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SMOOTHNESS RESULTS OF SINGLE AND DOUBLE LAYER SOLUTIONS OF THE HELMHOLTZ EQUATIONS

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ABSTRACT. In this paper, we prove the smoothness results of single and double layer solutions for Helmholtz's equation in two and three dimensions. For the most part, results on the differentiability of single and double layer solutions of Laplace's equation extend to the corresponding results for the Helmholtz equation.

1. Introduction. It is well known that smoothness results of an integral operator are closely related to the rates of convergence of the approximate numerical solutions to the true solution of the corresponding integral equation. Atkinson [**1, 2**] applied a particular Galerkin method to the Laplace equation and gave a complete convergence and error analysis. In [**4** or **5**], the author applied the same Galerkin method to the exterior Dirichlet problem for the Helmholtz equation in three dimensions. The convergence and error analysis of this required smoothness results of single and double layer potentials. These results are well known for Laplace's equation (see [**3**]), but the analogous results for Helmholtz's equation are not available. In this paper, we prove smoothness results of single and double layer solutions of the Helmholtz equation in two and three dimensions. For the most part, results on the differentiability of single and double layer solutions of Laplace's equation extend to the corresponding results for the Helmholtz equation.

2. Definitions. We first introduce the following definitions in $\mathbf{R}^3$ (see [**3**, p. 97]).

DEFINITION 2.1. Let a function $f(x,y,z) = f(M)$, defined in a region D, be bounded and possess bounded and continuous derivatives up to order $\ell(\ell \geq 0)$, and let the derivatives of order ℓ be Hölder continuous. Thus

$$(2.1) \qquad \left|\frac{\partial^p f}{\partial x^{p_1}\partial y^{p_2}\partial z^{p_3}}\right| < A, \quad \begin{pmatrix} p_1+p_2+p_3=p \\ p=0,1,2,\dots,\ell \end{pmatrix},$$

and for any pair of points M_1 and M_2 of D a distance r_{12} apart less than a certain number $r_0 \leq 1$, the inequality

$$(2.2) \qquad \left| \left(\frac{\partial^\ell f}{\partial x^{\ell_1} \partial y^{\ell_2} \partial z^{\ell_3}} \right)_{M_1} - \left(\frac{\partial^\ell f}{\partial x^{\ell_1} \partial y^{\ell_2} \partial z^{\ell_3}} \right)_{M_2} \right| < A r_{12}^\lambda, \quad (0 < \lambda \leq 1)$$

holds, where the number A and λ are independent of the choice of the point M. We shall say that f belongs to the class $C^{\ell,\lambda}$ in D and write $f \in C^{\ell,\lambda}(D)$. If f satisfies only (2.1), we shall say that f belongs to the class C^ℓ in D and write $f \in C^\ell(D)$.

DEFINITION 2.2. Let S be a closed and bounded surface in R^3. At each $Q \in S$, assume there is a tangent plane to S. We use this plane to introduce a local rectangular coordinate system with coordinates (ξ, η, ς), choose Q as the origin, let the ς-axis be perpendicular to the plane, and let the tangent plane be the $\varepsilon\eta$-pane. Using this coordinate system, we assume that there is some small $d > 0$ and a spherical neighborhood S_d of Q of radius d such that the part of the surface S within S_d can be represented by a function

$$\varsigma = F(\xi, \eta), \quad (\xi, \eta) \in D_d,$$

where D_d is the domain of F, yielding the portion of S within S_d. We shall say that S belongs to the class $C^{\ell,\lambda}, 0 < \lambda \leq 1$, if $F(\xi, \eta) \in C^{\ell,\lambda}$. We shall say that S belongs to the class C^ℓ if $F(\varsigma, \eta) \in C^\ell$. If the surface $S \in C^{1,\lambda}$ we call it a Lyapunov surface.

We shall say that a function f defined on S belongs to the class $C^{\ell,\lambda}$ on S and write $f \in C^{\ell,\lambda}(S)$ if $f(\xi, \eta) \in C^{\ell,\lambda}$ in D_d and the constants A and λ are independent of the choice of the point Q. We shall say that a function f defined on S belongs to the class C^ℓ on S and write $f \in C^\ell(S)$ if $f(\xi, \eta) \in C^\ell$ in D_d and the constant A is independent of the choice of the point Q.

REMARK. If $f \in C^\ell$, then $f \in C^{\ell-1,1}$. Also, if $f \in C^{\ell,\lambda}$, then $f \in C^\ell, 0 < \lambda \leq 1$.

NOTATION 2.3. Let M_0 be a fixed point of the surface S, M_2 a variable point of this surface, and r_{20} the distance between M_0 and M_2. Further let ν be the outer normal to S at the point M_2 and $(r_{20}\nu)$ the angle between the direction of $r_{20} = M_2 - M_0$ and ν (Figure 2.1). Sometimes we denote $\mu(M_2)$ by $\mu(2)$. If M_1 is another point on S, we have similar notation: $r_{21}, \mu(1), (r_{21}\nu)$, etc.

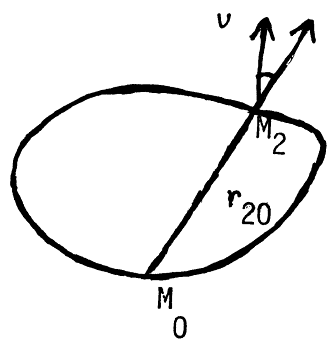

Figure 2.1

DEFINITION 2.4. (see **[8]** or **[9, 10]**). We call

$$L_k\mu(p) = \int_S \mu(q)\frac{e^{ik|p-q|}}{|p-q|}d\sigma_q, \quad p \in R^3,$$

a single layer function, and $\mu(q)$ is called the single layer density function. We call

$$M_k\mu(p) = -\int_S \mu(q)\frac{\partial}{\partial \nu_q}\frac{e^{ik}|p-q|}{|p-q|}d\sigma_q, p \in R^3,$$

a double layer function, and μ is called the double layer density function. For simplicity, sometimes we write $L\mu$ and $M\mu$ only. We note that, when $k = 0$, these are the single and double layer potentials satisfying Laplace's equation.

If we make the following changes we have the corresponding definitions in two dimensions for $f \in C^\ell(D), f \in C^\ell(S), S \in C^{\ell,\lambda}$, Lyapunov curve, and the notations ν, $(r_{20}\nu)$, etc.

Three dimensions	Two dimensions
$f(x,y,z)$	$f(x,y)$
$\frac{\partial^p f}{\partial x^{p_1} \partial y^{p_2} \partial z^{p_3}}$	$\frac{\partial^p f}{\partial x^{p_1} \partial y^{p_2}}$
$p_1 + p_2 + p_3 = p$	$p_1 + p_2 = p$
surface	curve
tangent plane	tangent line
local coordinates (ξ, η, ς)	(ξ, η)
ς-axis	η-axis
$\xi\eta$-plane	ξ line (or line)
$\varsigma = F(\xi, \eta)$	$\eta = F(\xi)$
$F(\xi, \eta)$	$F(\xi)$

DEFINITION 2.5. Let J_0 be the Bessel function of order zero,

$$J_0(z) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \left(\frac{z}{2}\right)^{2n},$$

N_0 the Neumann function of order zero,

$$N_0(z) = \frac{2}{\pi} J_0(z) (\log \frac{z}{2} + C) - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\sum_{j=1}^{n} \frac{1}{j} \right) \left(\frac{z}{2}\right)^{2n}$$

with

$$C = \lim_{m \to \infty} \left(\sum_{j=1}^{m} \frac{1}{j} - \log m \right) \approx 0.5772156649,$$

and let $H_0^{(1)}$ be the Hankel function of first kind of order zero,

$$H_0^{(1)} = J_0 + iN_0.$$

We call

$$L_k\mu(p) = \int_S \mu(q)\frac{i}{2}H_0^{(1)}(k|p-q|)d\sigma_q, \quad p \in R^2,$$

a single layer function, and $\mu(q)$ is called the single layer density function. We call

$$M_k\mu(p) = -\int_S \mu(q)\frac{\partial}{\partial\nu_q}\frac{i}{2}H_0^{(1)}(k|p-q|)d\sigma_q, \quad p \in \mathbf{R}^2,$$

a double layer function, and μ is called the double layer density function. For simplicity, we sometimes write only $L\mu$ and $M\mu$.

REMARK. We can write

$$\frac{i}{2}H_0^{(1)}(k|p-q|) = \frac{1}{\pi}\log\frac{1}{|p-q|} + E(r),$$

where $r = |p-q|$ and E is a continuously differentiable function with respect to r.

3. Smoothness results of Helmholtz single and double layer solutions in three dimensions.

We prove the following smoothness results in three dimensions.

THEOREM 3.1. *Let $S \in C^{\ell+2}$ and $\mu \in C^{\ell}(\ell \geq 0)$ on S, and assume $E(r_{20})$ is an infinitely differentiable function with respect to r_{20}. If*

$$W(\mu) = \int_S \mu(2)E(r_{20})d\sigma_2,$$

then $W(\mu) \in C^{\ell+2}$ on S.

PROOF. We partially adopt the idea and notations used by Günter [**3**, pp. 312-325] to prove this theorem. Let M_0 be some point of the surface S; let $(\xi, \eta\varsigma)$ be a local coordinate system about M_0. Let Σ be a subregion of the surface S, laying inside S_d about M_0 and having a projection Λ on the (ξ, η) plane that is a circle about M_0 of radius $\geq d/2$. Let the radius d_0 of the circle Λ_0 about M_0 in the (ξ, η) plane be so small that the circle Λ_1 of radius $2d_0$ in the (ξ, η) plane, concentric with Λ_0, is contained in the projection Λ of Σ. Let Σ_0 and Σ_1 be parts of S corresponding to Λ_0 and Λ_1 under the mapping $F(\xi, \eta)$. (For its two dimensional picture, see §4, Figure 4.1).

We shall start with the fact that, in $\Lambda_1, \mu(\xi, \eta) \in C^\ell$ and $F(\xi, \eta) \in C^{\ell+2}$, and show that $W(\mu) \in C^{\ell+2}$ in Λ_0. We have

$$W(\mu) = \int_{S-\Sigma} \mu(2)E(r_{21})d\sigma_2 + \int_\Sigma \mu(2)E(r_{21})d\sigma_2 \tag{1}$$

The integral over $S - \Sigma$ is a function of $M_1 = (\xi, \eta, \varsigma)$. In some region containing the surface Σ_0, it has bounded and continuous derivatives of arbitrary order with respect to ξ, η and ς. If we replace ς by $F(\xi, \eta)$ we obtain the value of this integral for points M_1 on Σ_0. Since $F(\xi, \eta)$ has derivatives with respect to ξ and η up to order $\ell + 2$, this is also the case for the first integral. Hence the first integral belongs to the class $C^{\ell+2}$ in Λ_0.

We denote the coordinates of the point M_1 by (ξ, η, ς) and those of the integration point M_2 by (x, y, z). Hence

$$\int_\Sigma \mu(2)E(r_{21})d\sigma_2 = \int\int_\Lambda \mu(x,y)E(r_{21})(1+F_\xi'^2(x,y)+F_\xi'^2(x,y))^{1/2}dx\,dy$$

where $r_{21} = ((x - \xi)^2 + (y - \eta)^2 + (F(x, y) - F(\xi, \eta))^2)^{1/2}$. Since $\mu(x, y)(1 + F_\xi'^2(x, y) + F_\eta'^2(x, y))^{1/2}$ is an element of the class C^ℓ if $\mu(x, y)$ is, it suffices to show that the integral

$$\int\int_\Lambda \mu(x,y)E(r_{21})dx\,dy \tag{2}$$

belongs to the class $C^{\ell+2}$ if $\mu \in C^\ell$ and $F \in C^{\ell+2}$.

Let $\omega(\gamma)$ be a function which has continuous derivatives up to order $\ell + 3$ for all $r \geq 0$ and which is equal to one for $r \leq \frac{3}{2}d_0$ and to zero for $r \geq 2d_0$. We put

$$\mu_1(x, y) = \mu(x, y)\omega(\sqrt{x^2 + y^2}),$$
$$F_1(x, y) = F(x, y)\omega(\sqrt{x^2 + y^2}),$$

where we shall assume that $\mu_1(x,y)$ and $F_1(x,y)$ are defined in the entire (x,y) plane and are equal to zero outside Λ_1.

In the circle $\sqrt{x^2+y^2} \leq \frac{3}{2}d_0$ it is clear that $\mu(x,y) = \mu_1(x,y)$ and $F(x,y) = F_1(x,y)$. Therefore, the integral (2) differs from the integral obtained upon replacing $\mu(x,y)$ by $\mu_1(x,y)$ and $F(x,y)$ by $F_1(x,y)$ by an integral which extends over the subregion outside this circle. Inside the circle Λ_0 this last integral also belongs to the class $C^{\ell+2}$.

Just as $\mu(x,y)$ and $F(x,y)$, the functions $\mu_1(x,y)$ and $F_1(x,y)$ also have continuous derivatives up to order ℓ and $\ell+2$ respectively in Λ_1. The functions $\mu_1(x,y)$ and $F_1(x,y)$, moreover, have these properties in the entire (x,y) plane, for on the boundary of the disk Λ_1 and outside of it, μ_1 and F_1 and their derivatives up to order ℓ and $\ell+2$ respectively are equal to zero. To prove our theorem it suffices to show that

$$\varphi(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu_1(x,y)E((x-\xi)^2+(y-\eta)^2 + (F_1(x,y)-F_1(\xi,\eta))^2)^{1/2})dx\,dy \tag{3}$$

belongs to the class $C^{\ell+2}$ in Λ_0. In place of $\mu_1(x,y)$ and $F_1(x,y)$ we shall henceforth write $\mu(x,y)$ and $F(x,y)$, again the region of integration will be the entire (x,y) plane. We denote

$$((x-\xi)^2+(y-\eta)^2+(F(x,y)-F(\xi,\eta))^2)^{1/2}$$

by r for the rest of the proof. Since

$$\frac{(\xi-x)-(F(x,y)-F(\xi,\eta))\cdot F'_\xi(\xi,\eta)}{r}$$

is bounded,

(4)

$$\frac{\partial\varphi(\xi,\eta)}{\partial\xi} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)E'(r)\frac{(\xi-x)-(F(x,y)-F(\xi,\eta))F'_\xi(\xi,\eta)}{r}dx\,dy.$$

We differentiate it again with respect to ξ,

(5)

$$\begin{aligned}
&\frac{\partial^2\varphi(\xi,\eta)}{\partial\xi^2}\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(x,y)E''(r)\Big(\frac{(\xi-x)-(F(x,y)-F(\xi,\eta))F'_\xi(\xi,\eta)}{r}\Big)^2\\
&\quad+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(x,y)E'(\gamma)\\
&\quad\cdot\Big(\frac{1-(F(x,y)-F(\xi,\eta))F''_{\xi\xi}(\xi,\eta)+F'^2_\xi(\xi,\eta)}{r}\\
&\quad-\Big(\frac{(\xi-x)-(F(x,y)-F(\xi,\eta))F'_\xi(\xi,\eta)}{r}\Big)^2\cdot\frac{1}{r}\Big)dx\,dy.
\end{aligned}$$

We note that all of the integrands in the above integrals have only weak singularities.

We go over to polar coordinates, with origin the point $M_1(\xi,\eta)$, and put $x=\xi+\rho\cos\theta, y=\eta+\rho\sin\theta$. Then

$$\begin{aligned}
F(M_2)-F(M_1)&=\int_0^1\frac{d}{dt}F(\xi+t\rho\cos\theta,\eta+t\rho\sin\theta)dt\\
&=\int_0^1\big(F'_\xi(M)\rho\cos\theta+F'_\eta(M)\rho\sin\theta\big)dt\\
&=\rho\int_0^1\big(F'_\xi(M)\cos\theta+F'_\eta(M)\sin\theta\big)dt,
\end{aligned}$$

where M denotes the point with coordinates $(\xi+t\rho\cos\theta,\eta+t\rho\sin\theta)$. The function

$$\begin{aligned}
\psi_1(\xi,\eta;\rho,\theta)&=\frac{F(M_2)-F(M_1)}{\rho}\\
&=\int_0^1(F'_\xi(M)\cos\theta+F'_\eta(M)\sin\theta)dt
\end{aligned}\tag{6}$$

has continuous derivatives with respect to all its arguments ξ,η,ρ,θ up to order $\ell+1$. We denote $F(\varepsilon+\rho\cos\theta,\eta+\rho\sin\theta)-F(\xi,\eta)$ by $[\tilde{F}]$ and

$F(x,y) - F(\xi,\eta)$ by $[F]$. Then

(7)

$$\frac{\partial^2\varphi(\xi,\eta)}{\partial\xi^2}$$
$$= \int_0^{2\pi}\int_0^\infty \mu(\xi+\rho\cos\theta,\eta+\rho\sin\theta)E''((\rho^2+[\tilde{F}]^2)^{1/2})$$
$$\cdot\Big(\frac{-\cos\theta-\psi_1(\xi,\eta;\rho,\theta)F'_\xi(\xi,\eta)}{(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}}\Big)^2\rho d\rho d\theta$$
$$+\int_0^{2\pi}\int_0^\infty \mu(\xi+\rho\cos\theta,\eta+\rho\sin\theta)E'((\rho^2+[\tilde{F}]^2)^{1/2})$$
$$\cdot\Big(\frac{1-\rho\psi_1(\xi,\eta;\rho,\theta)F''_{\xi\xi}(\xi,\eta)+F'^2_\xi(\xi,\eta)}{(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}}$$
$$-\Big(\frac{-\cos\theta-\psi_1(\xi,\eta;\rho,\theta)F'_\xi(\xi,\eta)}{(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}}\Big)^2\cdot\frac{1}{(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}}\Big)d\rho d\theta.$$

By induction, we see that the derivative of order k of the function $E(\sqrt{\rho^2+[\tilde{F}]^2})$ with respect to ξ,η is a finite linear combination of functions of the form

$$E^{(s)}(\sqrt{\rho^2+[\tilde{F}]^2}).\Big(\frac{[\tilde{F}][\tilde{F}^{(1)}]}{\sqrt{\rho^2+[\tilde{F}]^2}}\Big)^b\cdot\prod_{j=1}^n\frac{\prod_{i=1}^{2p_j}[\tilde{F}^{(\nu_i)}]}{(\rho^2+[\tilde{F}]^2)^{p_j-\frac{1}{2}}},\tag{8}$$

with n,s,b,p_j,ν_i integers and $0\le n,s,b,p_j,\nu_i\le k$, ν_i depends on p_j, $b+n=s$. If $p_j=0$, we let $\prod_{i=1}^{2p_j}[\tilde{F}^{(\nu_i)}]=1$ and $(\rho^2+[\tilde{F}])^{p_j-\frac{1}{2}}=1$. If $n=0$, we let

$$\prod_{j=1}^n\frac{\prod_{i=1}^{2p_j}[\tilde{F}^{(\nu_i)}]}{(\rho^2+[\tilde{F}]^2)^{p_j-\frac{1}{2}}}=1.$$

We note that the above expression is also true for $E'((\rho^2+[\tilde{F}]^2)^{1/2})$ and $E''(\rho^2+[\tilde{F}]^2)^{1/2}$, if we replace $b+n=s$ by $b+n=s-1, b+n=s-2$ respectively. If $0\le\nu_i\le\ell+1$, we have $F^{(\nu_i)}\in C^1$ and hence

$$|[F^{(\nu_i)}]|=|F^{(\nu_i)}(x,y)-F^{(\nu_i)}(\xi,\eta)|\le c\rho$$

for some constant c. Since $\rho^2+[F]^2\ge\rho^2$ we also have

$$\left|\frac{\prod_{i=1}^{2p}[F^{(\nu_i)}]}{(\rho^2+[F]^2)^{p-\frac{1}{2}}}\right|\le\frac{(c\rho)^{2p}}{\rho^{2p-1}}=c^{2p}\rho.$$

Thus the expression (8) is bounded if $0 \leq \nu_i \leq \ell + 1$, each integrand of (7) has continuous derivatives up to order ℓ with respect to ξ, η. So $\partial^2\varphi(\xi,\eta)/\partial\xi^2$ has continuous derivatives up to order ℓ with respect to ξ, η. Similarly, we can show that $\frac{\partial^2\varphi}{\partial\eta^2}$, $\frac{\partial^2\varphi}{\partial\eta\partial\xi}$ have continuous derivatives up to order ℓ with respect to ξ, η. Thus, $\varphi \in C^{\ell+2}$. □

THEOREM 3.2. *Let $S \in C^{\ell+2}$ and $\mu \in C^{\ell}(\ell \geq 0)$ on S, and let $E(r_{20})$ be an infinitely differentiable function with respect to r_{20}. If $W(\mu) = \int_S (2)E(r_{20})\cos(r_{20}\nu_2)d\sigma_2$, then $W(\mu) \in C^{\ell+2}$ on S.*

PROOF. Let $M_0, (\xi,\eta,\varsigma), \Lambda_0, \Lambda_1, d_0, \Sigma$ be defined as in the proof of Theorem 3.1. We have

$$\begin{aligned} W(\mu) = &\int_{S-\Sigma} \mu(2)E(r_{21})\cos(r_{21}\nu_2)d\sigma_2 \\ &+ \int_\Sigma \mu(2)E(r_{21})\cos(r_{21}\nu_2)d\sigma_2 \end{aligned} \tag{1}$$

For the same reason as before, we only need consider the second integral on the right-hand side of (1).

We denote the coordinates of the point M_1 by (ξ,η,ς) and those of the integration point M_2 by (x,y,z). From the relations

$$r_{12} = ((x-\xi)^2 + (y-\eta)^2 + (F(x,y) - F(\xi,\eta))^2)^{1/2}$$

and

$$\cos(r_{12}\nu_2) = \frac{F(\xi,\eta) - F(x,y) + (x-\xi)F'_\xi(x,y) + (y-\eta)F'_\eta(x,y)}{r_{12}(1 + F'^2_\xi(x,y) + F'^2_\eta(x,y))^{1/2}},$$

$$\begin{aligned} &\int_\Sigma \mu(2)E(r_{12})\cos(r_{12}\nu_2)d\sigma_2 \\ &= \int_\Lambda\int \mu(x,y)E(r_{12}) \\ &\quad\cdot \frac{F(\xi,\eta) - F(x,y) + (x-\xi)F'_\xi(x,y) + (y-\eta)F'_\eta(x,y)}{((x-\xi)^2 + (y-\eta)^2 + (F(x,y) - F(\xi,\eta))^2)^{1/2}} dx\,dy. \end{aligned} \tag{2}$$

We define $\omega(r), \mu_1(x,y), F_1(x,y)$ the same as in the proof of Theorem 3.1. To prove our theorem it suffices to show that $\varphi \in C^{\ell+2}$ on Λ_0, where

(3)
$$\varphi(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu_1(x,y)E(((x-\xi)^2+(y-\eta)^2+(F_1(x,y)-F_1(\xi,\eta))^2)^{1/2}) \cdot \frac{F_1(\xi,\eta)-F_1(x,y)+(x-\xi)F'_{1\xi}(x,y)+(y-\eta)F'_{1\eta}(x,y)}{((x-\xi)^2+(y-\eta)^2+(F_1(x,y)-F_1(\xi,\eta))^2)^{1/2}}\,dx\,dy.$$

In place of $\mu_1(x,y)$ and $F_1(x,y)$ we shall henceforth write $\mu(x,y)$ and $F(x,y)$, and again we take the region of integration to be the entire (x,y) plane.

Our approach to this theorem is the same as in Theorem 3.1. We first differentiate twice, and then change to polar coordinates to show that the second derivative is in C^ℓ.

For simplicity, we denote $F(\xi,\eta) - F(x,y) + (x-\xi)F'_\xi(x,y) + (y-\eta)F'_n(x,y)$ by $\{F\}$ and $F(x,y)-F(\xi,\eta)$ by $[F]$.

We denote r_{12} by r for the rest of the proof. Applying the mean value theorem, we see that

$$\frac{F'_\xi(\xi,\eta)-F'_\xi(x,y)}{r} \text{ and } \frac{\{F\}}{r^2}$$

are bounded. So

$$\begin{aligned}\frac{\partial\varphi(\xi,\eta)}{\partial\xi} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)E'(r)\frac{\xi-x-[F]F'_\xi(\xi,\eta)}{r}\frac{\{F\}}{r}\,dx\,dy \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)E(r)\frac{F'_\xi(\xi,\eta)-F'_\xi(x,y)}{r}\,dx\,dy \\ &- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)E(r)\frac{\{F\}}{r^2}\frac{\xi-x-[F]F'_\xi(\xi,\eta)}{r}\,dx\,dy \\ &= I_1+I_2-I_3.\end{aligned} \tag{4}$$

We differentiate it again,

$$
\begin{aligned}
&\frac{\partial I_1}{\partial \xi} = \\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)E''(r)\Big(\frac{\xi - x - [F]F'_\xi(\xi,\eta)}{r}\Big)^2 \frac{\{F\}}{r} dx\,dy \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)E'(r)\frac{\xi - x - [F]F'_\xi(\xi,\eta)}{r} \\
&\cdot\Big(\frac{F'_\xi(\xi,\eta) - F'_\xi(x,y)}{r} - \frac{\{F\}}{r^2}\cdot\frac{\xi - x - [F]F'_\xi(\xi,\eta)}{r}\Big) dx\,dy \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)E'(r)\frac{\{F\}}{r}\Big(\frac{1 - [F]F''_{\xi\xi}(\xi,\eta) + F'^2_\xi(\xi,\eta)}{r} \\
&\qquad - \frac{(\xi - x - [F]F'_\xi(\xi,\eta))^2}{r^3}\Big) dx\,dy,
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
&\frac{\partial I_2}{\partial \xi} = \\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)E'(r)\cdot\frac{\xi - x - [F]F'_\xi(\xi,\eta)}{r}\cdot\frac{F'_\xi(\xi,\eta) - F'_\xi(x,y)}{r} dx\,dy \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)E(r)\Big(\frac{F''_{\xi\xi}(\xi,\eta)}{r} - \frac{(F'_\xi(\xi,\eta) - F'_\xi(x,y))}{r^2} \\
&\qquad \cdot\frac{\xi - x - [F]F'_\xi(\xi,\eta)}{r}\Big) dx\,dy,
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&\frac{\partial I_3}{\partial \xi} = \\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)E'(r)\frac{\{F\}}{r^2}\Big(\frac{\xi - x - [F]F'_\xi(\xi,\eta)}{r}\Big)^2 dx\,dy
\end{aligned} \tag{7}
$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(x,y)E(r)\frac{\{F\}}{r^2}\Big(\frac{1-[F]F''_{\xi\xi}(\xi,\eta)+F'^2_{\xi}(\xi,\eta)}{r}$$
$$-\frac{((\xi-x)-[F]F'_{\xi}(\xi,\eta))^2}{r^3}\Big)dx\,dy$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(x,y)E(r)\frac{\xi-x-[F]F'_{\xi}(\xi,\eta)}{r}$$
$$\cdot\Big(\frac{F'_{\xi}(\xi,\eta)-F'_{\xi}(x,y)}{r^2}-\frac{2\{F\}}{r^3}\cdot\frac{\xi-x-[F]F'_{\xi}(\xi,\eta)}{r}\Big)dx\,dy.$$

We go over to polar coordinates with origin the point $M_1(\xi,\eta)$ and put

$$x=\xi+\rho\cos\theta,\quad y=\eta+\rho\sin\theta.$$

Then

$$F(M_2)-F(M_1)=\int_0^1\frac{d}{dt}F(\xi+t\rho\cos\theta,\eta+t\rho\sin\theta)dt$$
$$=\rho\int_0^1(F'_{\xi}(M)\cos\theta+F'_{\eta}(M)\sin\theta)dt,$$

where M denotes the point with coordinates $(\xi+t\rho\cos\theta,\eta+t\rho\sin\theta)$.

The function

$$\psi_1(\xi,\eta;\rho,\theta)=\frac{F(M_2)-F(M_1)}{\rho}$$
$$(8)\qquad =\int_0^1(F'_{\xi}(M)\cos\theta+F'_{\eta}(M)\sin\theta)dt$$

therefore has continuous derivatives up to order $\ell+1$. Integrating by parts, we further obtain

$$F(M_2)-F(M_1)=\int_0^1\frac{d}{dt}F(M)dt$$
$$=t\frac{d}{dt}F(M)\Big|_0^1-\int_0^1 t\frac{d^2}{dt^2}F(M)dt$$
$$=F'_{\xi}(M_2)\rho\cos\theta+F'_{\eta}(M_2)\rho\sin\theta$$
$$-\rho^2\int_0^1 t(F''_{\xi\xi}(M)\cos^2\theta+2F''_{\xi\eta}(M)\cos\theta\sin\theta$$
$$+F''_{\eta\eta}(M)\sin^2\theta)dt.$$

From this it follows that the function

$$
\begin{aligned}
&\psi_2(\xi,\eta;\rho,\theta)\\
(9)\qquad &= \frac{F(M_1)-F(M_2)+\rho\cos\theta F'_\xi(M_2)+\rho\sin\theta F'_\eta(M_2)}{\rho^2}\\
&= \int_0^1 t(F''_{\xi\xi}(M)\cos^2\theta + 2F''_{\xi\eta}(M)\cos\theta\sin\theta + F''_{\eta\eta}(M)\sin^2\theta)dt
\end{aligned}
$$

has continuous derivatives with respect to all the arguments ξ,η,ρ,θ up to order ℓ.

$$
\begin{aligned}
F'_\xi(M_2)-F'_\xi(M_1) &= \int_0^1 \frac{d}{dt}F'_\xi(\xi+t\rho\cos,\eta+t\rho\sin\theta)dt\\
&= \rho\int_0^1 (F''_{\xi\xi}(M)\cos\theta + F''_{\xi\eta}(M)\sin\theta)dt,
\end{aligned}
$$

where M denote the points with coordinates $(\xi+t\rho\cos\theta,\eta+t\rho\sin\theta)$. Let

$$
\begin{aligned}
(10)\qquad \psi_3(\xi,\eta;\rho,\theta) &= \frac{F'_\xi(M_2)-F'_\xi(M_1)}{\rho}\\
&= \int_0^1 (F''_{\xi\xi}(M)\cos\theta + F''_{\xi\eta}(M)\sin\theta)dt
\end{aligned}
$$

have continuous derivatives with respect to ξ,η,ρ,θ up to order ℓ. In the expression of $\{F\}$ and $[F]$, if we replace x and y by $\xi+\rho\cos\theta$ and $\eta+\rho\sin\theta$ respectively, we obtain a function of ξ,η,ρ,θ which we denote by $\{\tilde{F}\}$ and $[\tilde{F}]$ respectively. So

(11)

$$\begin{aligned}
\frac{\partial I_1}{\partial \xi} &= \int_0^{2\pi}\int_0^{\infty} \mu(\xi+\rho\cos\theta,\eta+\rho\sin\theta)E''((\rho^2+[\tilde{F}]^2)^{1/2}) \\
&\quad \cdot \Big(\frac{-\cos\theta-\psi_1(\xi,\eta;\rho,\theta)F'_\xi(\xi,\eta)}{(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}}\Big)^2 \frac{\rho\psi_2(\xi,\eta;\rho,\theta)}{(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}}\rho d\rho d\theta \\
&\quad + \int_0^{2\pi}\int_0^{\infty} \mu(\xi+\rho\cos\theta,\eta+\rho\sin\theta)E'((\rho^2+[\tilde{F}]^2)^{1/2}) \\
&\quad \cdot \frac{-\cos\theta-\psi_1(\xi,\eta;\rho,\theta)F'_\xi(\xi,\eta)}{(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}} \\
&\quad \cdot \Big(\frac{-\psi_3(\xi,\eta,\rho,\theta)}{(1+\psi_1^2(\xi,\eta;\rho,\theta)^{1/2}} \\
&\qquad - \frac{\psi_2(\xi,\eta;\rho,\theta)(-\cos\theta-\psi_1(\xi,\eta;\rho,\theta)F'_\xi(\xi,\eta))}{((1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2})^3}\rho d\rho d\theta \\
&\quad + \int_0^{2\pi}\int_0^{\infty} \mu(\xi+\rho\cos\theta,\eta+\rho\sin\theta)E'((\rho^2+[\tilde{F}]^2)^{1/2}) \\
&\quad \cdot \frac{\rho\psi_2(\xi,\eta;\rho,\theta)}{(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}}\Big(\frac{1-\rho\psi_1(\xi,\eta;\rho,\theta)F''_{\xi\xi}(\xi,\eta)+F'_\xi(\xi,\eta)}{\rho(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}} \\
&\qquad - \frac{(-\rho\cos\theta-\rho\psi_1(\xi,\eta;\rho,\theta)F'_\xi(\xi,\eta))^2}{(\rho(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2})^3}\Big)\rho d\rho d\theta,
\end{aligned}$$

(12)

$$\begin{aligned}
\frac{\partial I_2}{\partial \xi} &= \int_0^{2\pi}\int_0^{\infty} \mu(\xi+\rho\cos\theta,\eta+\rho\sin\theta)E'((\rho^2+[\tilde{F}]^2)^{1/2}) \\
&\quad \cdot \frac{-\cos\theta-\psi_1(\xi,\eta,\rho,\theta)F'_\xi(\xi,\eta)}{(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}} \cdot \frac{-\psi_3(\xi,\eta;\rho,\theta)}{(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}}\rho d\rho d\theta \\
&\quad + \int_0^{2\pi}\int_0^{\infty} \mu(\xi+\rho\cos\theta,\eta+\rho\sin\theta)E((\rho^2+[\tilde{F}]^2)^{1/2}) \\
&\quad \cdot \Big(\frac{F''_{\xi\xi}(\xi,\eta)}{\rho(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}} - \frac{-\psi_3(\xi,\eta;\rho,\theta)}{\rho(1+\psi_1^2(\xi,\eta;\rho,\theta))} \\
&\qquad \frac{-\cos\theta-\psi_1(\xi,\eta;\rho,\theta)F'_\xi(\xi,\eta)}{(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}}\Big)\rho d\rho d\theta,
\end{aligned}$$

(13)

$$\frac{\partial I_3}{\partial \xi} =$$

$$\int_0^{2\pi}\int_0^{\infty} \mu(\xi+\rho\cos\theta,\eta+\rho\sin\theta)E'((\rho^2+[\tilde{F}]^2)^{1/2})$$

$$\cdot\frac{\psi_2(\xi,\eta;\rho,\theta)}{1+\psi_1^2(\xi,\eta;\rho,\theta)}\Big(\frac{-\cos\theta-\psi_1(\xi,\eta;\rho,\theta)F'_\xi(\xi,\eta)}{(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}}\Big)^2\rho d\rho d\theta$$

$$+\int_0^{2\pi}\int_0^{\infty} \mu(\xi+\rho\cos\theta,\eta+\rho\sin\theta)E((\rho^2+[\tilde{F}]^2)^{1/2}).\frac{\psi_2(\xi,\eta;\rho,\theta)}{1+\psi_1^2(\xi,\eta;\rho,\theta)}$$

$$\cdot\Big(\frac{1-\rho\psi_1(\xi,\eta;\rho,\theta)F''_{\xi\xi}(\xi,\eta)+F'^2_\xi(\xi,\eta)}{\rho(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}}$$

$$-\frac{(-\rho\cos\theta-\rho\psi_1(\xi,\eta;\rho,\theta)F'_\xi(\xi,\eta))^2}{(\rho(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2})^3}\Big)\rho d\rho d\theta$$

$$+\int_0^{2\pi}\int_0^{\infty} \mu(\xi+\rho\cos\theta,\eta+\rho\sin\theta)E((\rho^2+[\tilde{F}]2)^{1/2})$$

$$\cdot\frac{-\cos\theta-\psi_1(\xi,\eta,\rho,\theta)F'_\xi(\xi,\eta)}{(1+\psi_1^2(\xi,\eta;\rho,\theta))^{1/2}}$$

$$\cdot\Big(\frac{-\psi_3(\xi,\eta;\rho,\theta)}{\rho(1+\psi_1^2(\xi,\eta;\rho,\theta))}$$

$$-\frac{2\psi_2(\xi,\eta;\rho,\theta)(-\cos\theta-\psi_1(\xi,\eta;\rho,\theta)F'_\xi(\xi,\eta))}{\rho(1+\psi_1^2(\xi,\eta;\rho,\theta))^2}\Big)\rho d\rho d\theta.$$

It is easy to see that $\partial I_1/\partial\xi$, $\partial I_2/\partial\xi$, and $\partial I_3/\partial\xi$ have continuous derivatives up to order ℓ with respect to ξ,η, so does $\partial^2\varphi/\partial\xi^2$. Similarly, we can prove that $\partial^2\varphi/\partial\eta^2$ and $\partial^2\varphi/\partial\eta\partial\xi$ belong to the class C^ℓ with respect to ξ,η. Thus $\varphi(\xi,\eta)$ belongs to the class $C^{\ell+2}$. □

Since smoothness results for the Laplacian single layer potential are not known, we prove the following theorem, following the same arguments as Günter [**3**, pp. 312-325] for the Laplacian double layer potential. For details of the proof of Theorem 3.3, see [**4**].

THEOREM 3.3. *If $S \in C^{\ell+2,\lambda}$, $\mu \in C^{\ell,\lambda}(\ell \geq 0)$ on S, and*

$$W(\mu) = \int_S \mu(2)\frac{1}{r_{20}}d\sigma_2,$$

then $W(\mu) \in C^{\ell+1,\lambda'}$, *where* λ' *is arbitrary in* $0 < \lambda' < \lambda$.

Now we prove our major result, Theorem 3.4, in this section.

THEOREM 3.4. *If* $S \in C^{\ell+2,\lambda}$ *and* $\mu \in C^{\ell,\lambda}(\ell \geq 0)$ *on* S, *then the Helmholtz single layer* $L_k(\mu)$ *and double layer* $M_k(\mu)$ *belong to the class* $C^{\ell+1,\lambda'}$, *with* λ' *arbitrary,* $0 < \lambda' < \lambda$.

PROOF. We split

$$\begin{aligned} L_k(\mu) &= \int_S \mu(2)\frac{e^{ikr_{20}}}{r_{20}}d\sigma_2 \\ &= \int_S \mu(2)\frac{1}{r_{20}}d\sigma_2 + \int_S \mu(2)\frac{e^{ikr_{20}}-1}{r_{20}}d\sigma_2, \\ M_k(\mu) &= -\int_S \mu(2)\frac{\partial}{\partial\nu_2}\frac{e^{ikr_{20}}}{r_{20}}d\sigma_2 \\ &= -\int_S \mu(2)\frac{\partial}{\partial\nu_2}\frac{1}{r_{20}}d\sigma_2 - \int_S \mu(2)\frac{\partial}{\partial\nu_2}\frac{e^{ikr_{20}}-1}{r_{20}}d\sigma_2. \end{aligned}$$

The functions $(e^{ikr_{20}}-1)/r_{20}$ and $\partial/\partial\nu_2\ ((e^{ikr_{20}}-1)/r_{20})$ can be written in the form $E(r_{20})$ and $E(r_{20})\cos(r_{20}\nu_2)$. Thus, by Theorems 3.1, 3.2, 3.3, and [**3**; Theorem 3., p. 106], we obtain the theorem. □

REMARK. For $\ell = 0$. Werner [**10**] proved the following stronger result under a weaker hypothesis on S:

> If $S \in C^2$ and $\mu \in C^{0,\lambda}$, with $0 < \lambda < 1$, then the Helmholtz single and double layer belong to the class $C^{1,\lambda}$.

We also prove the following theorems, if μ is bounded and integrable on S.

THEOREM 3.5. *Let* $S \in C^1$, μ *be bounded by a constant* A *and integrable on* S, *and* $E(r_{20})$ *be an infinitely differentiable function with respect to* r_{20}. *If* $W(\mu) = \int_S \mu(2)E(r_{20})\cos(r_{20}\nu_2)d\sigma_2$, *then* $W(\mu) \in C^{0,1}$ *on* S.

PROOF. With the same notation as Theorem 3.1, we know that it suffices to consider only

$$\varphi(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu_1(x,y)E(((x-\xi)^2+(y-\eta)^2+(F_1(x,y)-F_1(\xi,\eta))^2)^{1/2}) \cdot \frac{\{F_1\}}{((x-\xi)^2+(y-\eta)^2+(F_1(x,y)-F(\xi,\eta))^2)^{1/2}}dx\,dy$$

We replace μ_1 and F_1 by μ and F, respectively, for simplicity. Let

$$\begin{aligned} R(\xi,\eta;x,y) =& E(((x-\xi)^2+(y-\eta)^2+(F(x,y)-F(\xi,\eta))^2)^{1/2}) \\ &\cdot \frac{\{F\}}{((x-\xi)^2+(y-\eta)^2+(F(x,y)-F(\xi,\eta))^2)^{1/2}} \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial R}{\partial \xi} =& E'(r)\frac{(\xi-x)-[F]F'_\xi(\xi,\eta)}{r}\cdot\frac{\{F\}}{r} \\ &+E(r)\Big(\frac{F'_\xi(\xi,\eta)-F'_\xi(x,y)}{r}-\frac{\{F\}}{r^2}\cdot\frac{\xi-x-[F]F'_\xi(\xi,\eta)}{r}\Big) \end{aligned}$$

We have a similar equality for $\partial R/\partial\eta$. Let $M_0(\xi,\eta)$ and $M_1(\xi,\eta_1)$ be two points a distance δ apart. We denote the distance of the points M_0, M_1 from the integration point $M_2(x,y)$ by ρ,ρ_1 respectively. And let $R(M_0,M_2)=R(0,2), R(M_1,M_2)=R(1,2)$. Since

$$\begin{aligned} &\mid \{F\} \mid \\ &=\mid F(\xi,\eta)-F(x,y)+(x-\xi)F'(x,y)+(y-\eta)F'_\eta(x,y) \mid \\ &=\mid (x-\xi)(F'_\xi(x,y)-F'_\xi(\xi',\eta'))+(y-\eta)\cdot(F'_\eta(x,y)-F_\eta(\xi',\eta')) \mid< c\rho, \end{aligned}$$

where (ξ',η') is some point in the interior of the segment joining the points (x,y) and (ξ,η), we have

$$\mid R \mid< c_1, \tag{1}$$

$$\Big|\frac{\partial R}{\partial \xi}\Big| < \frac{c_2}{\rho}, \quad \Big|\frac{\partial_R}{\partial \eta}\Big| < \frac{c_2}{\rho}. \tag{2}$$

Then

$$
\begin{aligned}
&\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(x,y)R(1,2)dx\,dy-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(x,y)R(0,2)dx\,dy\right|\\
&\le\iint_{\rho>2\delta}\mid\mu(x,y)(R(1,2)-R(0,2))\mid dx\,dy\\
(3)\qquad&+\iint_{\rho\le 2\delta}\mid\mu(x,y)\mid\mid R(1,2)\mid dx\,dy\\
&+\iint_{\rho\le 2_\delta}\mid\mu(x,y)\mid\mid R(0,2)\mid dx\,dy.
\end{aligned}
$$

From (1),

$$
\begin{aligned}
\iint_{\rho\le 2_\delta}\mid\mu(x,y)\mid\mid R(0,2)\mid dx\,dy &\le 2\pi Ac_1\int_0^{2\delta}\rho d\rho\\
&=2\pi Ac_1\frac{(2\delta)^2}{2}<c_3\delta.
\end{aligned}
$$

Since the circle $\rho \le 2\delta$ is contained in the circle $\rho_1 \le 3\delta$ we obtain for the second integral on the right-hand side of (3)similarly the estimate $c_4\delta$. From the triangle inequality, we see that $\frac{1}{2}\rho < \rho' < \frac{3}{2}\rho$ is valid for the region $\rho > 2\delta$, where ρ' is the distance of the point M_2 from an arbitrary point of the segment M_0M_1. From the inequality (2),

$$
\begin{aligned}
\mid R(1,2)-R(0,2)\mid = &\left|\left(\frac{\partial}{\partial\xi}R\right)_{M'}(\xi_1-\xi)+\left(\frac{\partial}{\partial\eta}R\right)_{M'}(\eta_1-\eta)\right|\\
&\le\delta\frac{2c_2}{\rho'}<\frac{c\delta}{\rho},
\end{aligned}
$$

M' here denotes some point of the segment M_0M_1. Recalling that R vanishes for all sufficiently large ρ, e.g., for $\rho \ge a$, we find

$$
\begin{aligned}
&\iint_{\rho>2\delta}|\mu(x,y)(R(1,2)-R(0,2))\mid dxdy\\
&\qquad\le 2\pi c\delta\int_{2\delta}^{a}d\rho\le c'\delta.
\end{aligned}
$$

This completes the proof.□

THEOREM 3.6. *Let $S \in C^1$, μ be bounded by a constant A and integrable on S, and $E(r_{20})$ be an infinitely differentiable function with respect to r_{20}. If $W(\mu) = \int_S \mu(2)E(r_{20})d\sigma_2$, then $W(\mu) \in C^{0,1}$ on S.*

PROOF. With the same notation as Theorem 3.1, we know that it suffices to show that

$$\varphi(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu_1(x,y)E(((x-\xi)^2+(y-\eta)^2+(F_1(x,y)-F_1(\xi,\eta))^2)^{1/2})dx\,dy$$

belongs to the class $C^{0,1}$. We replace μ_1 and F_1 by μ and F for simplicity. Let

$$R(\xi,\eta;x,y) = E(((x-\xi)^2 + (y-\eta)^2 + (F(x,y) - F(\xi,\eta))^2)^{1/2});$$

then

$$\begin{aligned}\frac{\partial R}{\partial \xi} =& E'(((x-\xi)^2 + (y-\eta)^2 + (F(x,y) - F(\xi,\eta))^2)^{1/2}) \\ &\cdot \frac{\xi - x - [F]F'_\xi(\xi,\eta)}{((x-\xi)^2 + (y-\eta)^2 + [F]^2)^{1/2}}.\end{aligned}$$

As before, it is clear that

$$| R | < c_1, \quad \left|\frac{\partial R}{\partial \xi}\right| < c_2, \quad \left|\frac{\partial R}{\partial \eta}\right| < c_2.$$

We use the same argument as in Theorem 3.5 to prove the theorem. □

With the same argument as Theorem 3.4 we can prove the following theorem by using Theorem 3.5, 3.6 and [**3**, p. 44, p.49].

THEOREM 3.7. *If $S \in C^{1,\lambda}$ and μ is bounded and integrable on S, then the Helmholz single layer $L_k(\mu)$ and double layer $M_k(\mu)$ belongs to the class $C^{0,\lambda'}$, with $\lambda' = \lambda$ if $0 \le \lambda < 1$; λ' arbitrary, $0 < \lambda' < \lambda$ if*

$\lambda = 1$.

REMARK. Werner [**9**] proved the same result under stronger assumptions that μ is continuous on S and $S \in C^2$.

4. Smoothness results of Helmholtz single and double layer solutions in two dimensions. In this section, we prove similar smoothness results of Helmholtz single and double layer solutions in two dimensions.

THEOREM 4.1. *Let $S \in C^{\ell+2}$ and $\mu \in C^\ell (\ell \geq 0)$ on S, and assume $E(r_{20}) = \frac{i}{2}H_0^{(1)}(kr_{20}) - \frac{1}{\pi}\log(1/r_{20})$. If $W(\mu) = \int_S \mu(2)E(r_{20})d\sigma_2$, then*

$$W(\mu) \in C^{\ell+1,\lambda'} \text{ on } S,$$

where λ' arbitrary in $0 < \lambda' < 1$.

PROOF. We adopt the same notations as §3. Let M_0 be some point of the curve S; let (ξ, η) be a local coordinate system about M_0. Let Σ be an arc of the curve S, laying inside S_d about M_0 and having a projection Λ on the ξ-axis that is an interval about M_0 of radius $\geq d/2$ (here radius means half length of the interval). Let the radius d_0 of the interval Λ_0 about M_0 in the ξ-axis be so small that the interval Λ_1 of radius $2d_0$ in the ξ-axis, concentric with Λ_0, is contained in the projection Λ of Σ (Figure 4.1).

We shall start with the facts that in $\Lambda_1, \mu(\xi) \in C^\ell$ and $F(\xi) \in C^{\ell+2}$, and show that $W(\mu) \in C^{\ell+1,\lambda'}$ in Λ_0. We have, for the value of $W(\mu)$ at M_1,

$$W(\mu) = \int_{S-\Sigma} \mu(2)E(r_{21})d\sigma_2 + \int_\Sigma \mu(2)E(r_{21})d\sigma_2. \tag{1}$$

The integral over $S - \Sigma$ is a function of $M_1 = (\xi, \eta)$ in some region containing the curve Σ_0 and away from the curve $S - \Sigma$, it has bounded and continuous derivatives of arbitrary order with respect to (ξ, η). If

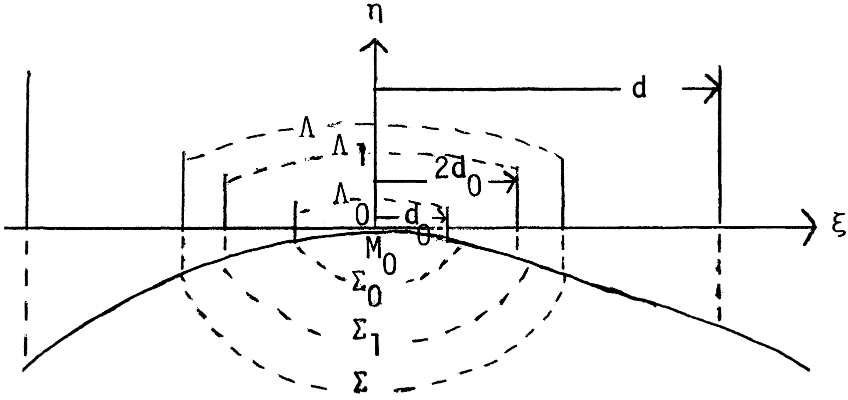

Figure 4.1

we replace η by $F(\xi)$ we obtain the value of this integral for points M_1 on Σ_0.

Since $F(\xi) \in C^{\ell+2}$, this is also the case for the first integral. Hence the first integral belongs to the class $C^{\ell+2}$ in Λ_0.

We denote the coordinates of the point M_1 by (ξ, η) and those of the integration point M_2 by (x, y). Hence

$$\int_\Sigma \mu(z)E(r_{21})d\sigma_2 = \int_\Lambda \mu(x)E(r_{21})(1 + F'^2(x))^{1/2}dx,$$

where $r_{21} = ((x-\xi)^2 + (F(x) - F(\xi))^2)^{1/2}$.

Since $\mu(x,y)(1+F'^2(x))^{1/2} \in C^\ell$ if $\mu \in C^\ell$, it suffices to show that the integral

$$\int_\Lambda \mu(x)E(r_{21})dx \tag{2}$$

belongs to the class $C^{\ell+1,\lambda'}$ if $\mu \in C^\ell$ and $F \in C^{\ell+2}$.

Let $\omega(r)$ be a function which has continuous derivatives up to order $\ell + 3$ for all $r \geq 0$, which equals one for $r \leq \frac{3}{2}d_0$ and equals zero for $r \geq 2d_0$. We put

$$\mu_1(x) = \mu(x)\omega(|\ x\ |),$$
$$F_1(x) = F(x)\omega(|\ x\ |),$$

where we shall assume that $\mu_1(x)$ and $F_1(x)$ are defined in the entire real line and are equal to zero outside Λ_1. In the interval $| x |\leq 3/2d_0$ it is clear that $\mu(x) = \mu_1(x)$ and $F(x) = F_1(x)$. Therefore, integral (2) differs from the integral obtained upon replacing $\mu(x)$ by $\mu_1(x)$ and $F(x)$ by $F_1(x)$ by an integral which extends over the subinterval outside the interval $| x |\leq 3.2d_0$. Inside the interval Λ_0, this last integral also belongs to the class $C^{\ell+1,\lambda}$.

Just as $\mu(x)$ and $F(x)$, the functions $\mu_1(x)$ and $F_1(x)$ also belong to the classes C^ℓ and $C^{\ell+2}$ in Λ_1, respectively; moreover, the functions $\mu_1(x)$ and $F_1(x)$ have these properties i the entire real line, for on the boundary of the interval Λ_1 and outside of it, μ_1 and F_1 and their derivatives up to order ℓ and $\ell + 2$ respectively are equal to zero.

To prove our theorem it suffices to show that

$$\varphi(\xi) = \int_{-\infty}^{\infty} \mu_1(x)E(((x-\xi)^2 + (F_1(x) - F_1(\xi))^2)^{1/2})dx \tag{3}$$

belongs to the class $C^{\ell+1,\lambda'}$ in Λ_0. In place of $\mu_1(x)$ and $F_1(x)$ we shall henceforth write $\mu(x)$ and $F(x)$, and again the interval of integration will be the entire real line. We will denote $((x-\xi)^2+(F(x)-F(\xi))^2)^{1/2}$ by r for the rest of the proof. Since $((\xi - x) - (F(x) - F(\xi))F'(\xi))/r$ is bounded,

$$\begin{aligned}\varphi'(\xi) &= \int_{-\infty}^{\infty} \mu(x)E'(r)\frac{(\xi - x) - (F(x) - F(\xi))F'(\xi)}{r}dx \\ &= \int_{-\infty}^{\xi} \mu(x)E'(r)\frac{(\xi - x) - (F(x) - F(\xi))F'(\xi)}{r}dx \\ &\quad + \int_{\xi}^{\infty} \mu(x)E'(r)\frac{(\xi - x) - (F(x) - F(\xi))F'(\xi)}{r}dx \\ &= \varphi_1(\xi) + \varphi_2(\xi).\end{aligned} \tag{4}$$

We will show $\varphi'(\xi) \in C^{\ell,\lambda'}$ in Λ_0. We first introduce the function

$$\psi_1(\xi, x) = \int_0^1 F'(\xi + (x-\xi)t)dt,$$

and we have

$$F(x) - F(\xi) = (x-\xi)\psi_1(\xi, x).$$

Let $x = \xi + \rho$, with $\rho = x - \xi \geq 0$; we denote $F(\xi+\rho) - F(\xi)$ by $[\tilde{F}]$ and $F(x) - F(\xi)$ by $[F]$.

By induction, we see that the derivative of order k of the function $E((\rho^2 + [\tilde{F}]^2)^{1/2})$ with respect to ξ is a finite linear combination of functions of the form

$$(5) \quad E^{(s)}((\rho^2 + [\tilde{F}]^2)^{1/2}) \cdot \left(\frac{[\tilde{F}][\tilde{F}^{(1)}]}{(\rho^2 + [\tilde{F}]^2)^{1/2}}\right)^b \prod_{j=1}^{n} \frac{\prod_{i=1}^{2p_j}[\tilde{F}^{(\nu_i)}]}{(\rho^2 + [\tilde{F}]^2)^{p_j - \frac{1}{2}}},$$

with n, s, b, p_j, ν_i integers and $0 \leq n, s, b, p_j, \nu_i \leq k$, $b + n = s$, ν_i depends on p_j. If$p_j = 0$, we let

$$\prod_{i=1}^{p_j}[\tilde{F}^{(\nu_i)}] = 1 \quad \text{and} \quad (\rho^2 + [\tilde{F}]^2)^{p_j - \frac{1}{2}} = 1.$$

If $n = 0$, we let

$$\prod_{j=1}^{n} \frac{\prod_{i=1}^{2p_j}[\tilde{F}^{(\nu_i)}]}{(p^2 + [\tilde{F}]^{p_j - \frac{1}{2}}} = 1.$$

Without loss of generality, we can assume that $n = 1$ if $n > 0$ and then denote p_j by p. We note that the above expression is also true for $E'((\rho^2 + [\tilde{F}]^2)^{1/2})$ and $E''((\rho^2 + [\tilde{F}]^2)^{1/2})$, if we replace $b + n = s$ by $b + n = s - 1$, $b + n = s - 2$, respectively. If $0 \leq \nu_i \leq \ell + 1$, we have $F^{(\nu_i)} \in C^1$ and hence

$$\mid [F^{(\nu_i)}] \mid = \mid F^{(\nu_i)}(x) - F^{(\nu_i)}(\xi) \mid \leq c\rho$$

for some constant c. Since $\rho^2 + [F]^2 \geq \rho^2$ we also have

$$\left| \frac{\prod_{i=1}^{2p}[F^{(\nu_i)}]}{(\rho^2 + [F]^2)^{p - \frac{1}{2}}} \right| \leq \frac{(c\rho)^{2p}}{\rho^{2p-1}} = c^{2p}\rho.$$

Thus the expression (5) is bounded if $0 \leq \nu_i \leq \ell + 1$. Since $\mu(\xi) \in C^\ell$ and $\psi_1 \in C^{\ell+1}$ with respect to ξ and ρ,

$$\varphi_2(\xi) = \int_0^\infty \mu(\xi + \rho) E'((\rho^2 + [\tilde{F}]^2)^{1/2}) \frac{-1 - \psi_1(\xi, \xi + \rho)F'(\xi)}{(1 + \psi_1^2(\xi, \xi + \rho))^{1/2}} d\rho$$

belongs to the class C^ℓ. Let $x = \xi - \rho$, $\rho \geq 0$, and $F(\xi-\rho) - F(\xi)$ by $[\tilde{F}]$. Applying the same argument, we can show

$$\varphi_1(\xi) = \int_0^\infty \mu(\xi-\rho)E'((\rho^2+[\tilde{F}]^2)^{1/2})\frac{1-\psi_1(\xi,\xi-\rho)F'(\xi)}{(1+\psi_1^2(\xi,\xi-\rho))^{1/2}}d\rho$$

also belongs to the class C^ℓ. Therefore $\varphi'(\xi) \in C^\ell$ and $\varphi(\xi) \in C^{\ell+1}$. It remains to show that $\varphi^{(\ell+1)}(\xi) \in C^{0,\lambda'}$ in Λ_0.

The derivative of order ℓ of $\varphi_2(\xi)$ is a certain linear combination of a finite number of integrals of the type

$$\int_0^\infty \mu^{(m)}(\xi+\rho)K^{(\ell-m)}(\xi,\rho)d\rho, \quad m = 0,1,\ldots,\ell, \tag{6}$$

where $\mu^{(m)}$ denotes some derivative of order m of $\mu(\xi)$ and $K^{(\ell-m)}$ some derivative of order $\ell - m$ with respect to ξ of $K(\xi,\rho)$, with

$$K(\xi,\rho) = E'((\rho^2+[\tilde{F}]^2)^{1/2}) \cdot \frac{-\rho - [\tilde{F}]\cdot F'(\xi)}{(\rho^2+[\tilde{F}]^2)^{1/2}}.$$

To prove the theorem, it suffices to show that an integral of type (6) belongs to $C^{0,\lambda'}$. We investigate $K^{(\ell-m)}(\xi,\rho)$ more closely. By induction we can show that a derivative of order k of the function

$$\frac{-\rho+[\tilde{F}]F'(\xi)}{(\rho^2+[\tilde{F}]^2)^{1/2}}$$

is a finite linear combination of the form

$$(\rho^2+[\tilde{F}]^2)^{-(\frac{1}{2}+p)}\prod_{i=1}^{2p}[\tilde{F}^{(v)}]\cdot[\tilde{F}^{(\nu_i)}]\cdot F^{(a)}(\xi)$$

or

$$(\rho^2+[\tilde{F}]^2)^{-(\frac{1}{2}+p)}\prod_{i=1}^{2p}[\tilde{F}^{(\nu_i)}]\cdot(-\rho+[\tilde{F}]F'(\xi)),$$

with p, ν_i, ν, a integers, $0 \leq p, \nu_i, \nu \leq k$ and $1 \leq a \leq k+1$. If $p = 0$, we let $\prod_{i=1}^{2p}[\tilde{F}^{(\nu_i)}] = 1$. Thus each of the derivatives $K^{(\ell-m)}(\xi,\rho)$ is some linear combination of a finite number of expressions of the form

$$
(7)\qquad
\begin{aligned}
&E^{(s)}((\rho^2+[\tilde F]^2)^{1/2})\left(\frac{[\tilde F][\tilde F^{(1)}]}{(\rho^2+[\tilde F]^2)^{1/2}}\right)^b\cdot\frac{\prod_{i=1}^{2p}[\tilde F^{(\nu_i)}]}{(\rho^2+[\tilde F]^2)^{p-\frac12}}\\
&\qquad\cdot\frac{\prod_{i=1}^{2q}[\tilde F^{(w_i)}]}{(\rho^2+[\tilde F]^2)^{1/2+q}}\cdot\widetilde{\mathrm{TERM}},
\end{aligned}
$$

where

$$
\begin{gathered}
\widetilde{\mathrm{TERM}}=-\rho+[\tilde F]F'(\xi)\ \text{or}\ [\tilde F^{(\nu)}]F^{(a)}(\xi),\\
0\le p,q,\nu,b,\nu_i,w_i\le \ell-m\le\ell,\\
1\le s\le \ell+1,\ \ 1\le a\le \ell+1-m\le\ell+1,\ b=\begin{cases} s-2, & \text{if } p>0\\ s-1, & \text{if } p=0\end{cases}.
\end{gathered}
$$

If $p=0$, we assume $\prod_{i=1}^{2p}[\tilde F^{(\nu_i)}]=1$ and $(\rho^2+[\tilde F]^2)^{p-\frac12}=1$, and similarly for $q=0$. We denote expression (7) by $\tilde R(\xi,\rho)$, and the function obtained on replacing in (7) the quantities $[\tilde F^{(\nu_i)}],[\tilde F^{(w_i)}],[\tilde F],\rho$ by $[F^{(\nu_i)}],[F],x-\xi$, by $R(\xi,x)$. Since $\mid[F^{(\nu_i)}]\mid< c\rho$ and $\rho^2+[F]^2\ge\rho^2$, it is easily seen that R is bounded. Now we claim that

$$
\mid\frac{\partial R}{\partial\xi}\mid\le\frac{c}{\rho}.
$$

Indeed, denote $F^{(\nu_i)}$ by $F_i,\prod_{i=1}^{n}[F^{(\nu_i)}]$ by $\psi_n,\ \psi_{2p}r^{-(2p-1)}$ by $\Omega_p,\psi_{2q}r^{-(2q+1)}$ by Ω_q. We have

$$
\begin{aligned}
\frac{\partial R}{\partial\xi}=&E^{(s+1)}((\rho^2+[F]^2)^{1/2})\cdot\frac{\xi-x-[F]F'(\xi)}{(\rho^2+[F]^2)^{1/2}}\\
&\left(\frac{[F][F^{(1)}]}{(\rho^2+[F]^2)^{1/2}}\right)^b\cdot\Omega_p\cdot\Omega_q\cdot\mathrm{TERM}\\
&+E^{(s)}(\sqrt{\rho^2+[F]^2})\Big[b\left(\frac{[F][F^{(1)}]}{(\rho^2+[F]^2)^{1/2}}\right)^{b-1}\\
&\cdot\left(\frac{-[F]\cdot F''(\xi)-F'(\xi)[F^{(1)}]}{(\rho^2+[F]^2)^{1/2}}-\frac{[F][F^{(1)}]}{\rho^2+[F]^2}\cdot\frac{\xi-x-[F]F'(\xi)}{(\rho^2+[F]^2)^{1/2}}\right)\\
&\cdot\Omega_p\cdot\Omega_q\cdot\mathrm{TERM}+\left(\frac{[F][F^{(1)}]}{(\rho^2+[F]^2)^{1/2}}\right)^b\\
&\cdot\left(\Omega_p\cdot\Omega_q\cdot\frac{\partial\mathrm{TERM}}{\partial\xi}+\mathrm{TERM}\cdot\left(\Omega_p\cdot\frac{\partial\Omega_q}{\partial\xi}+\Omega_q\cdot\frac{\partial\Omega_p}{\partial\xi}\right)\right)\Big].
\end{aligned}
$$

Since

$$\frac{\partial\Omega_q}{\partial\xi} = -(2q+1)\cdot\frac{(\xi-x)-[F]F'(\xi)}{r^{2q+3}}\psi_{2q} + \frac{1}{r^{2q+1}}\cdot\sum_{k=1}^{2q} F'_k\cdot\psi_{2q-1,k},$$

with $\psi_{2q-1,k} = \psi_{2q}/F_k$, it is easy to see that $\left|\partial\Omega_q/\partial\xi\right| < B_1/\rho^2$ for some constant B_1. Similarly we can show that

$$\left|\frac{\partial\Omega_p}{\partial\xi}\right| < B_2, \quad \left|\frac{\partial\mathrm{TERM}}{\partial\xi}\right| < B_3.$$

Also it is clear that

$$|\Omega_p| \le c_1 P, \quad \Omega_q\cdot\mathrm{TERM}\,| \le c_2,$$
$$|\Omega_q| \le \frac{c_3}{\rho}, \quad |\mathrm{TERM}\,| \le c_4\rho.$$

Therefore $|\partial R/\partial\xi| < c/\rho$ for some constant c. For derivatives of $\varphi_1(\xi)$ we have the same expressions of $\tilde{R}(\xi,\rho), R(\xi,x), |R| < c$ and $|\partial R/\partial\xi| < c/\rho$, if we replace ρ by $\xi - x$. Thus, in general,

$$|R| < c, \quad \left|\frac{\partial R}{\partial\xi}\right| < \frac{c}{|x-\xi|}. \tag{8}$$

Now we will show that

$$\mathring{\varphi}(\xi) = \int_{-\infty}^{\infty}\mu^{(m)}(x)R(\xi,x)dx$$

belongs to $C^{0,\lambda'}$ on Λ_0. In place of $\mu^{(m)}$ we shall simply write μ, where it is assumed that $\mu\in C^0$. Let $M_0(\xi)$ and $M_1(\xi_1)$ be two points a distance δ apart, then

(9)

$$\begin{aligned}\Big|\int_{-\infty}^{\infty}&\mu(x)R(\xi_1,x)dx - \int_{-\infty}^{\infty}\mu(x)R(\xi,x)dx\Big|\\ &\le \int_{|x-\xi|\le 2_\delta}|\mu(x)|\,|R(\xi_1,x)|dx + \int_{|x-\xi|\le 2_\delta}|\mu(x)|\,|R(\xi,x)|dx\\ &+\int_{|x-\xi|>2_\delta}|\mu(x)(R(\xi_1,x)-R(\xi,x))|dx.\end{aligned}$$

Let μ be bounded by a constant, say A, then

$$\int_{|x-\xi|\leq 2\delta} |\mu(x)|\,|R(\xi,x)|dx < Ac\cdot 4\delta.$$

Since the interval $|x-\xi|\leq 2\delta$ is contained in the interval $|x-\xi_1|\leq 3\delta$,

$$\int_{|x-\xi|\leq 2\delta} |\mu(x)|\,|R(\xi_1,x)|dx < 6Ac\delta.$$

Now we consider the third integral on the right-hand side of (9). Using the mean value theorem and (8),

$$|R(\xi_1,x)-R(\xi,x)| = \delta\left|\left(\frac{\partial R}{\partial \xi}\right)_{M'}\right| \leq \frac{\delta c}{|x-\xi'|},$$

where $M'(\xi')$ lies on M_0, M_1. We note that $|x-\xi'|\geq |x-\xi|/2$ for $|x-\xi|\geq 2\delta$. We can also assume that $R=0$ for $|x-\xi|>a$, then

$$\begin{aligned}
&\int_{|x-\xi|>2_\delta} |\mu(x)(R(\xi_1,x)-R(\xi,x))|dx \\
&\qquad < Ac\delta \int_{a\geq|x-\xi|>2_\delta} \frac{2}{|x-\xi|}dx \\
&\qquad = 2Ac\delta \int_{\xi+2_\delta}^{\xi+a} \frac{1}{x-\xi}dx + \int_{\xi-a}^{\xi-2_\delta} \frac{1}{\xi-x}dx \\
&\qquad = 4Ac\delta \log\frac{a}{2\delta} \leq c_1 A\delta^{\lambda'},
\end{aligned}$$

where λ' arbitrary in $0<\lambda'<1$.□

THEOREM 4.2. *Let $S\in C^{\ell+2}$, and $\mu\in C^{\ell}$ $(\ell\geq 0)$ on S, and assume $E(r_{20})$ is defined the same as Theorem* 4.1. *At $M_2\in S$, define*

$$W(\mu) = \int_S \mu(2)E(r_{20})\cos(r_{20}\nu_2)d\sigma_2.$$

Then $W(\mu)\in C^{\ell+1,\lambda'}$ on S, where λ' is arbitrary in $0<\lambda'<1$.

PROOF. Since the proof is about the same as Theorem 4.1, we only mention the following differences:

$$\varphi(\xi) = \int_{-\infty}^{\infty} \mu_1(x) E(((x-\xi)^2 + (F_1(x) - F_1(\xi))^2)^{1/2}) \cdot \frac{F_1(\xi) - F_1(x) + (x-\xi)F_1'(x)}{((x-\xi)^2 + (F_1(x) - F_1(\xi))^2)^{1/2}} dx.$$

As before, we still denote μ_1, F_1 by μ, F, respectively. In addition to ψ_1 defined in the proof of Theorem 4.1, let

$$\psi_2(\xi, x) = \int_0^1 tF''(\xi + (x-\xi)t)dt,$$
$$\psi_3(\xi, x) = \int_0^1 F''(\xi + (x-\xi)t)dt.$$

Then

$$F'(x) - F'(\xi) = (x-\xi)\psi_3(\xi, x),$$

and, applying integration by parts to ψ_2,

$$F(x) - F(\xi) + (\xi - x)F'(x) = -(\xi - x)^2 \psi_2(\xi, x).$$

We denote r_{12} by r for the rest of the proof.

$$\begin{aligned} \varphi'(\xi) = &\int_{-\infty}^{\infty} \mu(x)E'(r)\frac{\xi - x - (F(x) - F(\xi))F'(\xi)}{r} \\ &\cdot \frac{F(\xi) - F(x) + (x-\xi)F'(x)}{r} dx \\ &+ \int_{-\infty}^{\infty} \mu(x)E(r) \cdot \frac{F'(\xi) - F'(x)}{r} dx \\ &- \int_{-\infty}^{\infty} \mu(x)E(r) \cdot \frac{F(\xi) - F(x) + (x-\xi)F'(x)}{r^2} \\ &\cdot \frac{\xi - x - (F(x) - F(\xi))F'(\xi)}{r} dx. \end{aligned} \tag{1}$$

We split each integral into $\int_{-\infty}^{\xi} + \int_{\xi}^{\infty}$ and change the variable to show these integrals belong to C^{ℓ}, as in the proof of Theorem 4.1. Therefore

$\varphi'(\xi) \in C^{\ell}$ and $\varphi \in C^{\ell+1}$. It remains to show that $\varphi^{(\ell+1)} \in C^{0,\lambda'}$. We call $\varphi_3(\xi)$ for the $\int_{\xi}^{\infty}$ part of the third integral of (1).

The derivative of order ℓ of $\varphi_3(\xi)$ is a certain linear combination of a finite number of integrals of the type

$$\int_0^{\infty} \mu^{(m)}(\xi+\rho)K^{(\ell-m)}(\xi,\rho)d\rho, \tag{2}$$

where

$$K(\xi,\rho) = E(r)\cdot\frac{F(\xi)-F(\xi+\rho)+\rho F'(\xi+\rho)}{\rho^2+[\tilde{F}]^2}\cdot\frac{-\rho-[\tilde{F}]\cdot F'(\xi)}{(\rho^2+[\tilde{F}]^2)^{1/2}}. \tag{3}$$

Let $\{F\} = F(\xi) - F(x) + (x-\xi)F'(x)$ and $\{\tilde{F}\} = F(\xi) - F(\xi+\rho) + \rho F'(\xi+\rho)$. We can show that each of the derivatives $K^{(\ell-m)}(\xi;\rho)$ is some linear combination of a finite number of expressions of the form

$$\begin{aligned} &E^{(s)}((\rho^2+[\tilde{F}]^2)^{1/2})\cdot\left(\frac{[\tilde{F}][\tilde{F}^{(1)}]}{(\rho^2+[\tilde{F}]^2)^{1/2}}\right)^{b}\prod_{j=1}^{n}\frac{\prod_{i=1}^{2p_j}[\tilde{F}^{(\nu_i)}]}{(\rho^2+[\tilde{F}]^2)^{p_j-\frac{1}{2}}} \\ &\cdot\frac{\prod_{i=1}^{2s_1}[\tilde{F}^{(\nu_i)}]}{(\rho^2+[\tilde{F}]^2)^{1+s_1}}\cdot\{\tilde{F}^{(s_2)}\}\cdot\frac{\prod_{i=1}^{2q}[\tilde{F}^{(\omega_i)}]}{(\rho^2+[\tilde{F}]^2)^{\frac{1}{2}+q}}\cdot\widetilde{\mathrm{TERM}}, \end{aligned} \tag{4}$$

$0 \le s, b, p_j, q, s_1, \nu_i, s_2, \omega_i \le \ell - m \le \ell$, $b+n = s$, ν_i depends on p_j and s_1. $\widetilde{\mathrm{TERM}}$ is the same as in Theorem 4.1. Without loss of generality, we can assume that $n = 1$, if $n > 0$, and then denote p_j by p. We denote the above expression by $\tilde{R}(\xi;\rho)$, and correspondingly, we have $R(\xi, x)$ as before. By the mean value theorem,

$$|\{F^{(s_2)}\}| \le B\rho^2$$

for some constant B. By a complicated (but not deep) algebraic computation, as in the proof of Theorem 4.1, we can show that

$$|R| \le c_1, \quad \left|\frac{\partial R}{\partial \xi}\right| \le \frac{c_2}{\rho}. \tag{5}$$

We can get the same estimate for the $\int_{-\infty}^{\xi}$ part of the third integral of (1). Proceeding as in Theorem 4.1, we can show that the third integral of (1) belongs to $C^{\ell+1,\lambda'}$. Similar arguments hold for the second and the first integrals of (1). We note they also have the same estimates of R and $\frac{\partial R}{\partial \xi}$. Therefore $\varphi'(\xi) \in C^{\ell,\lambda'}$, i.e., $\varphi(\xi) \in C^{\ell+1,\lambda'}$. □

THEOREM 4.3. *If $S \in C^{\ell+2,\lambda}$ and $\mu \in C^{\ell,\lambda}$ $(\ell \geq 0)$ on S, and*

$$\begin{aligned} W(\mu) &= -\int_S \mu(2) \frac{\partial}{\partial \nu_2} \log r_{20} d\sigma_2 \\ &= -\int_S \mu(2) \frac{\cos(r_{20}\nu_2)}{r_{20}} d\sigma_2, \end{aligned}$$

then $W(\mu) \in C^{\ell+1,\lambda'}$ on S, where λ' arbitrary in $0 < \lambda' < \lambda$.

PROOF. The proof is quite similar to [**3**, p. 312-325] for the Laplacian double layer potential in three dimensions. We omit the proof, for details see [**4**].□

REMARK. For the case $\ell = 0$, $0 < \lambda < 1$, Schippers [**7**] showed $W(\mu) \in C^{1,\lambda}$ on S under a weaker assumption: μ is bounded and integrable on S.

For the Laplacian single layer potential, we give a completely different proof using trigonometric series.

THEOREM 4.4. *Let $S \in C^{\ell+3}$ and $\mu \in C^{\ell,\lambda}(\ell \geq 0)$ on S. If*

$$W(\mu)(p) = \int_S \mu(q) \log |p-q| d\sigma_q, \quad p \in S,$$

then $W(\mu) \in C^{\ell+1,\lambda'}$ on S, where

$$\lambda' = \begin{cases} \lambda, & \text{if } 0 < \lambda < 1 \\ \text{arbitrary in } 0 < \lambda' < 1, & \text{if } \lambda = 1. \end{cases}$$

PROOF. Let the curve S be parametrized by $(f(s), g(s)), 0 \leq s \leq L$, where s is the arc length, f and $g \in C^{\ell+3}$. Let $p = (f(t), g(t))$, then considering $W(\mu)(p)$ is equivalent to considering $W(\mu)(t)$ where

$$W(\mu)(t) = \int_0^L \mu(s) \log((f(s) - f(t))^2 + (g(s) - g(t))^2)^{1/2} ds.$$

Let $(f_C(s), g_C(s)) = (L/2\pi)(\cos(2\pi s/L), \sin(2\pi s/L))$, $0 \leq s \leq L$, be a point on a circle of radius $L/2\pi$. We split W into two parts,

(1)

$$\begin{aligned} W(\mu)(t) =& \frac{1}{2} \int_0^L \mu(s) \log \frac{(f(s) - f(t))^2 + (g(s) - g(t))^2}{(f_C(s) - f_C(t))^2 + (g_C(s) - g_C(t))^2} ds \\ &+ \int_0^L \mu(s) \log((f_C(S) - f_C(t))^2 + (g_C(s) - g_C(t))^2)^{\frac{1}{2}} ds \\ =& W_1(\mu)(t) + W_2(\mu)(t). \end{aligned}$$

If μ is continuous on S, then $W_1(\mu)(t) \in C^{\ell+2}, \forall\ 0 \leq t \leq L$. Indeed, let

$$\psi_1(s,t) = \int_0^1 f'(s + (t-s)u) du,$$
$$\psi_2(s,t) = \int_0^1 g'(s + (t-s)u) du,$$
$$\psi_3(s,t) = \int_0^1 f'_C(s + (t-s)u) du,$$
$$\psi_4(s,t) = \int_0^1 g'(s + (t-s)u) du.$$

Then $\psi_1, \psi_2, \psi_3, \psi_4 \in C^{\ell+2}, \forall 0 \leq t \leq L$, and

$$\begin{aligned} f(s) - f(t) &= (s-t)\psi_1(s,t), \\ g(s) - g(t) &= (s-t)\psi_2(s,t), \\ f_C(s) - f_C(t) &= (s-t)\psi_3(s,t), \\ g_C(s) - g_C(t) &= (s-t)\psi_4(s,t). \end{aligned}$$

Therefore

$$W_1(\mu)(t) = \frac{1}{2} \int_0^L \mu(s) \log \frac{\psi_1^2(s,t) + \psi_2^2(s,t)}{\psi_3^2(s,t) + \psi_4^2(s,t)} ds$$

also belongs to $C^{\ell+2}$.

Now we consider $W_2(\mu)(t)$. By [**6**, p. 517]

$$\log((f_C(s)-f_C(t))^2+(g_C(s)-g_C(t))^2)^{\frac{1}{2}} \\ = \log\frac{L}{2\pi}-\sum_{n=1}^{\infty}\frac{1}{n}\Big(\cos\frac{2\pi ns}{L}\cdot\cos\frac{2\pi nt}{L} \\ +\sin\frac{2\pi ns}{L}\cdot\sin\frac{2\pi nt}{L}\Big). \tag{2}$$

which is convergent in the L^2 sense. We write

$$\mu(s)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi ns}{L}+b_n\sin\frac{2\pi ns}{L}\right).$$

where

$$a_n=\frac{2}{L}\int_0^L\mu(t)\cos\frac{2\pi nt}{L}dt$$
$$b_n=\frac{2}{L}\int_0^L\mu(t)\sin\frac{2\pi nt}{L}dt$$
$$n=0,1,2,\dots.$$

Then

$$W_2(\mu)(t)=\int_0^L\Big(\log\frac{L}{2\pi}-\sum_{n=1}^{\infty}\frac{1}{n}\Big(\cos\frac{2\pi ns}{L}\cdot\cos\frac{2\pi nt}{L} \\ +\sin\frac{2\pi ns}{L}\cdot\sin\frac{2\pi nt}{L}\Big)\Big) \\ \cdot\Big(\frac{a_0}{2}+\sum_{m=1}^{\infty}\Big(a_m\cos\frac{2\pi ms}{L}+b_m\sin\frac{2\pi ms}{L}\Big)\Big)ds \\ =\frac{a_0L}{2}\cdot\log\frac{L}{2\pi}-\frac{L}{2}\sum_{n=1}^{\infty}\frac{1}{n}\Big(a_n\cos\frac{2\pi nt}{L}+b_n\sin\frac{2\pi nt}{L}\Big) \tag{3}$$

If μ is a constant c, then

$$W_2(\mu)(t)=cL\log\frac{L}{2\pi}, \tag{4}$$

is still a constant. Assuming $\ell \geq 1$, we have $\mu^{(\ell)}(s) \in C^{0,\lambda}$, $\forall\, 0 \leq s \leq L$. By the periodicity of $\mu^{(\ell-1)}$,

$$\mu^{(\ell)}(s) = \sum_{n=1}^{\infty} \Big(a_n \cos \frac{2\pi ns}{L} + b_n \sin \frac{2\pi ns}{L}) \tag{5}$$

in L^2-sense, where

$$a_n = \frac{2}{L}\int_0^L \mu^{(\ell)}(t) \cos \frac{2\pi nt}{L} dt,$$
$$b_n = \frac{2}{L}\int_0^L \mu^{(\ell)}(t) \sin \frac{2\pi nt}{L} dt,$$
$$n = 0, 1, 2, \dots.$$

Integrate (5) and use the periodicity of $\mu^{(i)}(s), 0 \leq i \leq \ell - 2$, to obtain (6)

$$\mu(s) = \begin{cases} (\frac{L}{2\pi})^\ell (-1)^{\ell/2} \sum_{n=1}^{\infty} (\frac{a_n}{n^\ell} \cos \frac{2\pi ns}{L} + \frac{b_n}{n^\ell} \sin \frac{2\pi ns}{L}) + c, & \ell \text{ even.} \\ (\frac{L}{2\pi})^\ell (-1)^{\frac{\ell-1}{2}} \sum_{n=1}^{\infty} (\frac{a_n}{n^\ell} \sin \frac{2\pi ns}{L} - \frac{b_n}{n^\ell} \cos \frac{2\pi ns}{L}) + c, & \ell \text{ odd.} \end{cases}$$

If ℓ is even, then

$$W_2(\mu)(t) = \Big(\frac{L}{2\pi}\Big)^\ell (-1)^{\ell/2}\Big(-\frac{L}{2}\Big) \cdot \sum_{n=1}^{\infty} \Big(\frac{a_n}{n^{\ell+1}} \cos \frac{2\pi nt}{L} + \frac{b_n}{n^{\ell+1}} \sin \frac{2\pi nt}{L}\Big) + cL \log \frac{L}{2\pi}.$$

Differentiating it $\ell + 1$ times,

$$(W_2\mu)^{(\ell)}(t) = -\frac{L}{2} \sum_{n=1}^{\infty} \Big(\frac{a_n}{n} \cos \frac{2\pi nt}{L} + \frac{b_n}{n} \sin \frac{2\pi nt}{L}\Big), \tag{8}$$

$$(W_2\mu)^{(\ell+1)}(t) = \pi \sum_{n=1}^{\infty} \Big(a_n \sin \frac{2\pi nt}{L} - b_n \cos \frac{2\pi nt}{L}\Big). \tag{9}$$

In fact, since $\mu^{(\ell)} \in C^{0,\lambda}$ (see [**11**, p. 221]), and

$$|a_n| \leq \frac{c}{n^\lambda}, \quad |b_n| \leq \frac{c}{n^\lambda},$$

then all the series expansions of $W_2\mu, (W_2\mu)^{(1)}, \ldots, (W_2\mu)^{(\ell)}$ are uniformly convergent, and hence differentiation is possible.

The last differentiation $(W_2\mu)^{(\ell+1)}$ can be justified for the following reason. Let $\varphi(t) = \pi \sum_{n=1}^{\infty}(a_n \sin(2\pi nt/L) - b_n \cos(2\pi nt/L))$. By the assumption that $\mu^{(\ell)}$ is in L^2, given in (5), let

$$\psi(t) = \int_0^t \phi(s)ds. \tag{10}$$

Then

$$\begin{aligned}\psi(t) &= \pi \int_0^t \Big(\sum_{n=1}^{\infty} a_n \sin\frac{2\pi ns}{L} - b_n \cos\frac{2\pi ns}{L}\Big)ds\\ &= -\frac{L}{2}\sum_{n=1}^{\infty}\Big(\frac{a_n}{n}\cos\frac{2\pi nt}{L} + \frac{b_n}{n}\sin\frac{2\pi nt}{L}\Big) + c\\ &= (W_2\mu)^{(\ell)}(t) + c.\end{aligned}$$

From (10), $\psi'(t) = \varphi(t)$. Therefore $(W_2\mu)^{(\ell+1)}(t)$ exists and

$$\begin{aligned}(W_2\mu)^{(\ell+1)}(t) &= \varphi(t)\\ &= \pi\sum_{n=1}^{\infty}\Big(a_n \sin\frac{2\pi nt}{L} - b_n\cos\frac{2\pi nt}{L}\Big).\end{aligned}$$

If ℓ is odd, we can obtain exactly the same expansion of $(W_2\mu)^{(\ell)}(t)$ and $(W_2\mu)^{(\ell+1)}(t)$.

We note that the series $\sum_{n=1}^{\infty}(a_n \sin nt - b_n \cos nt)$ is called the conjugate of the trigonometric series $(1/2)a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$. We note that the series expansion of $(W_2\mu)^{(\ell+1)}/\pi$ is the conjugate of the Fourier series expansion of $\mu^{(\ell)}$. By [**11**, p. 242], the series expansion of $(W_2\mu)^{(\ell+1)}/\pi$ is also the Fourier series expansion for $\tilde{u}^{(\ell)}$, the conjugate function of $\mu^{(\ell)}$. For definitions, see [**11**, p. 225]. From [**11**, p. 259], $(1/\pi)(W_2\mu)^{(\ell+1)} = \tilde{u}^{(\ell)}$. From [**12**, p. 121], $\tilde{f} \in c^{0,\lambda}$, if $f \in C^{0,\lambda}, 0 < \lambda < 1$. Thus $(W_2\mu)^{(\ell+1)}(t) \in C^{0,\lambda'}$, $\forall\, 0 \leq t \leq L$, and therefore

$$W_2(\mu)(t) \in C^{\ell+1,\lambda'}, \quad \forall\, 0 \leq t \leq L.$$

For the case $\ell = 0$, let

$$\mu(s) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n \cos\frac{2\pi ns}{L} + b_n \sin\frac{2\pi ns}{L}\Big).$$

Then

$$W_2\mu(t) = \frac{a_0}{2} L \log \frac{L}{2\pi} - \frac{L}{2} \sum_{n=1}^{\infty} \left(\frac{a_n}{n} \cos \frac{2\pi nt}{L} + \frac{b_n}{n} \sin \frac{2\pi nt}{L} \right),$$

$$(W_2\mu)^{(1)}(t) = \pi \sum_{n=1}^{\infty} \left(a_n \sin \frac{2\pi nt}{L} - b_n \cos \frac{2\pi nt}{L} \right).$$

By the same reasoning as with $\ell \geq 1$, we can show $(W_2\mu)^{(1)}(t) \in C^{0,\lambda'}$, and hence $W_2\mu(t) \in C^{1,\lambda'}, 0 \leq t \leq L$. □

THEOREM 4.5. *If $S \in C^{\ell+2,\lambda}$ and $\mu \in C^{\ell,\lambda}(\ell \geq 0)$ on S, then $M\mu \in C^{\ell+1,\lambda'}$ on S, where λ' is arbitrary in $0 < \lambda' < \lambda$. If, in addition, $S \in C^{\ell+3}$, then $L\mu \in C^{\ell+1,\lambda'}$ on S where*

$$\lambda' = \begin{cases} \lambda, & \text{if } 0 < \lambda < 1 \\ \text{arbitrary in } 0 < \lambda' < 1, & \text{if } \lambda = 1 \end{cases}$$

PROOF. Since $(i/2)H_0^{(1)}(k|p-q|) = (1/\pi)\log \frac{1}{|p-q|} + E(|p-q|)$, combining Theorem 4.1 and Theorem 4.4, we obtain $L\mu \in C^{\ell+1,\lambda'}$ on S. Similarly, combining Theorem 4.2 and Theorem 4.3, we obtain $M\mu \in C^{\ell+1,\lambda'}$ on S. □

Using the same arguments as in §3, Theorem 3.5-Theorem 3.7, but changing to the two-dimensional analogues, we can obtain the following theorems.

THEOREM 4.6. *If $S \in C^1, \mu$ is bounded and integrable on S, and*

$$W(\mu) = \int_S \mu(2)E(r_{20})d\sigma_2,$$

then $W(\mu) \in C^{0,1}$ on S.

THEOREM 4.7. *If $S \in C^{1,\lambda}, \mu$ is bounded and integrable on S, and*

$$W(\mu) = \int_S \mu(2)E(r_{20})\cos(r_{20}\nu_2)d\sigma_2,$$

then $W(\mu) \in C^{0,1}$ *on* S.

REMARK. If $S \in C^1$, μ is bounded and integrable on S in Theorem 4.7, then $W(\mu) \in C^{0,\lambda}$, where λ arbitrary in $0 < \lambda < 1$.

THEOREM 4.8. *If* $S \in C^{1,\lambda}$, μ *is bounded and integrable on* S, *and*

$$W(\mu) = -\int_S \mu(2) \frac{\partial}{\partial \nu_2} \log r_{20} d\sigma_2,$$

then $W(\mu) \in C^{0,\lambda'}$, *where*

$$\lambda' = \begin{cases} \lambda, & \text{if } 0 < \lambda < 1 \\ \text{arbitrary in } 0 < \lambda' < 1, & \text{if } \lambda = 1. \end{cases}$$

THEOREM 4.9. *Let* $S \in C^2$ *and* μ *be continuous on* S. *If*

$$W(\mu)(p) = \int_S \mu(q) \log |p - q| d\sigma_q, p \in S,$$

then $W(\mu) \in C^{0,\lambda}$ *on* S, *where* λ *arbitrary in* $0 < \lambda < 1$.

PROOF. Let $f, g, W_1, W_2, \psi_1, \psi_2, \psi_3, \psi_4$ be defined as in the proof of Theorem 4.4. Applying the same argument as Theorem 4.4, we can show that $W_1(\mu)(t) \in C^{0,\lambda}$, $\forall\ 0 \le t \le L$. Now we consider $W_2(\mu)(t)$. We write

$$\mu(s) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi ns}{L} + b_n \frac{2\pi ns}{L} \right),$$

convergent in L^2-sense, where

$$a_n = \frac{2}{L} \int_0^L \mu(t) \cos \frac{2\pi nt}{L} dt,$$
$$b_n = \frac{2}{L} \int_0^L \mu(t) \sin \frac{2\pi nt}{L} dt,$$
$$n = 0, 1, 2, \ldots.$$

Since $\mu(s)$ is continuous, so is $\mu(s) - \frac{a_0}{2}$. Let

$$\varphi(t) = \int_0^t \Big(\mu(s) - \frac{a_0}{2}\Big) ds. \tag{1}$$

Then

$$\varphi(t) = \frac{L}{2\pi} \sum_{n=1}^{\infty} \Big(\frac{a_n}{n} \sin \frac{2\pi n t}{L} - \frac{b_n}{n} \cos \frac{2\pi n t}{L}\Big) + c.$$

From (1), $\varphi'(t) = \mu(t) - a_0/2$, and therefore $\varphi(t) - c \in C^1$. Since $W_2(\mu)(t) = \frac{a_0}{2} L \log \frac{L}{2\pi} - \frac{L}{2} \sum_{n=1}^{\infty} (\frac{a_n}{n} \cos \frac{2\pi n t}{L} + \frac{b_n}{n} \sin \frac{2\pi n t}{L})$, the series expansion of $(\frac{-2}{L})(W_2(\mu)(t) - (a_0/2) L \log(L/2\pi))$ and the series expansion of $(2\pi/L)(\varphi(t) - c)$ are conjugate. Since $(\varphi(t) - c)(2\pi/L) \in C^{0,\lambda}$, for the same reason as in the proof of Theorem 2.4, $W_2(\mu)(t) \in C^{0,\lambda}, 0 \le t \le L$. □

THEOREM 4.10. *If $S \in C^{1,\lambda}, \mu$ is bounded and integrable on S, then $M\mu \in C^{0,\lambda'}$ where*

$$\lambda' = \begin{cases} \lambda, & \textit{if } 0 < \lambda < 1, \\ \textit{arbitrary in } 0 < \lambda' < 1, & \textit{if } \lambda = 1. \end{cases}$$

If, in addition, $S \in C^2$ and μ is continuous on S, then $L\mu \in C^{0,\lambda'}$ on S where λ' arbitrary in $0 < \lambda' < 1$.

PROOF. Combining Theorem 4.6-Theorem 4.9, we obtain our results. □

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