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ANALYTIC SIGNALS WITH NONNEGATIVE INSTANTANEOUS FREQUENCY

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ABSTRACT. In this paper we investigate instantaneous frequency as it applies to the Hardy spaces on the disc ($H^p(D)$), and on the upper half-plane ($H^p(\mathbf{C}_+)$). The results obtained can then be applied to any sufficiently smooth analytic signal, as the boundary functions of elements of a Hardy space correspond naturally to analytic signals.

Using only basic results from complex analysis, a more thorough understanding of instantaneous frequency is obtained. This allows the construction of analytic signals with nonnegative instantaneous frequency (ASNIFs) that have a prescribed amplitude. Resulting parallels are then drawn between the concepts of instantaneous frequency and Fourier frequency.

1. Introduction.

Instantaneous frequency has been used in many different contexts, especially in signal processing. It is commonly defined as the time-derivative of the phase of a complex valued signal (usually an analytic signal). In most applications, one does not have complex valued data. The imaginary part is usually obtained via the Hilbert transform, producing an analytic signal.

Instantaneous frequency makes available some interesting forms of analysis to time-frequency problems which do not yield readily to Fourier or wavelet methods. One recently developed method is the empirical mode decomposition (EMD), developed by Huang et al. [**5**]. This method aims to decompose signals into monocomponent functions whose analytic signals have nonnegative instantaneous frequency. (These component functions are called *intrinsic mode functions*, or

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IMFs.) Unfortunately, there have been $H^\infty(D)$ functions identified which satisfy the definition of an IMF, but which have negative instantaneous frequency [**7**]. In order to better understand EMD, there have been alternative decomposition algorithms proposed, see [**1**], which may help investigate the causes of these "pathological" IMFs.

Clearly, it is of interest to EMD research to better understand those analytic signals that have nonnegative instantaneous frequency. Some important steps in this direction have been taken by [**6**]. In this paper we consider analytic signals as being boundary value functions of elements of a Hardy space. (They may also be defined for more general settings, i.e., in the sense of distribution.) Periodic analytic signals can be mapped to the unit circle and extended to elements of a Hardy space for the disc. Nonperiodic analytic signals can be extended to elements of a Hardy space for the upper half-plane. In this paper we investigate the instantaneous frequency of smooth analytic signals using tools from Hardy spaces. These spaces are a natural setting in which to consider analytic signals. They are one of the most general settings in which one might encounter analytic signals, and the body of theory available contains powerful tools.

Section 2 lays out some definitions and preliminaries. Section 3 explores the consequences of Hardy spaces on the disc for instantaneous frequency. A characterization of instantaneous frequency is found for sufficiently smooth elements of $H^p(D)$. Parallel results of this analysis are then stated for the Hardy spaces on the upper half plane. Finally, these results are used to construct analytic signals with nonnegative instantaneous frequency (ASNIFs) in Section 4. The insights gained are used to compare and contrast instantaneous frequency with the classical (Fourier based) understanding of frequency.

2. Preliminaries, definitions and notation. In this section we introduce the notation used and state classical results which are needed to develop our results. Let z denote a complex number, whose real and imaginary parts are given by x and y, respectively, i.e., $z = x+iy$. Whenever polar notation is used, the complex variable z will be expressed as either $z = re^{i\theta}$ or as $z = \rho e^{i\phi}$ to indicate the argument and output value of a function, respectively, i.e., $\rho e^{i\phi} = f(re^{i\theta})$.

D denotes the unit disc $D = \{z : |z| < 1\}$.

$\mathbf{C}_+$ denotes the upper half of the complex plane: $\mathbf{C}_+ = \{z = x + iy \in \mathbf{C} : y > 0\}$.

$H^p(D)$ denotes the Hardy space of functions f, which are holomorphic on D and for which the following holds.

$$\sup_{1>r>0} \frac{1}{2\pi} \int |f(re^{i\theta})|^p d\theta = \|f\|^p_{H^p} < \infty.$$

$H^p(\mathbf{C}_+)$, $0 < p < \infty$, denotes the Hardy space of functions f, which are holomorphic on $\mathbf{C}_+$ and for which the following holds.

$$\sup_{y>0} \int_{-\infty}^{+\infty} |f(x+iy)|^p \, dx = \|f\|^p_{H^p} < \infty. \tag{1}$$

$H^\infty(\mathbf{C}_+)$ denotes the Hardy space of functions f, which are holomorphic and uniformly bounded on $\mathbf{C}_+$. The norm is given by

$$\|f\|^{H^\infty} = \sup_{z \in \mathbf{C}_+} |f(z)| < \infty. \tag{2}$$

The results in this paper apply to functions in $H^p(\mathbf{C}_+)$, $0 < p \leq \infty$.

One well-known result from Hardy spaces will be used. For the spaces $H^p(D)$, we have:

Theorem 1. *If $f \in H^p(D)$, $p > 0$, and f is not identically 0, then $f(z)$ has a unique decomposition*

$$f(z) = B(z)S(z)G(z),$$

where $B(z)$ is a Blaschke product in $H^p(D)$, and $S(z)$ is a singular inner function in $H^p(D)$, and $G(z)$ is an outer function in $H^p(D)$.

On the half plane, an analogous result holds:

Theorem 2. *If $f(z) \in H^p(\mathbf{C}_+)$, $p > 0$, then $f(z)$ has a unique decomposition*

$$f(z) = e^{i\alpha z} B(z)S(z)G(z), \tag{3}$$

where $\alpha \geq 0$, $B(z)$ is a Blaschke product, $S(z)$ is a singular function, and $G(z)$ is an outer function in $H^p(\mathbf{C}_+)$.

For definitions of Blaschke products, singular inner functions and outer functions for the classes $H^p(D)$ and $H^p(\mathbf{C}_+)$, and for further details about the canonical factorization theorem the reader is referred to [**5, 6**]. In this paper we use the notation from [**3**] for the zeros of Blaschke products, the singular measure of singular inner functions, etc.

3. Instantaneous frequency of analytic signals.

Definition 1. *Time* will denote the arc-length parameterization of the boundary of the Hardy domain in question, i.e., for $f \in H^p(\mathbf{C}_+)$ time denotes the real valued coordinate $x = \Re(z)$; for $f \in H^p(D)$, time is given by the angle $\theta = \operatorname{Arg}(z)$ (ignoring for now the optional multiples of 2π).

Definition 2. The *instantaneous frequency* of a complex valued function f is defined to be the time-derivative of its phase, whenever it exists. It is referred to here as $\omega_f(*)$, or as $\omega[f](*)$. This may also be written in terms of the complex logarithm:

$$\omega_f(z) = \frac{d}{dt} \operatorname{Im}(\operatorname{Log}(f)).$$

For $H^p(D)$ time is parameterized by $t = \theta$; for $H^p(\mathbf{C}_+)$, time is parameterized by the real axis $t = x$.

The following result applies to instantaneous frequency of a complex valued function f independent of what space f is in.

Theorem 3. *If f is analytic on an open set $O \subset \mathbf{C}$ and has a representation $f = \prod_n g_n(z)$ on O with convergence in the sense of uniform convergence on compacta of the partial products, then the series representation*

$$\omega_f(z) = \sum_n \omega_{g_n}(z)$$

converges pointwise on O*, and uniformly on compacta in* $O\backslash\{z : f(z) = 0\}$.

Proof. Let A be a compactum contained in $O\backslash\{z : f(z) = 0\}$. Let $f_m = \prod_{n=1}^{m} g_n$ on A. Clearly $\mathrm{Log}\,(f_m)$ is analytic on A, and

$$\mathrm{Log}\,(f_m) \longrightarrow \mathrm{Log}\,(f) + 2k\pi i, \quad \text{for some} \quad k \in \mathbf{Z}$$

uniformly on A. The result then follows from the analytic convergence theorem. □

Corollary 1. *If* $f(z) = Cg(z)$*,* C *constant, then* $\omega_f(z) = \omega_g(z)$.

Note that these results apply only on the open set $O \setminus \{z : f(z) = 0\}$. ($O$ may be D or $\mathbf{C}^+$ as appropriate.) Establishing strong results for the boundary values is beyond the scope of this paper. The boundary value results used in this paper require that f belong to some smoothness class within H^p.

3.1. *Periodic analytic signals.* Using the Cauchy-Riemann equations and Definition 2, we have

$$\begin{aligned}\lim_{|z|\to 1-} \omega_f(z) &= \lim_{|z|\to 1-} \mathrm{Im}\left(\frac{\partial}{\partial\theta}\mathrm{Log}\, f(z)\right) = \lim_{|z|\to 1-} \mathrm{Im}\left(iz\frac{d}{dz}\mathrm{Log}\, f(z)\right) \\ &= \lim_{|z|\to 1-} \mathrm{Re}\left(z\frac{f'(z)}{f(z)}\right).\end{aligned}$$

(The limits here are understood to be taken nontangentially. See [**3, 4**].) This limit does not necessarily converge for $f \in H^p$. General conditions for boundary convergence of the first derivative are still an open question for the Hardy spaces. To guarantee the existence of these limits on the boundary, further conditions will be imposed when this formula is used to ensure convergence.

We restrict ourselves to functions $f \in H^p(D)$ for which the boundary value function $f(\theta)$ is differentiable, and satisfies

$$\omega_f(\theta) = \lim_{|z|\to 1-} \omega_f(z). \tag{4}$$

Sufficient conditions used to guarantee this are given in Theorem 5.

Lemma 1. *The instantaneous frequency of a finite Blaschke product is nonnegative on* ∂D.

This result is not new, see [**7**]. It is easily shown, see [**6**], for Blaschke products that the instantaneous frequency function $\omega_B(\theta)$ is given by a summation of Poisson kernels:

$$\omega_B(\theta) = \sum_n \frac{1 - r_n^2}{r_n^2 - 2r_n \cos(\theta_n - \theta) + 1} > 0.$$

Here $z_n = r_n e^{i\theta_n}$ are the zeros of $B(z)$.

Lemma 2. *For a singular function* $S(z)$*, whose singular support* $\mathrm{supp}\,(d\nu)$ *is a discrete set, instantaneous frequency is nonnegative.*

Proof. The instantaneous frequency of a singular function in $H^p(D)$ is given by:

$$\begin{aligned} \omega_S(z) &= \mathrm{Re}\left(z\frac{S'(z)}{S(z)}\right) \\ &= \mathrm{Re}\left(-\int_{-\pi}^{\pi} \frac{2ze^{iu}}{(e^{iu}-z)^2}\, d\mu(u)\right) \\ &= -\int_{-\pi}^{\pi} \frac{2\mathrm{Re}\,(ze^{-iu} - 2|z|^2 + \bar{z}|z|^2 e^{iu})}{|e^{iu}-z|^4}\, d\mu(u) \\ &= \int_{-\pi}^{\pi} \frac{4|z|^2 + 2(1+|z|^2)|z|\cos(\theta - u)}{|e^{iu}-z|^4}\, d\mu(u) \end{aligned} \tag{5}$$

If $\mathrm{supp}\,(d\nu)$ is a discrete set, then limits can be taken almost everywhere on the boundary, and we have:

$$\begin{aligned}
\omega_S(\theta) &= \lim_{|z|\to 1-} \int_{-\pi}^{\pi} \frac{4|z|^2 + 2(1+|z|^2)|z|\cos(\theta-u)}{|e^{iu}-z|^4}\, d\mu(u) \\
&= \int_{-\pi}^{\pi} \frac{4(1+\cos(\theta-u))}{(|e^{iu}|^2+|z|^2-2|e^{iu}z|\cos\theta-u))^2}\, d\mu(u), \quad z \notin \operatorname{supp}(\mu) \qquad (6) \\
&= \int_{-\pi}^{\pi} \frac{4(1+\cos(\theta-u))}{(2-2\cos(\theta-u))^2}\, d\mu(u), \quad z \notin \operatorname{supp}(\mu) \\
\omega_S(\theta) &= \int_{-\pi}^{\pi} \frac{1+\cos(\theta-u)}{(1-\cos(\theta-u))^2}\, d\mu(u) \geq 0, \quad z \notin \operatorname{supp}(\mu)
\end{aligned}$$

$$\omega_S(\theta) \geq 0 \text{ almost everywhere.} \qquad \square$$

Theorem 4. *An outer function $G(z)$ which has a meromorphic continuation to some open set S containing $D \cup \partial D$ has mean instantaneous frequency given by*:

$$\frac{1}{2\pi}\int_0^{2\pi} \omega_G(\theta)\, d\theta = \pi \sum_{z_m \in \partial D} \operatorname{Re}\left[\operatorname{Res}\left(\frac{G'(z)}{G(z)}; z_m\right)\frac{1}{2\pi}\right].$$

Proof. If G has a meromorphic continuation to an open set O which contains $D \cup \partial D$, then $G'(z)/G(z)$ is also meromorphic on O and has no poles inside D. Let $P = \{z \in O$ such that $G'(z)/G(z)$ has a pole at $z\}$. Because $G(z)$ is meromorphic on O, P must be a finite set of isolated points. Let $\{z_m\} = P \cap \partial D$. Choose $\varepsilon < 1/2 \min_{k\neq l}|z_k - z_l|$. Let $Q_\varepsilon = \cup_{z_k} B(z_k;\varepsilon)$ be the union of all the open ε-balls around the points z_k. Let Γ_ε be the closed contour consisting of arcs on ∂D, and let arcs on ∂Q_ε be given by $\Gamma_\varepsilon = (\partial D \backslash Q_\varepsilon) \cup (\partial Q_\varepsilon \cap D)$. Clearly $G'(z)/G(z)$ has no poles on $\Gamma_\varepsilon \cup \operatorname{int}(\Gamma_\varepsilon)$. Integrating counterclockwise around the countour Γ_ε gives

$$0 = \int_{\Gamma_\varepsilon} \frac{G'(z)}{G(z)}\, dz.$$

Taking the limit as $\varepsilon \to 0$ gives

$$
\begin{aligned}
0 &= \lim_{\varepsilon\to 0+} \int_{\Gamma_\varepsilon} \frac{G'(z)}{G(z)}\, dz \\
&= \lim_{\varepsilon\to 0+} \left(\int_{\Gamma_\epsilon \cap \partial D} \frac{G'(z)}{G(z)}\, dz \right) + \lim_{\varepsilon\to 0+} \left(\int_{\Gamma_\varepsilon \cap D} \frac{G'(z)}{G(z)}\, dz \right) \\
&= \int_{\partial D} \frac{G'(z)}{G(z)}\, dz - \frac{1}{2} \sum_{z_m} 2\pi i \mathrm{Res} \left[\frac{G'(z)}{G(z)}; z_m \right] \\
&= \int_0^{2\pi} \frac{G'(e^{i\theta})}{G(e^{i\theta})}\; i\, e^{i\theta}\, d\theta - \pi i \sum_{z_m} \mathrm{Res} \left[\frac{G'(z)}{G(z)}; z_m \right] \\
&= \int_0^{2\pi} \frac{G'(e^{i\theta})}{G(e^{i\theta})}\; e^{i\theta}\, d\theta - \pi \sum_{z_m} \mathrm{Res} \left[\frac{G'(z)}{G(z)}; z_m \right] \\
&= \int_0^{2\pi} \mathrm{Re} \left(\frac{G'(e^{i\theta})}{G(e^{i\theta})}\; e^{i\theta} \right) d\theta - \mathrm{Re} \left(\pi \sum_{z_m} \mathrm{Res} \left[\frac{G'(z)}{G(z)}; z_m \right] \right) \\
&= \int_0^{2\pi} \omega_G(\theta)\, d\theta - \mathrm{Re} \left(\pi \sum_{z_m} \mathrm{Res} \left[\frac{G'(z)}{G(z)}; z_m \right] \right). \qquad \square
\end{aligned}
$$

Corollary 2. *An outer function $G(z)$ which is nonzero on ∂D, and which has an analytic continuation to some open set S containing $D \cup \partial D$ has zero mean instantaneous frequency. Particularly, any nonconstant outer function which has such an analytic continuation has negative instantaneous frequency at some point on ∂D.*

Theorem 5. *If $f \in H^p(D)$, and f has a meromorphic continuation to some open set S containing $D \cup \partial D$, then $\omega_f(\theta)$ can be decomposed into*

$$\omega_f(\theta) = \omega_B(\theta) + \omega_S(\theta) + \omega_G(\theta),$$

where B, S and G are determined by the canonical factorization of f.

Proof. The existence of a meromorphic continuation is a very strong condition. It implies that the singular support $\mathrm{supp}\,(d\nu)$ of $S(z)$ is discrete, and also that the Blaschke product is finite (because any infinite Blaschke product has infinitely many poles inside any such open set S). The result then follows from the canonical factorization theorem and Theorem 3. $\square$

3.2. *Nonperiodic analytic signals.* Nonperiodic analytic signals are elements of $H^p(\mathbf{C}_+)$ spaces, which $h = h$ shares many characteristics with the $H^p(D)$ spaces. (The most obvious difference is that nonzero constant functions are not elements of $H^p(\mathbf{C}_+)$, while they are elements of $H^p(D)$.) The results are stated without proof, as they are almost identical to the results for $H^p(D)$.

We restrict ourselves to functions $F(z) \in H^p(\mathbf{C}_+)$ which are "nice" on the boundary, giving:

$$\omega_F(x) = \lim_{\mathrm{Im}(z)\to 0+} \mathrm{Im}\left(\frac{F'(z)}{F(z)}\right). \tag{7}$$

This is not true in general; the boundary value function $F(x)$ may not be differentiable, meaning the lefthand side of equation (7) is not defined. In order to use equation (7), it is sufficient to assume that $F(z)$ has a meromorphic continuation to an open domain containing $\mathbf{R} \cup \mathbf{C}_+$.

The canonical factorization theorem, Theorem 2, for $H^p(\mathbf{C}_+)$ is slightly different from that for $H^p(D)$. (See [**3, 4**].)

Following the terminology for $H^p(D)$, we will refer to the parts of instantaneous frequency in $H^p(\mathbf{C}_+)$ given by the factors in equation (3) as the Fourier, Blaschke, singular and outer parts, respectively. Under similar conditions to $H^p(D)$, the instantaneous frequency of the Fourier, Blaschke and singular parts in $H^p(\mathbf{C}_+)$ are all nonnegative on $\mathbf{R}$. There are some differences worth noting:

1. It is sufficient to require that the zero set of the Blaschke product be a discrete set. This ensures that the set of zeros $\{z_n\}$ has no cluster points on $\mathbf{R}$. This then guarantees that $\omega_B(z)$ is well defined on $\mathbf{R}$. Note that in the case of $H^p(D)$, the discreteness condition implies finiteness of the Blaschke product. Thus, it is more appropriate to think of discreteness as the general rule being applied, rather than finiteness.

2. Similarly, it is sufficient to require that the singular support $\mathrm{supp}\,(d\nu)$ be a discrete set.

Under these conditions, we have

$$\omega[e^{i\alpha z}](x) = \alpha \geq 0 \tag{8}$$

$$\omega_B(x) = \sum_n \frac{2}{y_n} \frac{1}{(x_n - x/y_n)^2 + 1} \geq 0$$

$$\omega_S(x) = \int_{-\infty}^{\infty} \frac{u^2+1}{(u-x)^2}\, d\nu(u) \geq 0 \tag{9}$$

When evaluating $\omega_B(x)$ above, we admit the possibility that $z_n = i$.

For outer functions in $H^p(\mathbf{C}_+)$, we have the following result.

Theorem 6. *Let $G(z)$ be an outer function in $H^p(\mathbf{C}_+)$ which has a meromorphic continuation to some open set S containing $\mathbf{C}_+ \cup \mathbf{R}$. Then*

$$P.V. \int_{\mathbf{R}} \omega_G(x)\, dx = \pi \sum_{z_k \in \mathbf{R}} \operatorname{Re}\left[Res\left(\frac{G'(z)}{G(z)}; z_k \right) \right],$$

where $P.V.$ denotes the Cauchy principal value of the integral.

This result may be proved in a similar manner to Theorem 4, with modifications to allow for the fact that the function $G'(z)/G(z)$ may have an infinite number of isolated poles on $\mathbf{R}$.

Corollary 3. *An outer function $G(z)$ which is nonzero on $\mathbf{R}$, and which has an analytic continuation to some open set S containing $\mathbf{C}_+ \cup \mathbf{R}$ has zero mean instantaneous frequency, in the sense of Cauchy principal value. Particularly, any outer function which is nonzero on $\mathbf{R}$ which has such an analytic continuation has negative instantaneous frequency at some point $x_0 \in \mathbf{R}$.*

For nonperiodic signals, an analogous result to (5) then follows:

Theorem 7. *If $f \in H^p(\mathbf{C}_+)$, and f has a meromorphic continuation to some open set S containing $\mathbf{C}_+ \cup \mathbf{R}$, then $\omega_f(t)$ can be decomposed into*

$$\omega_f(t) = \alpha + \omega_B(t) + \omega_S(t) + \omega_G(t),$$

where B, S, and G are determined by the canonical factorization of f.

Proof. Clearly $\omega[e^{i\alpha t}] = \alpha$. The existence of a meromorphic continuation ensures that $\omega[e^{i\alpha t}]$, $\omega_B(z)$, $\omega_S(z)$, and $\omega_G(z)$ all exist and are well-defined on $\mathbf{C}_+ \cup \mathbf{R}$. The result then follows from the canonical factorization theorem and Theorem 3. $\square$

4. Constructing ASNIFs. In recent literature [**2, 8**] it is of particular interest how, given an amplitude function $a(x)$, one can construct an analytic signal with nonnegative instantaneous frequency (ASNIF) with prescribed amplitude. Using the results found here, we can give a more complete answer to this problem. In this section, we state each result for only one of $H^p(\mathbf{C}_+)$ and $H^p(D)$. (In each instance, analogous results for the other case are readily available with appropriate modification.)

From Theorem 2 it is clear that an analytic signal $F(x)$ has amplitude $|F(x)| = a(x)$ if and only if its outer factor $G(z)$ is equal to

$$G(z) = \exp\left\{\frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{(1+uz)\log a(u)}{(u-z)(1+u^2)}\,du\right\}. \tag{10}$$

Thus, choosing a given amplitude function $a(x)$ completely determines the outer factor of any ASNIF $F(x)$ for which $|F(x)| = a(x)$. (In the rest of this section, any reference to $G(z)$ or $G(x)$ will refer to the function given by equation (10), i.e., the outer function with amplitude equal to $a(x)$.)

We are then free to choose three other factors in the canonical factorization as needed in order to ensure that $F(z) = e^{i\alpha z}B(z)S(z)G(z)$ has nonnegative frequency on the whole real line. It remains to be shown under what conditions this is possible.

4.1. *Outer functions with locally bounded instantaneous frequency.* The easiest outer function to make into an ASNIF is one which satisfies the hypotheses of Theorem 6. For such an outer function, instantaneous frequency is negative somewhere, but it is locally bounded for all x. If it is uniformly bounded (by some M), all that is needed is a factor of the form e^{iMx}, giving $F(x) = e^{iMx}G(x)$ as an ASNIF with the required amplitude.

This is not the only way to construct an ASNIF $F(x)$, only the simplest. Multiplying two complex signals is equivalent to the pointwise addition of their instantaneous frequencies. Singular inner functions or Blaschke products may be used instead of e^{iMx}. There are many such choices available, and many different criteria that one might use to make that choice. Those criteria are not dictated by the theoretical results presented here. That choice would best be governed by application-specific concerns (such as minimizing the total variation of phase, for example).

If $\omega_G(x)$ is locally bounded but not uniformly bounded, we need to use an inner function $I(z)$ whose instantaneous frequency $\omega_I(x)$ is everywhere greater than $-\omega_G(x)$.[1] Such a function can easily be shown to exist. One way is to iteratively "build" a Blaschke product or singular function by adding zeros, or adding point masses to the singular measure, respectively, so that $\omega_B(x)$ or $\omega_S(x)$ is large enough on each successive pair of intervals $[-(k+1), -k] \cup [k, k+1]$. This can be done by a fairly simple induction.

4.2. *Outer functions whose instantaneous frequency is not locally bounded.* For signals with nonlocally bounded instantaneous frequency, we look at signals from $H^p(D)$. If the outer function $G(z)$ has zeros and/or poles on ∂D, then $\omega_G(t)$ is not locally bounded. This is easily checked by letting $G = (z - z_0)^k G_1(z)$ where $G_1(z)$ is an outer function which is analytic at z_0. If $k > 0$, then G has a zero of order k at z_0; if $k < 0$, then G has a pole of order $|k|$ at z_0. In either case we have

$$\begin{aligned}\omega_G(z) &= \omega\left[(z - z_0)^k\right](z) + \omega\left[G_1(z)\right](z) \\ &= \frac{kz}{z - z_0} + \omega\left[G_1(z)\right](z).\end{aligned}$$

Clearly the first piece $kz/(z - z_0)$ is not locally bounded near z_0, whether k is positive or negative, i.e., whether G has a zero or a pole at z_0, respectively. Even so, it may still be possible to construct an inner function $I(z)$ such that the analytic signal $I(t)G(t)$ has nonnegative instantaneous frequency almost everywhere.

[1] This is both necessary and sufficient, as $\omega_1(x) \geq 0$ by definition. Thus $\omega_1(x) \geq -\omega_G(x) \iff \omega_1(x) \geq \min\,(0, -\omega_G(x))$

We restrict ourselves to signals $G(t)$ for which $G(z)$ has an analytic continuation to some open set O containing $D \cup \partial D$, and for which the function $G'(z)/G(z)$ has a finite number of zeros and poles on ∂D. For such signals, $G(z)$ can be factorized into the form

$$G(z) = G_1(z)F(z), \tag{11}$$

where $G_1(z)$ is an outer function which satisfies Corollary 2, and $F(z)$ is of the form $F(z) = \prod_{k=1}^{N}(z - z_k)^{p_k}$, $N < \infty$, $p_k \in \mathbf{Z}$. ($F(z)$ cannot have any essential singularities as this would cause $G(z)$ not to be an outer function. See [**3, 4**].) We need only concern ourselves with the instantaneous frequency of $F(z)$, as $G_1(z)$ was addressed in the previous section. We then have

$$\begin{aligned}\omega_F(t) &= \sum_{k=1}^{N} \frac{d}{dt}\mathrm{Im}\left[\mathrm{Log}\left((t - x_k)^{p_k}\right)\right]\\ &= \sum_{k=1}^{N} \frac{d}{dt}(p_k\pi \cdot H(t - x_k)),\ \ (H(t) \quad \text{is the Heaviside function.})\\ &= \sum_{k=1}^{N} p_k\pi \cdot \delta(t - x_k) = 0 \quad \text{almost everywhere.}\end{aligned}$$

It is not clear how a jump in phase relates to instantaneous frequency. It is tempting to simply treat these jumps as being components of a "weak" instantaneous frequency. (Hence the use of the δ function above.) Upon closer inspection, it is clear that the value of a phase jump is not well defined. A (nonzero) phase jump value is only unique up to addition of $2n\pi$, $n \in \mathbf{Z}$. Thus, it is not clear what is meant by a positive, or negative, jump in instantaneous frequency.

In this paper we simply ignore the phase jumps, as there is currently no clear way to reconcile them with the concept of instantaneous frequency. This simplifies things greatly, as $\omega_H(t) = 0$ at every other point. We continue with this simplified analysis, but it is an open question how the phase jumps should be properly understood. If a more precise framework for treating phase jumps is desired, one might be tempted to define the phase jump at z_0 as "the nontangential limit of phase jump as one approaches z_0." However, this does not produce a well-defined quantity.

We now have everything we need to construct an ASNIF whose amplitude is given by an outer function $G(z)$ which has a meromorphic continuation. For any such $G(z)$, find the factorization $G(z) = G_1(z)H(z)$ prescribed in equation (11). Let $B^*(t)$ be a Blaschke product such that $\phi_{B^*}(t) \geq |\phi_{G_1}(t)|$. Then one ASNIF with outer part G is given by

$$F(t) = B^*(t)G(t).$$

5. Conclusions. It is important to note once more that the results presented in this paper only apply to a fairly or meromorphic exclusive If a more precise framework for treating phase jumps is desired, one might be tempted to define the phase jump at z_0 as "the non-tangential limit of phase jump as one approaches z_0". However, this does not produce a well-defined quantity."subclass of analytic signals: those elements of $H^p(D)$ or $H^p(\mathbf{C}_+)$ which admit an analytic continuation which is "nice enough" on a suitably large open set. The results as stated do not apply to the entire Hardy space in question. There is significant work required to establish results which properly address the questions of convergence for signals where the boundary values of the first derivative (and thus the concept of instantaneous frequency itself) are not readily available. However, for analytic signals which are generated by real-world time series data the results in this paper are often sufficient to conduct a meaningful analysis.

As progress is made towards a complete characterization of ASNIFs, we see more clearly the parallels and differences between how one should understand "instantaneous frequency" (IF) and the classical concept of "Fourier frequency" (FF).

The components of instantaneous frequency in the IF setting have natural parallels in the FF setting. Blaschke products for $H^p(D)$ are natural generalizations of the individual modes of the Fourier series.[2] Taking this one step further, the set of inner functions is an even broader generalization of the Fourier modes. In the case of $H^p(\mathbf{C}_+)$ and $H^p(D)$, the set of inner functions is a family of "pure instantaneous frequency" functions which is a natural generalization of the "pure Fourier frequency" functions.

The notion of what a "pure amplitude" function should be illustrates a divergence between the IF and FF perspectives. In the FF understanding, an amplitude function is understood to be purely real (and nonnegative). In the IF world, amplitude functions correspond to outer functions, which are never purely real valued.[3] In fact, they always have nonzero instantaneous frequency if they have any amplitude modulation. At first glance, this seems like a stark contrast between the IF and FF perspectives. A more careful read on the situation brings to light a surprising parallel: Outer functions (that are reasonably nice) have zero mean instantaneous frequency; the Fourier transform of a purely amplitude modulated (AM) signal has symmetric support. This leads to analogous procedures for making signals with positive frequency in both the IF and FF realms. In the FF realm, a band limited signal is made to have positive Fourier frequency by modulating it with a carrier frequency (a pure FF function) high enough to shift the Fourier frequencies above zero. (This is the well-known procedure for making an analytic signal out of an arbitrary band-limited signal.) In the IF realm, an analytic signal is made to have positive instantaneous frequency by modulating it with an inner function (a pure IF function) appropriately chosen to shift the instantaneous frequencies above zero. The analogy is quite clear: ASNIFs are to analytic signals as analytic signals are to "left band-limited signals."[4]

One problem of interest in applications is how to construct an ASNIF with a given amplitude function. This is solved here for a reasonably general class of outer functions. The restriction of the amplitude function to having only a finite number of zeros and poles (of only finite order) on the boundary is general enough for many applications. The focus of this paper is not so much on obtaining a particularly "good" ASNIF, but rather to outline the general approach, leaving the choice of method to be decided by the needs of the particular application.

Using tools from Hardy spaces to investigate analytic signals, a characterization of sufficiently smooth ASNIFs is a fairly straightforward exercise. The canonical factorization theorem gives telling insight into the order of developments made towards characterizing ASNIFs: First the Fourier part was described [**8**], then the Blaschke part [**2**], and finally the outer and singular parts. For sufficiently smooth signals the results presented here are sufficient. However, there are significant open questions remaining:

1. Can instantaneous frequency be defined meaningfully for classes of nonsmooth analytic signals? If so, how?

2. Do such results (or the possible lack thereof) illustrate significant

parallels or differences between the concepts of Fourier and instantaneous frequency?

ENDNOTES

1. This is both necessary and sufficient, as $\omega_I(x) \geq 0$ by definition. Thus $\omega_I(x) \geq -\omega_G(x) \Leftrightarrow \omega_I(x) \geq \min(0, -\omega_G(x))$.

2. The Fourier modes for periodic functions are in fact Blaschke products which have no zeros other than at the origin. For nonperiodic signals, the Fourier transform modes correspond to singular inner functions which have only one point mass at infinity. "At infinity" is used loosely here. The meaning is clear when looking at a conformal mapping of $D \to H$. The singular inner functions for D which have a point mass at the preimage of ∞ become the Fourier modes $e^{i\omega t}$.

3. In the $H^p(D)$ setting, outer functions may be constant. Thus, $G(z) = c \in \mathbf{R}$ is an outer function for $H^p(D)$. In $H^p(\mathbf{C}_+)$, outer functions may not be constant, ruling out this possibility. In either case, there are no nontrivial real-valued outer functions.

4. That is to say, signals whose Fourier transform vanishes for $\omega < K \in \mathbf{R}$.

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