|_c = \sup_{t\in[0,1]} \|x(t)\|$. Let $L^1([0,1],E)$ be the Banach space of measurable functions $x : [0,1] \to E$ which are Lebesgue integrable, equipped with the norm $\|x\|_{L^1} = \int_0^1 \|x(s)\|\, ds$. Let

$$\mu = \int_0^1 g(s)\, ds, \quad R^+ = (0,\infty), \quad R_+ = [0,\infty).$$

A function $x \in C([0,1],E)$ is called a solution of (1.1) if it satisfies (1.1).

Definition 2.1. A real function f is said to be in the space C_α, $\alpha \in R$, if there exists a real number $p > \alpha$ such that $f(t) = t^p g(t)$ for some $g \in C[0, \infty)$, and f is said to be in the space C_α^m if $f^{(m)} \in C_\alpha$, $m \in N$.

Definition 2.2. The fractional integral of the function $f \in L^1([a,b], R_+)$ of order $q \in R_+$ is defined by

$$I_a^q f(t) = \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\, ds,$$

where Γ is the Gamma function. When $a = 0$, we write $I^q f(t) = f(t) * \varphi_q(t)$, where $\varphi_q(t) = t^{q-1}/\Gamma(q)$ for $t > 0$ and $\varphi_q(t) = 0$ for $t \leq 0$. Note that $\varphi_q(t) \to \delta(t)$ as $q \to 0$, where δ is the delta function.

Definition 2.3. The Riemann-Liouville fractional integral of order $q > 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\, ds, \quad \text{for } q > 0 \text{ and } t > 0,$$

and in the case $q = 0$ we put $I^0 f(x) = f(x)$.

Definition 2.4. The Riemann-Liouville fractional derivative of order $q > 0$, of a function f, is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{q-n+1}}\, ds,$$

for $n - 1 < q < n$ and $n \in N$, where the function $f(t)$ has absolutely continuous derivatives up to order $n - 1$.

Definition 2.5. The Caputo derivative of fractional order q for a function $f(t)$ is defined by

$$({}^c D^q f)(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q-n+1}}\, ds,$$

for $n-1<q<n$ and $n=[q]+1$, where $[q]$ denotes the integer part of the real number q.

Remark 2.1. The Caputo derivative of a constant is equal to 0.

Lemma 2.1. *Let $q>0$. Then we have ${}^cD^q(I^q f(t)) = f(t)$.*

Lemma 2.2. *Let $q>0$ and $n=[q]+1$. Then*

$$I^q({}^cD^q f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}\, t^k.$$

Lemma 2.3. *If $Q(\tau) = \int_\tau^1 g(s)(s-\tau)^{q-1}\, ds$ for $\tau \in [0,1]$, and if $g \in L^1([0,1], R_+)$ satisfies $0 \le g(s) \le 1$ for $0 \le s \le 1$, then*

$$\frac{Q(\tau)}{\Gamma(q)} < e \quad \textit{and} \quad \frac{\int_0^t (t-s)^{q-1}\, ds}{\Gamma(q)} < e.$$

Proof. A direct computation shows

$$\begin{aligned} \frac{Q(\tau)}{\Gamma(q)} &= \frac{\int_\tau^1 g(s)(s-\tau)^{q-1} ds}{\int_0^\infty s^{q-1}e^{-s} ds} \le \frac{\int_\tau^1 (s-\tau)^{q-1} ds}{\int_0^\infty s^{q-1}e^{-s} ds} \\ &= \frac{\int_0^{1-\tau} s^{q-1} ds}{\int_0^\infty s^{q-1}e^{-s} ds} \le \frac{e\int_0^{1-\tau} s^{q-1}e^{-s} ds}{\int_0^\infty s^{q-1}e^{-s} ds} < e \end{aligned}$$

and

$$\frac{\int_0^t (t-s)^{q-1} ds}{\Gamma(q)} = \frac{\int_0^t s^{q-1} ds}{\int_0^\infty s^{q-1}e^{-s} ds} \le \frac{e\int_0^t s^{q-1}e^{-s} ds}{\int_0^\infty s^{q-1}e^{-s} ds} < e. \qquad \square$$

Theorem 2.4 (Krasnoselkii). *Let M be a closed convex and nonempty subset of a Banach space X. Let A and B be two operators such that*

1. *$Ax + By \in M$ whenever $x,\ y \in M$;*
2. *A is compact and continuous;*
3. *B is a contraction mapping.*

Then there exists a $z \in M$ such that $z = Az + Bz$.

3. Main results. Before stating and proving the main results, we introduce the notation

$$\Delta = \{(t, s) : 0 \le s \le t \le 1\},$$

and make the following hypotheses.

(H1) $k : \Delta \times E \to E$ is continuous, and there exists a constant $K_1 > 0$ such that

$$\|k(t, s, x_1) - k(t, s, x_2)\| \le K_1 \|x_1 - x_2\| \quad \text{for } x_1, x_2 \in E.$$

(H2) For any positive number r there exists an $h_r \in L^1(I)$ such that

$$\sup_{\|x\| \le r} \|k(t, s, x)\| \le h_r(t) \quad \text{for all } (t, s, x) \in I \times \Delta \times E.$$

Lemma 3.1. *If* (H1) *holds, then problem* (1.1) *is equivalent to the following integral equation*:

$$\begin{aligned} x(t) = {} & \frac{1}{(1-\mu)\Gamma(q)} \int_0^1 Q(\tau) \int_0^\tau k(\tau, \eta, x(\eta))\, d\eta\, d\tau \\ & + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s k(s, \tau, x(\tau))\, d\tau\, ds \end{aligned}$$

for $t \in [0, 1]$.

Proof. By Lemma 2.2 and (1.1), we have

$$x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s k(s, \tau, x(\tau))\, d\tau\, ds,$$

so

$$\begin{aligned}
x(0) &= \int_0^1 g(s)x(s)\,ds \\
&= \int_0^1 g(s)\left[x(0) + \frac{1}{\Gamma(q)}\int_0^s (s-\tau)^{q-1}\int_0^\tau k(\tau,\eta,x(\eta))\,d\eta\,d\tau\right] ds \\
&= \int_0^1 g(s)\,ds\,x(0) \\
&\quad + \frac{1}{\Gamma(q)}\int_0^1 g(s)\int_0^s (s-\tau)^{q-1}\int_0^\tau k(\tau,\eta,x(\eta))\,d\eta\,d\tau\,ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
x(0) &= \frac{1}{(1-\int_0^1 g(s)\,ds)\Gamma(q)} \\
&\quad \times \int_0^1 g(s)\int_0^s (s-\tau)^{q-1}\int_0^\tau k(\tau,\eta,x(\eta))\,d\eta\,d\tau\,ds \\
&= \frac{1}{(1-\mu)\Gamma(q)}\int_0^1 \\
&\quad \times \int_0^\tau k(\tau,\eta,x(\eta))\,d\eta\left[\int_\tau^1 g(s)(s-\tau)^{q-1}\,ds\right] d\tau \\
&= \frac{1}{(1-\mu)\Gamma(q)}\int_0^1 Q(\tau)\int_0^\tau k(\tau,\eta,x(\eta))\,d\eta\,d\tau,
\end{aligned}$$

and then

$$\begin{aligned}
x(t) &= \frac{1}{(1-\mu)\Gamma(q)}\int_0^1 Q(\tau)\int_0^\tau k(\tau,\eta,x(\eta))\,d\eta\,d\tau \\
&\quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\int_0^s k(s,\tau,x(\tau))\,d\tau\,ds.
\end{aligned}$$

Conversely, if x is a solution of (3.1), then for every $t \in [0,1]$, according to Lemma 2.1 and Remark 2.1, we have

$$\begin{aligned}
{}^cD^q x(t) &= {}^cD^q\bigg[\frac{1}{(1-\mu)\Gamma(q)}\int_0^1 Q(\tau)\int_0^\tau k(\tau,\eta,x(\eta))\,d\eta\,d\tau \\
&\qquad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\int_0^s k(s,\tau,x(\tau))d\,\tau\,ds\bigg] \\
&= {}^cD^q\bigg[\frac{1}{(1-\mu)\Gamma(q)}\int_0^1 Q(\tau)\int_0^\tau k(\tau,\eta,x(\eta))\,d\eta\,d\tau\bigg] \\
&\quad + {}^cD^q\bigg[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\int_0^s k(s,\tau,x(\tau))\,d\tau\,ds\bigg] \\
&= \theta + {}^cD^q\bigg(I^q\int_0^t k(t,s,x(s))\,ds\bigg) \\
&= \int_0^t k(t,s,x(s))\,ds.
\end{aligned}$$

It is obvious that $x(0) = \int_0^1 g(s)x(s)\,ds$. This completes the proof. ◻

Theorem 3.2. *If* (H1) *and* (H2) *hold with*

$$K_1 \le \frac{\Gamma(q+1)}{2T^q},$$

then (1.1) *has a unique solution.*

Proof. Define $F: C \to C$ by

$$\begin{aligned}
(Fx)(t) &= \frac{1}{(1-\mu)\Gamma(q)}\int_0^1 Q(\tau)\int_0^\tau k(\tau,\eta,x(\eta))\,d\eta\,d\tau \\
&\quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\int_0^s k(s,\tau,x(\tau))\,d\tau\,ds,
\end{aligned}$$

for $t \in [0,1]$, and recall that $\Delta = \{(s,t) : 0 \le \tau \le s \le t \le 1\}$. Choose

$$r \ge 2\bigg(\frac{eM}{(1-\mu)} + \frac{K_2T^q}{\Gamma(q+1)}\bigg),$$

and let $K_2 = \max\{\|k(s,\tau,0)\| : (s,\tau) \in \Delta\}$. To show that $FB_r \subset B_r$, where $B_r := \{x \in C : \|x\| \leq r\}$, let $x \in B_r$. Applying Lemma 2.3, we get

$$\begin{aligned}
\|Fx(t)\| &\leq \frac{1}{(1-\mu)\Gamma(q)} \int_0^1 Q(\tau) \int_0^\tau \|k(\tau,\eta,x(\eta))\|\, d\eta\, d\tau \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s \|k(s,\tau,x(\tau))\|\, d\tau\, ds \\
&\leq \frac{1}{(1-\mu)\Gamma(q)} \int_0^1 Q(\tau) \int_0^\tau \|k(\tau,\eta,x(\eta))\|\, d\eta\, d\tau \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s \big(\|k(s,\tau,x(\tau)) - k(s,\tau,0)\| \\
&\qquad\qquad + \|k(s,\tau,0)\|\big)\, d\tau\, ds \\
&\leq \frac{e}{(1-\mu)} \int_0^1 \int_0^\tau \|k(\tau,\eta,x(\eta))\|\, d\eta\, d\tau \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s (K_1\|x(\tau)\| + K_2)\, d\tau\, ds \\
&\leq \frac{eM}{(1-\mu)}\|x(\tau)\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}(K_1\|x(s)\| + K_2)\, ds \\
&\leq \frac{eM}{(1-\mu)}\|x(\tau)\| + (K_1 r + K_2)\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\, ds \\
&\leq \frac{erM}{(1-\mu)} + (K_1 r + K_2)\frac{T^q}{\Gamma(q+1)} \leq r,
\end{aligned}$$

by the choice of K_1, K_2 and r. Now we take x, $y \in C$. Then we get

$$\begin{aligned}
\|(Fx)(t) - (Fy)(t)\| &\leq \frac{eM}{(1-\mu)} \int_0^1 K_1\|x(\tau) - y(\tau)\|\, d\tau \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \\
&\quad\times \int_0^s \|k(s,\tau,x(\tau)) - k(s,\tau,y(\tau))\|\, d\tau\, ds \\
&\leq \Omega_{M,K_1,T,q}\|x-y\| \quad \text{where} \\
\Omega_{M,K_1,T,q} &:= \frac{eM}{(1-\mu)} + \frac{K_1 T^q}{\Gamma(q+1)},
\end{aligned}$$

and $\Omega_{M,K_1,T,q}$ depends only upon the parameters of the problem. The result follows by the contraction mapping principle, because $\Omega_{M,K_1,T,q} < 1$. □

Theorem 3.3. *If* (H1) *and* (H2) *hold with* $eM < 1 - \mu$, *then the* IBVP (1.1) *has at least one solution.*

Proof. Choose
$$r \geq \frac{eM}{(1-\mu)} + \frac{T^q \|h_r\|_{L^1}}{\Gamma(q+1)},$$
and consider $B_r := \{x \in C : \|x\| \leq r\}$. Now define on B_r the operators A and B by

$$(Ax)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s k(s, \tau, x(\tau))\, d\tau\, ds,$$

and

$$(Bx)(t) = \frac{1}{(1-\mu)\Gamma(q)} \int_0^1 Q(\tau) \int_0^\tau k(\tau, \eta, x(\eta))\, d\eta\, d\tau.$$

Let us observe that, if x, $y \in B_r$, then $Ax + By \in B_r$. Indeed, it is easy to check the inequality

$$\|Ax + By\| \leq \frac{eM}{(1-\mu)} + \frac{T^q \|h_r\|_{L^1}}{\Gamma(q+1)} \leq r.$$

We have to show that B is a contraction mapping. If x, $y \in B_r$, then

$$\begin{aligned} \|(Bx)(t) - (By)(t)\| &\leq \frac{eM}{(1-\mu)} \int_0^\tau K_1 \|x(\eta) - y(\eta)\|\, d\tau \\ &\leq \Omega_{M,K_1} \|x - y\|, \end{aligned}$$

where $\Omega_{M,K_1} := eMK_1/(1-\mu) < 1$ depends only upon the parameters of the problem, and hence B is contraction. Since x is continuous, so is Ax in view of (H1). Let us now note that A is uniformly bounded on B_r. This follows from the inequality

$$\|(Ax)(t)\| \leq \frac{T^q \|h_r\|_{L^1}}{\Gamma(q+1)}.$$

Now let us prove that $(Ax)(t)$ is equicontinuous. Let t_1, $t_2 \in I$ and $x \in B_r$. Using the fact that f is bounded on the compact set $I \times B_r$ (thus $\sup_{(s,\tau)\in I\times B_r} \|k(s,\tau,x(\tau))\| := c_0 < \infty$), we will get

$$\begin{aligned}
\|Ax(t_1) - Ax(t_2)\| &= \frac{1}{\Gamma(q)}\bigg\| \int_0^{t_1} (t_1 - s)^{q-1} \int_0^s k(s,\tau,x(\tau))\,d\tau\,ds \\
&\qquad - \int_0^{t_2} (t_2 - s)^{q-1} \int_0^s k(s,\tau,x(\tau))\,d\tau\,ds \bigg\| \\
&= \frac{1}{\Gamma(q)}\bigg\| \int_{t_2}^{t_1} (t_1 - s)^{q-1} \int_0^s k(s,\tau,x(\tau))\,d\tau\,ds \\
&\qquad - \int_0^{t_2} \big((t_2 - s)^{q-1} - (t_1 - s)^{q-1}\big) \\
&\qquad\qquad \times \int_0^s k(s,\tau,x(\tau))\,d\tau\,ds \bigg\| \\
&\le \frac{c_0}{\Gamma(q+1)} |2(t_1 - t_2)^q + t_2^q - t_1^q| \\
&\le \frac{2c_0}{\Gamma(q+1)} |t_1 - t_2|^q,
\end{aligned}$$

which does not depend upon x. So $A(B_r)$ is relatively compact. By the Arzela-Ascoli theorem, A is compact, and the result of the theorem follows by the Krasnoselkii theorem above. $\square$

4. Example. Consider the following fractional integrodifferential equation:

$$\begin{aligned}
{}^cD^q x(t) &= \int_0^t \frac{e^{-(t-s)}}{49} x(s)\,ds, \quad t \in I = [0,1], \\
x(0) &= \int_0^1 \frac{|x(s)|}{5 + |x(s)|}\,ds,
\end{aligned} \tag{4.1}$$

where $q = 1/5 \in (0,1]$. Set

$$k(t,s,x(s)) = \frac{e^{-(t-s)}}{49}\,x(s) \quad \text{and} \quad g(x) = \frac{|x|}{5 + |x|},$$

and let x, $y \in X$ and $t \in I$. Then we have

$$\|k(t,s,x) - k(t,s,y)\| \le \frac{1}{49}\,|x - y|,$$

and hence conditions (H1)–(H2) hold with $K_1 = 1/49$. Choose $M = 1/20$ and $\mu = 1/5$, so that

$$\frac{eM}{(1-\mu)} + \frac{K_1}{\Gamma(q+1)} < \frac{e}{16} + \frac{1}{49\Gamma(6/5)} = 0.19211964327988085 < 1,$$

and so, by Theorem 3.2, problem (4.1) has a unique solution on [0,1].

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