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ANALYSIS OF DIRECT BOUNDARY-DOMAIN INTEGRAL EQUATIONS FOR A MIXED BVP WITH VARIABLE COEFFICIENT, II: SOLUTION REGULARITY AND ASYMPTOTICS

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ABSTRACT. Mapping and invertibility properties of some parametrix-based surface and volume potentials are studied in Bessel-potential and Besov spaces. These results are then applied to derive regularity and asymptotics of the solution to a system of boundary-domain integral equations associated with a mixed BVP for a variable-coefficient PDE, in a vicinity of the curve of change of the boundary condition type.

1. Introduction. This paper is the second part of the paper [**6**], where we analyzed four versions of Boundary-Domain Integral Equation Systems (BDIES) to which a mixed (Dirichlet-Neumann) boundary value problem for the heat transfer equation with a variable heat conductivity coefficient can be reduced, and gave a full description of existence, uniqueness, and operator invertibility in appropriate Sobolev spaces.

In the present paper, we first discuss properties of surface and volume potentials, constituting the BDIES, in the Bessel potential spaces H_p^s and in the Besov spaces. Then we use these properties to analyze regularity and asymptotic behavior of the BDIES solutions.

A motivation for analysis of boundary-domain integral equations and notations used can be found in [**6**]. To simplify references, we will precede numbers of sections, equations and statements from [**6**] by I.

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2. Boundary value problem with variable coefficient and parametrix-based potentials. Here we recall some necessary material from [**6**]. Let Ω^+ be a bounded open three-dimensional region of $\mathbb{R}^3$ and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. For simplicity, we assume that the boundary $S := \partial\Omega^+ = \partial\Omega^-$ is a simply connected, closed, infinitely smooth surface. Moreover, $S = \overline{S}_D \cup \overline{S}_N$, where S_D and S_N are nonempty, nonintersecting ($S_D \cap S_N = \varnothing$), simply connected submanifolds of S with infinitely smooth boundary curve $\ell := \partial S_D = \partial S_N \in C^\infty$.

In this paper we will continue investigation of the Boundary-Domain Integral Equation Systems (BDIESs) introduced in Part I, which are equivalent to the following *mixed boundary value problem*:

Find a function $u \in H_2^1(\Omega^+)$ satisfying the conditions

$$L(x,\partial_x)\,u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a(x)\,\frac{\partial u(x)}{\partial x_i}\right) = f \quad \text{in } \Omega^+, \tag{2.1}$$

$$r_{S_D}\, u^+ = \varphi_0 \quad \text{on } S_D, \tag{2.2}$$

$$r_{S_N}\, T^+ u = \psi_0 \quad \text{on } S_N, \tag{2.3}$$

where $r_{\mathcal{M}}$ denotes the restriction operator on $\mathcal{M}$; $\varphi_0 \in H_2^{\frac{1}{2}}(S_D)$, $\psi_0 \in H_2^{-\frac{1}{2}}(S_N)$ and $f \in L_2(\Omega^+)$; $a \in C^\infty(\mathbb{R}^3)$, $a(x) > 0$ for $x \in \mathbb{R}^3$; $(\cdot)^+$ denotes the trace, and $T^+(x, n(x), \partial_x)$ is the conormal derivative operator correctly defined in the functional sense (see Section I.2).

To investigate the regularity and asymptotics of the BDIES solutions we will need the following spaces: the Sobolev-Slobodetski spaces $W_p^r(\Omega^+)$, $W_{p,\mathrm{loc}}^r(\Omega^-)$; the Bessel potential spaces $H_p^s(\Omega^+)$, $H_{p,\mathrm{loc}}^s(\Omega^-)$, $H_p^s(S)$; and the Besov spaces $B_{p,q}^s(\Omega^+)$, $B_{p,q,\mathrm{loc}}^s(\Omega^-)$, $B_{p,q}^s(S)$, where $r \geq 0$, $s \in \mathbb{R}$ and $1 < p, q < \infty$ (see e.g., [**13, 20**]). We recall that $H_2^r = W_2^r = B_{2,2}^r$ for $r \geq 0$, $H_2^s = B_{2,2}^s$ for any $s \in \mathbb{R}$, $W_p^t = B_{p,p}^t$ and $H_p^k = W_p^k$ for any positive and noninteger t, for any nonnegative integer k and for any $p > 1$.

For $S_1 \subset S$, we will use the subspace $\widetilde{H}_p^s(S_1) = \{g : g \in H_p^s(S),\ \operatorname{supp} g \subset \overline{S_1}\}$ of $H_p^s(S)$, while $H_p^s(S_1) = \{r_{S_1} g : g \in H_p^s(S)\}$ denotes the space of restriction on S_1 of functions from $H_p^s(S)$, where r_{S_1} denotes the restriction operator on S_1. The subspaces $\widetilde{B}_{p,q}^s(S_1)$ and $B_{p,q}^s(S_1)$ of $B_{p,q}^s(S)$ are defined similarly.

In Section I.4.1 we derived the following third Green identity for arbitrary function $u \in H_2^{1,0}(\Omega^+; L) := \{g \in H^1(\Omega) : L\,g \in L_2(\Omega)\}$,

$$u(y) + \mathcal{R}u(y) - VT^+u(y) + Wu^+(y) = \mathcal{P}Lu(y), \quad y \in \Omega^+, \tag{2.4}$$

where

$$Vg(y) := -\int_S P(x,y)\,g(x)\,dS_x, \quad y \notin S, \tag{2.5}$$

$$Wg(y) := -\int_S \big[T(x,n(x),\partial_x)\,P(x,y)\big]\,g(x)\,dS_x, \quad y \notin S, \tag{2.6}$$

$$\mathcal{P}g(y) := \int_{\Omega^+} P(x,y)\,g(x)\,dx, \quad y \in \Omega^\pm, \tag{2.7}$$

$$\mathcal{R}g(y) := \int_{\Omega^+} R(x,y)\,g(x)\,dx, \quad y \in \Omega^\pm \tag{2.8}$$

with

$$P(x,y) = \frac{-1}{4\pi\,a(y)\,|x-y|}, \quad x,y \in \mathbb{R}^3, \tag{2.9}$$

$$R(x,y) = \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi\,a(y)\,|x-y|^3}\,\frac{\partial\,a(x)}{\partial x_i}, \quad x,y \in \mathbb{R}^3. \tag{2.10}$$

If $Lu = f \in L_2(\Omega^+)$, then (2.4) gives (see Section I.4)

$$u(y) + \mathcal{R}u(y) - VT^+u(y) + Wu^+(y) = \mathcal{P}f(y), \quad y \in \Omega^+, \tag{2.11}$$

$$\mathcal{G}u(y) := \frac{1}{2}\,u^+(y) + \mathcal{R}^+u(y) - \mathcal{V}T^+u(y) + \mathcal{W}u^+(y) = [\mathcal{P}f]^+(y), \quad y \in S, \tag{2.12}$$

(2.13)
$$\mathcal{T}u(y) := \frac{1}{2}\,T^{+}u(y) + T^{+}\mathcal{R}u(y) - \mathcal{W}'T^{+}u(y) + \mathcal{L}^{+}u^{+}(y) = T^{+}\mathcal{P}f(y), \quad y \in S.$$

We recall that $\mathcal{R}^{+}u(y) := [\mathcal{R}u]^{+}(y)$,

$$\mathcal{V}\,g(y) := -\int_S P(x,y)\,g(x)\,dS_x, \tag{2.14}$$

$$\mathcal{W}\,g(y) := -\int_S \big[T(x,n(x),\partial_x))\,P(x,y)\big]\,g(x)\,dS_x, \tag{2.15}$$

$$\mathcal{W}'\,g(y) := -\int_S \big[T(y,n(y),\partial_y))\,P(x,y)\big]\,g(x)\,dS_x, \tag{2.16}$$

$$\mathcal{L}^{\pm}g(y) := [T(y,n(y),\partial_y))\,Wg(y)]^{\pm}, \tag{2.17}$$

where $y \in S$.

As in Part I, from definitions (2.5)–(2.10) and (2.14)–(2.17), one can obtain representations of the parametrix-based surface potential boundary operators in terms of their counterparts for $a = 1$, i.e., associated with the Laplace operator Δ,

$$Vg = \frac{1}{a}V_{\Delta}g, \qquad Wg = \frac{1}{a}W_{\Delta}(ag), \tag{2.18}$$

$$\mathcal{V}g = \frac{1}{a}\mathcal{V}_{\Delta}g, \qquad \mathcal{W}g = \frac{1}{a}\mathcal{W}_{\Delta}(ag), \tag{2.19}$$

$$\mathcal{W}'g = \mathcal{W}'_{\Delta}g + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]\mathcal{V}_{\Delta}g, \tag{2.20}$$

$$\mathcal{L}^{\pm}g = \mathcal{L}_{\Delta}(ag) + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]W^{\pm}_{\Delta}(ag) \tag{2.21}$$

$$\mathcal{P}\,g = \frac{1}{a}\,\mathcal{P}_{\Delta}\,g, \tag{2.22}$$

$$\mathcal{R}\,g = -\frac{1}{a}\,\sum_{j=1}^{3}\partial_j\left[\mathcal{P}_{\Delta}\left(g\,\partial_j a\right)\right], \tag{2.23}$$

where the subscript Δ (the Laplace operator) means that the corresponding surface potentials are constructed by means of the harmonic fundamental solution $P_{\Delta} = -(4\,\pi\,|x-y|)^{-1}$.

3. Properties of the potentials in Bessel-potential and Besov spaces.

3.1. Mapping properties of surface potentials. The mapping and jump properties of potentials of the type (2.5)–(2.8) and the corresponding boundary integral and pseudo-differential operators (3.3)–(3.7) in the Bessel potential (H_p^s) and Besov ($B_{p,q}^s$) spaces are well studied nowadays (for details see, e.g., [**7, 8, 16, 17**]; see also [**14, 15**], where the coerciveness properties of the boundary operators and also the case of Lipschitz domains are considered).

Theorems 3.1–3.2 below generalize their counterparts formulated in Part I for Sobolev spaces. They are well known, see e.g., the above references, for the case $a = \text{const}$. Using (2.18)–(2.21), one can easily prove they hold true also for the variable positive coefficient $a \in C^\infty(\mathbb{R}^3)$.

Theorem 3.1. *Let $s \in \mathbb{R}$, $1 < p < +\infty$ and $1 \le q \le +\infty$. The following operators are continuous*

$$\begin{aligned}
V : B^s_{p,p}(S) &\longrightarrow H_p^{s+1+\frac{1}{p}}(\Omega^+) \quad \Big[B^s_{p,p}(S) \longrightarrow H_{p,\mathrm{loc}}^{s+1+\frac{1}{p}}(\overline{\Omega^-})\Big],\\
: B^s_{p,q}(S) &\longrightarrow B_{p,q}^{s+1+\frac{1}{p}}(\Omega^+) \quad \Big[B^s_{p,q}(S) \longrightarrow B_{p,q,\mathrm{loc}}^{s+1+\frac{1}{p}}(\overline{\Omega^-})\Big];\\
W : B^s_{p,p}(S) &\longrightarrow H_p^{s+\frac{1}{p}}(\Omega^+) \quad \Big[B^s_{p,p}(S) \longrightarrow H_{p,\mathrm{loc}}^{s+\frac{1}{p}}(\overline{\Omega^-})\Big],\\
: B^s_{p,q}(S) &\longrightarrow B_{p,q}^{s+\frac{1}{p}}(\Omega^+) \quad \Big[B^s_{p,q}(S) \longrightarrow B_{p,q,\mathrm{loc}}^{s+\frac{1}{p}}(\overline{\Omega^-})\Big].
\end{aligned}$$

Theorem 3.2. *Let $s \in \mathbb{R}$, $1 < p < +\infty$ and $1 \le q \le +\infty$. The following pseudo-differential operators are continuous*

$$\begin{aligned}
\mathcal{V} : B^s_{p,q}(S) &\longrightarrow B^{s+1}_{p,q}(S);\\
\mathcal{W}, \mathcal{W}' : B^s_{p,q}(S) &\longrightarrow B^{s+1}_{p,q}(S);\\
\mathcal{L}^\pm : B^s_{p,q}(S) &\longrightarrow B^{s-1}_{p,q}(S).
\end{aligned}$$

Theorem 3.3. *Let $s \in \mathbb{R}$, $1 < p < +\infty$ and $1 \le q \le +\infty$. Let S_1 and S_2 with $\partial S_1, \partial S_2 \in C^\infty$ be nonempty open sub-manifolds of S. The operators*

$$r_{S_2}\,\mathcal{V} : \widetilde{B}^s_{p,q}(S_1) \longrightarrow B^s_{p,q}(S_2), \tag{3.1}$$

$$r_{S_2}\,\mathcal{W} : \widetilde{B}^s_{p,q}(S_1) \longrightarrow B^s_{p,q}(S_2), \tag{3.2}$$

$$r_{S_2}\mathcal{W}' : \widetilde{B}^s_{p,q}(S_1) \longrightarrow B^s_{p,q}(S_2) \tag{3.3}$$

are compact.

Proof. Theorem 3.2 implies that the operators $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{W}'$ have the following mapping properties

$$\begin{aligned} r_{S_2}\,\mathcal{V} :&\widetilde{B}^s_{p,q}(S_1) \longrightarrow B^{s+1}_{p,q}(S_2), \\ r_{S_2}\,\mathcal{W} :&\widetilde{B}^s_{p,q}(S_1) \longrightarrow B^{s+1}_{p,q}(S_2), \\ r_{S_2}\,\mathcal{W}' :&\widetilde{B}^s_{p,q}(S_1) \longrightarrow B^{s+1}_{p,q}(S_2). \end{aligned}$$

Since the embedding $B^{s+1}_{p,q}(S_2) \subset B^s_{p,q}(S_2)$ is compact, the proof follows. □

3.2. Fredholm properties and invertibility of some surface potentials. In our analysis we essentially apply the following assertion about elliptic pseudo-differential operators on manifolds with boundary (for general theory, see e.g., [**1, 3, 10, 12, 18**]).

Lemma 3.4. *Let $\overline{S}_1 \in C^\infty$ be a compact, two-dimensional, non selfintersecting, two-sided surface with boundary $\partial S_1 \in C^\infty$, and $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$. Further, let $\mathcal{A}$ be a strongly elliptic pseudo-differential operator of order $\alpha \in \mathbb{R}$ on S_1 having a uniformly positive principal homogeneous symbol, i.e., $\sigma(\mathcal{A}; y, \xi) \ge c_0 > 0$ for $y \in \overline{S}_1$, $\xi \in \mathbb{R}^2$ with $|\xi| = 1$, where c_0 is a constant.*

Then the operators

$$\mathcal{A} : \widetilde{H}^s_p(S_1) \ \longrightarrow H^{s-\alpha}_p(S_1) \tag{3.4}$$

$$: \widetilde{B}^s_{p,q}(S_1) \longrightarrow B^{s-\alpha}_{p,q}(S_1) \tag{3.5}$$

are Fredholm operators of index zero if

$$\frac{1}{p} - 1 < s - \alpha/2 < \frac{1}{p}. \tag{3.6}$$

Moreover, the null-spaces of operators (3.4) *and* (3.5) *are the same* (*for all values of the parameter $q \in [1, +\infty]$*) *provided p and s satisfy inequality* (3.6).

This assertion is a particular case of a more general Theorem 2.19 in [**18**].

Now we can prove

Theorem 3.5. *Let S_1 be a nonempty, simply connected submanifold of S with infinitely smooth boundary curve, $s \in \mathbb{R}$, $1 < p < +\infty$, $1 \leq q \leq +\infty$ and*

$$\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2}.$$

Then the pseudo-differential operators

$$r_{S_1}\,\mathcal{V} : \widetilde{H}_p^{s-1}(S_1) \longrightarrow H_p^s(S_1) \tag{3.7}$$

$$: \widetilde{B}_{p,q}^{s-1}(S_1) \longrightarrow B_{p,q}^s(S_1) \tag{3.8}$$

have order -1 and are invertible, while the pseudo-differential operators

$$r_{S_1}\,\mathcal{L}^{\pm} : \widetilde{H}_p^s(S_1) \longrightarrow H_p^{s-1}(S_1) \tag{3.9}$$

$$: \widetilde{B}_{p,q}^s(S_1) \longrightarrow B_{p,q}^{s-1}(S_1) \tag{3.10}$$

have order $+1$ and are Fredholm operators of index zero.

Proof. It is easy to show that

$$\sigma(\mathcal{V}; y, \xi) = \frac{1}{2}\,[a(y)\,|\xi|]^{-1}, \quad y \in S_1,\ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

is the principal homogeneous symbol of the operator $\mathcal{V}$, while the function

$$\sigma(-\mathcal{L}^{\pm}; y, \xi) = \frac{1}{2}\,a(y)\,|\xi|, \quad y \in S_1,\ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

is the principal homogeneous symbol of the operator $-\mathcal{L}^{\pm}$.

Therefore, $r_{S_1}\,\mathcal{V}$ and $r_{S_1}\,\mathcal{L}^{\pm}$ are strongly elliptic pseudo-differential operators on the submanifold S_1 with positive homogeneous principal symbols of order -1 and $+1$, respectively. Due to Lemma 3.4 we conclude that the operators (3.7)–(3.10) are Fredholm operators of index zero.

Further, let us note that $\langle a(y)\, r_{S_1}\, \mathcal{V}\, g,\, g\rangle_{S_1} > 0$ for arbitrary nonzero $g \in \widetilde{H}^{-\frac{1}{2}}(S_1)$. Therefore, the operator (3.7) is invertible for $s = \frac{1}{2}$, $p = 2$. In turn this implies that the operator (3.7) is invertible for all $s \in \frac{1}{p} - \frac{1}{2}, \frac{1}{p} + \frac{1}{2}$ due to Lemma 3.4. □

Theorem 3.6. *Let S_1 and $S\backslash\overline{S}_1$ be nonempty, open simply connected submanifolds of S with an infinitely smooth boundary curve, $1 < p < +\infty$, $1 \le q \le +\infty$ and $\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2}$. Then the pseudo-differential operators*

$$r_{S_1}\widehat{\mathcal{L}} : \widetilde{H}^s_p(S_1) \longrightarrow H^{s-1}_p(S_1) \tag{3.11}$$

$$: \widetilde{B}^s_{p,q}(S_1) \longrightarrow B^{s-1}_{p,q}(S_1) \tag{3.12}$$

where

$$\widehat{\mathcal{L}} := \Big[\mathcal{L}^{\pm} + \frac{\partial a}{\partial n}\Big(\mp\frac{1}{2}\, I + \mathcal{W}\Big)\Big] \ \ on\ S, \tag{3.13}$$

are invertible, while the operators

$$\begin{aligned} r_{S_1}(\mathcal{L}^{\pm} - \widehat{\mathcal{L}}) : \widetilde{H}^s_p(S_1) &\longrightarrow H^s_p(S_1) \\ : \widetilde{B}^s_{p,q}(S_1) &\longrightarrow B^s_{p,q}(S_1) \end{aligned} \tag{3.14}$$

are bounded and the operators

$$\begin{aligned} r_{S_1}(\mathcal{L}^{\pm} - \widehat{\mathcal{L}}) : \widetilde{H}^s_p(S_1) &\longrightarrow H^{s-1}_p(S_1) \\ : \widetilde{B}^s_{p,q}(S_1) &\longrightarrow B^{s-1}_{p,q}(S_1) \end{aligned} \tag{3.15}$$

are compact.

Proof. By Theorem I.3.6,

$$\widehat{\mathcal{L}} g = \mathcal{L}^{+} g + \frac{\partial a}{\partial n}\Big(-\frac{1}{2}\, I + \mathcal{W}\Big)\, g = \mathcal{L}^{-} g + \frac{\partial a}{\partial n}\Big(\frac{1}{2}\, I + \mathcal{W}\Big)\, g, \tag{3.16}$$

and the operator $r_{S_1}\widehat{\mathcal{L}} : \widetilde{H}^{\frac{1}{2}}(S_1) \to H^{-\frac{1}{2}}(S_1)$ is invertible. Then Lemma 3.5 and (3.16) implies the invertibility of the operators (3.11) and (3.12).

Since

$$\mathcal{L}^{\pm} - \widehat{\mathcal{L}} = \frac{\partial a}{\partial n}\Big(\pm \frac{1}{2}\, I + \mathcal{W}\Big),$$

the operators (3.14) are bounded due to Theorem 3.3. To prove the compactness of $\mathcal{L}^{\pm} - \widehat{\mathcal{L}}$, we remark that the imbeddings $H^s_p(S_1) \subset H^{s-1}_p(S_1)$ and $B^s_{p,q}(S_1) \subset B^{s-1}_{p,q}(S_1)$ are compact, which completes the proof. $\square$

Remark 3.7. By the same arguments as in the proof of Theorem 3.5, one can show that the operators

$$\mathcal{V} : H^{s-1}_p(S) \longrightarrow H^s_p(S) \quad [B^{s-1}_{p,q}(S) \longrightarrow B^s_{p,q}(S)]$$

are invertible for all $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \le q \le \infty$, cf. [**16**].

3. Mapping properties of volume potentials.

Theorem 3.8. *Let Ω^+ be a bounded open three-dimensional region of $\mathbb{R}^3$ with a simply connected, closed, infinitely smooth boundary $S = \partial\Omega^+$ and $1 < p, q < \infty$. The following operators are continuous*

(3.17)
$$\mathcal{P} : \widetilde{H}^s_p(\Omega^+) \longrightarrow H^{s+2}_p(\Omega^+) \quad \Big[\widetilde{B}^s_{p,q}(\Omega^+) \longrightarrow B^{s+2}_{p,q}(\Omega^+)\Big], \qquad s \in \mathbb{R},$$

(3.18)
$$: H^s_p(\Omega^+) \longrightarrow H^{s+2}_p(\Omega^+) \quad \Big[B^s_{p,q}(\Omega^+) \longrightarrow B^{s+2}_{p,q}(\Omega^+)\Big], \qquad s > -1 + \frac{1}{p};$$

(3.19)
$$\mathcal{R} : \widetilde{H}^s_p(\Omega^+) \longrightarrow H^{s+1}_p(\Omega^+) \quad \Big[\widetilde{B}^s_{p,q}(\Omega^+) \longrightarrow B^{s+1}_{p,q}(\Omega^+)\Big], \qquad s \in \mathbb{R},$$

(3.20)
$$: H^s_p(\Omega^+) \longrightarrow H^{s+1}_p(\Omega^+) \quad \Big[B^s_{p,q}(\Omega^+) \longrightarrow B^{s+1}_{p,q}(\Omega^+)\Big] \qquad s > -1 + \frac{1}{p};$$

(3.21)
$$\mathcal{P}^+ : \widetilde{H}_p^s(\Omega^+) \longrightarrow B_{p,p}^{s+2-\frac{1}{p}}(S) \quad \Big[\widetilde{B}_{p,q}^s(\Omega^+) \longrightarrow B_{p,q}^{s+2-\frac{1}{p}}(S)\Big], \qquad s > -2 + \frac{1}{p},$$

(3.22)
$$: H_p^s(\Omega^+) \longrightarrow B_{p,p}^{s+2-\frac{1}{p}}(S) \quad \Big[B_{p,q}^s(\Omega^+) \longrightarrow B_{p,q}^{s+2-\frac{1}{p}}(S)\Big], \qquad s > -1 + \frac{1}{p};$$

(3.23)
$$\mathcal{R}^+ : \widetilde{H}_p^s(\Omega^+) \longrightarrow B_{p,p}^{s+1-\frac{1}{p}}(S) \quad \Big[\widetilde{B}_{p,q}^s(\Omega^+) \longrightarrow B_{p,q}^{s+1-\frac{1}{p}}(S)\Big], \qquad s > -1 + \frac{1}{p},$$

(3.24)
$$: H_p^s(\Omega^+) \longrightarrow B_{p,p}^{s+1-\frac{1}{p}}(S) \quad \Big[B_{p,q}^s(\Omega^+) \longrightarrow B_{p,q}^{s+1-\frac{1}{p}}(S)\Big], \qquad s > -1 + \frac{1}{p};$$

(3.25)
$$T^+\mathcal{P} : \widetilde{H}_p^s(\Omega^+) \longrightarrow B_{p,p}^{s+1-\frac{1}{p}}(S) \quad \Big[\widetilde{B}_{p,q}^s(\Omega^+) \longrightarrow B_{p,q}^{s+1-\frac{1}{p}}(S)\Big], \qquad s > -1 + \frac{1}{p},$$

(3.26)
$$: H_p^s(\Omega^+) \longrightarrow B_{p,p}^{s+1-\frac{1}{p}}(S) \quad \Big[B_{p,q}^s(\Omega^+) \longrightarrow B_{p,q}^{s+1-\frac{1}{p}}(S)\Big], \qquad s > -1 + \frac{1}{p};$$

(3.27)
$$T^+\mathcal{R} : \widetilde{H}_p^s(\Omega^+) \longrightarrow B_{p,p}^{s-\frac{1}{p}}(S) \; \Big[\widetilde{B}_{p,q}^s(\Omega^+) \longrightarrow B_{p,q}^{s-\frac{1}{p}}(S)\Big], \qquad s > \frac{1}{p},$$

(3.28)
$$: H_p^s(\Omega^+) \longrightarrow B_{p,p}^{s-\frac{1}{p}}(S) \Big[B_{p,q}^s(\Omega^+) \longrightarrow B_{p,q}^{s-\frac{1}{p}}(S) \Big], \qquad s > \frac{1}{p}.$$

Proof. Similar to Theorem I.3.8, continuity of the operators (3.17), (3.19), (3.21), (3.23), (3.25) and (3.27) follows from (2.22), (2.23) and the corresponding properties of the operator $\mathcal{P}_\Delta$ due to the mapping properties of pseudodifferential operators on $\mathbb{R}^n$, see e.g., [**10, 18**], and the trace theorems, see e.g., [**20**]. Recall that if $u \in H_p^s(\Omega^+)$ then $[u]^+ \in B_{p,p}^{s-\frac{1}{p}}(S)$ for $1 < p < \infty$ and $s > \frac{1}{p}$.

To prove the remaining items of the theorem we consider in detail operator (3.18) on the scale of Bessel potential spaces (all arguments on the scale of Besov spaces are word for word). First let us assume that $-1 + \frac{1}{p} < s < \frac{1}{p}$. In this case $H_p^s(\Omega^+) = \widetilde{H}_p^s(\Omega^+)$, and the continuity of operator (3.18) is evident due to the above arguments.

Now let $\frac{1}{p} < s < 1+\frac{1}{p}$. For $g \in H_p^s(\Omega^+)$, clearly, $\partial_j g \in H_p^{s-1}(\Omega^+)$ and $g^+ \in B_{p,p}^{s-\frac{1}{p}}(S)$, due to the continuity of the operator $\partial_j : H_p^s(\Omega^+) \to H_p^{s-1}(\Omega^+)$ and the trace theorem, see e.g., [**11**]. For the Newton potential of the Laplace operator we have the following representation

$$\partial_j \, \mathcal{P}_\Delta \, g\,(y) = \mathcal{P}_\Delta \left(\partial_j g\right)(y) + V_\Delta \left(n_j g^+\right)(y) \quad \text{for } y \in \Omega^+, \tag{3.29}$$

where n_j, $j = 1, 2, 3$, are the components of the outward unit normal vector to S. Due to (3.29) and the mapping properties of the single layer potential, cf., Theorem 3.1, we conclude that $\partial_j \, \mathcal{P}_\Delta : H_p^s(\Omega^+) \to H_p^{s+1}(\Omega^+)$ is continuous for $j = 1, 2, 3$, which, along with formula (2.22) implies the continuity of operator (3.18) for $\frac{1}{p} < s < 1 + \frac{1}{p}$.

Further, with the help of these results and the representation (3.29), we can easily verify by induction that the operator (3.18) is continuous for $k - 1 + \frac{1}{p} < s < k + \frac{1}{p}$, where k is an arbitrary nonnegative integer. For the values $s = k+\frac{1}{p}$ (with $k = 0, 1, 2, \dots$) the continuity of operator (3.18) then follows due to the complex interpolation property of Bessel potential and Besov function spaces, see e.g., [**20**, Chapter 4].

It is evident that (3.22) and (3.26) are then the direct consequences of the trace theorem.

The word for word arguments show that the claims of the theorem concerning the operator $\mathcal{R}$ hold as well, which completes the proof. □

4. Regularity and asymptotic properties of solutions. In subsection 4.1 we will establish some regularity results for solutions of the mixed BVP and the BDIEs considered in Part I. In subsection 4.2 we will apply these results (in particular, inclusions (4.14)) in the study of asymptotic behavior of solutions near curve ℓ. Note that solution asymptotics for the boundary integral equations associated with the mixed BVP for *constant-coefficient* PDEs were considered in [**19**].

We will deal with the BDIES $(\mathcal{GT})$ introduced in Section I.5.1 for the unknowns $(u, \psi, \varphi) \in H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N)$:

$$u + \mathcal{R}u - V\psi + W\varphi = \mathcal{F}_1^{\mathcal{GT}} \quad \text{in } \Omega^+, \tag{4.1}$$

$$r_{S_D} \mathcal{R}^+ u - r_{S_D} \mathcal{V}\psi + r_{S_D} \mathcal{W}\varphi = \mathcal{F}_2^{\mathcal{GT}} \quad \text{on } S_D, \tag{4.2}$$

$$r_{S_N} T^+ \mathcal{R}u - r_{S_N} \mathcal{W}'\psi + r_{S_N} \mathcal{L}^+\varphi = \mathcal{F}_3^{\mathcal{GT}} \quad \text{on } S_N. \tag{4.3}$$

Throughout this section we assume that the righthand side of BDIES (4.1)–(4.3) is more smooth than in Part I, namely,

$$\mathcal{F}^{\mathcal{GT}} = (\mathcal{F}_1^{\mathcal{GT}}, \mathcal{F}_2^{\mathcal{GT}}, \mathcal{F}_3^{\mathcal{GT}})^\top \in H^3(\Omega^+) \times H^{\frac{5}{2}}(S_D) \times H^{\frac{3}{2}}(S_N).$$

By Theorems 3.1 and 3.8 it will be particularly the case if $\mathcal{F}^{\mathcal{GT}}$ is generated by the righthand sides of BVP (2.1)–(2.3) as

$$\mathcal{F}^{\mathcal{GT}} := \big[F_0,\, r_{S_D} F_0^+ - \varphi_0,\, r_{S_N} T^+ F_0 - \psi_0\big]^\top, \tag{4.4}$$

cf. (I.5.5), where

$$F_0 := \mathcal{P}f + V\Psi_0 - W\Phi_0 \quad \text{in } \Omega^+, \tag{4.5}$$

Φ_0 is a fixed extension of φ_0 from submanifold S_D to the whole of S, Ψ_0 is a fixed extension of ψ_0 from submanifold S_N to the whole of S, and the following enhanced smoothness conditions are satisfied

$$\begin{aligned} &f \in H_2^1(\Omega^+), \qquad \varphi_0 \in H_2^{\frac{5}{2}}(S_D), \qquad \psi_0 \in H_2^{\frac{3}{2}}(S_N), \\ &\Phi_0 \in H_2^{\frac{5}{2}}(S), \qquad \Psi_0 \in H_2^{\frac{3}{2}}(S). \end{aligned} \tag{4.6}$$

4.1. Some auxiliary smoothness results. By Corollary I.5.4, the system of BDIEs (4.1)–(4.3) with righthand side given by (4.6) has a unique solution

$$(u,\psi,\varphi)\in H^1_2(\Omega^+)\times\widetilde{H}_2^{-\frac{1}{2}}(S_D)\times\widetilde{H}_2^{\frac{1}{2}}(S_N). \tag{4.7}$$

From equation (4.2) it follows that

$$\begin{gathered} r_{S_D}\mathcal{V}(\psi)=\Psi \text{ on } S_D,\\ \Psi:=-\mathcal{F}_2^{\mathcal{GT}}+r_{S_D}\Big(\mathcal{R}^+u+\mathcal{W}\varphi\Big)\in H_2^{\frac{3}{2}}(S_D) \end{gathered} \tag{4.8}$$

due to Theorem 3.2 and mapping property (3.24).

Quite similarly, from equation (4.3) we get

$$\begin{gathered} r_{S_N}\widehat{\mathcal{L}}\varphi=\Phi \text{ on } S_N,\\ \Phi:=\mathcal{F}_3^{\mathcal{GT}}+r_{S_N}\Big\{-T^+\mathcal{R}u+(\widehat{\mathcal{L}}-\mathcal{L}^+)\varphi+\mathcal{W}'\psi\Big\}\in H_2^{\frac{1}{2}}(S_N), \end{gathered} \tag{4.9}$$

where the operator $\widehat{\mathcal{L}}$ is defined by (3.13).

Recall that, see [**20**, Theorem 4.6.2],

$$\begin{gathered} H_2^{\frac{3}{2}}(S_D)\subset B^s_{p,p}(S_D),\qquad H_2^{\frac{1}{2}}(S_N)\subset B^{s-1}_{p,p}(S_N)\\ \text{for } s\le\min\left\{\frac{3}{2},\,\frac{1}{2}+\frac{2}{p}\right\},\quad 1<p<+\infty. \end{gathered}$$

Applying Theorem 3.5 and Corollary 3.6 for

$$s\in\left(\frac{1}{p}-\frac{1}{2},\frac{1}{p}+\frac{1}{2}\right)$$

and then extending the result for smaller s due to the embedding theorem, we derive that if $(\psi,\varphi)\in\widetilde{H}_2^{-\frac{1}{2}}(S_D)\times\widetilde{H}_2^{\frac{1}{2}}(S_N)$ satisfies equations (4.8) and (4.9), then

$$\psi\in\widetilde{B}^{s-1}_{p,p}(S_D),\qquad\varphi\in\widetilde{B}^{s}_{p,p}(S_N)$$

for any p and s such that

$$1<p<+\infty,\qquad s<\frac{1}{p}+\frac{1}{2}. \tag{4.10}$$

By Theorem 3.1 we see that

$$V\psi \in H_p^{s+\frac{1}{p}}(\Omega^+), \qquad W\varphi \in H_p^{s+\frac{1}{p}}(\Omega^+) \tag{4.11}$$

with s and p as in (4.10).

Since $\mathcal{R}u \in H_2^2(\Omega^+)$ for $u \in H_2^1(\Omega^+)$, then from equation (4.1) and inclusions (4.11) along with the embedding theorems for the Besov space, see [**20**, subsection 4.6], we obtain the following Hölder continuity of the solution to the mixed BVP,

$$u = \mathcal{F}_1^{\mathcal{G}\mathcal{T}} - \mathcal{R}u + V\psi - W\varphi \in H_p^{s+\frac{1}{p}}(\Omega^+) \subset C^{s-\frac{2}{p}}(\overline{\Omega^+}), \tag{4.12}$$

where s and p satisfy conditions (4.10) and $s - \frac{2}{p} > 0$. If we take here p sufficiently large and s close to $\frac{1}{2}$ such that the above restrictions on parameters p and s are satisfied, then we see that the inclusion $u \in C^{\frac{1}{2}-\delta}(\overline{\Omega^+})$ holds with arbitrarily small $\delta > 0$.

Further, from (4.12) due to (3.20) and the trace theorem, it also follows that

$$\mathcal{R}u \in H_p^{s+1+\frac{1}{p}}(\Omega^+),\ r_{S_D}\mathcal{R}^+u \in B_{p,p}^{s+1}(S_D),\ r_{S_N}T^+\mathcal{R}u \in B_{p,p}^{s}(S_N),$$

where s and p satisfy conditions (4.10). Then, from formulas (4.8) and (4.9) for Ψ and Φ, we get the following inclusions

$$\Psi \in B_{p,p}^{s+1}(S_D), \qquad \Phi \in B_{p,p}^{s}(S_N), \tag{4.13}$$

with s and p as in (4.10).

For any $\sigma < \frac{1}{2}$, one can find s satisfying (4.10) and $\varepsilon > 0$ such that $s = \frac{1}{p} + \sigma + \varepsilon$. Bearing in mind that $B_{p,p}^{t+\varepsilon}(S_D) \subset H_p^t(S_D)$ for all $t \in (-\infty, +\infty)$, $p \in (1, +\infty)$ and $\varepsilon > 0$, we arrive from (4.13) at the relation

$$\Psi \in H_p^{1+\frac{1}{p}+\sigma}(S_D), \qquad \Phi \in H_p^{\frac{1}{p}+\sigma}(S_N), \tag{4.14}$$

where $\sigma < \frac{1}{2}$.

4.2. Asymptotics of solution. In what follows we derive asymptotic expansion formulas in local coordinates for the components of the solution vector (u, ψ, φ). To this end, in the normal plane $\Pi_{y'}$ to the curve $\ell := \partial S_D = \partial S_N$, we consider a local polar coordinate system (r, ϑ) such that $y = (y', r, \vartheta)$, $r \geq 0$, $\vartheta \in [0, \pi]$. The pole of the local coordinate system, $r = 0$, belongs to the curve ℓ; $\vartheta = \pi$ corresponds to the Dirichlet part of the boundary, while $\vartheta = 0$ corresponds to the Neumann part of the boundary. Moreover, the interior domain corresponds locally to the interval $0 < \vartheta < \pi$. Actually, y' defines some parameterization of the curve ℓ.

Further, we apply the theory of asymptotic expansions of solutions to elliptic pseudodifferential equations on manifolds with boundary, developed in [**10**], see also [**2, 5, 9**].

Note that the principal homogeneous symbols

$$\sigma(\mathcal{V}; y', \xi) = \frac{1}{2}\,[a(y')\,|\xi|]^{-1} \quad \text{and} \quad \sigma(-\mathcal{L}^+; y', \xi) = \frac{1}{2}\,a(y')\,|\xi|,$$

corresponding to operators $\mathcal{V}$ and $-\mathcal{L}^+$, are positive and even functions in ξ for $|\xi| = 1$. Therefore, from (4.8) and (4.9) along with the embedding (4.14) we obtain, similar to [**9**, Theorems 4.1 and 4.2],

$$\psi(y', r) = c_0(y')\,\chi(r)\,r^{-\frac{1}{2}} + \psi_1(y', r), \tag{4.15}$$

$$\varphi(y', r) = b_0(y')\,\chi(r)\,r^{\frac{1}{2}} + \varphi_1(y', r), \tag{4.16}$$

where $\chi \in C_0^\infty(\overline{\mathbb{R}^+})$ is a cut-off function with compact support and $\chi(r) = 1$ for $0 \leq r \leq \varepsilon$ with a suitable $\varepsilon > 0$, while

$$\begin{aligned} c_0 &\in H_p^{\sigma+\frac{1}{2}}(\ell), \quad \psi_1 \in \widetilde{H}_p^{\sigma+\frac{1}{p}}(S_D), \\ b_0 &\in H_p^{\sigma+\frac{3}{2}}(\ell), \quad \varphi_1 \in \widetilde{H}_p^{\sigma+1+\frac{1}{p}}(S_N) \end{aligned}$$

for any $2 < p < +\infty$, $\sigma < \frac{1}{2}$. More detailed analysis, based on the factorization technique, shows that, cf., [**19**, Theorem 4.9],

$$c_0(y') = -\frac{a(y')}{2} b_0(y'). \tag{4.17}$$

Now the asymptotic behavior of u in a spatial vicinity of l can be found from (4.12) with the help of formulae (4.15) and (4.16). In fact, we have

$$u = V\psi - W\varphi + G \quad \text{in } \Omega^+, \tag{4.18}$$

where ψ and φ have the structure given by (4.15) and (4.16), and

$$G := \mathcal{F}_1^{\mathcal{GT}} - \mathcal{R}u \in H_p^{1+\sigma+\frac{2}{p}}(\Omega^+) \subset C^{1+\sigma-\frac{1}{p}}(\overline{\Omega^+}) \tag{4.19}$$

with $2 < p < +\infty$ and $\frac{1}{p} - 1 < \sigma < (1/2)$.

Note that, for $t \in \mathbb{R}$ and $2 < p < +\infty$, we have the embedding (see [**20**, Theorem 4.6.1 (b)]) $\widetilde{H}_p^t(S_1) \subset H_p^t(S) \subset B_{p,p}^t(S)$ for any subsurface S_1 of S. Therefore by Theorem 3.1,

$$V\psi_1 \in H_p^{1+\sigma+\frac{2}{p}}(\Omega^+), \qquad W\varphi_1 \in H_p^{1+\sigma+\frac{2}{p}}(\Omega^+)$$

for $2 < p < +\infty$ and $\sigma < 1/2$. Consequently,

$$V\psi_1 \in C^{1+\sigma-\frac{1}{p}}(\overline{\Omega^+}), \qquad W\varphi_1 \in C^{1+\sigma-\frac{1}{p}}(\overline{\Omega^+}). \tag{4.20}$$

for any p, σ such that $2 < p < +\infty$ and $\frac{1}{p} - 1 < \sigma < \frac{1}{2}$.

Applying the results obtained by Chkadua and Duduchava (see [**4**, Theorem 2.2 and Remark 2.11]) for potential type functions (4.18) with densities (4.15) and (4.16) whose asymptotic expansions are known, we arrive at the following representation near the curve l due to (4.19) and (4.20)

$$u(y', r, \vartheta) = d_0(y', \vartheta)\, \chi(r)\, r^{\frac{1}{2}} + u_1(y', r, \vartheta), \tag{4.21}$$

where

$$\begin{gathered} d_0(y', \vartheta) = d_1(y') \cos\frac{\vartheta}{2} + d_2(y') \sin\frac{\vartheta}{2}, \\ d_1, d_2 \in H_p^{\frac{1}{2}+\sigma}(\ell), \quad u_1 \in C^{\frac{1}{2}+\sigma-\frac{1}{p}}(\overline{\Omega^+}) \end{gathered} \tag{4.22}$$

for any p, σ such that $2 < p < +\infty$ and $\frac{1}{p} - \frac{1}{2} < \sigma < \frac{1}{2}$, and d_1 and d_2 are real.

Membership (4.22) implies $u_1 \in C^{1-\delta}(\overline{\Omega^+})$ for arbitrarily small $\delta > 0$, and we get from (4.21) the best regularity result for the solution of the mixed BVP, $u \in C^{\frac{1}{2}}(\overline{\Omega^+})$.

Subtracting (4.2) from the trace of (4.1) on S_D, we obtain,

$$r_{S_D} u^+ = r_{S_D} (\mathcal{F}_1^{\mathcal{GT}})^+ - \mathcal{F}_2^{\mathcal{GT}} \in H^{\frac{5}{2}}(S_D) \quad \text{on } S_D. \tag{4.23}$$

Taking into account that $\vartheta = \pi$ on S_D near l, inclusion (4.23) implies $d_2(y') = 0$ in (4.22). Moreover, if the BDIES' righthand side $\mathcal{F}^{\mathcal{GT}}$ is generated by the BVP righthand side according to (4.4)–(4.6), then by Theorem I.5.2,

$$\psi = T^+ u - \Psi_0, \qquad \varphi = u^+ - \Phi_0 \quad \text{on } S. \tag{4.24}$$

Substituting in (4.24) asymptotics (4.15)–(4.17) and (4.21)–(4.22) with $d_2 = 0$, and comparing participating terms and their smoothness, we arrive at the following asymptotics of the BDIE solution,

$$u(y', r, \vartheta) = b_0(y') \cos \frac{\vartheta}{2}\, \chi(r)\, r^{\frac{1}{2}} + u_1(y', r, \vartheta), \tag{4.25}$$

$$\psi(y', r) = -\frac{b_0(y')}{2}\, a(y')\, \chi(r)\, r^{-\frac{1}{2}} + \psi_1(y', r), \tag{4.26}$$

$$\varphi(y', r) = b_0(y')\, \chi(r)\, r^{\frac{1}{2}} + \varphi_1(y', r), \tag{4.27}$$

where

$$b_0 \in H_p^{\sigma+\frac{3}{2}}(\ell), \qquad u_1 \in C^{\frac{1}{2}+\sigma-\frac{1}{p}}(\overline{\Omega^+}),$$
$$\psi_1 \in \widetilde{H}_p^{\sigma+\frac{1}{p}}(S_D), \qquad \varphi_1 \in \widetilde{H}_p^{\sigma+1+\frac{1}{p}}(S_N).$$

The smoothness and asymptotic results obtained above for BDIEs (4.1)–(4.3) with the righthand side associated with the BVP righthand sides as in (4.4)–(4.5), will hold true also for the other three BDIE systems considered in Part I (with their righthand sides associated with the same BVP), due to their equivalence to the BVP and thus to each other.

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