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AN INTEGRAL EQUATION METHOD FOR THE TIME-HARMONIC MAXWELL EQUATIONS WITH BOUNDARY CONDITIONS FOR THE NORMAL COMPONENTS

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ABSTRACT. The reflection of electromagnetic waves at an anistropic medium leads to boundary conditions for the normal components. A new integral equation approach is developed for multiply connected domains. The existence of a solution is established by using the second part of Fredholm's alternative.

Introduction. The mathematical description of the scattering of time-harmonic electromagnetic waves with frequency $\omega > 0$ by an obstacle, say D, surrounded by a homogeneous isotropic medium in $\mathbf{R}^3$ leads to exterior boundary-value problems for the reduced Maxwell equations

$$\mathrm{curl\,E - ikH = 0, \ \ curl\,H + ikE = 0}$$

for the electric field E and the magnetic field H. Here, the wave number k is given in terms of ω, the electric permittivity ε, the magnetic permeability μ and the electric conductivity σ by

$$k^2 = \Big(\varepsilon + \frac{i\sigma}{\omega}\Big)\mu\omega^2,$$

and the sign of k is chosen such that $\mathrm{Im\,k} \geq 0$. The scattering of a given incoming electromagnetic wave E^i, H^i by a perfect conducting body gives rise to a boundary condition of the form

$$[\nu, E] = 0 \text{ on } \partial D \tag{0.1}$$

describing vanishing tangential components of the electric fields for the total wave $E = E^i + E^s, H = H^i + H^s$ where E^s, H^s are the scattered fields.

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In addition to the classical boundary condition (0.1) Rumsey [**12**] suggested to consider a boundary condition of the form

$$(0.2) \qquad (\nu, E) = 0, \ (\nu, H) = 0 \text{ on } \partial D$$

for the normal components of both the electric and magnetic field. This problem may be considered as the limiting case of the scattering from a dielectric domain with both very large electric permittivity and magnetic permeability.

The same type of boundary conditions occurs in the theory of force-free fields, arising in plasmaphysics and astrophysics [**6, 7, 8**]. In contrast to the classical boundary condition (0.1), uniqueness for the boundary condition (0.2) depends on the connectedness of D. Uniqueness for a solution of Maxwell equations with boundary conditions (0.2) for simply connected domains D was established by Yee [**14**]. His results were extended by Kress [**5**] for multiply connected domains by prescribing circulations for the electric and magnetic fields. Further Kress [**5**] proved existence of a solution by using an integral equation method for an auxiliary problem.

In this paper we choose a more direct integral equation approach to establish existence results. Our method leads to integral equations of the first kind which have to be regularized in order to apply the classical Riesz-Fredholm theory. In the first part of the paper we describe our method for a simply connected domain in order to state the main ideas. In the second part we briefly discuss the extension of our ideas to the case of multiply connected domains. The reader interested in more details is referred to the author's thesis [**2**].

1. The boundary-value problems, uniqueness. Let D denote a simply connected bounded open domain in $\mathbf{R}^s$ with boundary ∂D of class C^2. We assume the complement $\mathbf{R}^3 \backslash D$ to be connected. Then the boundary ∂D also is connected. By ν we denote the unit normal to ∂D directed into the exterior of D.

We consider the following

Exterior boundary-value problem. Find two vector fields $E, H \in C^1(\mathbf{R}^3 \backslash \bar{D}) \cap \cup(\mathbf{R}^3 \backslash D)$ satisfying the time-harmonic Maxwell's equations

$$(1.1) \qquad \mathrm{curl\ E - ikH = 0, \ curl\ H + ikE = 0 \ in\ } \mathbf{R}^3 \backslash \bar{\mathrm{D}}$$

the Silver-Mueller radiation condition

$$\left[H(x), \frac{x}{|x|}\right] - E(x) = o\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \tag{1.2}$$

uniformly for all directions $x \backslash |x|$, and the boundary conditions

$$(\nu, E) = f, \ (\nu, H) = g \text{ on } \partial D \tag{1.3}$$

where $f, g \in C^{0,\alpha}(\partial D)$ are given functions.

REMARK 1.1. By Stokes' theorem, the conditions

$$\int_{\partial D} f ds = 0 \text{ and } \int_{\partial D} g ds = 0 \tag{1.4}$$

are necessary for solvability. Therefore, in the subsequent analysis, we will assume these conditions to be fulfilled.

The following uniqueness result was given by Yee [**14**] and Kress [**5**].

THEOREM 1.2. *The exterior boundary-value problem has no more than one solution.*

We also want to consider the

Interior boundary-value problem. Find two vector fields $E, H \in C^1(D) \cap C(\bar{D})$ satisfying the time-harmonic Maxwell's equations

$$\text{curl E} - \text{ikH} = 0, \ \text{curl H} + \text{ikE} = 0 \text{ in D} \tag{1.5}$$

and the boundary conditions

$$(\nu, E) = f, \quad (\nu, H) = g \text{ on } \partial D \tag{1.6}$$

where $f, g \in C^{0,\alpha}(\partial D)$ are given functions.

Analogously to Theorem 1.2 we have

THEOREM 1.3. *If* $\operatorname{Im} \mathrm{k} > 0$ *then the interior boundary-value problem has no more than one solution.*

For k real, this is not generally the case; instead, as shown by Kress [**5**] we have

THEOREM 1.4. *There exists a countable set of positive wave numbers k, called interior eigenvalues, accumulating only at infinity for which the homogeneous interior boundary-value problem has nontrivial solutions.*

2. Existence for the interior boundary-value problem. We try to find a solution to the interior boundary-value problem in a simply connected domain in the form

$$
\begin{aligned}
E(x) = & \operatorname{curl} \int_{\partial \mathrm{D}} \lambda(\mathrm{y}) \nu(\mathrm{y}) \Phi(\mathrm{x}, \mathrm{y}) \mathrm{ds}(\mathrm{y}) \\
& + \operatorname{curl} \operatorname{curl} \int_{\partial \mathrm{D}} \mu(\mathrm{y}) \nu(\mathrm{y}) \Phi(\mathrm{x}, \mathrm{y}) \mathrm{ds}(\mathrm{y}), \quad \mathrm{x} \in \mathrm{D},
\end{aligned} \tag{2.1}
$$

with scalar densities $\lambda, \mu \in C^{1,\alpha}(\partial D)$. Here

$$
\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y,
$$

denotes the fundamental solution to the Helmholtz equation.

In order to formulate an integral equation for the unknown densities we introduce the following integral operators S, K, K' and T by

$$
(S\varphi)(x) = 2 \int_{\partial D} \varphi(y) \Phi(x, y) ds(y), \quad \varphi \;\; x \in C(\partial D), \; x \in \partial D,
$$

$$
(K\varphi)(x) = 2 \int_{\partial D} \varphi(y) \frac{\partial}{\partial \nu(y)} \Phi(x, y) ds(y), \quad \psi \in C(\partial D), \; x \in \partial D,
$$

$$
(K'\varphi)(x) = 2 \int_{\partial D} \varphi(y) \frac{\partial}{\partial \nu(x)} \Phi(x, y) ds(y), \quad \psi \in C(\partial D), \; x \in \partial D,
$$

and

$$(T\psi)(x) = 2\frac{\partial}{\partial\nu(x)}\int_{\partial D}\psi(y)\frac{\partial}{\partial\nu(y)}\Phi(x,y)ds(y), \psi \in C^{1,\alpha}(\partial D), \ \ x \in \partial D.$$

The operators S, K, K' are compact in $C(\partial D)$ and $C^{0,\alpha}(\partial D)$ for $0 < \alpha < 1$. In addition S and K are compact in $C^{1,\alpha}(\partial D)$. The normal derivative of the double layer potential $T : C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ is only continuous because of the strong singularity. For proofs of these statements we refer to Colton, Kress [**1**].

In addition, we define the operator P by

$$(P\lambda)(x) = 2\left(\nu(x), \operatorname{curl}\int_{\partial \mathrm{D}}\nu(\mathrm{y})\lambda(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds}(\mathrm{y})\right),$$
$$\lambda \in^{1,\alpha} (\partial D), \ \ x \in \partial D.$$

By Stokes' theorem, Müller [**11**], we derive

$$\begin{aligned} &\operatorname{curl}\int_{\partial \mathrm{D}}\lambda(\mathrm{y})\nu(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds}(\mathrm{y}) \\ (2.2) \qquad &= -\int_{\partial D}[\lambda(y)\operatorname{grad}_y\Phi(x,y),\nu(y)]ds(y) \\ &= -\int_{\partial D}[\nu(y), \operatorname{Grad}\lambda(y)]\Phi(x,y)ds(y), \ \ x \in \mathbf{R}^3\backslash\partial D. \end{aligned}$$

Hence, passing to the limit $x \in \partial D$, we obtain $P\lambda = -(\nu, S[\nu, \operatorname{Grad}\lambda])$. In particular this implies that $P : C^{1,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ is continuous. Furthermore, we define the operator Q by $(Q\lambda)(x) := -(\nu(x), S(\lambda\nu)(x))$ and finally $A : (C^{1,\alpha}(\partial D))^2 \to (C^{0,\alpha}(\partial D))^2$ by

$$(2.3) \qquad A := \begin{bmatrix} -T - k^2Q & k^2P \\ P & -T - k^2Q \end{bmatrix}.$$

THEOREM 2.1. *The fields E defined by* (2.1) *and* $H := 1\backslash ik$ curl E *solve the interior boundary-value problem provided* $(\lambda,\mu)^T \in (C^{1,\alpha}(\partial D))^2$ *satisfies the integral equation*

$$(2.4) \qquad A\binom{\lambda}{\mu} = 2\binom{ik\,g}{f}.$$

of the first kind.

PROOF. Let $(\lambda,\mu)^T$ be a solution of (2.4) and define E by (2.1). Obviously $E \in C^2(D)$ and $H := 1\backslash ik, \mathrm{curl\ E} \in \mathrm{C^2(D)}$ solve the Maxwell equations. Using the vector formula curl curl A = grad div A$-$ ΔA we derive

(2.5)

$$\mathrm{curl\ curl} \int_{\partial \mathrm{D}} \lambda(\mathrm{y})\nu(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds}(\mathrm{y}) = k^2 \int_{\partial D} \lambda(y)\nu(y)\Phi(x,y)ds(y) - \mathrm{grad} \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \lambda(y)ds(y).$$

From this, in particular, using Theorems 2.17 and 2.23 of Colton, Kress [**1**], we deduce that $E, H \in C^1(\bar{D})$ since $\lambda, \mu \in C^{1,\alpha}(\partial D)$.

Since $(\lambda,\mu)^T$ is a solution of (2.4), from the jump relations (By the indices $+$ and $-$ we distinguish limits obtained by approaching ∂D from $\mathbf{R}^3\backslash\bar{D}$ and D respectively.), we get

$$ik(\nu(x), H(x))_- = \Big(\nu(x), \mathrm{curl\ curl} \int_{\partial \mathrm{D}} \lambda(\mathrm{y})\nu(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds}(\mathrm{y})\Big) + \Big(\nu(x), k^2 \mathrm{curl} \int_{\partial \mathrm{D}} \mu(\mathrm{y})\nu(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds}(\mathrm{y}) = \mathrm{ikg(x)},$$

$x \in \partial D$, and

$$(\nu(x), E(x))_- = \Big(\nu(x), \mathrm{curl} \int_{\partial \mathrm{D}} \lambda(\mathrm{y})\nu(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds}(\mathrm{y})\Big) + \Big(\nu(x), \mathrm{curl\ curl} \int_{\partial \mathrm{D}} \mu(\mathrm{y})\nu(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds}(\mathrm{y})\Big) = \mathrm{f(x)},$$

$x \in \partial D$. Here we have used (2.5) and the transformation

$$\Big(\nu(x), \mathrm{curl\ curl} \int_{\partial \mathrm{D}} \lambda(\mathrm{y})\nu(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds}(\mathrm{y})\Big) = -(\mathrm{T}\lambda)(\mathrm{x}) - \mathrm{k}^2(\mathrm{Q}\lambda)(\mathrm{x}),$$

$x \in \partial D$. □

Similarly, to solve the exterior boundary-value problem, we choose

$$E(x) = \mathrm{curl\ curl} \int_{\partial \mathrm{D}} \lambda'(\mathrm{y})\nu(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds}(\mathrm{y}) + \mathrm{curl} \int_{\partial \mathrm{D}} \mu'(\mathrm{y})\nu(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds}(\mathrm{y}), \tag{2.6}$$

$x \in \mathbf{R}^3\backslash\bar{D}$, with unknown scalar densities $\lambda', \mu' \in C^{1,\alpha}(\partial D)$. We define an operator $A' : (C^{1,\alpha}(\partial D))^2 \to (C^{0,\alpha}(\partial D))^2$ by

$$A' := \begin{bmatrix} -T - k^2 Q & P \\ k^2 P & -T - k^2 Q \end{bmatrix} \tag{2.7}$$

THEOREM 2.2. *The fields E defined by* (2.6) *and* $H := 1\backslash ik,$ curl E *solve the exterior boundary-value problem provided* $(\lambda', \mu')^T \in (C^{1,\alpha}(\partial D))^2$ *satisfies the integral equation*

$$A'\binom{\lambda'}{\mu'} = 2\binom{f}{ik\,g}. \tag{2.8}$$

of the first kind.

PROOF. This is analogous to the proof of Theorem 2.1. The radiation condition (1.2) for (2.6) follows from Chapter 4 in Knauff, Kress [**3**].

Both of the integral equation systems (2.4) and (2.8) are of the first kind. We shall establish existence of the solution by the Riesz-Fredholm Theory for compact operators after equivalently regularizing both equations, see Michlin [**10**].

Choose a wave number $\tilde{k}$ which is not an eigenvalue of the interior Dirichlet problem for the Laplace equation, that is, the Dirichlet problem $\Delta u + \tilde{k}^2 u = 0$ in D with homogeneous boundary conditions $u = 0$ on ∂D admits only the trivial solution $u = 0$. Define the operator $\tilde{S} : C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ as the operator S for this wave number $\tilde{k}$. Then $\tilde{S}$ is bijective and thus the operator $R : (C^{0,\alpha}(\partial D))^2 \to (C^{1,\alpha}(\partial D))^2$, defined by

$$R := \begin{bmatrix} \tilde{S} & 0 \\ 0 & \tilde{S} \end{bmatrix}, \tag{2.9}$$

is also bijective. □

THEOREM 2.3. *R is an equivalent right regularizer of A in $(C^{0,\alpha}(\partial D))^2$ and an equivalent left regularizer of A' in $(C^{1,\alpha}(\partial D))^2$.*

PROOF. Since Q and P are bounded and $\tilde{S}$ is compact in $C^{0,\alpha}(\partial D)$ and in $C^{1,\alpha}(\partial D)$, the products are compact in these spaces. So we only have to take care of $\tilde{S}T$ and $T\tilde{S}$. From Colton, Kress [**1**, p. 90], we see that

$$TS = I + K'^2 \text{ and } ST = -I + K^2,$$

that is, S regularizes T from the right in $C^{0,\alpha}(\partial D)$ and from the left in $C^{1,\alpha}(\partial D)$. For the special wave number $\tilde{k}$, the regularizations are equivalent because of

$$-T\tilde{S} = -\tilde{T}\tilde{S} + (\tilde{T} - T)\tilde{S} = I - \tilde{K}'^2 + (\tilde{T} - T)\tilde{S}.$$

By regularization from the right, the solution of the original equation (2.4) is given by

$$\binom{\lambda}{\mu} = R\binom{\lambda^*}{\mu^*} \text{ if } AR\binom{\lambda^*}{\mu^*} = \binom{ik\,g}{f}.$$

Before we introduce an appropriate bilinear form, we consider the nullspaces $N(A)$ and $N(A')$ of the operators A and A'. □

THEOREM 2.4. *Let $(\lambda, \mu)^T \in N(A)$ and define E by (2.1) in $\mathbf{R}^3\backslash\partial D$. Then E and $H := 1\backslash ik, \operatorname{curl} \mathrm{E}$ vanish in $\mathbf{R}^3\backslash D$ and solve the homogeneous interior boundary-value problem in D. There holds*

$$(E_-)_{\tan} = \operatorname{Grad}\mu, \quad ik(H_-)_{\tan} = \operatorname{Grad}\lambda \text{ on } \partial D. \tag{2.10}$$

PROOF. By Theorem 2.1, the fields E and H solve the homogeneous interior boundary-value problem in D. From (2.2) we have

$$\begin{aligned} E(x) = &-\int_{\partial D} [\nu(y),\ \operatorname{Grad}\lambda(y)]\Phi(x,y)ds(y) \\ &- \operatorname{curl}\int_{\partial D} [\nu(y),\ \operatorname{Grad}\mu(y)]\Phi(x,y)ds(y) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} H(x) = &-\frac{1}{ik}\operatorname{curl}\int_{\partial D} [\nu(y),\ \operatorname{Grad}\lambda(y)]\Phi(x,y)ds(y) \\ &+ ik\int_{\partial D} [\nu(y),\ \operatorname{Grad}\mu(y)]\Phi(x,y)ds(y). \end{aligned}$$

Hence from the jump relations for single-layer potentials, see Theorem 2.24 in Colton, Kress [**1**], we see that E and H solve the homogeneous exterior boundary-value problem. Therefore, by the uniqueness Theorem 1.2 we see that $E = H = 0$ in $\mathbf{R}^3 \backslash D$. Now (2.10) is a consequence of the jump-relations, i.e.,

$$[\nu, [\nu, E]]_- = -\operatorname{Grad} \mu \text{ and } ik[\nu, [\nu, H]]_- = -\operatorname{Grad} \lambda \quad \text{on } \partial D.$$

COROLLARY 2.5. *Let k be an eigenvalue of the interior boundary-value problem with multiplicity m_k. Then, for the dimension of $N(A)$, there holds*

$$\dim N(A) \leq m_k + 2.$$

PROOF. Let $(\lambda_1, \mu_1), \ldots, (\lambda_{m_k+1}, \mu_{m_k+1}) \in N(A)$ and let $E_1, \ldots, E_{m_k+1}$ be the corresponding fields given by (2.1). Then there exists numbers $\alpha_1, \ldots \alpha_{m_k+1} \in \mathbf{C}$ such that

$$E := \sum_{j=1}^{m_k+1} \alpha_j E_j$$

vanishes in D. Hence for $\lambda := \sum_{j=1}^{m_k+1} \alpha_j \lambda_j$ and $\mu := \sum_{j=1}^{m_k+1} \alpha_j \mu_j$ we have $\operatorname{Grad} \lambda = \operatorname{Grad} \mu = 0$, that is, $\lambda = \text{const}$ and $\mu = \text{const}$ on ∂D. Therefore it follows that

$$\sum_{j=1}^{m_k+1} \alpha_j \binom{\lambda_j}{\mu_j} \in \operatorname{span} \left\{ \binom{1}{0}, \binom{0}{1} \right\},$$

Now the statement follows from the observation that the elements on the right at the last equation belong to $N(A)$. □

The nullspace of A' is described by

THEOREM 2.6.

i) *Let E, H be a solution to the homogeneous interior boundary-value problem. Then there exist unique scalar functions $\lambda', \mu' \in C^{1,\alpha}(\partial D)$ with $\int_{\partial D} \lambda' ds = \int_{\partial D} \mu' ds = 0$ such that*

(2.12) $$ikH_{\tan} = \operatorname{Grad} \mu', \quad E_{\tan} = \operatorname{Grad} \lambda' \textit{ on } \partial D$$

and $\binom{\lambda'}{\mu'}$ *solve the homogeneous integral equation* (2.8).

ii) $\binom{\lambda'}{\mu'} \in \operatorname{span}\left\{\binom{1}{0}, \binom{0}{1}\right\}$ *solves the homogeneous integral equation* (2.8).

PROOF.

i). Let (E, H) be a solution to the homogeneous interior boundary-value problem. Then

$$\operatorname{Curl} \mathrm{E}_{\tan} = -\operatorname{Div}[\nu, \mathrm{E}_{\tan}] = (\nu, \operatorname{curl} \mathrm{E}) = 0$$

and

$$\operatorname{Curl} \mathrm{H}_{\tan} = -\operatorname{Div}[\nu, \mathrm{H}_{\tan}] = (\nu, \operatorname{curl} \mathrm{H}) = 0.$$

From this, since ∂D is simply connected, we observe that the tangential fields $E_{\tan}$ and $H_{\tan}$ are circulation free. Therefore there exists scalar functions $\lambda', \mu' \in C^{1,\alpha}(\partial D)$, normalized by $\int_{\partial D} \lambda' ds = 0$ and $\int_{\partial D} \mu' ds = 0$, such that

$$\operatorname{Grad} \lambda' = E_{\tan}, \ \operatorname{Grad} \mu' = ikH_{\tan} \quad \text{on } \partial D.$$

From the Stratton-Chue representation theorem, see Colton, Kress [**1**], Theorem 4.1, we now derive

$$\begin{aligned} -E(x) &= \operatorname{curl} \int_{\partial \mathrm{D}} [\nu(\mathrm{y}), \mathrm{E}(\mathrm{y})] \Phi(\mathrm{x}, \mathrm{y}) \mathrm{ds}(\mathrm{y}) \\ &\quad - \frac{1}{ik} \operatorname{curl} \operatorname{curl} \int_{\partial \mathrm{D}} [\nu(\mathrm{y}), \mathrm{H}(\mathrm{y})] \Phi(\mathrm{x}, \mathrm{y}) \mathrm{ds}(\mathrm{y}) \\ &= \operatorname{curl} \int_{\partial \mathrm{D}} [\nu(\mathrm{y}), \ \operatorname{Grad} \lambda'(\mathrm{y})] \Phi(\mathrm{x}, \mathrm{y}) \mathrm{ds}(\mathrm{y}) \\ &\quad + \frac{1}{k^2} \operatorname{curl} \operatorname{curl} \int_{\partial \mathrm{D}} [\nu(\mathrm{y}), \ \operatorname{Grad} \mu'(\mathrm{y})] \Phi(\mathrm{x}, \mathrm{y}) \mathrm{ds}(\mathrm{y}) \\ &= -\operatorname{curl} \operatorname{curl} \int_{\partial \mathrm{D}} \nu(\mathrm{y}) \lambda'(\mathrm{y}) \Phi(\mathrm{x}, \mathrm{y}) \mathrm{ds}(\mathrm{y}) \\ &\quad - \operatorname{curl} \int_{\partial \mathrm{D}} \nu(\mathrm{y}) \mu'(\mathrm{y}) \Phi(\mathrm{x}, \mathrm{y}) \mathrm{ds}(\mathrm{y}), \ \mathrm{x} \in \mathrm{D}. \end{aligned}$$

The jump relations and the homogeneous boundary conditions for E and H now imply that

$$A' \binom{\lambda'}{\mu'} = 0.$$

Part ii) is easily seen from (2.11). □

As a consequence of Theorem 2.6 we have

COROLLARY 2.7.

$$\dim N(A') \geq m_k + 2.$$

We now introduce the nondegenerate bilinear form $\langle \cdot, \cdot \rangle : (C^{1,\alpha}(\partial D))^2 \times (C^{1,\alpha}(\partial D))^2 \to \mathbf{C}$ by

$$\left\langle \begin{pmatrix} \lambda \\ \mu \end{pmatrix}, \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} \right\rangle = \int_{\partial D} (\lambda\lambda' + \mu\mu') ds, \tag{2.13}$$

i.e., $(C^{0,\alpha}(\partial D))^2 \times (C^{0,\alpha}(\partial D))^2$ is a dual system with respect to (2.13). By interchanging the integration, we derive

THEOREM 2.8. *A is the adjoint operator to A' with respect to* (2.13).

In this sense, the interior and exterior boundary-value problems are adjoint. Since A and A' can be equivalently regularized, from Fredholm's alternative, we derive that the dimensions of the null spaces of A and A' are the same.

LEMMA 2.9.

$$\dim N(A) = m_k + 2, \dim N(A') = m_k + 2.$$

PROOF. By Corollary 2.5 and 2.7, and, since A and A' are adjoint, we have

$$m_k + 2 \geq \dim N(A) = \dim N(A') \geq m_k + 2.$$

□

Theorem 2.4 and 2.6 define two mappings j and j' from the space $ME := \{(E, H) | (E, H) \text{ is an interior eigen solution}\}$ into $(C^{1,\alpha}(\partial D))^2$. For abbreviation we set $W := j(ME)$ and $W' := j'(ME)$ and are now in the position to state our first main result of this chapter.

THEOREM 2.10. *The interior boundary-value problem in a simply connected domain is solvable if and only if*

$$\left\langle \begin{pmatrix} ik\,g \\ f \end{pmatrix}, \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} \right\rangle = 0 \text{ for all } \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} \in W' \oplus \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}. \tag{2.14}$$

PROOF. Necessity. In the following, let E be a solution of the inhomogeneous, F of the homogeneous interior boundary-value problem. As shown in the proof of Theorem 2.6, there exist $\lambda', \mu' \in C^{1,\alpha}(\partial D)$ with

$$F_{\tan} = \operatorname{Grad} \lambda', \quad (\operatorname{curl} \mathrm{F})_{\tan} = \operatorname{Grad} \mu' \text{ on } \partial \mathrm{D}.$$

By the second Green's Theorem we find

$$\begin{aligned}
\int_{\partial D} ik\,g\lambda' + f\mu' ds &= \int_{\partial D} (\nu, \operatorname{curl} \mathrm{E})\lambda' + \frac{1}{\mathrm{k}^2}(\nu, \operatorname{curl} \operatorname{curl} \mathrm{E})\mu' \mathrm{ds} \\
&= \int_{\partial D} \operatorname{Div}\{[\nu, E]\lambda'\} - \lambda' \operatorname{Div}[\nu, E] ds \\
&\quad + \frac{1}{k^2} \int_{\partial D} \operatorname{Div}\{[\nu, \operatorname{curl} \mathrm{E}]\mu'\} - \mu' \operatorname{Div}[\nu, \operatorname{curl} \mathrm{E}] \mathrm{ds} \\
&= \int_{\partial D} (\nu, E, \operatorname{Grad} \lambda') - \frac{1}{k^2}(\nu, \operatorname{Grad} \mu', \operatorname{curl} \mathrm{E}) \mathrm{ds} \\
&= \int_{\partial D} (\nu, E, F) - \frac{1}{k^2}(\nu, \operatorname{curl} \mathrm{F}, \operatorname{curl} \mathrm{E}) \mathrm{ds} \\
&= \frac{1}{k^2} \int_{\partial D} (\nu, E, \operatorname{curl} \operatorname{curl} \mathrm{F}) - (\nu, \operatorname{curl} \mathrm{F}, \operatorname{curl} \mathrm{E}) \mathrm{ds} \\
&= 0.
\end{aligned}$$

Sufficiency: Let (2.12) be fulfilled. Bye the Riesz-Fredholm theory, the condition is sufficient for the solvability of the system of integral equations (2.4). □

REMARK 2.11. We can also formulate a similar theorem for the exterior boundary-value problem. The condition is only sufficient but not necessary for solvability, since the exterior boundary-value problem has no eigenvalues.

In order to fill this gap we add some appropriate volume potentials to the ansatz (2.5). Let m_k be the multiplicity of the eigenvalue k and let $E_i, i = 1, \dots, m_k$, be a basis of the corresponding eigensolutions. Define

$$F_i(x) := \int_D \bar{E}_i(y)\Phi(x,y)dy, \ x \in \mathbf{R}^3, i = 1, \dots, m_k. \tag{2.15}$$

Since $E_i \in C^{0,\alpha}(\bar{D})$ these potentials F_i are of the class $C^1(\mathbf{R}^3)$, in particular, $(\nu, F_i) \in C^{0,\alpha}(\partial D)$. By Gauss' Theorem, using $\mathrm{curl}\ \mathrm{E_j} \in \mathrm{C}^{0,\alpha}(\bar{\mathrm{D}})$, we get

$$\begin{aligned}\mathrm{curl}\ \mathrm{F_i}(\mathrm{x}) = &-\int_{\partial D} [\nu(y), \bar{E}_i(y)]\Phi(x,y)ds(y) \\ &-\int_D \mathrm{curl}\ \bar{\mathrm{E}}_\mathrm{i}(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{dy}, \quad \mathrm{x} \in \mathbf{R}^3 \backslash \bar{\mathrm{D}},\ \mathrm{i} = 1, \dots, \mathrm{m_k}.\end{aligned}$$

Therefore $(\nu, \mathrm{curl}\ \mathrm{F_j}) \in \mathrm{C}^{0,\alpha}(\partial \mathrm{D})$.

We now try to solve the exterior boundary-value problem in the form

$$\begin{aligned}E(x) =&\mathrm{curl}\ \mathrm{curl} \int_{\partial \mathrm{D}} \lambda'(\mathrm{y})\nu(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds}(\mathrm{y}) \\ &+ \mathrm{curl} \int_{\partial \mathrm{D}} \mu'(\mathrm{y})\nu(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds}(\mathrm{y}) \\ &+ \sum_{i=1}^{m_k} \alpha_i F_i(x); \ x \in \mathbf{R}^3 \backslash \bar{D}\end{aligned}$$

Analogously to Theorem 2.2 we find

THEOREM 2.12. *The fields E and $H := 1\backslash ik$ curl E solve the exterior boundary-value in a simply connected domain if the densities $(\lambda', \mu')^T \in (C^{1,\alpha}(\partial D))^2$ satisfy*

$$A'\binom{\lambda'}{\mu'} = 2\binom{f}{ik\, g} - \sum_{i=1}^{m_k} \alpha_i \begin{pmatrix} (\nu, F_i)_+ \\ (\nu, \mathrm{curl}\ \mathrm{F_i})_+ \end{pmatrix}.$$

LEMMA 2.13. *Let k be an interior eigenvalue and let $(\lambda, \mu)^T \in W$ be an eigensolution of the homogeneous integral equation* (2.4). *Let*

E be the related interior eigensolution given by (2.1) *and let F be the corresponding field. Then we have*

$$\left\langle \begin{pmatrix} (\nu, F)_+ \\ (\nu, \text{curl F})_+ \end{pmatrix}, \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \right\rangle = \frac{1}{k^2} \int_D |E|^2 dx \neq 0.$$

PROOF. By Green's and Stokes' Theorems we derive

$$\begin{aligned}
\left\langle \begin{pmatrix} (\nu, F)_+ \\ (\nu, \text{ curl F})_+ \end{pmatrix}, \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \right\rangle &= \int_{\partial D} (\nu, \mu \text{ curl } \mathrm{F}_+) + (\nu, \mathrm{F})_+ \lambda \mathrm{ds} \\
&= \int_{\partial D} (\nu, \mu \text{ curl } \mathrm{F}_+) + \frac{1}{\mathrm{k}^2} (\nu, \text{ curl curl F})_+ \lambda \mathrm{ds} \\
&= \int_{\partial D} (\nu, F_+, \text{Grad}\, \mu) + \frac{1}{k^2} (\nu, \text{curl } \mathrm{F}_+, \text{Grad}\, \lambda) \mathrm{ds} \\
&= \int_{\partial D} (\nu, F_+, E_-) + \frac{1}{k^2} (\nu, \text{curl } \mathrm{F}_+, \text{curl } \mathrm{E}_-) \mathrm{ds} \\
&= \int_{\partial D} (\nu, F_-, E_-) - \frac{1}{k^2} (\nu, \text{curl } \mathrm{E}_-, \text{curl } \mathrm{F}_-) \mathrm{ds} \\
&= \frac{1}{k^2} \int_D (F, \Delta E) - (E, \Delta F) dx \\
&= \frac{1}{k^2} \int_D (F, k^2 E) - (E, k^2 F) + (E, \bar{E}) dx \\
&= \frac{1}{k^2} \int_D |E|^2 dx \neq 0,
\end{aligned}$$

where we have used the continuity of F and curl F. □

From the preceding Lemma 2.13 we see that if k is an interior eigenvalue, the coefficients $\alpha_i, i = 1, \ldots, m_k$, can be chosen so that

$$\left\langle \sum_{i=1}^{m_k} \alpha_i \begin{pmatrix} (\nu, F_i)_+ \\ (\nu, \text{curl } \mathrm{F_i})_+ \end{pmatrix}, \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} g \\ f \end{pmatrix}, \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \right\rangle \text{ for all } \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \in W.$$

Therefore as the second result we get the existence

THEOREM 2.14. *For all wave numbers k the exterior boundary-value problem in a simply connected domain is solvable if and only if*

$$\left\langle \begin{pmatrix} f \\ ik\, g \end{pmatrix}, \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \right\rangle = 0, \text{ for all } \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

PROOF. The necessity follows from Remark 1.1, the sufficiency from the choice of the coefficients α_i by the Riesz-Fredholm Theory. □

3. The interior and exterior boundary-value problem in a multiply connected domain. In the final chapter we want to give the reader a short impression of how to consider the boundary-value problems in multiply connected domains. The connectedness of D can be described by its topological genus p. Since ∂D is topologically equivalent to a sphere with p handles we can choose two sets of orientated surfaces $S_1, \dots, S_p$ in $\mathbf{R}^3 \backslash D$ with orientated boundaries $C_i' := \partial S_i, i = 1, \dots, p$, and $S_1', \dots, S_p'$ with orientated boundaries $C_i := \partial S_i', i = 1, \dots, p$, such that $\mathbf{R}^3 \backslash D \cup_{i=1}^P S_i$ and $D \backslash \cup_{i=1}^P S_i'$ are simply connected.

In the potential theoretic case $k = 0$, the time-harmonic Maxwell equations separate into the system

$$\operatorname{div} E = 0 \text{ and } \operatorname{curl} \mathrm{E} = 0 \tag{3.1}$$

for the electric field and the same system for the magnetic field H. Solutions to the system (3.1) are called harmonic vector fields. The radiation condition has to be replaced by

$$E(x) = o(1), \ |x| \to \infty, \ \text{ uniformly for all directions } \frac{x}{|x|}. \tag{3.2}$$

From condition (3.2) at infinity for harmonic fields E it follows that

$$E(x) = O\Big(\frac{1}{|x|^2}\Big), \ |x| \to \infty \text{ uniformly for all directions.} \tag{3.3}$$

Harmonic fields with vanishing normal components on the boundary (and satisfying (3.2) in unbounded domains) are called Neumann vector fields. If D is simply connected there exists only the trivial Neumann field $E = 0$ in D since any curl free field can be represented as the gradient of a harmonic function. If D is multiply connected, then as shown by Martensen [**9**] and Werner [**13**], there exist exactly p linearly independent Neumann fields in D and exactly p linearly independent Neumann fields in $\mathbf{R}^3 \backslash \bar{D}$. Let $Z_1', \dots, Z_p'$ denote a basis of Neumann fields in D. These can be normalized by the circulations

$$\int_{C_j} (T, Z_l')ds = \delta_{jl}, \ j, l = 1, \dots, p. \tag{3.4}$$

By τ we denote the unit tangent vector to curves.

In order to treat the interior boundary-value problem in multiply connected domains, we reformulate the

Interior boundary-value problem. Find two vector fields $E, H \in C^1(D) \cap C(\bar{D})$ satisfying the time-harmonic Maxwell equations (1.5) and the boundary conditions (1.6). In addition the fields E and H are required to have circulations

$$(3.5) \quad \int_{\partial D} (\nu, E, Z_j')ds = e_j' \text{ and } \int_{\partial D} (\nu, H, Z_j')ds = h_j', j = 1, \dots, p$$

with given complex numbers $\tilde{e} := (e_1', \dots, e_p')^T, \tilde{h} := (h_1', \dots, h_p')^T$

In the same way, for the exterior boundary-value problem circulations have to be prescribed. Under these additional conditions as shown by Kress [**5,4**] the uniqueness Theorems 1.2, 1.3 and 1.4 are still valid.

In order to show existence, let $Z_1', \dots, Z_p'$ be a basis of interior Neumann fields, normalized by their circulations (3.4) and define

$$(3.5) \quad X_j(x) := \int_{\partial D} [\nu(y), Z_j'(y)]\Phi(x, y)ds(y), \ x \in D, j = 1, \dots, p.$$

Since Neumann fields are Hölder-continuous up to the boundary we have $X_j \in C^{1,\alpha}(\bar{D}), j = 1, \dots, p$. By Gauss' Theorem we derive $\operatorname{div} X_j = 0$ in D, that is, the X_j solve the Maxwell equations. We now try to find a solution for the interior boundary-value problem in the form

$$(3.6) \quad \begin{aligned} E(x) =& \operatorname{curl} \int_{\partial \mathrm{D}} \lambda(\mathrm{y})\nu(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds}(\mathrm{y}) \\ &+ \operatorname{curl}\operatorname{curl} \int_{\partial \mathrm{D}} \mu(\mathrm{y})\nu(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds}(\mathrm{y}) \\ &+ \sum_{j=1}^{p} a_j \operatorname{curl} \mathrm{X_j(x)} + \sum_{\mathrm{j=1}}^{\mathrm{p}} \mathrm{b_j X_j(x)}; \ \mathrm{x} \in \mathrm{D}. \end{aligned}$$

In addition to the unknown densities $\lambda, \mu \in C^{1,\alpha}(\partial D)$ we have to determine the coefficients $a := (a_1, \dots, a_p)^T$ and $b := (b_1, \dots, b_p)^T \in \mathbf{C}$.

Let's define the linear operators

$$(Mb)(x) := 2\sum_{j=1}^{p}(\nu(x), X_j(x))b_j,$$
$$(Na)(x) := 2\sum_{j=1}^{p}(\nu(x), \mathrm{curl}\ \mathrm{X_j(x)})\mathrm{a_j}, \quad \mathrm{x} \in \partial\mathrm{D}$$

and the vector valued linear operators H and J by their i-th component

$$(H\mu)_i := 2\int_{\partial D}(\nu(x), (S\mu\nu)(x), Z_i(x))ds(x),$$

$$(J\lambda)_i := 2\int_{\partial D}(\nu(x), \mathrm{curl}\int_{\partial \mathrm{D}}\lambda(\mathrm{y})\nu(\mathrm{y})\Phi(\mathrm{x},\mathrm{y})\mathrm{ds(y)}, \mathrm{Z_i(x)})\mathrm{ds(x)},$$

$i = 1, \dots, p$. Further we declare the finite dimensional operators C and D by the $p \times p$ matrices with the (i, j)-element

$$C_{ij} := 2\int_{\partial D}(\nu(x), X_{j-}, Z_i(x))ds(x),$$
$$D_{ij} := 2\int_{\partial D}(\nu(x), \mathrm{curl}\ \mathrm{X_{j-}}, \mathrm{Z_i(x)})\mathrm{ds(x)}.$$

These operators clearly have the following mapping properties: $M, N : \mathbf{C}^p \to \mathbf{C}^{0,\alpha}(\partial D)$ are bounded, $H, J : \mathbf{C}^{1,\alpha}(\partial D) \to \mathbf{C}^p$ are bounded and $\mathbf{C}, D : \mathbf{C}^p \to \mathbf{C}^p$ are bounded. We modify the operator A for multiply connected domains by

$$A := \begin{bmatrix} -T - k^2Q & P & k^2M & N \\ k^2P & -T - k^2Q & k^2N & k^2M \\ J & H & D & C \\ k^2H & J & k^2C & D \end{bmatrix} \tag{3.7}$$

Then $A : (\mathbf{C}^{1,\alpha}(\partial D))^2 \times \mathbf{C}^{2p} \to (\mathbf{C}^{0,\alpha}(\partial D))^2 \times \mathbf{C}^{2p}$ is continuous. Analogously to Theorem 2.1 we can formulate

THEOREM 3.1. *The fields E, defined by* (3.2) *and* $H := 1\backslash ik, \mathrm{curl}\ \mathrm{E}$ *solve the interior boundary-value problem, if the unknown densities*

$\lambda, \mu \in \mathbf{C}^{1,\alpha}(\partial D)$ and the coefficients $a, b \in C^p$ are solutions of the integral equations

$$A \begin{bmatrix} \lambda \\ \mu \\ a \\ b \end{bmatrix} = \begin{bmatrix} g \\ f \\ \tilde{e} \\ \tilde{h} \end{bmatrix}. \tag{3.8}$$

REMARK 3.2. If we change the interior to the exterior Neumann fields in the additional fields in (3.5), we can modify the ansatz (2.6) for the exterior problem in the same manner. This leads to the corresponding operator A', which is still the adjoint to A if we complete the bilinear form (2.13) correspondingly.

Since the additional integral equations have a finite dimensional image, we can regularize as above by

$$R := \begin{bmatrix} \tilde{S} & 0 & 0 & 0 \\ 0 & \tilde{S} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Now, by the same procedure but with more technical difficulties we can determine the nullspaces of A and A'. For details the reader is referred to Gülzow [**2**]. Again from the Riesz-Fredholm theory we get

THEOREM 3.3. *The interior boundary-value problem is solvable if and only if*

$$\int_{\partial D} (\lambda' g + \mu' f) ds + \sum_{j=1}^{p} \{e_{F_j} h_j + h_{F_j} e_j\} = 0$$

for all eigensolutions F of the homogeneous interior boundary-value problem with

$$e_{F_j} := -\int_{\partial D} (\nu, F, Z_j') ds, \quad h_{F_j} := -\int_{\partial D} (\nu, \mathrm{curl}\, \mathrm{F}, \mathrm{Z_j'}) \mathrm{ds}$$

and

$$\mathrm{Grad}\, \mu' = F_{\tan} - \sum_{j=1}^{p} e_{F_j} Z_j, \quad \mathrm{Grad}\, \lambda' = (\mathrm{curl}\, \mathrm{F})_{\tan} - \sum_{j=1}^{p} \mathrm{h_{F_j} Z_j}.$$

Similarly, for the exterior boundary-value problem we can derive

THEOREM 3.4. *The exterior boundary-value problem is solvable if and only if*

$$\int_{\partial D} g ds = 0, \int_{\partial D} f ds = 0. \tag{3.9}$$

REMARK 3.5. We have only discussed the case of one scattering object. If there are more, the condition (3.5) has to be fulfilled on each component. This is easily seen, if we remember that the solvability condition arises from the fact that in the null space of A' we have two free constants which may differ on each connectivity component.

REMARK. This paper is a short version of the author's thesis.

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