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ON NONLINEAR INTEGRAL EQUATIONS IN THE SPACE OF FUNCTIONS OF BOUNDED GENERALIZED φ-VARIATION

DARIA BUGAJEWSKA, DARIUSZ BUGAJEWSKI AND GRZEGORZ LEWICKI

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ABSTRACT. The purpose of this paper is to deal with the superposition operator as well as with solutions to nonlinear integral equations in spaces of functions of bounded generalized φ-variation.

1. Introduction. The notion of φ-variation of a real function was introduced by L.C. Young [**13**] (see also [**14**]) in connection with the investigation of the behaviour of Fourier series. This concept seems to be one of the most important generalizations of the classical variation in the sense of Jordan. It is noteworthy to recall that the space of functions of bounded φ-variation from the point of view of functional analysis and some applications were studied by J. Musielak and W. Orlicz [**11**], and R. Leśniewicz and W. Orlicz [**9**]. Additionally, composing functions of bounded φ-variation were investigated by J. Ciemnoczołowski and W. Orlicz [**6**]; in particular, they proved a generalization of the result by M. Josephy [**8**] with regard to composing functions of bounded variation in the sense of Jordan.

We recall that basic results concerning the superposition operator in different spaces, in particular in the space of functions of bounded variation in the sense of Jordan as well as exhaustive references on this topic can be found in [**2**].

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We recall also that the parameter t in $\varphi(t,u)$, in connection with spaces of functions of bounded φ-variation, was introduced and investigated in papers by S. Gniłka (see e.g. [**7**]). Such spaces are called spaces of functions of generalized bounded φ-variation.

This paper serves two main purposes. Firstly, it will function to further our interest in the superposition operator acting in the space of functions of generalized bounded φ-variation. In particular, for the large class of functions $\varphi(t,u)$, we will formulate the conditions which ensure the composition operator maps the space of functions of generalized bounded φ-variation into itself (see Corollary 1). Our results expand on the results proved by Ciemnoczołowski and Orlicz [**6**].

Secondly, it will function as a medium by which we can observe solutions, particularly continuous solutions, to nonlinear integral equations which are functions of generalized bounded φ-variation.

Our results generalize the previous ones from the papers [**3, 4, 5**]. Let us draw a reader's attention to Remark 1. In a sense, this remark explains the significance of our results. In particular, for some class of functions $\varphi(t,u)$ we obtain solutions to equations under consideration, which are functions of bounded variation in the sense of Jordan, constant on each interval of continuity.

The paper is organized as follows. In Section 2 we will collect a few definitions and facts which will be needed in the sequel. Section 3 contains results about the performana of the autonomous superposition operator in the space of functions of generalized bounded φ-variation. Finally, in Sections 4-6 we will deal with solutions to the nonlinear Hammerstein as well as the Volterra-Hammerstein integral equation which are functions of generalized bounded φ-variation. We will prove a number of existence results concerning local and global solutions to these equations.

The proofs of the theorems from Section 6 are based on the Leray-Schauder alternative for contractions from [**12**].

2. Preliminaries. In this section we will collect some definitions and results which will be needed in the sequel. Throughout this paper we assume that $\varphi : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$, $a < +\infty$ satisfies the following conditions:

(i) $\varphi(t,u)$ is a continuous, nondecreasing function of $u \geq 0$ for every $t \in [0,a]$, $\varphi(t,u) \to +\infty$ as $u \to +\infty$;

(ii) $\varphi(t,0) = 0$ for every $t \in [0,a]$ and $\varphi(0,u) = 0$ implies $u = 0$.

Let $X = \{x : [0,a] \to \mathbb{R}\}$. Recall that for a function $x \in X$, the number

$$V_\varphi(x) = \sup_\Pi \sum_{i=1}^{n} \varphi(s_i, |x(t_i) - x(t_{i-1})|),$$

where the supremum is taken over all partitions $\Pi : 0 = t_0 < t_1 < \cdots < t_n = a$ with intermediate points $s_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, n$, is called the generalized φ-variation of the function x in $[0,a]$. Let us denote

$$BV_\varphi = BV_\varphi(I) = \{x \in X : V_\varphi(\lambda x) < +\infty \text{ for some } \lambda > 0\},$$

where $I = [0,a]$. It is well-known that, if φ satisfies the condition:

(iii) $\varphi(t,u)$ is a convex function of u for all $t \in [0,a]$,

then $BV_\varphi(I)$ with the norm

$$\|x\|_{V_\varphi} = \inf\left\{\varepsilon > 0 : V_\varphi\left(\frac{x}{\varepsilon}\right) \leq 1\right\}$$

is a Banach space (see [**10**], Theorem 10.8, p.71 and Theorem 1.5, pp. 2-3). Elements of this space will be called generalized BV_φ-functions and solutions to integral equations belonging to this space will be called generalized BV_φ-solutions.

Let us denote $\psi(u) = \sup_{0 \leq s \leq a} \varphi(s,u)$ and we will assume that the following condition is satisfied

(iv) if $\psi(u) = 0$, then $u = 0$.

For other basic concepts concerning modular spaces (as e.g. φ-function, s-convexity, the condition Δ_2 for small u) a reader is referred to [**10**].

3. Superposition operator in $BV_\varphi(I)$. We start with the following

Lemma 1. *Let $\varphi : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy conditions* (i) *and* (ii). *Let $F_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions such that $F_n(0) = 0$. Assume* *that for any $v>0$ there exists $K_v>0$ such that for any $u_1, u_2 \in [-v, v]$ and $n \in \mathbb{N}$*

$$|F_n(u_1) - F_n(u_2)| \leq K_v |u_1 - u_2|.$$

Then for any $x \in BV_\varphi$ there exists $\lambda > 0$ such that

$$\sup_{n \in \mathbb{N}} V_\varphi(\lambda(F_n \circ x)) < +\infty.$$

Proof. Fix $x \in BV_\varphi$ and $\lambda > 0$ with $V_\varphi(\lambda x) < +\infty$. By [**10**] Theorem 10.7, (a), p. 69, there exists $v > 0$ such that for any $t \in [0, a]$, $|x(t)| < v$. By our assumptions, for any partition $\Pi = \{t_0, t_1, ..., t_l\}$ of $[0, a]$ and $s_i \in [t_{i-1}, t_i]$, $i = 1, ..., l$, we have

$$\sum_{i=1}^{l} \varphi(s_i, \frac{\lambda}{K_v} |F_n(x(t_i)) - F_n(x(t_{i-1}))|)$$
$$\leq \sum_{i=1}^{l} \varphi(s_i, \lambda |x(t_i) - x(t_{i-1})|) \leq V_\varphi(\lambda x) < +\infty.$$

Hence $\sup_{n \in \mathbb{N}} V_\varphi(\frac{\lambda}{K_v}(F_n \circ x)) < +\infty$, as required. □

Before presenting the next results, we firstly introduce some notation. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, nondecreasing function such that

(v) $\lim_{u \to +\infty} g(u) = +\infty$;

(vi) $g(u) = 0$ if and only if $u = 0$;

(vii) g satisfies Δ_2 condition for small u, that is there exists a constant $\tilde{k} > 0$ such that $g(2u) \leq \tilde{k} g(u)$ for $0 \leq u \leq \tilde{u}$, where $\tilde{u} > 0$ is fixed.

Furthermore, we assume that there exist positive constants M, m and $u_0 > 0$ such that for any $u \in [0, u_0]$ and $t \in [0, a]$

$$mg(u) \leq \varphi(t, u) \leq Mg(u). \tag{1}$$

Now we can state

Therorem 1. *Let $\varphi : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the conditions* (i),(ii), (v)-(vii) *and* (1). *Let $F_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions such that $F_n(0) = 0$. Then the following conditions are equivalent:*

(a) *For any $x \in BV_\varphi$ there exists $k > 0$ such that*

$$\sup_{n\in\mathbb{N}} V_\varphi(k(F_n \circ x)) < +\infty;$$

(b) *For any $v > 0$ there exists $K_v > 0$ such that for any $u_1, u_2 \in [-v, v]$ and $n \in \mathbb{N}$*

$$g(|F_n(u_1) - F_n(u_2)|) \leq K_v g(|u_1 - u_2|).$$

Proof. Assume that condition (a) is satisfied and fix $x \in BV_\varphi$. By [**10**] Theorem 10.7, (b), p. 69, there exists $M > 0$ such that for any $n \in \mathbb{N}$ and $t \in [0,a]$,

$$|F_n(x(t))| < M. \tag{2}$$

Hence there exists $k_1 > 0$ such that $k_1|F_n(x(t))| < \frac{u_0}{2}$ for any $t \in [0,a]$ and $n \in \mathbb{N}$. Without loss of generality, we can assume that $k < 1$ and $k_1 < 1$. Note that for any partition $\Pi = \{t_0, t_1, ..., t_l\}$ of $[0,a]$, by (1) and (a),

$$\begin{aligned}\sum_{i=1}^{l} g(k_1 k|F_n(x(t_i)) &- F_n(x(t_{i-1}))|)\\ &\leq \sum_{i=1}^{l} \varphi(s_i, k_1 k|F_n(x(t_i)) - F_n(x(t_{i-1}))|)/m\\ &\leq \left(\sup_{n\in\mathbb{N}} V_\varphi(k_1 k(F_n \circ x))\right)/m < +\infty.\end{aligned}$$

Consequently for any $x \in BV_\varphi$

$$\sup_{n\in\mathbb{N}} V_g(k_1 k(F_n \circ x)) < +\infty. \tag{3}$$

Observe that by (1) and [**10**], Theorem 10.11, p. 74, $BV_\varphi = BV_g$. Hence for any $x \in BV_g$ (3) holds true. Now we show that

$$\sup_{n\in\mathbb{N}} V_g(F_n \circ x) < +\infty \tag{4}$$

for any $x \in BV_g$. Since g satisfies local Δ_2 condition and g is nondecreasing, for any $M > 0$ and $u \in [0, M]$, there exists $L_M > 0$

$$g(2u) \leq L_M g(u).$$

Fix $M > 0$ satisfying (2) and $w \in \mathbb{N}$ such that $2^{-w} < kk_1$. Note that for any partition $\Pi = \{t_0, t_1, ..., t_l\}$ of $[0, a]$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{i=1}^{l} g(|F_n(x(t_i))-F_n(x(t_{i-1}))|) \\
&\qquad \leq (L_M)^w \sum_{i=1}^{l} g(2^{-w}|F_n(x(t_i)) - F_n(x(t_{i-1}))|) \\
&\qquad \leq (L_M)^w (\sum_{i=1}^{l} g(kk_1|F_n(x(t_i)) - F_n(x(t_{i-1}))|)) \\
&\qquad < (L_M)^w \sup_{n\in\mathbb{N}} V_g(kk_1(F_n \circ x)) < +\infty,
\end{aligned}$$

which shows our claim. By (4) and [**6**], Theorem 1 applied to g, for any $v > 0$ there exists $K_v > 0$ such that for any $u_1, u_2 \in [-v, v]$ and $n \in \mathbb{N}$

$$g(|F_n(u_1) - F_n(u_2)|) \leq K_v g(|u_1 - u_2|), \tag{5}$$

which shows (b).

Now assume that (b) is satisfied and fix $x \in BV_\varphi$. Let $k > 0$ be such that $V_\varphi(kx) < +\infty$. By [**10**], Theorem 10.7 a), p. 69 x is a bounded function. Choose $v > 0$ such that $|x(t)| < v$ for any $t \in [0, a]$. Since g is nondecreasing, $\lim_{u\to+\infty} g(u) = +\infty$, and $F_n(0) = 0$, by (5),

$$\sup\{|F_n(u)| : n \in \mathbb{N}, u \in [-v, v]\} < +\infty.$$

Hence making k smaller, if necessary, we can assume that $k|x(t)| < u_0$ and $k|F_n(x(t))| \leq \frac{u_0}{2}$ for any $t \in [0, a]$ and $n \in \mathbb{N}$. Since g satisfies local Δ_2 condition, by (b) there exists $L_v > 0$ such that for any $u_1, u_2 \in [-v, v]$ and $n \in \mathbb{N}$,

$$g(k|F_n(u_1) - F_n(u_2)|) \leq L_v g(k|u_1 - u_2|). \tag{6}$$

Note that for any partition $\Pi = \{t_0, t_1, ..., t_l\}$ of $[0, a]$ and $s_i \in [t_{i-1}, t_i]$, $i = 1, ..., l$, by (1) and (6),

$$\begin{aligned}
&\sum_{i=1}^{l} \varphi(s_i, k|F_n(x(t_i)) - F_n(x(t_{i-1}))|) \\
&\qquad \leq M \left(\sum_{i=1}^{l} g(k|F_n(x(t_i)) - F_n(x(t_{i-1}))|) \right) \\
&\qquad \leq (M L_v) \left(\sum_{i=1}^{l} g(k|x(t_i) - x(t_{i-1})|) \right) \\
&\qquad \leq \frac{M L_v}{m} \left(\sum_{i=1}^{l} \varphi(s_i, k|x(t_i) - x(t_{i-1})|) \right) \\
&\qquad \leq \frac{M L_v}{m} V_\varphi(kx) < +\infty.
\end{aligned}$$

Hence

$$\sup_{n \in \mathbb{N}} V_\varphi(k(F_n \circ x)) < +\infty,$$

which completes the proof. $\square$

Theorem 2. *Let φ and g be as in Theorem 1. Furthermore, we assume that g is s-convex for some $s \in (0, 1]$ or there exists $t \in [0, a]$ such that $\varphi(t, .)$ is s-convex for some $s \in (0, 1]$. Then* (a) *is equivalent to*

(c) *For any $v > 0$ there exists $K_v > 0$ such that for any $u_1, u_2 \in [-v, v]$ and $n \in \mathbb{N}$*

$$|F_n(u_1) - F_n(u_2)| \leq K_v |u_1 - u_2|.$$

Proof. First we assume that g is s-convex. By Theorem 1, (a) implies (b). Fix $v > 0$ and positive constant L_v corresponding to v

by (b). Without loss of generality we can assume that $L_v > 1$. By s-convexity,

$$g(\frac{|F_n(u_1) - F_n(u_2)|}{(L_v)^{1/s}}) \leq g(|F_n(u_1) - F_n(u_2)|)/L_v \leq g(|u_1 - u_2|).$$

Since g is nondecreasing, this implies that

$$|F_n(u_1) - F_n(u_2)| \leq (L_v)^{1/s}|u_1 - u_2|$$

as required.

Now let us assume that $\varphi(t, .)$ is an s-convex function for some $t \in [0, a]$ and $s \in (0, 1]$. We show that (b) implies (c). Fix $v > 0$. Using the same logic as in the proof of Theorem 1 we can show that

$$\sup\{|F_n(u)| : u \in [-v, v], n \in \mathbb{N}\} < +\infty.$$

Hence we can find $k \in (0, 1)$ such that $k|F_n(u)| < \frac{u_0}{2}$ and $k|u| < \frac{u_0}{2}$ for any $n \in \mathbb{N}$ and $u \in [-v, v]$. Since g satisfies local Δ_2 condition, by (b) and (1)

$$\begin{aligned} &\varphi(t, k|F_n(u_1) - F_n(u_2)|) \leq Mg(k|F_n(u_1) - F_n(u_2)|) \\ &\leq (L_vM)g(|u_1 - u_2|) \leq M_vMg(k|u_1 - u_2|) \leq \frac{M_vM}{m}\varphi(t, k|u_1 - u_2|) \end{aligned}$$

with some constant $M_v > 0$. Since $\varphi(t, .)$ is s-convex, reasoning as in the previous case, we get

$$\frac{k|F_n(u_1) - F_n(u_2)|}{(M_vM/m)^{1/s}} \leq k|u_1 - u_2|,$$

which immediately give us (c) with the constant $K_v = (\frac{M_vM}{m})^{1/s}$. By Lemma 1, (c) implies (a). The proof is complete. □

Corollary 1. *Let φ, g be as in Theorem 2. Assume that $F : \mathbb{R} \to \mathbb{R}$ satisfies $F(0) = 0$. Then the composition operator $x \to F \circ x$ maps BV_φ into BV_φ if and only if F satisfies local Lipschitz condition.*

Proof. Follows immediately from Lemma 1 and Theorem 2, taking $F_n = F$ for any $n \in \mathbb{N}$.

Corollary 2. *Let φ, g and F_n be as in Theorem 2. Assume that there exists $s \in (0,1]$ such that $\varphi(t,\cdot)$ is an s-convex function for any $t \in [0,a]$. Then* (a) *is equivalent to*

(d) *There exists $C > 0$ such that*

$$\sup\{\|F_n \circ x\|_{\varphi,s} : x \in V_\varphi, \|x\|_{\varphi,s} = 1, n \in \mathbb{N}\} \leq C.$$

Proof. By Theorem 2, (a) implies (c). By s-convexity of $\varphi(t,\cdot)$, (c) implies (d). Conversely, (d) implies

(e) For any $x \in V_\varphi$, there exists $C_x > 0$ with $\sup\limits_n \|F_n \circ x\|_{\varphi,s} < C_x$.

It is clear that (e) implies (a), which completes the proof.

4. Hammerstein integral equation. For simplicity assume that $a = 1$. Assume also that φ satisfies conditions (i)-(iv) from Section 2. Consider the Hammerstein integral equation

$$x(t) = g(t) + \nu \int_I K(t,s) f(x(s)) ds \quad \text{for} \quad t \in I,\ \nu \in \mathbb{R}. \tag{7}$$

Assume that

1^0 $g : I \to \mathbb{R}$ is a generalized BV_φ-function ($g(0) = 0$);

2^0 $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function;

3^0 $K : I \times I \to \mathbb{R}$ is a function such that $K(t,\cdot)$ is integrable in the Lebesgue sense (briefly: L-integrable) for every $t \in I$, $K(0,s) = 0$ and there exists a number $\alpha > 0$ such that $V_\varphi\left(\frac{K(\cdot,s)}{\alpha}\right) \leq M(s)$ for a.e. $s \in I$, where $M : I \to \mathbb{R}_+$ is an L-integrable function.

Theorem 3. *Under the above assumptions there exists a number $\rho > 0$ such that for every ν with $|\nu| < \rho$, equation* (7) *has a unique generalized BV_φ-solution, defined on I.*

Proof. First, let us observe that from 3^0 it follows that

$$\inf\{\varepsilon > 0 : \int_I V_\varphi\left(\frac{K(\cdot,s)}{\varepsilon}\right) ds \leq 1\} =: c < +\infty.$$

Indeed, by 3^0 we have $\int\limits_I V_\varphi\left(\frac{K(\cdot,s)}{\alpha}\right)ds < +\infty$. Let $\beta = \int\limits_I V_\varphi\left(\frac{K(\cdot,s)}{\alpha}\right)ds$ and $\gamma = \max(1,\beta)\alpha$. Now, we have

$$\int\limits_I V_\varphi\left(\frac{K(\cdot,s)}{\gamma}\right)ds \leq \frac{1}{\max(1,\beta)}\int\limits_I V_\varphi\left(\frac{K(\cdot,s)}{\alpha}\right)ds = \frac{\beta}{\max(1,\beta)} \leq 1,$$

so $\int\limits_I V_\varphi\left(\frac{K(\cdot,s)}{\gamma}\right)ds \leq 1$. Thus
$\inf\left\{\varepsilon > 0 : \int_I V_\varphi\left(\frac{K(\cdot,s)}{\varepsilon}\right)ds \leq 1\right\} =: c \leq \gamma.$

Choose a positive number $r > 0$ such that $\|g\|_{V_\varphi} < r$. Denote by L_r the Lipschitz constant which corresponds to the function f and the interval $[-r,r]$. Choose a number $\rho > 0$ such that $\|g\|_{V_\varphi} + c\rho \sup\limits_{[-r,r]} |f(t)| < r$ and $\rho L_r c\tilde{c} < 1$, where $\tilde{c} > 0$ is the infimum of all positive numbers $\tilde{\tilde{c}}$ such that $\|x\|_{\sup} \leq \tilde{\tilde{c}}\|x\|_{V_\varphi}$. The existence of such a number $\tilde{c}$ follows from [**10**], 10.7c, p. 69 and [**1**], Theorem 4.1, p. 119. Let $\bar{B}_r$ denote the closed ball of center zero and radius r in the space $BV_\varphi(I)$. Fix ν such that $|\nu| < \rho$. Define the operators

$$\begin{aligned} F(x)(t) &= \int\limits_I K(t,s)f(x(s))ds, \\ G(x)(t) &= g(t) + \nu F(x)(t), \end{aligned}$$

where $x \in \bar{B}_r$ and $t \in I$. By Lemma 1, $f(x) \in BV_\varphi(I)$, so in view of [**10**], 10.7a, p.69 and Theorem 10.9, p.71 it is Lebesgue measurable and bounded. Thus the mappings F and G are well defined. Now, we verify that G maps $\bar{B}_r$ into itself. Indeed, for any $x \in \bar{B}_r$, we have

$$\begin{aligned} \|G(x)\|_{V_\varphi} &\leq \|g\|_{V_\varphi} + \|\nu F(x)\|_{V_\varphi} \\ &= \|g\|_{V_\varphi} + \inf\{\varepsilon > 0 : V_\varphi\left(\frac{\nu F(x)}{\varepsilon}\right) \leq 1\}. \end{aligned}$$

The sign " $\sup\limits_{\Pi,\{s_i\}}$ " below denotes that the supremum is taken over all partitions Π with all possible intermediate points $s_i \in [t_{i-1}, t_i]$,

$i = 1, \ldots, n$. By the Jensen inequality, we have

$$
\begin{aligned}
&V_\varphi\left(\frac{\nu F(x)}{\varepsilon}\right) = \sup_{\Pi,\{s_i\}} \sum_{i=1}^{n} \varphi\left(s_i, \frac{|\nu|}{\varepsilon}|F(x)(t_i) - F(x)(t_{i-1})|\right) \\
&\le \sup_{\Pi,\{s_i\}} \sum_{i=1}^{n} \varphi\left(s_i, \int_0^1 \frac{|\nu|}{\varepsilon}|K(t_i, s) - K(t_{i-1}, s)||f(x)(s)|ds\right) \\
&\le \sup_{\Pi,\{s_i\}} \sum_{i=1}^{n} \varphi\left(s_i, \int_0^1 \frac{|\nu|}{\varepsilon} \sup_{s\in I}|f(x)(s)||K(t_i, s) - K(t_{i-1}, s)|ds\right) \\
&\le \sup_{\Pi,\{s_i\}} \sum_{i=1}^{n} \varphi\left(s_i, \int_0^1 \frac{|\nu|}{\varepsilon} \sup_{t\in[-r,r]}|f(t)||K(t_i, s) - K(t_{i-1}, s)|ds\right) \\
&\le \sup_{\Pi,\{s_i\}} \sum_{i=1}^{n} \int_0^1 \varphi\left(s_i, \frac{|\nu|}{\varepsilon} \sup_{t\in[-r,r]}|f(t)||K(t_i, s) - K(t_{i-1}, s)|\right) ds \\
&\le \int_0^1 V_\varphi\left(|\nu| \sup_{t\in[-r,r]}|f(t)|\frac{K(\cdot, s)}{\varepsilon}\right) ds
\end{aligned}
$$

and

$$
\begin{aligned}
\inf&\left\{\varepsilon > 0 : V_\varphi\left(\frac{\nu F(x)}{\varepsilon}\right) \le 1\right\} \\
&\le \inf\left\{\varepsilon > 0 : \int_0^1 V_\varphi\left(|\nu| \sup_{t\in[-r,r]}|f(t)|\frac{K(\cdot, s)}{\varepsilon}\right) ds \le 1\right\} \\
&= |\nu| \sup_{t\in[-r,r]}|f(t)| \inf\left\{\varepsilon > 0 : \int_0^1 V_\varphi\left(\frac{K(\cdot, s)}{\varepsilon}\right) ds \le 1\right\} \\
&= |\nu| \sup_{t\in[-r,r]}|f(t)|c.
\end{aligned}
$$

Therefore, we conclude that

$$
\|G(x)\|_{V_\varphi} \le \|g\|_{V_\varphi} + |\nu| \sup_{[-r,r]}|f(t)|c < r,
$$

which means that G maps $\bar{B}_r$ into itself.

Similarly, for any $x, y \in \bar{B}_r$ we have

$$\begin{aligned}
&V_\varphi\left(\frac{\nu(F(x)-F(y))}{\varepsilon}\right)\\
&\leq \sup_{\Pi,\{s_i\}} \sum_{i=1}^{n} \varphi\left(s_i, \int_0^1 \frac{|\nu|}{\varepsilon}|K(t_i,s)-K(t_{i-1},s)||f(x(s))-f(y(s))|ds\right)\\
&\leq \sup_{\Pi,\{s_i\}} \sum_{i=1}^{n} \int_0^1 \varphi\left(s_i, \frac{|\nu|}{\varepsilon}\sup_{s\in I}|f(x(s))-f(y(s))||K(t_i,s)-K(t_{i-1},s)|\right)ds\\
&\leq \sup_{\Pi,\{s_i\}} \sum_{i=1}^{n} \int_0^1 \varphi\left(s_i, \frac{|\nu|}{\varepsilon}L_r\sup_{s\in I}|x(s)-y(s)||K(t_i,s)-K(t_{i-1},s)|\right)ds\\
&\leq \int_0^1 V_\varphi\left(|\nu|L_r\sup_{s\in I}|x(s)-y(s)|\frac{K(\cdot,s)}{\varepsilon}\right)ds,
\end{aligned}$$

and

$$\begin{aligned}
\|G(x)-G(y)\|_{V_\varphi} &= \inf\left\{\varepsilon>0 : V_\varphi\left(\frac{\nu(F(x)-F(y))}{\varepsilon}\right)\leq 1\right\}\\
&\leq \inf\left\{\varepsilon>0 : \int_0^1 V_\varphi(|\nu|L_r\sup_{s\in I}|x(s)-y(s)|\frac{K(\cdot,s)}{\varepsilon})ds \leq 1\right\}\\
&= |\nu|L_r c\tilde{c}\|x-y\|_{V_\varphi},
\end{aligned}$$

so G is a contraction. Now, applying the Banach contraction principle we infer that G has a unique fixed point in $\bar{B}_r$, which is a generalized BV_φ-solution to equation (7).

Remark 1. (1) Let φ be a φ-function without parameter. Then Theorem 3 reduces to Theorem 1 from [**3**].

(2) Again, let φ be a φ-function without parameter satisfying the conditions $u^{-1}\varphi(u) \to +\infty$ as $u \to 0+$ and $\varphi(u_1+\cdots+u_n) \leq k(\varphi(\lambda u_1)+\ldots+\varphi(\lambda u_n))$ for $n \in \mathbb{N}$ with some constants $k, \lambda > 0$. Let us denote

$$s_x(t) = x(0+0)-x(0)+\sum_{t_i<t}(x(t_i+0)-x(t_i-0))+x(t)-x(t-0)$$

for $0 < t \leq a$ for every $x \in V_\varphi$, where $t_1, t_2, \ldots$ are all points of discontinuity of x. It can be shown that in this case $x(t) = s_x(t)$ for every $t \in [0, a]$ (see [**10**], pp. 73-74 for details).

(3) Now, assume that $\varphi(t,u)$ satisfies the condition $u\psi^{-1}(u) \to +\infty$, as $u \to 0+$, where ψ is the function defined in (iv) and let $x \in V_\varphi$. One can easily verify then that the function x is of bounded variation in $[0,a]$, in the usual sense. Moreover, it can be shown that x is constant in each interval of continuity (see again [**10**], p. 73 for details).

To finish this section we indicate a class of functions which satisfy the assumption 3^0.

Example 1. Let $K : I \times I \to \mathbb{R}$ be defined as follows: $K(t,s) = K_1(t)K_2(s)$ for $(t,s) \in I^2$, where K_2 is an L-measurable and bounded function on I, $V_\varphi(K_1) < +\infty$ and $K_1(0) = 0$. Obviously $K(t,\cdot)$ is L-integrable on I for every $t \in I$. Moreover, we have

$$\begin{aligned} V_\varphi\Big(\frac{K(\cdot,s)}{\alpha}\Big) &= \sup_{\Pi,\{s_i\}} \sum_{i=1}^{n} \varphi\left(s_i, \frac{|K(t_i,s)-K(t_{i-1},s)|}{\alpha}\right) \\ &\le \frac{|K_2(s)|}{\alpha} \sup_{\Pi,\{s_i\}} \sum_{i=1}^{n} \varphi(s_i, |K_1(t_i)-K_1(t_{i-1})|) \\ &= \frac{|K_2(s)|}{\alpha} V_\varphi(K_1), \end{aligned}$$

where $\alpha = \sup\limits_{s\in I} |K_2(s)|$ (obviously, we can assume $\alpha > 0$ to avoid triviality) and the function $s \to \frac{|K_2(s)|}{\alpha} V_\varphi(K_1)$ is L-integrable on I. Hence the function K satisfies assumption 3^0.

5. Volterra-Hammerstein integral equation. Throughout this section we assume that φ-function φ is convex and satisfies following Δ_2-condition:

$$\varphi(t,2u) \le k\varphi(t,u) \quad \text{for } \ 0 \le u \le u_0, \ \ t \in [0,a],$$

where $u_0 > 0$ is fixed and k is a positive constant. For simplicity we assume again that $a = 1$. For $x \in X$, we shall denote by $\bigvee\limits_{s}^{1}{}_{\varphi}(x)$ the φ-variation of x on the interval $[s,1]$, where $0 \le s < 1$.

Consider the following Volterra-Hammerstein integral equation

$$x(t) = g(t) + \int_0^t K(t,s)f(x(s))ds \quad \text{for} \quad t \in I. \tag{8}$$

Let us define $\tilde{K}(t,s) = \begin{cases} K(t,s)\ , & 0 \le s \le t, \\ 0 \quad , & t < s \le 1. \end{cases}$

To continue we will need the following assumption

4^0 Let $T = \{(t,s) : 0 \le t \le 1,\ 0 \le s \le t\}$ and $K : T \to \mathbb{R}$ be a function such that $K(t,\cdot)$ is an L-integrable on $[0,t]$ for every $t \in I$, and there exists a number $\alpha > 0$ such that $\bigvee\limits_0^1 {}_\varphi \left(\frac{\tilde{K}(\cdot,s)}{\alpha}\right) \le m(s)$ for a.e. $s \in I$, where $m : I \to \mathbb{R}_+$ is an L-integrable function.

Now, we prove the following

Theorem 4. *Suppose conditions 1^0, 2^0 and 4^0 are satisfied. Then there exists an interval $J \subset I$ such that the equation (8) has a unique generalized BV_φ-solution, defined on J.*

Proof. Let r, L_r and $\tilde{c}$ be as in the proof of Theorem 3. Choose a positive integer N such that $\sup\limits_{t\in[-r,r]} |f(t)|_{\frac{\alpha}{2^N}} + \|g\|_{V_\varphi} < r$ and $L_r \tilde{c} \frac{\alpha}{2^N} < 1$. Further, let $0 < d \le \min\{u_0, 1\}$ be such that

$$\int_0^d \bigvee_0^d {}_\varphi \left(\frac{2^N \tilde{K}(\cdot,s)}{\alpha}\right) ds \le k^N \int_0^d m(s)ds \le 1. \tag{9}$$

Indeed, by 4^0 and the absolute continuity of the Lebesgue integral, there exists $0 < d \le \min\{u_0, 1\}$ such that

$$\int_0^d \bigvee_0^d {}_\varphi \left(\frac{\tilde{K}(\cdot,s)}{\alpha}\right) ds \le \int_0^d m(s)ds \le 1.$$

In view of the Δ_2-condition, we obtain

$$\int_0^d \bigvee_0^d {}_\varphi \left(\frac{2\tilde{K}(\cdot,s)}{\alpha}\right) ds \le k \int_0^d m(s)ds,$$

so one can choose d such that $k\int_0^d m(s)ds \le 1$. Using arguments from above we deduce that for every $N \in \mathbb{N}$ there exists a number $0 < d \le \min\{u_0, 1\}$ which satisfies (9). From (9) we get the inequality

$$\inf \left\{\varepsilon > 0 : \int_0^d \bigvee_0^d {}_\varphi \left(\frac{\tilde{K}(\cdot,s)}{\varepsilon}\right) ds \le 1\right\} \le \frac{\alpha}{2^N}. \tag{10}$$

Define $J=[0,d]$ and $G(x)(t)=g(t)+F(x)(t)$, where

$$F(x)(t)=\int_0^t K(t,s)f(x(s))ds, \quad \text{for } x\in\bar{B}_r,\ t\in J,$$

($\bar{B}_r$ denotes here the same ball as in the proof of Theorem 3). Now, we verify that G maps $\bar{B}_r$ into itself. We have obviously

$$\|G(x)\|_{V_\varphi}\le\|g\|_{V_\varphi}+\|F(x)\|_{V_\varphi}=\|g\|_{V_\varphi}+\inf\left\{\varepsilon>0:\bigvee_0^d\varphi(\frac{F(x)}{\varepsilon})\le 1\right\}.$$

Since

$$\begin{aligned}
\bigvee_0^d\varphi\left(\frac{F(x)}{\varepsilon}\right)&=\sup_{\Pi,\{s_i\}}\sum_{i=1}^n\varphi\left(s_i,\frac{1}{\varepsilon}\left|F(x)(t_i)-F(x)(t_{i-1})\right|\right)\\
&=\sup_{\Pi,\{s_i\}}\sum_{i=1}^n\\
&\quad\varphi\left(s_i,\left|\int_0^{t_i}\frac{1}{\varepsilon}K(t_i,s)f(x(s))ds-\int_0^{t_{i-1}}\frac{1}{\varepsilon}K(t_{i-1},s)f(x(s))ds\right|\right)\\
&=\sup_{\Pi,\{s_i\}}\sum_{i=1}^n\varphi\left(s_i,\left|\int_0^d\frac{1}{\varepsilon}(\tilde{K}(t_i,s)-\tilde{K}(t_{i-1},s))f(x(s))ds\right|\right)\\
&\le\sup_{\Pi,\{s_i\}}\sum_{i=1}^n\varphi\left(s_i,\frac{1}{d}\int_0^d d\frac{1}{\varepsilon}\sup_{s\in J}|f(x(s))||\tilde{K}(t_i,s)-\tilde{K}(t_{i-1},s)|ds\right)\\
&\le\sup_{\Pi,\{s_i\}}\sum_{i=1}^n\frac{1}{d}\int_0^d\varphi\left(s_i,\frac{d}{\varepsilon}\sup_{t\in[-r,r]}|f(t)||\tilde{K}(t_i,s)-\tilde{K}(t_{i-1},s)|\right)ds\\
&\le\int_0^d\bigvee_0^d\varphi\left(\sup_{t\in[-r,r]}|f(t)|\frac{\tilde{K}(\cdot,s)}{\varepsilon}\right)ds,
\end{aligned}$$

so

$$\begin{aligned}
&\inf\left\{\varepsilon>0:\bigvee_0^d\varphi\left(\frac{F(x)}{\varepsilon}\right)\le 1\right\}\\
&\quad\le\inf\left\{\varepsilon>0:\int_0^d\bigvee_0^d\varphi\left(\sup_{t\in[-r,r]}|f(t)|\frac{\tilde{K}(\cdot,s)}{\varepsilon}\right)ds\le 1\right\}\\
&\quad=\sup_{t\in[-r,r]}|f(t)|\inf\left\{\varepsilon>0:\int_0^d\bigvee_0^d\varphi\left(\frac{\tilde{K}(\cdot,s)}{\varepsilon}\right)ds\le 1\right\}.
\end{aligned}$$

Further, by (10), we get

$$\inf\left\{\varepsilon > 0 : \bigvee_0^d {}_\varphi(\frac{F(x)}{\varepsilon}) \leq 1\right\} \leq \sup_{t\in[-r,r]} |f(t)|\frac{\alpha}{2^N}.$$

Thus $\|G(x)\|_{V_\varphi} < r$ which means that $G(\bar{B}_r) \subset \bar{B}_r$.

Now, for any $x, y \in \bar{B}_r$ we have

$$\|G(x) - G(y)\|_{V_\varphi} = \inf\left\{\varepsilon > 0 : \bigvee_0^d {}_\varphi(\frac{F(x) - F(y)}{\varepsilon}) \leq 1\right\}$$

and

$$\begin{aligned}
&\bigvee_0^d {}_\varphi\left(\frac{F(x) - F(y)}{\varepsilon}\right)\\
&= \sup_{\Pi,\{s_i\}} \sum_{i=1}^n \varphi\left(s_i, \frac{1}{\varepsilon}\left|F(x)(t_i) - F(x)(t_{i-1}) - F(y)(t_i) + F(y)(t_{i-1})\right|\right)\\
&\leq \sup_{\Pi,\{s_i\}} \sum_{i=1}^n \varphi\left(s_i, \int_0^d \frac{1}{\varepsilon}|\tilde{K}(t_i, s) - \tilde{K}(t_{i-1}, s)||f(x(s)) - f(y(s))|ds\right)\\
&\leq \sup_{\Pi,\{s_i\}} \sum_{i=1}^n \int_0^d \varphi\left(s_i, \frac{1}{\varepsilon}\sup_{s\in J}|f(x(s)) - f(y(s))||\tilde{K}(t_i, s) - \tilde{K}(t_{i-1}, s)|\right)ds\\
&\leq \sup_{\Pi,\{s_i\}} \sum_{i=1}^n \int_0^d \varphi\left(s_i, \frac{1}{\varepsilon}L_r \sup_{s\in J}|x(s) - y(s)||\tilde{K}(t_i, s) - \tilde{K}(t_{i-1}, s)|\right)ds\\
&\leq \int_0^d \bigvee_0^d {}_\varphi\left(L_r \sup_{s\in J}|x(s) - y(s)|\frac{\tilde{K}(\cdot, s)}{\varepsilon}\right)ds,
\end{aligned}$$

so, by (10), we get

$$\begin{aligned}
&\|G(x) - G(y)\|_{V_\varphi}\\
&\qquad\leq \inf\left\{\varepsilon > 0 : \int_0^d \bigvee_0^d {}_\varphi(L_r \sup_{s\in J}|x(s) - y(s)|\frac{\tilde{K}(\cdot, s)}{\varepsilon})ds \leq 1\right\}\\
&\qquad\leq L_r \sup_{s\in J}|x(s) - y(s)|\frac{\alpha}{2^N} \leq L_r \tilde{c}\frac{\alpha}{2^N}\|x - y\|_{V_\varphi}.
\end{aligned}$$

By the Banach contraction principle, we infer that G has a unique fixed point in $\bar{B}_r$, which is a generalized BV_φ-solution of the equation (8).

6. Global solutions of equations (7) and (8). Let us begin with the Hammerstein integral equation of the form

$$x(t) = g(t) + \int_I K(t,s) f(x(s))ds, \quad \text{for} \quad t \in I, \tag{11}$$

where $I = [0,1]$ for simplicity. Assume that

5^0 $f : \mathbb{R} \to \mathbb{R}$;

6^0 there exists $\Psi : [0,+\infty) \to [0,+\infty)$ with $\Psi(u) > 0$ for $u > 0$ and $\sup_{s\in[0,1]} |f(x(s))| \leq \Psi(\|x\|_{V_\varphi})$ for any $x \in BV_\varphi(I)$;

7^0 there exists $M_0 > 0$ with $M_0/\|g\|_{V_\varphi} + \Psi(M_0)c > 1$, where c is the constant defined in the proof of Theorem 3;

8^0 there exists a continuous and nondecreasing function φ_{M_0} : $[0,+\infty) \to [0,+\infty)$ such that $c\varphi_{M_0}(\tilde{c}z) < z$ for $z > 0$ and $|f(x) - f(y)| < \varphi_{M_0}(|x-y|)$, for $|x|,\ |y| \leq M_0$, where $\tilde{c}$ is the constant defined in the proof of Theorem 3.

Now we prove the following existence result for equation (11).

Theorem 5. *Under the assumptions* 1^0, 3^0, 5^0-8^0, *equation* (11) *has a generalized* BV_φ*-solution, defined on* I.

Proof. Let $\bar{B}_{M_0}$ denote the closed ball of center zero and radius M_0 in the space $BV_\varphi(I)$. Define

$$G(x)(t) = g(t) + \int_I K(t,s) f(x(s))ds, \quad \text{for } \ x \in \bar{B}_{M_0} \text{ and } \ t \in I.$$

For any $x, y \in \bar{B}_{M_0}$ we have

$$\begin{aligned} &\|G(x) - G(y)\|_{V_\varphi} \\ &\qquad \leq \inf\left\{\varepsilon > 0 : \int_I V_\varphi\left(\frac{K(\cdot,s)}{\varepsilon} \sup_{s\in I} \varphi_{M_0}(|x(s) - y(s)|)\right) ds \leq 1\right\} \\ &\qquad \leq c\varphi_{M_0}(\tilde{c}\|x-y\|_{V_\varphi}). \end{aligned}$$

From the above inequality it follows, in particular, that $G(\bar{B}_{M_0})$ is a bounded set. Now suppose that $x \in BV_\varphi(I)$ with $\|x\|_{V_\varphi} = M_0$ is a solution of

$$x(t) = \lambda \left(g(t) + \int_I K(t,s) f(x(s)) ds \right) \quad \text{for} \quad t \in I,$$

where $\lambda \in (0,1]$. By 6^0 and 7^0, we have

$$\|x\|_{V_\varphi} \leq \|g\|_{V_\varphi} + \sup_{s \in I} |f(x(s))| \cdot c \leq \|g\|_{V_\varphi} + c\Psi(\|x\|_{V_\varphi}),$$

so

$$\frac{\|x\|_{V_\varphi}}{\|g\|_{V_\varphi} + c\Psi(\|x\|_{V_\varphi})} \leq 1. \tag{12}$$

Since $\|x\|_{V_\varphi} = M_0$, (12) implies that

$$\frac{M_0}{\|g\|_{V_\varphi} + c\Psi(M_0)} \leq 1$$

which contradicts 7^0. Applying the nonlinear alternative of Leray-Schauder type (see [**12**] Theorem 3.9) we infer that G has a fixed point in the open ball B_{M_0}, which obviously is a global generalized BV_φ solution of (11).

Now, consider again equation (8), and write it in the following form

$$x(t) = g(t) + \int_I \tilde{K}(t,s) f(x(s)) ds, \quad \text{for} \quad t \in I. \tag{13}$$

Hence, as a corollary from Theorem 5 we obtain the following result for equation (8).

Theorem 6. *Suppose 1^0, 4^0, 5^0 and 6^0 are satisfied. Moreover, assume that*

9^0 *there exists $M_0 > 0$ with $M_0/\|g\|_{V_\varphi} + \Psi(M_0)\bar{c} > 1$, where* $\bar{c} = \inf \left\{ \varepsilon > 0 : \int_0^1 \bigvee_0^1 \varphi\left(\frac{\tilde{K}(\cdot,s)}{\alpha}\right) ds \leq 1 \right\}$

and condition 8^0 *with* $\bar{c}$ *instead of* c *holds. Then equation* (13) *has a generalized* BV_φ*-solution, defined on* I.

Remark 2. Note that the inequality $\|x\|_{\sup} \le \tilde{\tilde{c}}\|x\|_{V_\varphi}$, $x \in BV_\varphi(I)$, mentioned in the proof of Theorem 3, in particular implies that continuous functions of bounded generalized φ-variation form a closed subspace of the space $BV_\varphi(I)$. Therefore, it is clear that if we assume additionally that g is continuous and impose a suitable continuity assumption on the kernel K, one can obtain the existence and uniqueness results concerning continuous generalized BV_φ solutions to equations (7) and (8).

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Faculty of Mathematics and Computer Science, A. Mickiewicz University, Umultowska, 87 61-614, Poznań, Poland
Email address: dbw@amu.edu.pl

School of Computer, Mathematical and Natural Sciences, Morgan State University, 1700 E. Cold Spring Lane, Baltimore, MD 21251 USA
Email address: Dariusz.Bugajewski@morgan.edu

Department of Mathematics, Jagiellonian University, Łojasiewicza 6, 30-348 Cracow, Poland,
Email address: Grzegorz.Lewicki@im.uj.edu.pl