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SOLVABILITY OF A QUADRATIC HAMMERSTEIN INTEGRAL EQUATION IN THE CLASS OF FUNCTIONS HAVING LIMITS AT INFINITY

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ABSTRACT. The goal of the paper is to prove that a quadratic Hammerstein integral equation has solutions in the class of real functions defined, bounded, continuous on the real half-axis and having limits at infinity. The main tools used in our investigations are the technique of measures of noncompactness and the Darbo fixed point theorem. We provide an example illustrating our theory.

1. Introduction. The principal goal of the paper is to study the solvability of the quadratic Hammerstein integral equation

$$x(t) = p(t) + f(t, x(t)) \int_0^\infty g(t,\tau) h(\tau, x(\tau))\, d\tau, \quad t \geq 0. \tag{1.1}$$

We will conduct our investigations concerning the above equation in the space of real functions which are defined, bounded and continuous on the real half-axis $\mathbf{R}_+ = [0, \infty)$. Moreover, we look for solutions of equation (1.1) which have limits at infinity, i.e., we look for any solution $x = x(t)$ of equation (1.1) having a limit $\lim_{t\to\infty} x(t)$. Obviously that limit is finite since the solution $x(t)$ is a member of the above described function space.

In order to realize our goal we will apply the technique of measures of noncompactness and a fixed point theorem of Darbo type. More precisely, we use such a measure of noncompactness in the mentioned

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function space to show that any fixed point of an operator associated with equation (1.1), obtained with help of the Darbo fixed point theorem, is a solution of equation (1.1) having the desired property.

It is worthwhile noticing that the quadratic integral equations of Hammerstein type (1.1) are a generalization of the classical Hammerstein integral equation on bounded intervals of the form

$$x(t) = p(t) + \int_a^b g(t,\tau)h(\tau, x(\tau))\,d\tau, \quad t \in [a,b] \tag{1.2}$$

(cf. [**10, 15, 17, 18**]). Also equation (1.1) generalizes the Hammerstein integral equation on an unbounded intervals having the form

$$x(t) = p(t) + \int_0^\infty g(t,\tau)h(\tau, x(\tau))\,d\tau, \quad t \geq 0, \tag{1.3}$$

which was considered in several papers and monographs (cf. [**1, 10, 12, 15, 16, 18**], for example).

Let us point out that the above-mentioned integral equations (1.1), (1.2) and (1.3) were investigated from miscellaneous points of view in [**1, 10, 12, 14, 15, 18**] and there the authors considered the existence of solutions of equation (1.2) in the classical space consisting of real functions being continuous or Lebesgue L^p-integrable on an interval $[a,b]$. In the papers [**7, 8, 9**] the authors studied the existence of solutions of equations (1.1) and (1.3) which vanish at infinity or are asymptotically stable (see also [**16**]).

The novelty of the present paper is that we consider the existence of solutions of equation (1.1) in the class of functions being bounded and continuous on $\mathbf{R}_+$ and tending to limits at infinity. Thus, from this point of view the result of this paper is new and original.

Let us notice that a result obtained in the paper [**8**] is a particular case of that proved in this paper. Indeed, in [**8**] it was shown that equation (1.1) has a solution in the class of functions tending to zero at infinity.

It is also worthwhile mentioning that in our considerations we impose different and more general assumptions than those utilized in [**8**].

In the last section of the paper we give an example illustrating our main result and showing how the technique developed in this paper can be applied in a concrete situation.

2. Measures of noncompactness and Darbo fixed point theorem. Assume that E is an infinite-dimensional real Banach space with norm $||\cdot||$ and zero element θ. Denote by $B(x,r)$ the closed ball centered at x and with radius r. We write B_r to denote the ball $B(\theta,r)$. For a subset X of E we denote by $\overline{X}$ and $\operatorname{Conv} X$ the closure and the convex closure of X, respectively. Moreover, we use the classical symbols $X+Y$ and λX to denote the usual operations on sets. Further, let us denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets. Any function $\mu : \mathfrak{M}_E \to \mathbf{R}_+$ will be called the *set quantity* in E. Obviously, the concept of a set quantity plays a very important role in nonlinear analysis and its applications (cf. [**2, 3, 4, 6**], for example). If μ is a set quantity in E, then the family $\ker\mu$ defined by putting

$$\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$$

will be called the *kernel* of μ.

In this paper we will use set quantities of special type which are called measures of noncompactness. Let us recall the definition of this concept presented in [**6**].

Definition 2.1. A set quantity μ is said to be *a measure of noncompactness* in E if it satisfies the following conditions:

1^o The family $\ker\mu$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$.

2^o $X \subset Y$ implies $\mu(X) \le \mu(Y)$.

3^o $\mu(\overline{X}) = \mu(X)$.

4^o $\mu(\operatorname{Conv} X) = \mu(X)$.

5^o $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.

6^o If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1,2,\dots)$ and if $\lim_{x\to\infty}\mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Let us pay attention to fact that the intersection set X_∞ occurring in axiom 6^o is a member of the kernel $\ker\mu$ [**6**]. In what follows we will use following fixed point theorem. This theorem was formulated first by Darbo [**11**] (see [**2, 4, 6, 12**]).

Theorem 2.2. *Let Ω be a nonempty, bounded, closed and convex subset of the Banach space E, and let $Q : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that $\mu(QX) \le k\mu(X)$ for any nonempty subset X of Ω. Then Q has a fixed point in the set Ω.*

Remark 2.3. It is worthwhile mentioning that under the hypotheses of Theorem 2.2 the set $\operatorname{Fix} Q$ of all fixed points of the operator Q in the set Ω belongs to the kernel $\ker \mu$ of the measure of noncompactness μ [**6**]. This simple observation will be crucial in our further considerations.

3. Some set quantities in the space BC($\mathbf{R}_+$) and superposition operators. Now $BC(\mathbf{R}_+)$ consists of all real functions defined, bounded and continuous on the real half-line $\mathbf{R}_+ = [0, \infty)$ and equipped with the standard norm

$$||x|| = \sup\{|x(t)| : t \in \mathbf{R}_+\}.$$

For a function $x \in BC(\mathbf{R}_+)$ and for a fixed number $T > 0$, let us define the quantity

$$\beta_T(x) = \sup\{|x(t) - x(s)| : t \ge T, s \ge T\}.$$

Next, if X is a nonempty and bounded subset of $BC(\mathbf{R}_+)$ (i.e., $X \in \mathfrak{M}_{BC(\mathbf{R}_+)}$), let us put

$$\beta_T(X) = \sup\{\beta_T(x) : x \in X\}$$

and

$$\beta(X) = \lim_{T\to\infty} \beta_T(X). \tag{3.1}$$

Observe that the quantity $\beta_T(x)$ represents the oscillation of the function $x = x(t)$ on the interval $[T, \infty)$. Obviously $\beta(x) = 0$ if and only if there exists a finite limit $\lim_{t\to\infty} x(t)$. It is a consequence of well-known facts from classical mathematical analysis. Moreover, if $X \in \mathfrak{M}_{BC(\mathbf{R}_+)}$, then $X \in \ker \beta$ if and only if for any $\varepsilon > 0$ there exists $T > 0$ such that $|x(t) - x(s)| \le \varepsilon$ for all $t, s \ge T$ and for each $x \in X$. Then, passing in the inequality $|x(t) - x(s)| \leq \varepsilon$ with s to infinity, we get $|x(t) - x_g| \leq \varepsilon$ for any $t \geq T$, where x_g denotes the limit $\lim_{t\to\infty} x(t)$. Since x was chosen arbitrarily in X this allows us to deduce that for any $\varepsilon > 0$ there exists $T > 0$ such that $|x(t) - x_g| \leq \varepsilon$ for each $t \geq T$ and for each $x \in X$. This assertion means that all functions from the set X tend to their limits at infinity uniformly with respect to the set X or, equivalently, that all functions from X tend to their limits with the same rate.

In what follows we give a few facts concerning the properties of the so-called superposition operator related to the quantity β defined above. Assume that $f : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ is a given function. Then, we may assign to every function $x : \mathbf{R}_+ \to \mathbf{R}$ the function Fx defined by the equality

$$(Fx)(t) = f(t, x(t)), \quad t \geq 0.$$

The operator F defined in this way is called the *superposition operator* generated by the function f (cf. [**3**]).

We have the following result.

Theorem 3.1. *Assume that the function* $f(t, x) = f : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ *satisfies the following conditions*:

(i) *For any* $r > 0$ *the following equality holds*

$$\lim_{T\to\infty} \{\sup\{|f(t, x) - f(s, x)| : t, s \geq T, |x| \leq r\}\} = 0.$$

(ii) *There exists a nondecreasing function* $k(r) = k : \mathbf{R}_+ \to \mathbf{R}_+$ *such that*

$$|f(t, x) - f(t, y)| \leq k(r)|x - y|$$

for each $t \in \mathbf{R}_+$ *and for all* $x, y \in [-r, r]$.

Then, for any function x *from the space* $BC(\mathbf{R}_+)$ *such that* $x \in B_r$, *the following inequality holds*

$$\beta(Fx) \leq k(r)\beta(x),$$

where F *is the superposition operator generated by the function* $f(t, x)$.

Proof. Fix $r > 0$, $T > 0$ and take $t, s \geq T$. Then, for an arbitrary function $x \in B_r$ we have:

$$\begin{aligned}|(Fx)(t) - (Fx)(s)| &\leq |f(t, x(t)) - f(s, x(t))| + |f(s, x(t)) - f(s, x(s))| \\ &\leq |f(t, x(t)) - f(s, x(t)) + k(r)|x(t) - x(s)||.\end{aligned}$$

Hence we obtain

$$\beta_T(Fx) \leq \sup\{|f(t, x) - f(s, x)| : t, s \geq T, |x| \leq r\} + k(r)\beta_T(x).$$

Now, keeping in mind assumption (i), from the above estimate we derive

$$\beta(Fx) \leq k(r)\beta(x)$$

and the proof is complete. □

Corollary 3.2. *Let X be a nonempty subset of the ball B_r in the space $BC(\mathbf{R}_+)$. Then, under the assumptions of Theorem* 3.1 *the following inequality is satisfied*

$$\beta(FX) \leq k(r)\beta(X).$$

Remark 3.3. If we additionally assume that there exists a number $r_0 > 0$ such that $k(r_0) < 1$ ($k(r_0)$ is the Lipschitz constant appearing in assumption (ii)) then in view of Corollary 3.2 we can say that the superposition operator F generated by f strictly improves the *ultimate oscillation* of functions from an arbitrary nonempty set X, $X \subset B_{r_0}$.

Below we provide a few examples of functions $f(t, x)$ generating superposition operators corresponding to Theorem 3.1 and Corollary 3.2.

Example 3.4. Let $f(t, x)$ have the form

$$f(t, x) = a(t)b(x),$$

where $a : \mathbf{R}_+ \to \mathbf{R}$ is continuous and has a finite limit at infinity, while $b : \mathbf{R} \to \mathbf{R}$ is locally Lipschitzian, i.e., there exists a nondecreasing function $k : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$|b(x) - b(y)| \leq k(r)|x - y|$$

for all $x, y \in [-r, r]$. Then the function f satisfies the assumptions of Theorem 3.1.

Indeed, it is easily seen that $a = a(t)$ is bounded on $\mathbf{R}_+$. Thus, if we put $A = \sup\{|a(t)| : t \in \mathbf{R}_+\}$ then for each fixed $r > 0$ and for all $t \in \mathbf{R}_+$, $x, y \in [-r, r]$ we have

$$|f(t,x) - f(t,y)| \leq |a(t)||b(x) - b(y)| \leq Ak(r)|x - y|.$$

On the other hand, in view of the existence of a finite limit $\lim_{t\to\infty} a(t)$ and the estimate

$$|b(x)| \leq |b(x) - b(0)| + |b(0)| \leq k(|x|)|x| + |b(0)|$$

it is easy to check that the function f satisfies assumption (i) of Theorem 3.1.

Assuming the same hypotheses as above it is also easy to verify that the function $f(t,x) = a(t) + b(x)$ satisfies the assumptions of Theorem 3.1.

Example 3.5. Let $f : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ be $f(t,x) = \arctan tx$.

Observe that this function does not satisfy assumption (i) of Theorem 3.1. In fact, for a fixed $x > 0$, we have

$$\begin{aligned} \sup\{|f(t,x) - f(s,x)| : t, s \geq T\} &= \sup\{|\arctan tx - \arctan sx| : t, s \geq T\} \\ &= \frac{\pi}{2} - \arctan Tx, \end{aligned}$$

where $T > 0$. Hence, we get

$$\begin{aligned} &\sup\{|f(t,x) - f(s,x)| : t, s \geq T, |x| \leq r\} \\ &\qquad \geq \sup\{|f(t,x) - f(s,x)| : t, s \geq T, 0 \leq x \leq r\} \\ &\qquad = \frac{\pi}{2}, \end{aligned}$$

as we claimed.

Example 3.6. Let $f(t,x) = \arctan(t + x)$ for $t \in \mathbf{R}_+$ and $x \in \mathbf{R}$. Then, for fixed $T > 0$ and $x \in \mathbf{R}$ we get

$$\sup\{|f(t,x) - f(s,x)| : t, s \geq T\} = \frac{\pi}{2} - \arctan(T + x).$$

This yields

$$\lim_{T\to\infty}\{\sup\{|f(t,x)-f(s,x)| : t,s \geq T, |x| \leq r\}\} = 0$$

which means that the function f satisfies assumption (i) of Theorem 3.1. Obviously this function satisfies also assumption (ii) with $k(r) = 1$ for $r \geq 0$.

Thus, in view of Theorem 3.1 we deduce that for any function $x \in BC(\mathbf{R}_+)$ we have

$$\beta(Fx) \leq \beta(x),$$

where F is the superposition operator generated by f.

Now we recall the definition of the measure of noncompactness in the space $BC(\mathbf{R}_+)$. Take an arbitrary nonempty and bounded subset X of the space $BC(\mathbf{R}_+)$. Next, fix $\varepsilon > 0$, $T > 0$ and choose $x \in X$. Denote by $\omega^T(x,\varepsilon)$ the *modulus of continuity* of the function x on the interval $[0,T]$ defined by the formula

$$\omega^T(x,\varepsilon) = \sup\{|x(t)-x(s)| : t,s \in [0,T], |t-s| \leq \varepsilon\}.$$

Further, let us put

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : x \in X\}, \quad \omega_0^T(X) = \lim_{\varepsilon\to 0}\omega^T(X,\varepsilon)$$

and

$$\omega_0(X) = \lim_{T\to\infty}\omega_0^T(X).$$

Finally, we define

$$\mu(X) = \omega_0(X) + \beta(X), \tag{3.2}$$

where β is the set quantity defined by formula (3.1).

The set quantity μ defined above is a measure of noncompactness in the space $BC(\mathbf{R}_+)$ in the sense of Definition 2.1 [**6**]. The kernel $\ker\mu$ of this measure consists of all sets $X \in \mathfrak{M}_{BC(\mathbf{R}_+)}$ such that functions belonging to X are locally equicontinuous on $\mathbf{R}_+$ and tend to their limits at infinity with the same rate (cf. the characterization of the kernel $\ker\beta$ given at the beginning of this section). Other properties of the measure of noncompactness μ can be found in [**5, 6**].

4. Main result. In this section we consider the quadratic Hammerstein integral equation (1.1). We assume the following:

(i) $p \in BC(\mathbf{R}_+)$ and there exists the limit $\lim_{t\to\infty} p(t)$.

(ii) The function $f : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ is continuous and there exists a nondecreasing function $k_1 : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$|f(t,x) - f(t,y)| \leq k_1(r)|x-y|$$

for any $t \in \mathbf{R}_+$ and for all $x, y \in [-r, r]$, where $r \geq 0$ is an arbitrarily fixed number. Moreover, the function $t \to f(t,0)$ is a member of the space $BC(\mathbf{R}_+)$.

(iii) For any $r > 0$ the following equality holds

$$\lim_{T\to\infty} \{\sup\{|f(t,x) - f(s,x)| : t, s \geq T, |x| \leq r\}\} = 0.$$

(iv) The function $g : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$ is continuous.

(v) The function $h : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ is continuous, and there exist a continuous function $a : \mathbf{R}_+ \to \mathbf{R}_+$, a nondecreasing function $k_2 : \mathbf{R}_+ \to \mathbf{R}_+$ and a continuous and nondecreasing function $b : \mathbf{R}_+ \to \mathbf{R}_+$ with $b(0) = 0$ such that

$$|h(t,x) - h(t,y)| \leq a(t)k_2(r)b(|x-y|)$$

for $t \in \mathbf{R}_+$ and for $x, y \in [-r, r]$, where $r > 0$ is arbitrarily a fixed number.

(vi) The functions $\tau \to a(\tau)|g(t,\tau)|$, $\tau \to |g(t,\tau)h(\tau,0)|$ are integrable over $\mathbf{R}_+$ for any fixed $t \in \mathbf{R}_+$. Moreover, the functions $G_a, G_h : \mathbf{R}_+ \to \mathbf{R}_+$ defined by the formulas

$$G_a(t) = \int_0^\infty a(\tau)|g(t,\tau)|\, d\tau,$$
$$G_h(t) = \int_0^\infty |g(t,\tau)h(\tau,0)|\, d\tau$$

are bounded on $\mathbf{R}_+$.

(vii) The following equalities hold:

$$\lim_{T\to\infty} \left\{ \sup \left\{ \int_0^\infty |g(t,\tau) - g(s,\tau)| a(\tau)\, d\tau : t, s \geq T \right\} \right\} = 0,$$
$$\lim_{T\to\infty} \left\{ \sup \left\{ \int_0^\infty |g(t,\tau) - g(s,\tau)||h(\tau,0)|\, d\tau : t, s \geq T \right\} \right\} = 0.$$

(viii) The integrals

$$\int_0^\infty a(\tau)|g(t,\tau)|\,d\tau, \qquad \int_0^\infty |g(t,\tau)h(\tau,0)|\,d\tau$$

are uniformly convergent with respect to $t \in \mathbf{R}_+$ [**13**], i.e., the following equalities are satisfied

$$\lim_{T\to\infty}\left\{\sup\left\{\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau : t \in \mathbf{R}_+\right\}\right\} = 0,$$
$$\lim_{T\to\infty}\left\{\sup\left\{\int_T^\infty |g(t,\tau)h(\tau,0)|\,d\tau : t \in \mathbf{R}_+\right\}\right\} = 0.$$

In what follows let us observe that taking into account assumption (vi) we may define the finite constants $\overline{G}_a$ and $\overline{G}_h$ by putting

$$\overline{G}_a = \sup\{G_a(t) : t \in \mathbf{R}_+\},$$
$$\overline{G}_h = \sup\{G_h(t) : t \in \mathbf{R}_+\}.$$

Moreover, in view of assumption (ii) the constant $F = \sup\{|f(t,0)| : t \in \mathbf{R}_+\}$ is also finite.

Now we formulate our last assumption.

(ix) There exists a positive solution r_0 of the inequality

$$||p|| + rk_1(r)k_2(r)b(r)\overline{G}_a + rk_1(r)\overline{G}_h + k_2(r)b(r)F\overline{G}_a + F\overline{G}_h \le r$$

such that $k_1(r_0)(k_2(r_0)b(r_0)\overline{G}_a + \overline{G}_h) < 1$.

Remark 4.1. Observe that if r_0 is a positive solution of the inequality from assumption (ix) then we can write

$$r_0k_1(r_0)k_2(r_0)b(r_0)\overline{G}_a + r_0k_1(r_0)\overline{G}_h \le r_0 - ||p|| - k_2(r_0)b(r_0)F\overline{G}_a - F\overline{G}_h.$$

This yields

$$k_1(r_0)(k_2(r_0)b(r_0)\overline{G}_a + \overline{G}_h) \le 1 - \frac{||p|| + k_2(r_0)b(r_0)F\overline{G}_a + F\overline{G}_h}{r_0}.$$

Consequently we deduce

$$k_1(r_0)(k_2(r_0)b(r_0)\overline{G}_a + \overline{G}_h) \leq 1.$$

Moreover, if we assume additionally that at least one of the terms $p(t)$, $k_2(r_0)b(r_0)f(t,0)G_a(t)$, $f(t,0)G_h(t)$ does not vanish identically on $\mathbf{R}_+$, then the second inequality from assumption (ix) is automatically satisfied.

The main result of the paper is contained in the following theorem.

Theorem 4.2. *Under assumptions* (i)–(ix) *equation* (1.1) *has at least one solution $x = x(t)$ belonging to the space $BC(\mathbf{R}_+)$ and tending to a finite limit at infinity.*

Proof. Consider the operator H defined on the space $BC(\mathbf{R}_+)$ by the formula

$$(Hx)(t) = p(t) + f(t,x(t)) \int_0^\infty g(t,\tau)h(\tau,x(\tau))\,d\tau, \quad t \in \mathbf{R}_+.$$

In view of our assumptions let us notice that the function Hx is well defined.

In what follows we show that Hx is continuous on $\mathbf{R}_+$ for each fixed function $x \in BC(\mathbf{R}_+)$.

To this end let us fix $T > 0$ and $\varepsilon > 0$. Next, take $t, s \in [0,T]$ such that $|t-s| \leq \varepsilon$. Then, invoking our assumptions we get

$$\begin{aligned}
|(Hx)(t) - (Hx)(s)| &\leq |p(t) - p(s)| \\
&\quad + \bigg| f(t,x(t)) \int_0^\infty g(t,\tau)h(\tau,x(\tau))\,d\tau \\
&\quad - f(s,x(s)) \int_0^\infty g(t,\tau)h(\tau,x(\tau))\,d\tau \bigg| \\
&\quad + \bigg| f(s,x(s)) \int_0^\infty g(t,\tau)h(\tau,x(\tau))\,d\tau \\
&\quad - f(s,x(s)) \int_0^\infty g(s,\tau)h(\tau,x(\tau))\,d\tau \bigg| \\
&\leq \omega^T(p,\varepsilon) + |f(t,x(t)) - f(s,x(s))|
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^\infty |g(t,\tau)|[|h(\tau,x(\tau))-h(\tau,0)|+|h(\tau,0)|]\,d\tau \\
&+|f(s,x(s))|\int_0^\infty |g(t,\tau)-g(s,\tau)| \\
&\qquad\qquad \times [|h(\tau,x(\tau))-h(\tau,0)|+|h(\tau,0)|]\,d\tau \\
&\le \omega^T(p,\varepsilon)+[|f(t,x(t))-f(t,x(s))| \\
&+|f(t,x(s))-f(s,x(s))|] \\
&\times \int_0^\infty |g(t,\tau)|[a(\tau)k_2(|x(\tau)|)b(|x(\tau)|)+|h(\tau,0)|]\,d\tau \\
&+[|f(s,x(s))-f(s,0)|+|f(s,0)|] \\
&\times \int_0^\infty |g(t,\tau)-g(s,\tau)|[a(\tau)k_2(|x(\tau)|)b(|x(\tau)|) \\
&+|h(\tau,0)|]\,d\tau \\
&\le \omega^T(p,\varepsilon)+[k_1(||x||)|x(t)-x(s)|+\omega^T_{||x||}(f,\varepsilon)] \\
&\times \left\{k_2(||x||)b(||x||)\int_0^\infty a(\tau)|g(t,\tau)|\,d\tau \right. \\
&\left. +\int_0^\infty |g(t,\tau)h(\tau,0)|\,d\tau\right\}+[||x||k_1(||x||)+|f(s,0)|] \\
&\times \left\{\int_0^\infty k_2(||x||)b(||x||)a(\tau)|g(t,\tau)-g(s,\tau)|\,d\tau \right. \\
&\qquad\qquad \left. +\int_0^\infty |g(t,\tau)-g(s,\tau)||h(\tau,0)|\,d\tau\right\} \\
&\le \omega^T(p,\varepsilon)+k_1(||x||)k_2(||x||)b(||x||)\omega^T(x,\varepsilon) \\
&\times \int_0^\infty a(\tau)|g(t,\tau)|\,d\tau+k_2(||x||)b(||x||)\omega^T_{||x||}(f,\varepsilon) \\
&\times \int_0^\infty a(\tau)|g(t,\tau)|\,d\tau+k_1(||x||)\omega^T(x,\varepsilon) \\
&\times \int_0^\infty |g(t,\tau)h(\tau,0)|\,d\tau+\omega^T_{||x||}(f,\varepsilon) \\
&\times \int_0^\infty |g(t,\tau)h(\tau,0)|\,d\tau+||x||k_1(||x||)k_2(||x||)b(||x||)
\end{aligned} \tag{4.1}
$$

$$
\begin{aligned}
&\times \int_0^\infty a(\tau)|g(t,\tau)-g(s,\tau)|\,d\tau + ||x||k_1(||x||)\\
&\times \int_0^\infty |g(t,\tau)-g(s,\tau)||h(\tau,0)|d\tau + k_2(||x||)b(||x||)F\\
&\times \int_0^\infty a(\tau)|g(t,\tau)-g(s,\tau)|\,d\tau + F\\
&\times \int_0^\infty |g(t,\tau)-g(s,\tau)||h(\tau,0)|\,d\tau \le \omega^T(p,\varepsilon)\\
&\ \ + k_1(||x||)k_2(||x||)b(||x||)\overline{G}_a\omega^T(x,\varepsilon)\\
&\ \ + k_2(||x||)b(||x||)\overline{G}_a\omega^T_{||x||}(f,\varepsilon)\\
&\ \ + k_1(||x||)\overline{G}_h\omega^T(x,\varepsilon) + \overline{G}_h\omega^T_{||x||}(f,\varepsilon)\\
&\ \ + k_2(||x||)b(||x||)(||x||k_1(||x||)+F)\\
&\times \left\{ \int_0^T a(\tau)|g(t,\tau)-g(s,\tau)|\,d\tau \right.\\
&\ \ \left. + \int_T^\infty a(\tau)[|g(t,\tau)|+|g(s,\tau)|]\,d\tau \right\}\\
&\ \ + (||x||k_1(||x||)+F)\\
&\ \ \times \left\{ \int_0^T |g(t,\tau)-g(s,\tau)||h,(\tau,0)|\,d\tau \right.\\
&\ \ \left. + \int_T^\infty [|g(t,\tau)|+|g(s,\tau)|]|h(\tau,0)|\,d\tau \right\}\\
&\le \omega^T(p,\varepsilon) + k_1(||x||)[k_2(||x||)b(||x||)\overline{G}_a + \overline{G}_h]\omega^T(x,\varepsilon)\\
&\ \ + [k_2(||x||)b(||x||)\overline{G}_a + \overline{G}_h]\omega^T_{||x||}(f,\varepsilon)\\
&\ \ + k_2(||x||)b(||x||)(||x||k_1(||x||)+F)\omega_1^T(g,\varepsilon)\\
&\times \int_0^T a(\tau)\,d\tau + (||x||k_1(||x||)+F)\omega_1^T(g,\varepsilon)\\
&\times \int_0^T |h(\tau,0)|\,d\tau + k_2(||x||)b(||x||)(||x||k_1(||x||)+F)\\
&\times \left\{ \int_T^\infty a(\tau)|g(t,\tau)|\,d\tau + \int_T^\infty a(\tau)|g(s,\tau)|\,d\tau \right\}\\
&\ \ + (||x||k_1(||x||)+F)
\end{aligned}
$$

$$\times\left\{\int_T^\infty |g(t,\tau)h(\tau,0)|\,d\tau+\int_T^\infty |g(s,\tau)h(\tau,0)|\,d\tau\right\},$$

where we denoted

$$\omega_\alpha^T(f,\varepsilon)=\sup\{|f(t,x)-f(s,x)|:t,s\in[0,T],|t-s|\le\varepsilon,x\in[-\alpha,\alpha]\},$$
$$\omega_1^T(g,\varepsilon)=\sup\{|g(t,\tau)-g(s,\tau)|:\tau,t,s\in[0,T],|t-s|\le\varepsilon\};$$

here $\alpha=||x||$.

Further, let us observe that taking into account the uniform continuity of the function $p(t)$ on the interval $[0,T]$ and the uniform continuity of functions $f(t,x)$, $g(t,\tau)$ on the sets $[0,T]\times[-||x||,||x||]$, $[0,T]\times[0,T]$, respectively, and assumption (viii), from estimate (4.1) we deduce that the function Hx is continuous on the interval $[0,T]$. In view of the arbitrariness of T this implies that Hx is continuous on $\mathbf{R}_+$.

Now we show that the function Hx is bounded on $\mathbf{R}_+$. To do this fix arbitrarily $t\in\mathbf{R}_+$. Then, keeping in mind our assumptions, we obtain:

$$\begin{aligned}
|(Hx)(t)|&\le|p(t)|+|f(t,x(t))|\int_0^\infty|g(t,\tau)||h(\tau,x(\tau))|\,d\tau\\
&\le||p||+[|f(t,x(t))-f(t,0)|+|f(t,0)|]\\
&\quad\times\int_0^\infty|g(t,\tau)|[|h(\tau,x(\tau))-h(\tau,0)|+|h(\tau,0)|]\,d\tau\\
&\le||p||+[k_1(|x(t)|)|x(t)|+|f(t,0)|]\\
&\quad\times\int_0^\infty|g(t,\tau)|[a(\tau)k_2(|x(\tau)|)b(|x(\tau)|)+|h(\tau,0)|]\,d\tau\\
&\le||p||+[||x||k_1(||x||)+F]\\
&\quad\times\int_0^\infty|g(t,\tau)|[a(\tau)k_2(||x||)b(||x||)+|h(\tau,0)|]\,d\tau\\
&=||p||+||x||k_1(||x||)k_2(||x||)b(||x||)\int_0^\infty a(\tau)|g(t,\tau)|\,d\tau\\
&\quad+||x||k_1(||x||)\int_0^\infty|g(t,\tau)h(\tau,0)|\,d\tau\\
&\quad+k_2(||x||)b(||x||)F\int_0^\infty a(\tau)|g(t,\tau)|\,d\tau
\end{aligned}\tag{4.2}$$

$$
\begin{aligned}
&+ F \int_0^\infty |g(t,\tau)h(\tau,0)|\, d\tau = ||p|| \\
&+ ||x||k_1(||x||)k_2(||x||)b(||x||)G_a(t) + ||x||k_1(||x||)G_h(t) \\
&+ k_2(||x||)b(||x||)FG_a(t) \\
&+ FG_h(t) \le ||p|| + ||x||k_1(||x||)k_2(||x||)b(||x||)\overline{G}_a \\
&+ ||x||k_1(||x||)\overline{G}_h + k_2(||x||)b(||x||)F\overline{G}_a + F\overline{G}_h.
\end{aligned}
$$

The above inequality implies that the function Hx is bounded on the interval $\mathbf{R}_+$. This fact in conjunction with the continuity of Hx on $\mathbf{R}_+$ implies that Hx belongs to the space $BC(\mathbf{R}_+)$, i.e., the operator H transforms the space $BC(\mathbf{R}_+)$ into itself.

Moreover, combining estimate (4.2) with assumption (ix), we deduce that there exists a positive number r_0 such that $k_1(r_0)(k_2(r_0)b(r_0)\overline{G}_a + \overline{G}_h) < 1$ and the operator H transforms the ball B_{r_0} into itself.

In what follows we show that the operator H is continuous on the ball B_{r_0}. To this end fix a number $\varepsilon > 0$ and take $x, y \in B_{r_0}$ with $||x-y|| \le \varepsilon$. Then for an arbitrarily fixed $t \in \mathbf{R}_+$ we obtain:

$$
\begin{aligned}
|(Hx)(t) - (Hy)(t)| \le &\bigg| f(t,x(t)) \int_0^\infty g(t,\tau)h(\tau,x(\tau))\, d\tau \\
&\qquad - f(t,y(t)) \int_0^\infty g(t,\tau)h(\tau,x(\tau))\, d\tau \bigg| \\
&+ \bigg| f(t,y(t)) \int_0^\infty g(t,\tau)h(\tau,x(\tau))\, d\tau \\
&\qquad - f(t,y(t)) \int_0^\infty g(t,\tau)h(\tau,y(\tau))\, d\tau \bigg| \\
\le\ & |f(t,x(t)) - f(t,y(t))| \\
&\times \int_0^\infty |g(t,\tau)|[|h(\tau,x(\tau)) - h(\tau,0)| + |h(\tau,0)|]\, d\tau \\
&+ [|f(t,y(t)) - f(t,0)| + |f(t,0)|] \\
&\times \int_0^\infty |g(t,\tau)||h(\tau,x(\tau)) - h(\tau,y(\tau))|\, d\tau \\
\le\ & k_1(r_0)|x(t) - y(t)| \\
&\times \int_0^\infty |g(t,\tau)|[a(\tau)k_2(r_0)b(r_0) + |h(\tau,0)|]\, d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ [k_1(r_0)|y(t)| + |f(t,0)|] \\
&\times \int_0^\infty |g(t,\tau)|a(\tau)k_2(r_0)b(|x(\tau) - y(\tau)|)\,d\tau \\
\le{}& \varepsilon k_1(r_0)k_2(r_0)b(r_0) \\
\times{}& \int_0^\infty a(\tau)|g(t,\tau)|\,d\tau + \varepsilon k_1(r_0) \\
\times{}& \int_0^\infty |g(t,\tau)h(\tau,0)|\,d\tau + r_0k_1(r_0)k_2(r_0)b(\varepsilon) \\
\times{}& \int_0^\infty a(\tau)|g(t,\tau)|\,d\tau + k_2(r_0)b(\varepsilon)F \\
\times{}& \int_0^\infty a(\tau)|g(t,\tau)|\,d\tau \\
\le{}& \varepsilon k_1(r_0)k_2(r_0)b(r_0)\overline{G}_a + \varepsilon k_1(r_0)\overline{G}_h \\
&+ r_0k_1(r_0)k_2(r_0)b(\varepsilon)\overline{G}_a + k_2(r_0)b(\varepsilon)F\overline{G}_a.
\end{aligned}
$$

From the above estimate the desired continuity follows.

Further, let us take a nonempty subset X of the ball B_{r_0}. Fix arbitrarily $\varepsilon > 0$, $T > 0$ and choose a function $x \in X$ and numbers $t, s \in [0, T]$ such that $|t - s| \le \varepsilon$. Then reasoning in the same way as in (4.1) and using the notation introduced in that evaluation, we get:

(4.3)

$$
\begin{aligned}
|(Hx)(t) - (Hx)(s)| \le{}& \omega^T(p,\varepsilon) \\
&+ k_1(r_0)(k_2(r_0)b(r_0)\overline{G}_a + \overline{G}_h)\omega^T(x,\varepsilon) \\
&+ (k_2(r_0)b(r_0)\overline{G}_a + \overline{G}_h)\omega^T_{r_0}(f,\varepsilon) \\
&+ k_2(r_0)b(r_0)(r_0k_1(r_0) + F)\omega_1^T(g,\varepsilon)\int_0^T a(\tau)\,d\tau \\
&+ (r_0k_1(r_0) + F)\omega_1^T(g,\varepsilon)\int_0^T |h(\tau,0)|\,d\tau \\
&+ k_2(r_0)b(r_0)(r_0k_1(r_0) + F) \\
\times{}& \left\{\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau + \int_T^\infty a(\tau)|g(s,\tau)|\,d\tau\right\} \\
&+ (r_0k_1(r_0) + F)
\end{aligned}
$$

$$\times \left\{ \int_T^\infty |g(t,\tau)h(\tau,0)|\, d\tau + \int_T^\infty |g(s,\tau)h(\tau,0)| d\tau \right\}.$$

In the same way as before, using the uniform continuity of the function $p(t)$ on the interval $[0,T]$ and of the functions $f(t,x)$, $g(t,\tau)$ on the sets $[0,T]\times[-r_0,r_0]$, $[0,T]\times[0,T]$, respectively, from estimate (4.3) we derive the following inequality

$$\begin{aligned}\omega_0^T(HX) \le{}& k_1(r_0)(k_2(r_0)b(r_0)\overline{G}_a + \overline{G}_h)\omega_0^T(X)\\ &+ k_2(r_0)b(r_0)(r_0k_1(r_0)+F)\\ &\times \left\{ \int_T^\infty a(\tau)|g(t,\tau)|\, d\tau + \int_T^\infty a(\tau)|g(s,\tau)|\, d\tau \right\}\\ &+ (r_0k_1(r_0)+F)\\ &\times \left\{ \int_T^\infty |g(t,\tau)h(\tau,0)|\, d\tau + \int_T^\infty |g(s,\tau)h(\tau,0)|\, d\tau \right\}.\end{aligned}$$

Hence, employing assumption (viii), we derive the following estimate

$$\omega_0(HX) \le k_1(r_0)(k_2(r_0)b(r_0)\overline{G}_a + \overline{G}_h)\omega_0(X), \tag{4.4}$$

where the quantities ω_0^T and ω_0 are defined in Section 3.

Next, take as before a nonempty set $X \subset B_{r_0}$ and fix a number $T>0$. Then, for an arbitrarily fixed function $x\in X$ and for arbitrary numbers t,s such that $t\ge T$, $s\ge T$, reasoning similarly as in (4.1) we obtain:

$$\begin{aligned}|(Hx)(t)-(Hx)(s)| \le{}& |p(t)-p(s)| + [|f(t,x(t)) - f(t,x(s))|\\ &+ |f(t,x(s)) - f(s,x(s))|]\\ &\times \int_0^\infty |g(t,\tau)|[a(\tau)k_2(r_0)b(r_0) + |h(\tau,0)|]\, d\tau\\ &+ [|f(s,x(s)) - f(s,0)| + |f(s,0)|]\\ &\times \int_0^\infty |g(t,\tau)-g(s,\tau)|[a(\tau)k_2(r_0)b(r_0)\\ &\qquad + |h(\tau,0)|]\, d\tau\\ \le{}& |p(t)-p(s)| + [k_1(r_0)|x(t)-x(s)|\\ &+ |f(t,x(s)) - f(s,x(s))|]\end{aligned}$$

$$\begin{aligned}
&\times \{k_2(r_0)b(r_0)G_a(t) + G_h(t)\} \\
&+ (r_0 k_1(r_0) + F)\{k_2(r_0)b(r_0) \\
&\times \int_0^\infty |g(t,\tau) - g(s,\tau)|a(\tau)\,d\tau \\
&\qquad + \int_0^\infty |g(t,\tau) - g(s,\tau)||h(\tau,0)|\,d\tau\}.
\end{aligned}$$

Hence, taking the supremum with respect to $s \geq T$, $t \geq T$ and $x \in X$, and letting $T \to \infty$, in view of assumptions (i), (iii) and (vii) we get

$$\beta(HX) \leq k_1(r_0)(k_2(r_0)b(r_0)\overline{G}_a + \overline{G}_h)\beta(X), \tag{4.5}$$

where the set quantity β was defined by formula (3.1).

Now, linking (4.4) and (4.5) we derive the following estimate

$$\mu(HX) \leq k_1(r_0)(k_2(r_0)b(r_0)\overline{G}_a + \overline{G}_h)\mu(X), \tag{4.6}$$

where μ is the measure of noncompactness defined by formula (3.2).

Finally, keeping in mind the second inequality appearing in assumption (ix) and applying Theorem 2.2, from estimate (4.6) we infer that the operator H has at least one fixed point in the ball B_{r_0} which is a solution of equation (1.1). This completes the proof. □

Remark 4.3. Taking into account Remark 2.3 and the description of the kernel $\ker\mu$ of the measure of noncompactness μ given at the end of Section 3, we conclude that all solutions of equation (1.1) belonging to the ball B_{r_0} tend to finite limits at infinity.

5. An example. In this section we give an example illustrating the main result of the paper contained in Theorem 4.2. Consider the quadratic Hammerstein integral equation

$$x(t) = \frac{e^t}{10(1+e^t)} + \alpha\frac{t^2x^2(t)}{1+t^2}\int_0^\infty \frac{\tau^2(\sqrt[3]{x^2(\tau)+\arctan\tau} + x(\tau)\sqrt{|x(\tau)|})}{1+t^2+\tau^4}\,d\tau, \tag{5.1}$$

where α is a positive constant and $t \in \mathbf{R}_+$.

Observe that equation (5.1) is a particular case of equation (1.1) if we put $p(t) = e^t/10(1+e^t)$, $f(t,x) = \alpha t^2 x^2/(1+t^2)$ and

$$g(t,\tau) = \frac{\tau^2}{1+t^2+\tau^4},$$
$$h(t,x) = \sqrt[3]{x^2 + \arctan t} + x\sqrt{|x|}.$$

In what follows we show that functions involved in equation (5.1) satisfy the assumptions imposed in Theorem 4.2. Indeed, assumption (i) is satisfied since $\lim_{t\to\infty} p(t) = 1/10$. Further, let us note that

$$|f(t,x) - f(t,y)| \leq \alpha \frac{t^2}{1+t^2}|x^2 - y^2| \leq \alpha |x+y||x-y|.$$

This implies that the function $f(t,x)$ satisfies assumption (ii) with $k_1(r) = 2\alpha r$. Moreover, we have that $f(t,0) = 0$ which implies that $F = 0$.

Next, fix $T > 0$ and take arbitrary numbers t, s such that $t \geq T$, $s \geq T$. Without loss of generality we may assume that $s < t$. Then we have:

$$\begin{aligned} |f(t,x) - f(s,x)| &\leq \alpha x^2 \left| \frac{t^2}{1+t^2} - \frac{s^2}{1+s^2} \right| \\ &\leq \alpha x^2 \frac{t^2}{1+t^2} \cdot \frac{1}{1+s^2}. \end{aligned}$$

This implies that for any fixed $r > 0$ we get

$$\lim_{T\to\infty} \{\sup\{|f(t,x) - f(s,x)| : t, s \geq T, |x| \leq r\}\} = 0,$$

which means that assumption (iii) is fulfilled.

Obviously the function $g = g(t,\tau)$ is continuous on the set $\mathbf{R}_+ \times \mathbf{R}_+$ (cf. assumption (iv)).

Next, we verify assumption (v). Notice that we have the following estimate

$$\begin{aligned} |h(t,x) - h(t,y)| &\leq \left| \sqrt[3]{x^2 + \arctan t} - \sqrt[3]{y^2 + \arctan t} \right| \\ &\quad + \left| x\sqrt{|x|} - y\sqrt{|y|} \right|. \end{aligned} \tag{5.2}$$

Now, fixing $r > 0$ and assuming that $|x| \le r$, $|y| \le r$, we have the following inequality:

$$\begin{aligned} \left|x\sqrt{|x|} - y\sqrt{|y|}\right| &\le \left|x\sqrt{|x|} - y\sqrt{|x|}\right| + \left|y\sqrt{|x|} - y\sqrt{|y|}\right| \\ &\le \sqrt{r}|x-y| + r\left|\sqrt{|x|} - \sqrt{|y|}\right| \\ &\le \sqrt{r}|x-y| + r\sqrt{|x-y|}. \end{aligned} \tag{5.3}$$

Next, let us notice that we have the estimate

$$\left|\sqrt[3]{x^2 + \arctan t} - \sqrt[3]{y^2 + \arctan t}\right| \le \sqrt[3]{(x-y)^2}. \tag{5.4}$$

The above estimate is a consequence of the following inequality

$$\left|\sqrt[3]{x^2 + a} - \sqrt[3]{y^2 + a}\right| \le \sqrt[3]{(x-y)^2} \tag{5.5}$$

which is satisfied for all $x, y \in \mathbf{R}$ and for any fixed a, $a \ge 0$.

Since (as far as we know) inequality (5.5) is not standard we provide the sketch of its proof.

In order to prove (5.5) it is sufficient to show that for all $x, y \in \mathbf{R}_+$ such that $y < x$ we have

$$\sqrt[3]{x^2 + a} - \sqrt[3]{y^2 + a} \le \sqrt[3]{(x-y)^2}. \tag{5.6}$$

To prove (5.6) fix arbitrarily $y \ge 0$. Put, for convenience, $b = y$. Then (5.6) can be written equivalently in the form

$$\sqrt[3]{(x+b)^2 + a} - \sqrt[3]{b^2 + a} \le \sqrt[3]{x^2} \tag{5.7}$$

for any $x \ge 0$, where $a \ge 0$ and $b \ge 0$ are arbitrarily fixed.

In what follows we prove inequality (5.7). To this end consider the auxiliary function $f : \mathbf{R}_+ \to \mathbf{R}$ defined by the formula

$$f(x) = \sqrt[3]{(x+b)^2 + a} - \sqrt[3]{x^2} - \sqrt[3]{b^2 + a}.$$

Then $f(0) = 0$ and, applying standard methods of differential calculus, we can check that $f'(x) < 0$ for $x > 0$. We omit details of calculations.

Hence we derive that $f(x) \leq 0$ for $x \geq 0$ which proves (5.7) and shows the validity of (5.5) and (5.4).

Now, observe that linking (5.2), (5.3) and (5.4) we get

$$|h(t,x) - h(t,y)| \leq \max\{1, r\} \left[\sqrt[3]{(x-y)^2} + |x-y| + \sqrt{|x-y|} \right].$$

Thus assumption (v) is satisfied with $k_2(r) = \max\{1, r\}$, $a(t) \equiv 1$ and $b(r) = \sqrt[3]{r^2} + r + \sqrt{r}$.

In order to verify assumption (vi) let us consider the functions $G_a(t)$, $G_h(t)$ which are defined here by the following formulas

$$\begin{aligned} G_a(t) &= \int_0^\infty a(\tau)|g(t,\tau)|\, d\tau \\ &= \int_0^\infty \frac{\tau^2}{1+t^2+\tau^4}\, d\tau, \\ G_h(t) &= \int_0^\infty |g(t,\tau)h(\tau,0)|\, d\tau \\ &= \int_0^\infty \frac{\tau^2}{1+t^2+\tau^4} \sqrt[3]{\arctan\tau}\, d\tau. \end{aligned}$$

Obviously these functions are well defined. Moreover, using standard methods of integral calculus we get:

$$\begin{aligned} G_a(t) &= \frac{\pi}{4\sqrt{2}} \cdot \frac{1}{\sqrt[4]{1+t^2}} \leq \frac{\pi}{4\sqrt{2}}, \\ G_h(t) &\leq \sqrt[3]{\frac{\pi}{2}}\, \frac{\pi}{4\sqrt{2}}\, \frac{1}{\sqrt[4]{1+t^2}} \leq \sqrt[3]{\frac{\pi}{2}}\, \frac{\pi}{4\sqrt{2}}. \end{aligned}$$

This shows that assumption (vi) is satisfied. Also we obtain

$$\overline{G}_a = \frac{\pi}{4\sqrt{2}}, \qquad \overline{G}_h \leq \sqrt[3]{\frac{\pi}{2}}\, \frac{\pi}{4\sqrt{2}}.$$

Next, fix arbitrarily $T > 0$ and choose $t, s \geq T$. Without loss of generality we can assume that $t > s$. Then, similarly as above we obtain:

$$\begin{aligned}\int_0^\infty &|g(t,\tau)-g(s,\tau)|a(\tau)\,d\tau\\ &= \int_0^\infty \left|\frac{\tau^2}{1+t^2+\tau^4}-\frac{\tau^2}{1+s^2+\tau^4}\right|a(\tau)\,d\tau\\ &= \int_0^\infty \left(\frac{\tau^2}{1+s^2+\tau^4}-\frac{\tau^2}{1+t^2+\tau^4}\right)d\tau\\ &= \frac{\pi}{4\sqrt{2}}\left[\frac{1}{\sqrt[4]{1+s^2}}-\frac{1}{\sqrt[4]{1+t^2}}\right].\end{aligned}$$

Hence, taking into account that

$$\sup\left\{\frac{\pi}{4\sqrt{2}}\left[\frac{1}{\sqrt[4]{1+s^2}}-\frac{1}{\sqrt[4]{1+t^2}}\right]: T\le s<t\right\} = \frac{\pi}{4\sqrt{2}}\cdot\frac{1}{\sqrt[4]{1+T^2}},$$

we get

$$\lim_{T\to\infty}\left\{\sup\left\{\int_0^\infty |g(t,\tau)-g(s,\tau)|a(\tau)\,d\tau : t,s\ge T\right\}\right\}=0. \tag{5.8}$$

In the same way we have

$$\begin{aligned}\int_0^\infty &|g(t,\tau)-g(s,\tau)||h(\tau,0)|\,d\tau\\ &= \int_0^\infty \left|\frac{\tau^2}{1+t^2+\tau^4}-\frac{\tau^2}{1+s^2+\tau^4}\right|\sqrt[3]{\arctan\tau}\,d\tau\\ &\le \sqrt[3]{\frac{\pi}{2}}\int_0^\infty \left|\frac{\tau^2}{1+t^2+\tau^4}-\frac{\tau^2}{1+s^2+\tau^4}\right|d\tau.\end{aligned}$$

From the above estimate we conclude that

$$\lim_{T\to\infty}\left\{\sup\left\{\int_0^\infty |g(t,\tau)-g(s,\tau)||h(\tau,0)|\,d\tau : t,s\ge T\right\}\right\}=0. \tag{5.9}$$

Combining (5.8) and (5.9) we infer that assumption (vii) is satisfied as well.

Also for a fixed $T > 0$, analogously as above we have:

$$\begin{aligned}\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau &= \int_T^\infty \frac{\tau^2}{1+t^2+\tau^4}\,d\tau \\ &= \frac{1}{2\sqrt{2}}\frac{1}{\sqrt[4]{1+t^2}}\Bigg\{-\frac{1}{2}\ln\left(\frac{\frac{T^2}{\sqrt{1+t^2}}-\frac{\sqrt{2}T}{\sqrt[4]{1+t^2}}+1}{\frac{T^2}{\sqrt{1+t^2}}+\frac{\sqrt{2}T}{\sqrt[4]{1+t^2}}+1}\right) \\ &\qquad -\arctan\left(\frac{\sqrt{2}T}{\sqrt[4]{1+t^2}}-1\right) \\ &\qquad +\arctan\left(\frac{\sqrt{2}T}{\sqrt[4]{1+t^2}}+1\right)\Bigg\}.\end{aligned}$$

Hence we obtain that

$$\lim_{T\to\infty}\left\{\sup\left\{\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau : t\in\mathbf{R}_+\right\}\right\} = 0.$$

Similarly we can prove the second equality from assumption (viii).

Finally we check that assumption (ix) is satisfied. Indeed, taking into account the above obtained estimates we see that the inequality from assumption (ix) is satisfied provided is satisfied the following inequality

$$\frac{1}{10}+\frac{\pi}{2\sqrt{2}}\alpha r^2\max\{1,r\}\left(\sqrt[3]{r^2}+r+\sqrt{r}\right)+\sqrt[3]{\frac{\pi}{2}}\frac{\pi}{2\sqrt{2}}\alpha r^2 \le r.$$

It is easily seen that if we take $\alpha = 4/21$ then the number $r_0 = 1$ satisfies the above inequality. Moreover, in this case we have

$$k_1(r_0)(k_2(r_0)b(r_0)\overline{G}_a+\overline{G}_h) \le \frac{8}{21}\left(\frac{3\pi}{4\sqrt{2}}+\sqrt[3]{\frac{\pi}{2}}\frac{\pi}{4\sqrt{2}}\right) = 0.87855\cdots < 1.$$

Thus assumption (ix) is satisfied.

Let us note here that we can choose other values of r_0 if we fix another value of α. For example, putting $\alpha = 9/20$ we can easily see that the number $r_0 = 1/2$ satisfies both inequalities from assumption (ix).

REFERENCES

1. R.P. Agarwal, D. O'Regan and P.J.Y. Wong, *Positive solutions of differential, difference and integral equation*, Kluwer Academic Publishers, Dordrecht, 1999.

2. R.R. Akmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina and B.N. Sadovskii, *Measures of noncompactness and condensing operators*, Birkhäuser Verlag, Basel, 1992.

3. J. Appell and P.P. Zabrejko, *Nonlinear superposition operators*, Cambridge University Press, Cambridge, 1990.

4. J.M. Ayerbe Toledano, T. Dominguez Benavides and G. Lopez Acedo, *Measures of noncompactness in metric fixed point theory*, Birkhäuser, Basel, 1997.

5. J. Banaś, *Measures of noncompactness in the space of continuous tempered functions*, Demon. Math. **14** (1981), 127–133.

6. J. Banaś and K. Goebel, *Measures of noncompactness in Banach spaces*, Lect. Notes Pure Appl. Math. **60** (1980), Dekker, New York.

7. J. Banaś, D. O'Regan and R.P. Agarwal, *Measures of noncompactness and asymptotic stablility of solutions of a quadratic Hammerstein integral equation*, Rocky Mountain J. Math., to appear.

8. J. Banaś, D. O'Regan and K. Sadarangani, *On solutions of a quadratic Hammerstein integral equation on an unbounded interval*, Dynamic Syst. Appl. **18** (2009), 251–264.

9. J. Banaś, J. Rocha Martin and K. Sadarangani, *On solutions of a quadratic integral equation of Hammerstein type*, Math. Comput. Model. **43** (2006), 97–104.

10. C. Corduneanu, *Integral equations and applications*, Cambridge University Press, Cambridge, 1991.

11. G. Darbo, *Punti uniti in transformazioni a codominio non compatto*, Rend. Sem. Math. Univ. Padova **24** (1955), 84–92.

12. K. Deimling, *Nonlinear functional analysis*, Springer Verlag, Berlin, 1985.

13. G.M. Fichtenholz, *Differential and integral calculus*, II, PWN, Warsaw, 1980 (in Polish).

14. J. Mawhin, *On an existence result of Leray for nonlinear integral equations*, Fixed Point Theory **7** (2006), 297–304.

15. D. O'Regan and M. Meehan, *Existence theory for nonlinear integral and integrodifferential equations*, Kluwer Academic Publishers, Dordrecht, 1998.

16. R. Stańczy, *Hammerstein equations with an integral over a noncompact domain*, Ann. Polon. Math. **69** (1998), 49–60.

17. Z. Yang, *Nontrivial solutions of nonlinear Hammerstein integral equations*, Nonlin. Funct. Anal. Appl. **10** (2005), 331–342.

18. P.P. Zabrejko, A.I. Koshelev, M.A. Krasnoselśkii, S.G. Mikhlin, L.S. Rakovschik and J. Stetsenko, *Integral equations*, Nordhoff, Leyden, 1975.

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