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REPRESENTATION THEORY OF TOPOLOGICAL SELECTIONS OF MULTIVALUED LINEAR MAPPINGS WITH APPLICATIONS TO INTEGRAL AND DIFFERENTIAL OPERATORS

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1. Introduction. This paper is concerned with the characterization and representation theory of topological selections of closed linear relations in Banach spaces, and with some applications to differential and integral operators.

We first introduce some definitions. Let X and Y be real or complex Banach spaces and let M be a subspace (i.e., a vector subspace) of $X \times Y$. We can also view M as the graph of a multivalued linear mapping, and call it a linear relation in $X \times Y$. We say that R is an *algebraic selection* (or *algebraic operator part*) of M if R is the graph of a single-valued linear operator on $\operatorname{Dom} M$ into $\operatorname{Range} M$ such that $R \subset M$. Equivalently, $R = \operatorname{Null} \mathbf{P} := \{a \in M : \mathbf{P}(a) = 0\}$ for some algebraic projector $\mathbf{P}$ on M with $\operatorname{Range} \mathbf{P} = \{0\} \times M(0)$. If $\mathbf{P}$ is continuous, then R is called a *topological selection* (or *topological operator part*) of M. If, in addition, $\mathbf{P}(x,y) = \mathbf{P}(z,y)$ for all $(x,y), (z,y)$ in M, then R is called a *principal* topological selection of M. Let M^+ be a subspace of $Y^\# \times X^\#$, where $X^\#$ is the dual of X. An algebraic selection of M^+ is called a $w^\star$-topological selection of M^+ if the corresponding projector is $w^\star$-continuous.

We now summarize briefly the contents of this paper. In §2 we consider a general closed subspace (linear relation) M of $X \times Y$ and a $w^\star$-closed subspace M^+ of $Y^\# \times X^\#$ such that $M(0)$ and $M^+(0)$ are both finite dimensional. In Theorem 2.1 and Corollary 2.2, any topological selection is expressed by an adjoint subspace, while in Theorem 2.3 and Corollary 2.4, any $w^\star$-topological selection is expressed by a preadjoint subspace. These theorems and corollaries generalize and complete the corresponding theorems of Coddington and Dijksma [**1**]. §3 is concerned with the problem of characterizing a topological selection for a subspace of a linear relation in terms of a known topological selection of that relation, and with some related consequences. More specifically, suppose that M_1 is a known closed linear relation in $X \times Y$ with $\dim \operatorname{Null} M_1 < \infty$. Let M be any closed subspace of M_1 with $\dim M_1/M < \infty$. In Theorem 3.5 we give a necessary and sufficient condition for the graph of an operator to be a topological selection of M^{-1}. This is expressed via an arbitrary, but fixed topological selection, say R_1, of M_1^{-1}. Using this theorem we prove in Theorem 3.6 that any topological selection of M^{-1} is necessarily a finite-rank "perturbation" of R_1 restricted to $\operatorname{Dom} R$, and hence if R_1 is compact, so is any topological selection of M^{-1}. In §4 we are concerned with M and M_1 when X and Y are L_p-type spaces or the space of continuous functions on a compact Hausdorff space. Using notions of general integral operators as in [**2**], we give in Lemmas 4.1 and 4.2 necessary conditions for certain functionals to be represented by integral operators. We prove in Theorem 4.3 that if a topological selection R_1 of M_1^{-1} is an integral operator with a suitable kernel, then any topological selection of M^{-1} is also an integral operator with a related kernel. In §5 we give an application of Theorem 3.6 to a concrete case when M_1 is a finite-dimensional "perturbation" of the graph of a regular ordinary differential operator in an L_p-type space. It is shown in Theorem 5.1 that if M is a closed subspace of M_1 with $\dim M_1/M < \infty$, then any topological selection of M^{-1} is a compact integral operator. This theorem, even in the special case when M_1 is the graph of a maximal ordinary differential operator, generalizes the corresponding theorems in the literature that are concerned with two-point boundary conditions.

2. Characterization of topological selections of multivalued linear operators by adjoints. When a linear ordinary differential operator is nondensely defined, its adjoint is a multivalued operator. It is sometimes desirable to describe the linear selections of its adjoint in terms of boundary conditions. In the present section we address this problem in an abstract setting. Thus we will describe all topological selections of certain linear relations via "boundary" conditions. The setting and results generalize and at the same time complete some results of Coddington and Dijksma [**1**]. Throughout this section X, Y are Banach spaces and $M \subset X \times Y$ is a *closed* linear relation. For a Banach space W with dual $W^{\#}$, the natural pairing on $W \times W^{\#}$ is denoted by $\langle \cdot, \cdot \rangle$. The adjoint (or adjoint subspace) of a linear relation $M \subset X \times Y$ is the linear relation $M^{\star} \subset Y^{\#} \times X^{\#}$ defined by

$M^\star = \{(y,x) : (x,-y) \in M^\perp\}$ where $M^\perp$ denotes the annihilator of M. In the case when $M^+ \subset Y^\# \times X^\#$, the preadjoint of M^+ is the linear relation ${}^\star(M^+) := \{(x,y) \in X \times Y : (y,-x) \in {}^\perp M^+\}$, where ${}^\perp M^+$ is the preannihilator of M^+.

THEOREM 2.1. *Assume that* $\dim M(0) =: n < \infty$. *Then* R *is a topological selection of* M *if and only if* $R = M \cap {}^\star Z^+$ *for some* n*-dimensional vector space*

$$Z^+ := \quad \text{linear span}\,\{(\phi_{2i}^+ + \psi_{2i}^+, \phi_{1i}^+) : 1 \le i \le n\},$$

where

(i) $\phi_{1i}^+ \in X^\#$,

(ii) $\phi_{2i}^+ \in (M(0))^\perp$,

(iii) $\psi_{2i}^+ \in Y^\#$, *and* $\{\psi_{2i}^+ : 1 \le i \le n\}$ *is a linearly independent set on* $M(0)$.

In the case when Y *is a Hilbert space, the condition* (iii) *is replaced by*

(iii)′ $\{\psi_{2i}^+, \dots, \psi_{2n}^+\}$ *is a basis for* $M(0)$.

Moreover, when R, Z^+ *are as above, the adjoint subspace of* R *is represented by*

$$R^\star = M^\star \dotplus Z^+ \quad (\textit{direct sum}).$$

PROOF. Suppose that R is a topological selection of M. Then $R = M \cap \operatorname{Null}\mathbf{P}$ for some continuous projector $\mathbf{P} : M \to \{0\} \times M(0)$ with $\operatorname{Range}\mathbf{P} = \{0\} \times M(0)$. Let $\{\phi_1, \dots, \phi_n\}$ be a basis for $M(0)$ and let $\psi_{2i}^+ \in (M(0))^\#$, the dual of $M(0)$, such that $\langle \phi_i, \psi_2^+ j\rangle = \delta_{ij} (1 \le i, j \le n)$, where δ_{ij} is the Kronecker delta. Define $P_0 : \operatorname{Range} M \to M(0)$ by

$$P_0(y) = \sum_{i=1}^{n} \langle y, \psi_{2i}^+\rangle \phi_i, \quad y \in \operatorname{Range} M. \tag{2.1}$$

Then P_0 is a continuous projector of $\operatorname{Range} M$ onto $M(0)$. Thus by Sobczyk's lemma, which characterizes all continuous projectors with a prescribed range (see [**7**]), there exists a continuous linear operator $A : M \to M(0)$ such that

$$\mathbf{P}(x,y) = (0, A(x,y) + P_0(y)), \quad (x,y) \in M, \tag{2.2}$$

and

$$A(0,y)=0 \quad \text{for all} \quad y\in M(0). \tag{2.3}$$

In our setting we can write

$$A(x,y)=\sum_{i=1}^{n}\langle A(x,y),\ \psi_{2i}^{+}\rangle\ \phi_i. \tag{2.4}$$

Since M is closed, $M\times M(0)$ is Banach space. Let $A_\star$ denote the graph of the adjoint operator of A in the Banach space $M\times M(0)$. Thus $\operatorname{Dom} A_\star=(M(0))^{\#}\times M^{\#}$ and

$$A_\star=\{(y^+,k^+) : y^+\in(M(0))^{\#}, k^+\in M^{\#}, \langle Ak,y^+\rangle=\langle k,k^+\rangle, \\ k\in\operatorname{Dom} A\}.$$

Now $A_\star(\psi_{2i}^+)\in M^{\#}$. By the Hahn-Banach extension theorem, we can extend $A_\star(\psi_{2i}^+)$ to $Y^{\#}\times X^{\#}$. Denote this extension again by $A_\star(\psi_{2i}^+)$. Thus $A_\star(\psi_{2i}^+)\in Y^{\#}\times X^{\#}$. Let us write $A_\star(\psi_{2i}^+)=(-\phi_{1i}^+,\phi_{2i}^+)$. Then (2.4) becomes

$$A(x,y)=\sum_{i=1}^{n}[\langle x,-\phi_{1i}^+\rangle+\langle y,\phi_{2i}^+\rangle]\ \phi_i. \tag{2.5}$$

Let

$$\mathbf{P}_2(x,y):=A(x,y)+P_0(y),\ \ (x,y)\in M. \tag{2.6}$$

Then substituting A in (2.5) and P_0 in (2.1) into (2.6) we obtain

$$\mathbf{P}_2(x,y)=\sum_{i=1}^{n}[\langle x,-\phi_{1i}^+\rangle+\langle y,\phi_{2i}^+ +\psi_{2i}^+\rangle]\ \phi_i. \tag{2.7}$$

Let Z^+ be the linear span of the set $\{(\phi_{2i}^+ +\psi_{2i}^+,\phi_{1i}^+) : 1\le i\le n\}$. Then it follows from (2.7) that $(x,y)\in M$, $\mathbf{P}_2(x,y)=0$ if and only if $(x,y)\in M\cap {}^\star Z^+$. Now the condition (2.3) implies, using (2.5), that $\phi_{2i}^+\in(M(0))^\perp$ for all $i\le n$. Since $\langle\phi_i,\psi_{2j}^+\rangle=\delta_{ij}$, we see easily that $\{\psi_{2i}^+|_{M(0)} : 1\le i\le n\}$ is linearly independent. Moreover, since

$\psi_{2i}^+ \in (M(0))^\#$, we can treat ψ_{2i}^+ as an element of $Y^\#$. Thus Z^+ satisfies the required conditions.

Conversely, suppose that Z^+ is defined as in the theorem. Let $\mathbf{P}_2$ be as in (2.7) and define $\mathbf{P} : M \to \{0\} \times M(0)$ by

$$\mathbf{P}(x, y) = (0, \mathbf{P}_2(x, y)).$$

Then it is easy to check that $\mathbf{P}$ is a continuous projector on M onto $\{0\} \times M(0)$. We now show that $\operatorname{Range} \mathbf{P}_2 = M(0)$. This would then imply that $\operatorname{Range} \mathbf{P} = \{0\} \times M(0)$. Since $\{\phi_1, \ldots, \phi_n\}$ is a basis for $M(0)$, it is sufficient to show that the map $\pi : M \to \mathbb{C}^n$ defined by

$$(\pi(x, y))_i = \langle x, -\phi_{1i}^+ \rangle + \langle y, \phi_{2i}^+ + \psi_{2i}^+ \rangle$$

is surjective, where $(y)_i$ denotes the i^{th} component of y. It is, in turn, sufficient to show that $(\operatorname{Range} \pi)^\perp = \{0\}$ since π is of finite-dimensional range. Suppose that $\alpha_i \in \mathbb{C}$ satisfy

$$\sum_{i=1}^{n} \alpha_i [\langle x, -\phi_{1i}^+ \rangle + \langle y, \phi_{2i}^+ + \psi_{2i}^+ \rangle] = 0$$

for all $(x, y) \in M$. This is true, in particular, for all $(x, y) \in \{0\} \times M(0)$. Thus $\sum_{i=1}^{n} \alpha_i \langle y, \phi_{2i}^+ + \psi_{2i}^+ \rangle = 0$ for all $y \in M(0)$. Since $\phi_{2i}^+ \in (M(0))^\perp$ and $\{\psi_{2i}^+|_{M(0)} : 1 \leq i \leq n\}$ is linearly independent, $\alpha_i = 0$ for all i. Thus $(\operatorname{Range} \pi)^\perp = \{0\}$ and the claim is proved. Therefore $\mathbf{P}$ is a continuous projector of M onto $\{0\} \times M(0)$ and so $M \cap \operatorname{Null} \mathbf{P}$ is a topological selection of M. Now $M \cap \operatorname{Null} \mathbf{P} = M \cap {}^\star Z^+$. Finally since $M^\star \dotplus Z^+$ is $w^\star$-closed, $R^\star$ is the direct sum of $M^\star$ and Z^+. □

COROLLARY 2.2. *Assume that* $\dim M(0) =: n < \infty$. *Then* R *is a principal topological selection of* M *if and only if* $R = M \cap^\star Z^+$ *for some* n*-dimensional vector subspace* Z^+ *of* $Y^\# \times X^\#$ *which is the same as in the above theorem with the additional condition:* $\phi_{1i}^+ = 0$ *for all* $1 \leq i \leq n$.

PROOF. Let P_0 be the projector defined as in (2.1). It follows from the definition of a principal selection of M that R is a principal selection of M if and only if there exists a continuous projector P on $\operatorname{Range} M$ onto $M(0)$ such that $R = \{(x,y) \in M : P(y) = 0\}$ (see [**5**]). Thus R is a principal topological selection of M if and only if $R = \{(x,y) \in M : B(y) + P_0(y) = 0\}$ for some continuous linear operator $B : \operatorname{Range} M \to M(0)$ such that $B(y) = 0$ for all $y \in M(0)$. We now argue similarly as we did in the proof of Theorem 2.1. □

REMARK 2.1. Suppose that X, Y are Hilbert spaces and M a *closed* linear relation in $X \times Y$. Then $M \cap (\{0\} \times M(0))^{\perp} =: R_0$ is called the *orthogonal selection* of M. This can be written as

$$R_0 = \{(x,y) \in M : y \in (M(0))^{\perp}\}.$$

Thus if $\dim M(0) =: n < \infty$, then we can write R_0 as

$$R_0 = M \cap Z^{\star}$$

for some $Z =:$ linear span $\{(\phi_i, 0) : 1 \leq i \leq n\}$ where $\{\phi_1, \dots, \phi_n\}$ is a basis for $M(0)$. □

The following is the dual of Theorem 2.1.

THEOREM 2.3. *Let M^+ be a $w^\star$-closed linear relation in $Y^\# \times X^\#$ with $\dim M^+(0) =: n^+ < \infty$. Then R^+ is a $w^\star$-topological selection of M^+ if and only if $R^+ = M^+ \cap Z^\star$ for some n^+-dimensional space*

$$Z := \quad \text{linear span}\ \{(\phi_{2i} + \psi_{2i}, \phi_{1i}) :\ 1 \leq i \leq n^+\}$$

where

(i) $\phi_{1i} \in X$, $1 \leq i \leq n^+$,

(ii) $\phi_{2i} \in {}^{\perp}(M^+(0))$,

(iii) $\{\psi_{2i} : 1 \leq i \leq n^+\}$ *is a subset of Y which is linearly independent* $\operatorname{mod}^{\perp}(M^+(0))$.

Moreover, when R^+ and Z are as above,

$${}^\star R^+ = {}^\star M^+ \dot{+} Z \quad \text{(direct sum)}.$$

The following is a dual of Corollary 2.2.

COROLLARY 2.4. *Let M^+ be as Theorem 2.3. Then R^+ is a $w^\star$-principal topological selection of M^+ if and only if*

$$R^+ = M^+ \cap Z^\star$$

for some n^+-dimensional space

$$Z := \quad \text{linear span}\ \{(\phi_{2i} + \psi_{2i}, 0) : 1 \leq i \leq n^+\}$$

where

(i) $\phi_{2i} \in^\perp (M^+(0))$,

(ii) $\{\psi_{2i} : 1 \leq i \leq n^+\}$ is linearly independent $\mathrm{mod}^\perp(M^+(0))$.

REMARK 2.1. Theorem 2.1 and Corollary 2.2 generalize and complete Theorem 3.1 of Coddington and Dijksma [**1**], while Theorem 2.3 and Corollary 2.4 generalize and complete Theorem 3.4 of [**1**].

3. Representation theory of topological selections of inverses of linear relations. Let X, Y be Banach spaces and let M_1 be a *closed* subspace of $X \times Y$. Suppose that M is a *closed* subspace of M_1 such that $\dim M_1/M < \infty$. Let R_1 be a given topological selection of M^{-1}. In this section we consider the following problems:

(i) Is it possible to express any topological selection of M^{-1} in terms of R_1 and possibly some other known quantities?

(ii) If R_1 is compact, then is every topological selection of M^{-1} also compact?

(iii) If R_1 is an "integral" operator, then is every topological selection of M^{-1} also an "integral" operator?

In the case when M_1 is single-valued, the problems (i), (ii) arise, for example, in inverting ordinary differential operators (see also §5 below). To address the above problems it is convenient to introduce a "minimal" *closed* subspace M_0 such that $M_0 \subset M \subset M_1$. In a concrete case M_0 and M_1 may be considered as the graphs of the minimal and maximal ordinary differential operators. Throughout this section unless otherwise mentioned M_0 is an arbitrary but fixed *closed* subspace of M_1, where M_0 and M_1 satisfy the conditions:

$$\begin{cases} \dim \operatorname{Null} M_1 =: n_1 < \infty \\ M_0 \subset M_1, \ \dim M_1/M_0 =: d < \infty. \end{cases} \tag{3.1}$$

For the rest of this section, let B be an arbitrary, but fixed bounded linear operator on M_1 onto $\mathbb{C}^d$ with $\operatorname{Null} B = M_0$. Such an operator clearly exists since M_1/M_0 is isometrically isomorphic to $\mathbb{C}^d$.

LEMMA 3.1. *Let M be a closed linear relation such that $M_0 \subset M \subset M_1$. Then* $\dim M_1/M = m$ *if and only if there exists a $m \times d$ ($m \le d$) constant matrix $\boldsymbol{\Gamma}$ such that*

$$M = \{a \in M_1 : \boldsymbol{\Gamma} B(a) = 0_{m\times 1}\}. \tag{3.2}$$

PROOF. See [**4**]. □

Let $\{\phi_1, \dots, \phi_{n_1}\}$ be a basis for $\operatorname{Null} M_1$ and let $\boldsymbol{\Gamma}$ be the $d \times n_1$ constant matrix defined by

$$\boldsymbol{\Gamma} = [B(\phi_1, 0), \dots, B(\phi_{n_1}, 0)].$$

Let R_1 be an arbitrary but fixed topological selection of M_1^{-1}. Let $\boldsymbol{\Gamma}$ be a $m \times d$ constant matrix. Let $(\boldsymbol{\Gamma} D)^\dagger$ denote the Moore-Penrose generalized inverse of $\boldsymbol{\Gamma} D$. For $y \in \operatorname{Range} M$, define $\eta_\Gamma(y)$ by

$$\eta_\Gamma(y) := R_1(y) - \sum_{i=1}^{n_1} \phi_i[(\boldsymbol{\Gamma}\mathbf{D})^\dagger\, \boldsymbol{\Gamma} B(R_1(y), y)]_i.$$

LEMMA 3.2. *Let M be as in* (3.2). *Then*

$$M = \left\{(\eta_\Gamma(y) + \sum_{i=1}^{n_1}(\alpha_0)_i\phi_i, y) : y \in \operatorname{Range}(M_1),\ \boldsymbol{\Gamma} D\alpha_0 = 0,\right.$$
$$\left. \boldsymbol{\Gamma} B(R_1(y), y) \in \operatorname{Range}(\boldsymbol{\Gamma}\mathbf{D})\right\}.$$

In particular,

$$\operatorname{Null} M = \left\{\sum_{i=1}^{n_1} \phi_i(\alpha_0)_i : \boldsymbol{\Gamma}\mathbf{D}\alpha_0 = 0\right\}.$$

PROOF. $(x, y) \in M$ if and only if $(x, y) \in M_1$ and $\boldsymbol{\Gamma} B(x, y) = 0$. Now, since R_1 is an algebraic selection of M_1^{-1}, $(x, y) \in M_1$ if and only if $y \in \operatorname{Range} M_1$ and $x = R_1(y) + k$ for some $k \in \operatorname{Null} M_1$. Let us write $k = \sum_{i=1}^{n_1} \phi_i(\alpha)_i$ for some $\alpha \in \mathbb{C}^{n_1}$. Then

$$M = \Big\{ (R_1(y) + \sum_{i=1}^{n_1} \phi_i(\alpha)_i, y) : y \in \operatorname{Range} M_1,\ \alpha \in \mathbb{C}^{n_1},$$

$$\boldsymbol{\Gamma} B(R_1(y), y) + \boldsymbol{\Gamma}\mathbf{D}\alpha = 0 \Big\}$$

$$= \Big\{ (\eta_\Gamma(y) + \sum_{i=1}^{n_1} \phi_i(\alpha_0)_i, y) : y \in \operatorname{Range} M_1,\ \alpha_0 \in \mathbb{C}^{n_1}, \boldsymbol{\Gamma}\mathbf{D}\alpha_0 = 0,$$

$$\text{and } \boldsymbol{\Gamma} P B(R_1(y), y) \in \operatorname{Range}(\boldsymbol{\Gamma}\mathbf{D}) \Big\}.$$

The last assertion of the theorem follows immediately from this characterization of M. □

In the following we express an arbitrary *principal* topological selection of M^{-1} in terms of a given selection of M_1^{-1}.

LEMMA 3.3. *Let M be as in* (3.2). *Let R_1 be a topological selection (principal or nonprincipal) of M^{-1}. If R_0 is a principal topological selection of M^{-1} given by*

$$R_0 = \{(y, x) : (y, x) \in M^{-1},\ \mathcal{P}_0(x) = 0\} \tag{3.3}$$

for some continuous projector $\mathcal{P}_0 : \operatorname{Dom} M \to \operatorname{Null} M$ with $\operatorname{Range} \mathcal{P}_0 = \operatorname{Null} M$, *then*

$$\begin{cases} \operatorname{Dom} R_0 = \{y : y \in \operatorname{Range} M_1, & \boldsymbol{\Gamma} B(R_1(y), y) \in \operatorname{Range}(\boldsymbol{\Gamma}\mathbf{D})\}, \\ R_0(y) = (I - \mathcal{P}_0)\eta_\Gamma(y), & y \in \operatorname{Dom} R_0. \end{cases} \tag{3.4}$$

Conversely, if R_0 is defined as in (3.3) *for some continuous projector $\mathcal{P}_0$ on* $\operatorname{Dom} M$ *onto* $\operatorname{Null} M$, *then R_0 is a principal topological selection of M^{-1}.*

PROOF. Assume that R_0 is given as in (3.3). Then $(y, x) \in R_0$ if and only if (i) $(x, y) \in M$, and (ii) $(x, y) \in M$, $\mathcal{P}_0(x) = 0$. By Lemma 3.2,

(i) holds if and only if

$$\text{(i)}' \quad \begin{cases} y \in \operatorname{Range} M_1, & \boldsymbol{\Gamma} B(R_1(y), y) \in \operatorname{Range}(\boldsymbol{\Gamma}\mathbf{D}), \\ x = \eta_\Gamma(y) + \sum_{i=1}^{n_1} \phi_i(\alpha_0)_i, & \boldsymbol{\Gamma}\mathbf{D}\alpha_0 = 0. \end{cases}$$

Thus (ii) holds if and only if (i)$'$ holds in addition to

$$\text{(ii)}' \qquad \mathcal{P}_0(\eta_\Gamma(y)) + \mathcal{P}_0 \left(\sum_{i=1}^{n_1} \phi_i(\alpha_0)_i \right) = 0.$$

Since $\boldsymbol{\Gamma}\mathbf{D}\alpha_0 = 0$, we see that $\sum_{i=1}^{n_1} \phi_i(\alpha_0)_i \in \operatorname{Null} M$, and so $\mathcal{P}_0 \left(\sum_{i=1}^{n_1} \phi_i(\alpha_0)_i\right) = \sum_{i=1}^{n_1} \phi_i(\alpha_0)_i$. Then, using (ii)$'$, we obtain

$$x = \eta_\Gamma(y) - \mathcal{P}_0(\eta_\Gamma(y)) = (I - \mathcal{P}_0)\eta_\Gamma(y).$$

Thus (3.4) holds. Suppose now that (3.4) holds for some continuous projector $\mathcal{P}_0$ on $\operatorname{Dom} M$ onto $\operatorname{Null} M$. We will show that

$$R_0 = \{(y, x) \in M^{-1} : \mathcal{P}_0(x) = 0\}.$$

First we will show that "$\subset$" holds. Note that, by Lemma 3.2, $\operatorname{Dom} R_0 = \operatorname{Range} M$. Thus $\eta_\Gamma(y)$ is well-defined for $y \in \operatorname{Dom} R_0$. Take any $(y, x) \in R_0$. Then $y \in \operatorname{Dom} M$ and

$$(y, x) = (y, (I - \mathcal{P}_0)\eta_\Gamma(y)) = (y, \eta_\Gamma(y)) + (0, -\mathcal{P}_0\eta_\Gamma(y)).$$

By Lemma 3.2, $(y, \eta_\Gamma(y)) \in M^{-1}$. Since $\operatorname{Range} \mathcal{P}_0 \subset \operatorname{Null} M$, we see that $(y, x) \in M^{-1}$. Now $\mathcal{P}_0(x) = \mathcal{P}_0 R_0(y) = \mathcal{P}_0[I - \mathcal{P}_0]\eta_\Gamma(y) = 0$. Thus

$$R_0 \subset \{(y, x) \in M^{-1}\ \mathcal{P}_0(x) = 0\}.$$

If $(y, x) \in M^{-1}$ and $\mathcal{P}_0(x) = 0$, then as in the first part of this proof, $(y, x) \in R$. Hence

$$R_0 = \{(y, x) \in M^{-1} : \mathcal{P}_0(x) = 0\}. \ \square$$

LEMMA 3.4. (*See* [**5**]) *Suppose that M is a linear relation in $X \times Y$. Assume that R_0 is a principal topological selection of M^{-1}. Then R is a topological selection of M^{-1} if and only if*

$$R(y) = R_0(y) - A(y, R_0(y)), \quad \textit{for all} \quad y \in \operatorname{Range} M$$

for some continuous linear operator A on M^{-1} into $\operatorname{Null} M$ *such that* $A(0, x) = 0$ *for all* $x \in \operatorname{Null} M$.

We now express an arbitrary (nonprincipal) topological selection of M^{-1} in terms of a given topological selection of M_1^{-1}.

THEOREM 3.5. *Let R_1 be a given topological selection of M_1^{-1}. Let M be as in (3.2) for some $m \times d$ constant matrix $\boldsymbol{\Gamma}$. Then R is a topological selection of M^{-1} if and only if*

$$\operatorname{Dom} R = \{y : y \in \operatorname{Range} M_1,\ \boldsymbol{\Gamma} B(R_1(y), y) \in \operatorname{Range}(\boldsymbol{\Gamma}\mathbf{D})\},$$

$$R(y) = (I - \mathcal{P}_0)\eta_\Gamma(y) - A(y, (I - \mathcal{P}_0)\eta_{\mathbf{P}}(y)),\ y \in \operatorname{Dom} R$$

for some continuous projector $\mathcal{P}_0$ on $\operatorname{Dom} M$ *onto* $\operatorname{Null} M$ *and a continuous linear operator $A : M^{-1} \to \operatorname{Null} M$ such that $A(0, x) = 0$ for all $x \in \operatorname{Null} M$.*

PROOF. This follows Lemma 3.4 together with Lemma 3.3. □

In the following we give a simpler necessary form for the topological selections of M^{-1}.

THEOREM 3.6. *Suppose that M_1 is a closed linear relation in $X \times Y$ with* $\dim \operatorname{Null} M_1 =: n_1 < \infty$. *Let R_1 be an arbitrary, but fixed topological operator of M_1^{-1}. If M is a closed linear relation in $X \times Y$ with $M \subset M_1$, $\dim M_1/M < \infty$ and if R is any topological selection of M^{-1}, then there exist $x_j^+ \in X^\#$ and $y_j^+ \in Y^\#(1 \leq j \leq n_1)$ such that*

$$R(y) = R_1(y) + \sum_{j=1}^{n_1}[\langle R_1(y), x_j^+\rangle + \langle y, y_j^+\rangle]\phi_j,\ y \in \operatorname{Dom} R.$$

PROOF. Let M be as in the theorem. Since $\dim M_1/M < \infty$, without loss of a generality we may assume that there exists a *closed* linear relation $M_0 \subset X \times Y$ such that $M_0 \subset M \subset M_1$. We may also assume that

$$\dim M_1/M =: m \leq \dim M_1/M_0 =: d.$$

Let B be a continuous linear operator on M_1 onto $\mathbb{C}^d$ annihilating M_0. Using Lemma 3.1, we may also assume that

$$M = \{a \in M_1 : \boldsymbol{\Gamma} B(a) = 0_{m \times 1}\},$$

for some $m \times d$ constant matrix $\boldsymbol{\Gamma}$. This means that M fits into the setting described at the beginning of this section. Thus we assume that $\phi_i, \mathbf{D}$ are as before. Take any topological selection R of M^{-1}. Then by Theorem 3.5,

$$\begin{aligned} &\operatorname{Dom} R = \{y \in \operatorname{Range} M_1 : \boldsymbol{\Gamma} B(R_1(y), y) \in \operatorname{Range}(\boldsymbol{\Gamma} D)\}, \\ &R(y) = (I - \mathcal{P}_0)\eta_\Gamma(y) - A(y, (I - \mathcal{P}_0)\eta_\Gamma(y)), \ y \in \operatorname{Dom} R, \end{aligned} \tag{3.5}$$

for some continuous projector $\mathcal{P}_0$ on $\operatorname{Dom} M$ onto $\operatorname{Null} M$ and a continuous linear operator $A : M^{-1} \to \operatorname{Null} M$ such that $A(0, x) = 0$ for all $x \in \operatorname{Null} M$, where

$$\eta_\Gamma(y) = R_1(y) - \sum_{i=1}^{n_1} \phi_i [(\boldsymbol{\Gamma} \mathbf{D})^\dagger \boldsymbol{\Gamma} B(R_1(y), y)]_i. \tag{3.6}$$

Since B is a bounded linear operator there exist $h_{j1}^+ \in X^\#$, $h_{j2}^+ \in Y^\#$ such that

$$(B(x, y))_j = \langle x, h_{j1}^+ \rangle + \langle y, h_{j2}^+ \rangle, \ (x, y) \in M_1. \tag{3.7}$$

Since $\mathcal{P}_0 : \operatorname{Dom} M \to \operatorname{Null} M$ and $A : M^{-1} \to \operatorname{Null} M$ are continuous and $\operatorname{Null} M \subset \operatorname{Null} M_1$, there exist $g_{i1}^+ \in X^\#$, $f_{i1}^+ \in X^\#$, $f_{j2}^+ \in Y^\#$ such that

$$\mathcal{P}_0(x) = \sum_{i=1}^{n_1} \langle x, g_{i1}^+ \rangle \phi_i, \qquad x \in \operatorname{Dom} M, \tag{3.8}$$

$$A(y, x) = \sum_{i=1}^{n_1} [\langle y, f_{i2}^+ \rangle + \langle x, f_{i1}^+ \rangle] \phi_i \tag{3.9}$$

for some $f_{i2}^+ \in Y^\#$, $f_{i1}^+ \in X^\#$. Substituting B in (3.6), $\mathcal{P}_0$ in (3.8) and A in (3.9) into (3.6) and then into (3.5), it follows, after a tedious calculation, that $R(y)$ has the form claimed in the above corollary. □

COROLLARY 3.7. *Suppose that* M_1 *is a closed linear relation in* $X \times Y$ *with* $\dim \mathrm{Null}\, M_1 < \infty$. *Suppose that* M *is a closed subspace of* M_1 *such that* $\dim M_1/M < \infty$. *If some topological selection of* M_1^{-1} *is compact, then all topological selections of* M^{-1} *are also compact.*

PROOF. The result follows immediately from Theorem 3.6. □

REMARK 3.1. By the above corollary, if R_1 is a compact topological selection of M_1^{-1} and if $\mathrm{Null}\, M_1$ is finite dimensional, then any topological selection of M_1^{-1} is also compact. It is proved in [**5**] that if M is a closed linear relation in $X \times Y$, then the continuity of a particular topological selection of M^{-1} implies the continuity of *any* topological selection of M^{-1}.

4. Applications to integral operators. In this section we consider the following question: Suppose that L and L_1 are closed linear operators from a Banach space X into a Banach space Y such that $\mathcal{G}r\, L \subset \mathcal{G}r\, L_1$, where $\mathcal{G}r\, L$ denotes the graph of L. Assume that $(\mathcal{G}r\, L_1)^{-1}$, as a linear relation, has a topological selection which is an "integral" operator. Is every topological selection of $(\mathcal{G}r\, L)^{-1}$ also an "integral" operator? The question is, of course, trivially true if L_1 is one-to-one. The above question arises naturally if L_1 is an ordinary differential operator. We will answer this question under a certain finiteness condition. We will consider the problem when X and Y are L^p-type spaces or the space of continuous functions. Let (S, Σ_S, μ) and (T, Σ_T, ν) be σ-finite nonnegative measure spaces, and for $1 \leq p \leq +\infty$, let $X^p := X^p(S, \Sigma_S, \mu)$ be the Banach space of all complex-valued μ-measurable functions x on S such that $\|x\|_p < \infty$, where

$$\begin{aligned}\|x\|_p &:= \left(\int_S |x(s)|^p d\mu(s)\right)^{1/p} \quad \text{if} \quad 1 \leq p < \infty,\\ &:= \operatorname*{ess\,sup}_{s \in S} |x(s)| \quad \text{if} \quad p = \infty.\end{aligned}$$

For $1 \leq q \leq \infty$, we define $Y^q := Y^q(T, \Sigma_T, \nu)$ in a similar way.

Let K be a compact Hausdorff space and let $C(K)$ denote the Banach space of all complex-valued functions f on K with $\|f\|_\infty := \sup\{|f(k)| :$

$k \in K\} < \infty$. As in §3, for a Banach space Z, the natural pairing on $Z \times Z^{\#}$ is denoted by $\langle \cdot, \cdot \rangle$.

DEFINITION 4.1. U is an integral operator from X^p into Y^q if $\operatorname{Dom} U$ is a vector subspace of X^p and there exists a complex-valued function $\mathcal{K}(s,t)$ $(s \in S, t \in T)$ such that

(i) $\mathcal{K}(s,t)$ is $\mu \times \nu$-measurable, and

(ii) for all $x \in \operatorname{Dom} U$ and for ν-almost all $t \in T$, $\mathcal{K}(s,t)x(s)$, considered as a function of s, is μ-integrable, and

$$(Ux)(t) := \int_S \mathcal{K}(s,t)x(s)\, d\mu(s) \in Y^q.$$

DEFINITION 4.2. U is an integral operator from X^p into $C(K)$ if $\operatorname{Dom} U$ is a vector subspace of X^p and there exists a complex-valued function $\mathcal{K}(s,k)(s \in S, k \in K)$ such that

(i) $\mathcal{K}(s,k)$ is $\Sigma_S \times \mathcal{B}$-measurable, where $\mathcal{B}$ is the σ-algebra of all Borel subset of K, and

(ii) for all $x \in \operatorname{Dom} U$ and *all* $k \in K$, $\mathcal{K}(s,k)x(s)$ as a function of s is μ-integrable, and

$$(Ux)(k) := \int_S \mathcal{K}(s,k)x(s)d\mu(s) \in C(K).$$

The function $\mathcal{K}$ is called the kernel of U.

We recall that the natural pairing $\langle \cdot, \cdot \rangle$ on $X^p \times X^{p'}$, where p' is the conjugate number to $p > 1$ $(p^{-1} + (p')^{-1} = 1)$, is given by

$$\langle x, y \rangle := \int_S x(s)\overline{y}(s)\, d\mu(s),$$

and that the natural pairing on $C(K) \times (C(K))^{\#}$ is given by

$$\langle x, \phi \rangle := \int_K x(k)\, d\phi(k),\ x \in C(K),\ \phi \in (C(K))^{\#}.$$

LEMMA 4.1. *Let U be an integral operator from X^p into Y^q with kernel $\mathcal{K}(x,t)$ $(s \in S, t \in T)$. Suppose that*

$$\int_S |\mathcal{K}(s,t)x(s)|\, d\mu(s) \in Y^q \tag{4.1}$$

for all $x \in \operatorname{Dom} U$. *Then for any* $y^+ \in (Y^q)^\#$, *the functional* $\langle U(x), y^+\rangle$, $x \in \operatorname{Dom} U$, *is an integral operator from* X^p *into* $\mathbb{C}$. *In particular,* (4.1) *is satisfied if*

$$\left\{\begin{array}{l} h(t) := \|\mathcal{K}(\cdot,t)\|_{p'} < \infty, \quad \text{and} \\ \quad h \in Y^q \end{array}\right. \tag{4.2}$$

where p' *is the conjugate number to* p.

LEMMA 4.2. *Let* U *be an integral operator from* X^p *into* $C(K)$ *with kernel* $\mathcal{K}(s,k)$ $(s \in S, k \in K)$. *Suppose that*

$$\int_S |\mathcal{K}(s,k)x(s)|\, d\mu(s) \in C(K). \tag{4.3}$$

Then for any $\phi \in (C(K))^\#$, *the functional*

$$\langle U(x), \phi\rangle, \ x \in \operatorname{Dom} U$$

is an integral operator from X^p *into* $\mathbb{C}$. *In particular,* (4.2) *is satisfied if*

$$\left\{\begin{array}{l} h(k) := \|\mathcal{K}(\,\cdot\,,k)\|_{p'} < \infty \quad \text{and} \\ \quad h \in C(K) \end{array}\right. . \tag{4.4}$$

PROOF OF LEMMA 4.1.

Case (i). $1 \le q < +\infty$. Assume (4.1) and take $x \in X^p$ and $y^+ \in (Y^q)^\star$. Let q' be the conjugate to q. Let

$$f(t)) = \int_S |\mathcal{K}(s,t)x(s)|\, d\mu(s).$$

Then

$$\begin{aligned} &\int_T \left[\int_S |\mathcal{K}(s,t)|\, |x(s)|\, d\mu(s)\right] |y^+(t)|\, d\nu(t) \\ &\quad = \int_T f(t)|y^+(t)|\, d\nu(t) \le \|f\|_{Y^q}\, \|y^+\|_{Y^{q'}} < \infty. \end{aligned}$$

Thus by Tonelli's theorem

$$\langle U(x), y^+\rangle = \int_S \left[\int_T \mathcal{K}(s,t)\overline{y^+}(t)\, d\nu(t)\right] x(s) d\mu(s).$$

Hence $\langle U(x), y^+\rangle$ is an integral operator.

Case (ii). $q = \infty$.

It is known (see Kantorovich & Akilov [**2**, p. 192] that $(Y^\infty)^\#$ is the Banach space of all "bounded additive" functions on Σ_T, that is, the set of all complex-valued additive functions ϕ on Σ_T such that (i) and (ii) hold:

(i) If $A \in \Sigma_T$ and $\nu(A) = 0$, then $\phi(A) = 0$.

(ii) The total variation of ϕ, $|\phi|(T)$, is finite.

The norm $\|\phi\|$ of ϕ is given by $\|\phi\| := |\phi|(T)$. Thus for any $y \in Y^\infty$ and $\phi \in (Y^\infty)^\#$,

$$\langle y, \phi\rangle := \int_T y(t)\, d\,\overline{\phi(t)},$$

where the integral is a Stieltjes integral. Take any $x \in X^p$ and $\phi \in Y^\infty$. Then

$$\langle Ux, \phi\rangle = \int_T (Ux)(t) d\,\overline{\phi}(t) = \int_T \int_S \mathcal{K}(s,t)x(s)\, d\mu(s)\, d\overline{\phi}(t),$$

and

$$\int_T \int_S |\mathcal{K}(s,t)|\, |x(s)|\, d\mu(s)\, d|\phi|(t)$$
$$\leq \left(\operatorname*{ess\,sup}_{t\in T} \int_S |\mathcal{K}(s,t)|\, |x(s)|\, d\mu(s)\right) \|\phi\| < \infty.$$

Thus the order of integration in the above expression for $\langle Ux, \phi\rangle$ may be interchanged, and so

$$\langle Ux, \phi\rangle = \int_S \left[\int_T \mathcal{K}(s,t)\, d\overline{\phi}(t)\right] x(s)\, d\mu(s).$$

Hence $\langle Ux, \phi\rangle$ is an integral operator. Combining the cases (i) and (ii), we see that when (4.1) holds, $\langle Ux, \phi\rangle$, $\phi \in (Y^q)^\#$, is an integral operator.

Suppose now that (4.2) holds. Take any $x \in X^p$. Suppose $1 \leq q < \infty$. Then

$$\int_T \left(\int_S |\mathcal{K}(s,t)|\, |x(x)|\, d\mu(s)\right)^q d\nu(t)$$
$$\leq \int_T (h(t)\|x\|_p)^q\, d\nu(t) = \|x\|_p^q\, \|h\|_q^q < \infty.$$

Suppose $q = +\infty$, Then

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\in T}\left(\int_S |\mathcal{K}(s,t)|\,|x(s)|\,d\mu(s)\right) \\
&\qquad \le \operatorname*{ess\,sup}_{t\in T}(h(t))\|x\|_p = \|x\|_p\,\|h\|_\infty < \infty.
\end{aligned}$$

Thus if (4.2) holds, then (4.1) holds. This proves Lemma 4.1. □

PROOF OF LEMMA 4.2. It is known that $(C(K))^{\#}$ is the Banach space of all regular countably additive complex-valued functions ϕ on the σ-algebra $\mathcal{B}$ of the Borel sets in K such that $|\phi|(K) < \infty$. In particular, for any $x \in C(K)$, and $\phi \in (C(K))^{\#}$, x is ϕ-integrable (ϕ is a σ-finite signed measure on $\mathcal{B}$). Assume (4.3). Take any $x \in C(K)$ and $\phi \in (C(K))^{\#}$. Then

$$\begin{aligned}
&\int_T\int_S |\mathcal{K}(s,t)x(s)|\,d\mu(s)\,d|\phi|(t) \\
&\le \int_K\left(\sup_{t\in K}\int_S |\mathcal{K}(s,t)x(s)|\,d\mu(s)\right)\,d|\phi|(t) \\
&\le \|\phi\|\,\sup_{t\in K}\int_S |\mathcal{K}(s,t)x(s)|\,d\mu(s) < +\infty.
\end{aligned}$$

Hence,

$$\begin{aligned}
\langle Ux,\phi\rangle &= \int_K\int_S \mathcal{K}(s,t)x(s)\,d\mu(s)\,d\phi(s) \\
&= \int_S\left[\int_K \mathcal{K}(s,t)\,d\phi(s)\right]x(s)\,d\mu(s).
\end{aligned}$$

Thus $\langle Ux,\phi\rangle$ is an integral operator in x from X^p into $C(K)$.

Assume now that (4.4) holds. Take any $x \in X^p$. Then

$$\int_S |\mathcal{K}(s,t)x(s)|\,d\mu(s) \le h(t)\|x\|_p$$

Thus

$$\sup_{t\in K}\int_S |\mathcal{K}(s,t)x(s)|\,d\mu(s) \le \sup_{t\in K} h(t)\,\|x\|_p < \infty.$$

Thus (4.3) holds. This completes the proof for Lemma 4.2. □

THEOREM 4.3 *Let Z be either Y^q or $C(K)$, as defined earlier. Let M_1 be a closed linear relation in $Z \times X^p$ such that* $\dim \operatorname{Null} M_1 < \infty$. *Let M be a closed subspace of M_1 such that* $\dim M_1/M < \infty$. *Suppose that there exists a topological selection U_1 of M_1^{-1} such that U_1 is an integral operator with kernel $\mathcal{K}(s,t)$ satisfying*

$$\int_S |\mathcal{K}(s,t)x(s)|\, d\mu(s) \in Z \tag{4.5}$$

for all $x \in$ $\operatorname{Dom} U_1$. *Then any topological selection U of M^{-1} is also an integral operator. Moreover, if U_1 is continuous, so is U.*

PROOF. Let $n_1 = \dim \operatorname{Null} M_1$ and let $\{\phi_1, \dots, \phi_{n_1}\}$ be a basis for $\operatorname{Null} M_1$. Take any topological selection U of M^{-1}. Then by Theorem 3.6, U has the form

$$U(x) = U_1(x) + \sum_{j=1}^{n_1} [\langle U_1(x), z_j^+\rangle + \langle x, x_j^+\rangle]\phi_j, \; x \in \operatorname{Dom} U \tag{4.6}$$

for some elements $x_j^+ \in X^{p'}$, $z_j^+ \in Z^{\#}$. By Lemmas 4.1 and 4.2, $\langle U_1(x), z_j^+\rangle$ as a function of x is an integral operator. Hence by (4.6), U is also an integral operator from X^p into Z. The rest follows easily. □

5. Applications to regular ordinary differential subspaces in $\mathbf{L_p[a,b] \times L_q[a,b]}$. In this section we consider concrete applications of Theorem 3.6 and Corollary 3.7 to ordinary differential subspaces. Let $[a,b]$ be a compact interval. Let $1 \le p \le \infty$, $1 \le q \le \infty$. Let $X_p (1 \le p \le \infty)$ be the Banach space of all functions $f : [a,b] \to \mathbb{C}^m$ which are Lebesgue measurable on $[a,b]$ and $\|f\|_p < \infty$, where $\|f\|_p^p = \int_a^b (f^\star(t)f(t))^p\, dt$ when $p < \infty$; $\|f\|_\infty :=$ ess sup$\{f^\star(t)f(t) : t \in [a,b]\}$. Let n be a natural integer. Let T_1 be the "maximal" differential operator from $\operatorname{Dom} T_1 \subset X_p$ into X_q defined by

$$x \in \operatorname{Dom} T_1 \quad \text{if and only if} \quad x \in X_p,\; x^{(n-1)} \in AC[a,b],\; x^{(n)} \in X_q,$$

$$T_1 x = Q_n(t)x^{(n)} + Q_{n-1}(t)x^{(n-1)} + \cdots + Q_1(t)x^{(1)} + Q_0(t)x.$$

Here each $Q_i(t)$ is a $m \times m$ matrix-valued function of $t \in [a,b]$ which is i-times continuously differentiable on $[a,b]$ and $Q_n(t)$ is invertible for all $t \in [a,b]$, and $x^{(i)}$ denotes the i^{th} derivative of x. Then it is well-known that T_1 is closed, i.e., $\mathcal{G}r(T_1)$ is closed in $X_p \times X_q$. Let

$$M_1 := \{a + (x, T_1 x) : a \in A,\ x \in \operatorname{Dom} T_1\},$$

where A is a *finite* dimensional subspace of $X_p \times X_q$. Then it is not difficult to see that $\operatorname{Null} M_1$ is finite dimensional and M_1 is closed. Thus, in particular, there exist topological selections of M_1^{-1}.

THEOREM 5.1. *Let T_1 and M_1 be as above. Let M be an arbitrary, but fixed closed subspace of M_1 with* $\dim M_1/M < \infty$. *Let R be any topological selection of M^{-1}. Then R is the graph of a compact integral operator on* $\operatorname{Range} M$. *Moreover,* $\operatorname{Range} M$ *is closed.*

PROOF. Since the differential equation $T_1 x = g$ has a solution for every $g \in X_q$, we see that $\operatorname{Range} T_1 = X_q$. Since T_1 is closed and $\dim \operatorname{Null} T_1 < \infty$, it is easy to see that $(\mathcal{G}r\, T_1)^{-1}$ has a topological selection which is a compact integral operator. Pick one and call it R_0. We will show that M_1^{-1} has a topological selection which is a compact integral operator defined on X_q. It is clear that M_1^{-1} has a principal topological selection. Call this R_1. Then $\operatorname{Dom} R_1 = X_q$ and

$$R_1(y) = (I - \mathcal{P})(x),\ x \in M_1^{-1}(y),\ y \in X_q \tag{5.1}$$

where $\mathcal{P}$ is a continuous projector on $\operatorname{Dom} M_1$ onto $\operatorname{Null} M_1$. Take any $y \in \operatorname{Dom} R_1$. Then $R_1(y)$ is given as in (5.1) for some $(x, y) \in M_1$. Write

$$x = z + f,\ y = T_1 z + g$$

for some $(f, g) \in A$. Then, in particular, $z = R_0(y - g) + k$ for some $k \in \operatorname{Null} T$. It then follows that $R_1(y) = (I - \mathcal{P})\,[R_0(y) + k - R_0(g) + f]$. Since

$$\operatorname{Null} M_1 = \{u + h : u \in \operatorname{Dom} T_1,\ T_1 u + g = 0 \text{ for some } (h, g) \in A\},$$

it follows that

$$k - R_0(g) + f \in \operatorname{Null} M_1.$$

Thus

$$R_1(y) = (I - \mathcal{P})R_0(y).$$

This shows that R_1 is a compact operator. Applying Fubini's theorem, if necessary, it follows that R_1 is also an integral operator since $\mathcal{P}$ has a finite-dimensional range. Hence we have shown that M_1^{-1} has a topological operator which is also a compact integral operator. Therefore using Theorems 3.6 and 4.1 we see that any topological selection R of M^{-1} is a compact integral operator. Finally $\mathrm{Dom}\, R :=$ $\mathrm{Range}\, M$ must be closed since R is a closed bounded linear operator. This completes the proof. □

REMARK 5.1. For a nonnegative integer m and complex constants α_{ij}, β_{ij} define a differential operator T by

$$\mathrm{Dom}\, T = \left\{ x \in \mathrm{Dom}\, T_1 : \sum_{i=0}^{n-1} (\alpha_{ij} x^{(i)}(a) + \beta_{ij} x^{(i)}(b)) = 0,\ 0 \le j \le m \right\},$$

$Tx = T_1 x$, $x \in \mathrm{Dom}\, T$, where T_1 is as defined earlier in this section. Then T is a closed operator, but is not necessarily one-to-one. Using Theorem 3.1 of [**5**] a linear operator $R : \mathrm{Range}\, T \to \mathrm{Dom}\, T$ is a *topological* selection of $(\mathcal{G}r\, T)^{-1}$ if and only if $TR = I$ on $\mathrm{Range}\, M$, and RT is continuous on $\mathrm{Dom}\, T$. Since $\dim(\mathcal{G}r\, T_1/\mathcal{G}r\, T) =$ $\dim(\mathrm{Dom}\, T_1/\mathrm{Dom}\, T) < +\infty$, it follows from the above theorem that any topological selection of $(\mathcal{G}r\, T)^{-1}$ is a compact integral operator. The kernel of this selection is called a generalized Green function when T is not invertible, and the Green function when T is invertible. This has been extensively studied in the literature.

REMARK 5.2. Let f and g be $m \times n$ matrix-valued functions on $[a, b]$ such that the columns of f are in $X_{p'}$ and the columns of g are in $X_{q'}$. Define an operator T by

$$\mathrm{Dom}\, T := \left\{ x \in \mathrm{Dom}\, T_1 : \int_z^b [x^\star(t) f(t) + (T_1 x)^\star g(t)]\, dt = 0_{1\times m} \right\},$$

$$Tx := T_1 x,\ x \in \mathrm{Dom}\, T.$$

Then $\mathcal{G}r\,T \subset \mathcal{G}r\,T_1$, T is closed. Moreover $\dim(\mathcal{G}r\,T_1/\mathcal{G}r\,T) \le m < \infty$. Thus by the above theorem any topological selection of $(\mathcal{G}r\,T)^{-1}$ is a compact integral operator.

REMARK 5.3. Let $\mu(1 \le i \le m < \infty)$ be $m \times n$ vector-valued functions of bounded variation on $[a, b]$. Define T by

$$\operatorname{Dom} T := \left\{ x \in \operatorname{Dom} T_1 : \int_a^b (d\mu(t))x(t) = 0_{m\times 1} \right\},$$

$$Tx := T_1 x, \ x \in \operatorname{Dom} T$$

The integral is taken as a Riemann-Stieltjes integral. Clearly $\mathcal{G}r\,T \subset \mathcal{G}r\,T_1$. *Assume* that T is closed. Then one can see easily that $\dim(\mathcal{G}r\,T_1/\mathcal{G}r\,T) < \infty$ and $\operatorname{Dom} T$ can be written as in Remark 5.2, and so, by the above theorem, any topological selection of $(\mathcal{G}r\,T)^{-1}$ is a compact integral operator. Special cases of this problem have been considered in the literature. See the survey paper by Krall [**3**].

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