JOURNAL OF COMMUTATIVE ALGEBRA
Volume 5, Number 3, Fall 2013

HILBERT COEFFICIENTS OF PARAMETER IDEALS

LORI MCCUNE

ABSTRACT. We consider the non-positivity of the Hilbert coefficients for a parameter ideal of a commutative Noetherian local ring. In particular, we show that the second Hilbert coefficient of a parameter ideal of depth at least $d-1$ is always non-positive and give a condition for the coefficient to be zero. With the added condition that the depth of the associated graded ring also be at least $d-1$ we show $e_i(q) \leq 0$ for $i = 1, \ldots, d$.

1. Introduction. Let $(R, \mathfrak{m})$ be a Noetherian local ring, $I \subseteq R$ an $\mathfrak{m}$-primary ideal and M a finitely generated R-module. We let $\lambda_R(-)$ denote the length of an R-module. The *Hilbert function* for I with respect to M is the function $H_{I,M} : \mathbf{Z} \to \mathbf{Z}$ given by $H_{I,M}(n) = \lambda_R(M/I^n M)$. Samuel showed that these functions agree with a polynomial $P_{I,M}(n)$ (called the *Hilbert-Samuel polynomial*) of degree $d = \dim M$ for n sufficiently large. We can always write $P_{I,M}(n)$ in the form

$$P_{I,M}(n) = \sum_{i=0}^{d} (-1)^i e_i(I, M) \binom{n+d-i-1}{d-i}$$

for unique numbers $e_i(I, M)$, known as the *Hilbert coefficients* for I with respect to M. The largest number for which $H_{I,M}(n)$ and $P_{I,M}(n)$ disagree is called the *postulation number* for I, denoted $n(I, M) := \min\{j \mid H_{I,M}(n) = P_{I,M}(n) \text{ for all } n > j\}$. Whenever $M = R$ we often suppress the M.

In this note, we focus our attention on non-positivity of the Hilbert coefficients for a parameter ideal. Recall that a ring R is *unmixed*

2010 AMS *Mathematics subject classification.* Primary 13D40.
Keywords and phrases. Hilbert-Samuel polynomial, Hilbert coefficients.
Received by the editors on November 8, 2011, and in revised form on January 24, 2012.
DOI:10.1216/JCA-2013-5-3-399

if $\dim \widehat{R}/p = \dim R$ for all $p \in \mathrm{Ass}_{\widehat{R}}\widehat{R}$ where $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of R. Our work was inspired by the following result of Ghezzi, et al. [**2**] which characterizes the Cohen-Macaulayness of a ring in terms of the first Hilbert coefficient of a parameter ideal. By a parameter ideal, we mean an ideal generated by a full system of parameters.

Theorem 1.1 [**2**]. *Suppose* $(R, \mathfrak{m})$ *is an unmixed local ring and* q *a parameter ideal. Then* $e_1(q) \leq 0$ *with equality if and only if* R *is Cohen-Macaulay.*

With the assumption that $\operatorname{depth} R \geq d-1$, we are able to prove the following:

Theorem 1.2. *Let* $(R, \mathfrak{m})$ *be a local Noetherian ring of dimension* $d \geq 2$*. Suppose that* $\operatorname{depth} R \geq d-1$*. If* q *is a parameter ideal of* R*, then the following hold*:

(1) $e_2(q) \leq 0$,

(2) $e_2(q) = 0$ *if and only if* $n(q) < 2-d$ *and* $\operatorname{grade} gr_q(R)_+ \geq d-1$,

(3) $e_2(q) = 0$ *implies* $e_3(q) = e_4(q) = \cdots = e_d(q) = 0$.

Here, $gr_q(R)$ denotes the associated graded ring of R with respect to q.

We also discuss some results with respect to the first difference function, Δ, defined in Section 4 and use this to examine the other Hilbert coefficients of a parameter ideal under the additional assumption that $\operatorname{depth} gr_q(R) \geq d-1$.

2. The Hilbert coefficients in dimension one. In this section we will discuss some results on the Hilbert coefficients of a parameter ideal in a one-dimensional ring.

Definition 2.1. Let $f : \mathbf{Z} \to \mathbf{Z}$. The first difference function, $\Delta(f)$, is defined by $\Delta(f(n)) = f(n+1) - f(n)$. We define the ith difference function inductively by $\Delta^i(f) = \Delta(\Delta^{i-1}(f))$. By convention, we define $\Delta^0(f) = f$.

We first give a formula for the Hilbert coefficients in a one-dimensional ring.

Proposition 2.2. *Suppose* $(R, \mathfrak{m})$ *is a one-dimensional local Noetherian ring and* $q = (x) \subseteq R$ *is a parameter ideal. Then* $((x^{i+1}) : x^i) = ((x^{i+2}) : x^{i+1})$ *for all* $i \gg 0$. *We set* $l = \min\{i \mid ((x^{n+1}) : x^n) = ((x^{i+1}) : x^i)$ *for all* $n \geq i\}$ *and* $\widetilde{x} = ((x^{l+1}) : x^l)$. *Then*

(1) (a) $e_0(q) = \lambda_R(R/\widetilde{x})$, *and*

(b) $e_1(q) = \sum_{i=0}^{l-1}(\lambda_R(R/\widetilde{x}) - \lambda_R(R/((x^{i+1}) : x^i)))$ for a fixed integer l.

(2) (a) $P_q(n) - H_q(n) = \sum_{i=n}^{\infty}(\lambda_R(R/((x^{i+1}) : x^i)) - \lambda(R/\widetilde{x}))$, *and*

(b) $P_q(n) \geq H_q(n)$ *for all* $n \geq 0$.

Proof. Write $q = (x)$. Note that $\lambda_R((x^i)/(x^{i+1})) = \lambda_R(R/((x^{i+1}) : x^i))$ for all i as $((x^{i+1}) : x^i)$ is the kernel of the surjective map $R \to (x^i)/(x^{i+1})$ defined by $1 \mapsto \overline{x^i}$. Then $\lambda(R/q^n) = \sum_{i=0}^{n-1} \lambda_R((x^i)/(x^{i+1})) = \sum_{i=0}^{n-1} \lambda_R(R/((x^{i+1}) : x^i))$. Note the ascending chain

$$((x) : x^0) \subseteq ((x^2) : x) \subseteq ((x^3) : x^2) \subseteq \cdots$$

must stabilize. Let

$$l = \min\{i \mid ((x^{n+1}) : x^n) = ((x^{i+1}) : x^i) \text{ for all } n \geq i\}$$

and set $\widetilde{x} = ((x^{l+1}) : x^l)$.

For $n \geq l$, we have $\lambda_R(R/(x^n)) = \sum_{i=0}^{l-1} \lambda_R(R/((x^{i+1}) : x^i)) + (n - l)\lambda_R(R/\widetilde{x})$. This gives that

$$P_q(n) = \sum_{i=0}^{l-1} \lambda_R(R/((x^{i+1}) : x^i)) + (n - l)\lambda_R(R/\widetilde{x}).$$

From this, we see that $e_0(q) = \lambda_R(R/\widetilde{x})$ and $e_1(q) = \sum_{i=0}^{l-1}[\lambda_R(R/\widetilde{x}) - \lambda_R(R/((x^{i+1} : x^i))]$. This proves (1).

Now, if $n \leq l-1$, then $H_q(n) = \sum_{i=0}^{n-1} \lambda_R(R/((x^{i+1}) : x^i))$, and

$$\begin{aligned} P_q(n) - H_q(n) &= \sum_{i=n}^{l-1} \lambda_R(R/((x^{i+1}) : x^i)) + (n-l)\lambda_R(R/\widetilde{x}) \\ &= \sum_{i=n}^{l-1} \left(\lambda_R(R/((x^{i+1}) : x^i)) - \lambda_R(R/\widetilde{x})\right) \\ &= \sum_{i=n}^{\infty} \left(\lambda_R(R/((x^{i+1}) : x^i)) - \lambda_R(R/\widetilde{x})\right), \end{aligned}$$

where the last equality holds since $((x^{i+1}) : x^i) = \widetilde{x}$ for all $i \geq l$. This gives 2 (a).

Note that, for all i, we have $((x^{i+1}) : x^i) \subset \widetilde{x}$, so $\lambda_R(R/((x^{i+1}) : x^i)) \geq \lambda_R(R/\widetilde{x})$, and we have $P_q(n) - H_q(n) \geq 0$. In fact, if $n \geq l$, we have $P_q(n) - H_q(n) = 0$. This gives part 2 (b) of the proposition. □

This proposition gives us a formula for the postulation number, $n(q)$, of a parameter ideal q in a one-dimensional ring.

Corollary 2.3. *Let $(R, \mathfrak{m})$ be a one-dimensional local Noetherian ring and $q = (x)$ a parameter ideal. Then*

$$n(q) = \min\{i \mid ((x^{i+1}) : x^i) = ((x^{j+1}) : x^j) \text{ for all } j \geq i\} - 1.$$

Proof. Let $l = \min\{i \mid ((x^{i+1}) : x^i) = ((x^{j+1}) : x^j) \text{ for all } j \geq i\}$ and $\widetilde{x} = ((x^{l+1}) : x^l)$. Then, using part 2 (a) of Proposition 2.2, clearly $n(q) \leq l-1$. If $n(q) < l-1$, then we have $P_q(l) = H_q(l)$ and, using 2 (a) again, this gives $\lambda_R(R/((x^l) : x^{l-1})) = \lambda_R(R/\widetilde{x})$. But this contradicts the minimality of l. Thus, we must have $n(q) = l-1$. □

Corollary 2.4. *Let $(R, \mathfrak{m})$ be a one-dimensional local Noetherian ring and $q = (x)$ a parameter ideal. Then*

(1) *For $k \in \mathbf{Z}$, if $P_q(k) - H_q(k) = 0$, then $P_q(n) - H_q(n) = 0$ for all $n \geq k$, i.e., $k > n(q)$.*

(2) $\Delta^2(P_q(n) - H_q(n)) = \lambda_R(((x^{n+2}) : x^{n+1})/((x^{n+1}) : x^n))$ *for all n.*

Proof. For the first statement, suppose $P_q(k) - H_q(k) = 0$. Let $\widetilde{x}$ be defined as in Proposition 2.2. Then $P_q(k) - H_q(k) = \sum_{i=k}^{\infty}(\lambda_R(R/((x^{i+1}) : x^i)) - \lambda_R(R/\widetilde{x})) = 0$. Since $\lambda_R(R/((x^{i+1}) : x^i)) - \lambda_R(R/\widetilde{x}) \geq 0$ for all $i \geq 0$, we must have equality for each $i \geq k$. It follows that $P_q(n) - H_q(n) = 0$ for all $n \geq k$, i.e., $k > n(q)$.

For (2), note by Proposition 2.2,

$$\Delta(P_q(n) - H_q(n)) = \lambda_R(R/\widetilde{x}) - \lambda_R(R/((x^{n+1}) : x^n)).$$

So,

$$\begin{aligned}\Delta^2(P_q(n) - H_q(n)) &= \Delta(\Delta(P_q(n) - H_q(n))) \\ &= \Delta(\lambda_R(R/\widetilde{x}) - \lambda_R(R/((x^{n+1}) : x^n))) \\ &= -\lambda_R(R/((x^{n+2}) : x^{n+1})) + \lambda_R(R/((x^{n+1}) : x^n)) \\ &= \lambda_R(((x^{n+2}) : x^{n+1})/((x^{n+1}) : x^n)).\end{aligned}$$

In particular, this shows $\Delta(P_q(n) - H_q(n)) \leq 0$ and $\Delta^2(P_q(n) - H_q(n)) \geq 0$ for all n. □

3. The second Hilbert coefficient. When working with Hilbert functions, a common technique is to reduce by a superficial sequence to obtain a ring of smaller dimension. The following proposition due to Nagata guarantees that, when we do this, the Hilbert coefficients behave nicely.

Proposition 3.1 (cf. [**9**, 22.6]). *Let $(R, \mathfrak{m})$ be a Noetherian local ring, I an $\mathfrak{m}$-primary ideal and M a nonzero finitely generated R-module of dimension d. Suppose that $y \in I$ is superficial with respect to M. Then $\lambda_R(0 :_M y)$ is finite and*

$$P_{\overline{I},\overline{M}}(n) = P_{I,M}(n) - P_{I,M}(n-1) + \lambda_R(0 :_M y).$$

In particular, we have

$$e_i(\overline{I}, \overline{M}) = \begin{cases} e_i(I, M) & \text{for } i = 0, \dots, d-2, \\ e_{d-1}(I, M) + (-1)^{d-1}\lambda_R(0 :_M y) & \text{for } i = d-1. \end{cases}$$

If $x \in I\backslash I^2$ is a non-zero-divisor and x^* is a regular element of $gr_I(R)$, it can be easily shown that $n(I/(x)) = n(I) + 1$, so the postulation number also behaves nicely when we reduce via a superficial non-zero-divisor.

We begin with a formula for the last Hilbert coefficient.

Lemma 3.2. *Suppose* $(R, \mathfrak{m})$ *has dimension* d. *Let* I *be an* $\mathfrak{m}$-*primary ideal and* $y \in I$ *a superficial element. Let* $\overline{I} = I/(y)$, $H_{\bar{I}}(k) = \lambda_R(R/(I^k, y))$ *and* $P_{\bar{I}}(k)$ *denote the Hilbert Samuel polynomial for* $\overline{I}$. *Then, for* $l \gg 0$,

$$(-1)^d e_d(I) = \sum_{k=1}^{l} (H_{\bar{I}}(k) - P_{\bar{I}}(k)) - \sum_{k=1}^{l} \lambda_R((I^k : y)/I^{k-1}) + l\lambda_R(0 : y).$$

Furthermore, if y *is also a non-zero-divisor on* R, *we have*

$$(-1)^d e_d(I) = \sum_{k=1}^{\infty} (H_{\bar{I}}(k) - P_{\bar{I}}(k)) - \sum_{k=1}^{\infty} \lambda_R((I^k : y)/I^{k-1}).$$

Proof. For $k \in \mathbf{Z}$, consider the exact sequence:

$$0 \longrightarrow \frac{I^k : y}{I^{k-1}} \longrightarrow R/I^{k-1} \xrightarrow{\ y\ } R/I^k \longrightarrow R/(I^k, y) \longrightarrow 0.$$

From this, we see that $\lambda_R(R/(y, I^k)) = \lambda_R(R/I^k) - \lambda_R(R/I^{k-1}) + \lambda_R(I^k : y/I^{k-1})$. Subtracting $P_{\bar{I}}(k)$ and summing both sides, we get, for $l \gg 0$,

$$\sum_{k=1}^{l} \big(\lambda(R/(y, I^k) - P_{\bar{I}}(k)\big) = \sum_{k=1}^{l} \Bigg(\lambda(R/I^k) - \lambda(R/I^{k-1}) + \lambda\bigg(\frac{I^k : y}{I^{k-1}}\bigg) - P_{\bar{I}}(k)\Bigg).$$

Thus,

$$\sum_{k=1}^{l} (H_{\bar{I}}(k) - P_{\bar{I}}(k)) = \lambda(R/I^l) - \sum_{k=1}^{l} \sum_{i=0}^{d-1} (-1)^i \binom{k+d-2-i}{d-1-i} e_i(\overline{I}) + \sum_{k=1}^{l} \lambda\bigg(\frac{I^k : y}{I^{k-1}}\bigg)$$

$$= \sum_{i=0}^{d}(-1)^i \binom{l+d-1-i}{d-i} e_i(I)$$
$$- \sum_{i=0}^{d-1}(-1)^i \binom{l+d-1-i}{d-i} e_i(\overline{I})$$
$$+ \sum_{k=1}^{l} \lambda\left(\frac{I^k : y}{I^{k-1}}\right)$$

where $\lambda(-) = \lambda_R(-)$.

By Proposition 3.1, we have $e_i(I) = e_i(\overline{I})$ for $i = 0, \dots, d-2$ and $e_{d-1}(I) = e_{d-1}(\overline{I}) - (-1)^{d-1}\lambda_R(0 : y)$. Hence,

$$\sum_{k=1}^{l}(H_{\bar{I}}(k) - P_{\bar{I}}(k))$$
$$= -l\lambda_R(0 : y) + (-1)^d e_d(I) + \sum_{k=1}^{l} \lambda_R((I^k : y)/I^{k-1}).$$

Rearranging, we get

$$(-1)^d e_d(I) = \sum_{k=1}^{l} (H_{\bar{I}}(k) - P_{\bar{I}}(k)) - \sum_{k=1}^{l} \lambda_R((I^k : y)/I^{k-1}) + l\lambda_R(0 : y),$$

and if y is also a non-zero-divisor on R, we have

$$(-1)^d e_d(I) = \sum_{k=1}^{\infty} (H_{\bar{I}}(k) - P_{\bar{I}}(k)) - \sum_{k=1}^{\infty} \lambda_R((I^k : y)/I^{k-1})$$

since, for $k \gg 0$, $H_{\bar{I}}(k) - P_{\bar{I}}(k) = 0$ and $\lambda_R((I^k : y)/I^{k-1}) = 0$. □

We define $\operatorname{grade} gr_I(R)_+$ to be the maximal length of a regular sequence for $gr_I(R)$ contained in $gr_I(R)_+$. Then $\operatorname{grade} gr_I(R)_+ = \operatorname{depth} gr_I(R)$. For $x \in I^n \backslash I^{n+1}$, let x^* denote its image in $I^n/I^{n+1} \subseteq gr_I(R)$. The grade of the associated graded ring also behaves nicely with respect to superficial sequences as evidenced by the following lemmas. Lemma 3.4 is also known as "Sally's Machine."

Lemma 3.3 [**5**, Lemma 2.1]. *Let $x_1, \dots, x_k$ be a superficial sequence for I. If* $\operatorname{grade} gr_I(R)_+ \geq k$, *then $x_1^*, \dots, x_k^*$ is a regular sequence.*

Lemma 3.4 [**5**, Lemma 2.2]. *Suppose $y_1, \dots, y_k$ is a superficial sequence for an ideal I. Let $\overline{R}$ and $\overline{I}$ denote $R/(y_1, \dots, y_k)$ and $I/(y_1, \dots, y_k)$, respectively. If* $\operatorname{grade} gr_{\bar{I}}(\overline{R})_+ \geq 1$, *then* $\operatorname{grade} gr_I(R)_+ \geq k+1$.

We now consider the second Hilbert coefficient, $e_2(q)$ for a parameter ideal q.

Theorem 3.5. *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring of dimension $d \geq 2$ and* $\operatorname{depth} R \geq d-1$. *Let $q \subseteq R$ be a parameter ideal. Then*

(1) $e_2(q) \leq 0$.

(2) $e_2(q) = 0$ *if and only if* $n(q) < 2-d$ *and* $\operatorname{depth} gr_q(R) \geq d-1$.

(3) $e_2(q) = 0$ *implies* $e_3(q) = \cdots = e_d(q) = 0$.

Proof. We may assume that R has an infinite residue field by passing to $R[x]_{\mathfrak{m}R[x]}$ if necessary. We will proceed by induction on $d = \dim R$. First suppose $d = 2$. Let $q = (y, x)$ where $y \in q \backslash \mathfrak{m}q$ is a superficial non-zero-divisor for R. Let $(\bar{\cdot})$ denote working modulo (y). Now, $\overline{q}$ is a parameter ideal in the one-dimensional ring $\overline{R}$, so by Proposition 2.2, $H_{\bar{q}}(k) - P_{\bar{q}}(k) \leq 0$ for all $k \geq 0$. In particular, Lemma 3.2 gives

$$\begin{aligned} e_2(q) &= \sum_{k=1}^{\infty}(H_{\bar{q}}(k) - P_{\bar{q}}(k)) - \sum_{k=1}^{\infty} \lambda_R((q^k : y)/q^{k-1}) \\ &\leq 0. \end{aligned}$$

Note that, if the left-hand side of the equation above is zero, we must have that $\lambda_R((q^k : y)/q^{k-1}) = 0$ and $P_{\bar{q}}(k) = H_{\bar{q}}(k)$ for all $k \geq 1$. In particular, the condition $\lambda_R((q^k : y)/q^{k-1}) = 0$ for all $k \geq 1$ implies that y^* is a non-zero-divisor in $gr_q(R)$, so $\operatorname{depth} gr_q(R) \geq 1$. Now, since y^* is a non-zero-divisor, $n(\bar{q}) = n(q) + 1$, i.e., $n(q) < 0$. This proves the corollary when $d = 2$.

Now, if $\dim R > 2$, then let $y_1, \dots, y_{d-2} \in q \backslash \mathfrak{m}q$ be a superficial sequence of non-zero-divisors for R. Then $\overline{q} = q/(y_1, \dots, y_{d-2})$ is a parameter ideal in the two-dimensional ring $\overline{R} = R/(y_1, \ldots, y_{d-2})$ which has depth $\overline{R} \geq 1$. Hence, by induction, we have $e_2(q) = e_2(\overline{q}) \leq 0$.

For (2), first suppose $e_2(q) = 0$. Then, by induction, grade $gr_{\overline{q}}(\overline{R})_+ \geq 1$. By Lemma 3.4, this implies grade $gr_q(R)_+ \geq d - 2 + 1 = d - 1$. Finally, this gives $y_1^*, \ldots, y_{d-2}^*$ is a regular sequence by Lemma 3.3. Hence, $n(\overline{q}) < 0$ if and only if $n(q) < 2 - d$. This gives the forward implications for (2).

For the backward implication of (2), suppose $n(q) < 2 - d$ (i.e., $H_q(n) = P_q(n)$ for all $n \geq 2 - d$) and grade $gr_q(R)_+ \geq d - 1$. Then $P_q(n) = 0$ for all $2 - d \leq n \leq 0$. Plugging the values $0, -1, -2, \ldots, 2 - d$ successively into $P_q(n)$, one can see that we get $e_d(q) = e_{d-1}(q) = \cdots = e_2(q) = 0$.

Finally, (3) follows from the proof of (2). □

Corollary 3.6. *Suppose* $(R, \mathfrak{m})$ *is a local Noetherian ring of dimension* $d \geq 2$ *and* depth $R \geq d - 1$. *Then, for any parameter ideal* $q \subseteq R$, *we have*

$$\lambda_R(R/q) \leq e_0(q) - e_1(q).$$

Proof. As before, we may assume that $R/\mathfrak{m}$ is infinite. From Proposition 2.2, we have that $H_{\overline{q}}(n) \leq P_{\overline{q}}(n)$ for all $n \geq 1$, where $\overline{q} = q/(y_1, \ldots, y_{d-1})$ for $y_1, \ldots, y_{d-1} \in q$, a superficial sequence which is part of a minimal generating set for q. Note that we may also choose $y_1, \ldots, y_{d-1}$ to be a regular sequence as depth $R \geq d - 1$. Now, letting $n = 1$ and using the fact that $e_i(q) = e_i(\overline{q})$ for $i = 0, 1$ since $y_1, \ldots, y_{d-1}$ is a superficial and regular sequence, the result follows. □

The assumption that depth $R \geq d - 1$ is necessary in Theorem 3.5, as evidenced by the following example. We use Macaulay2 [**4**] to compute the example.

Example 3.7. Let $R = k[x, y, z, u, v, w]/I$ where I is the intersection of ideals $I = (x + y, z - u, w) \cap (z, u - v, y) \cap (x, u, w)$ and $q = (u - y, z + w, x - v)$. Then R is an unmixed ring of dimension three and depth one and q is a parameter ideal with

$$P_q(n) = 3\binom{n+2}{3} + 2\binom{n+1}{2} + n.$$

In particular, $e_2(q) = 1 > 0$.

Note that, in the example above, one could mod out the ring R by a superficial non-zero-divisor in $q \backslash \mathfrak{m}q$ to obtain an example of a two-dimensional ring $\overline{R}$ of depth zero with parameter ideal $\overline{q}$ satisfying $e_2(\overline{q}) = 1 > 0$.

The upper bound for $e_2(q)$ in Theorem 3.5 can be achieved even if R is not Cohen-Macaulay. We also provide an example below with a negative second Hilbert coefficient. In both examples, we use the software system Macaulay2 [**4**] to compute the Hilbert-Samuel functions.

Example 3.8. Let $R = k[[x^5, xy^4, x^4y, y^5]] \cong k[[t_1, t_2, t_3, t_4]]/J$ where J is the ideal $J = (t_2t_3 - t_1t_4, t_2^4 - t_3t_4^3, t_1t_2^3 - t_3^2t_4^2, t_1^2t_2^2 - t_3^3t_4, t_1^3t_2 - t_3^4, t_3^5 - t_1^4t_4)$. Then R is a two-dimensional complete domain with depth one. The parameter ideal $q = (x^5, y^5)$ has Hilbert-Samuel polynomial

$$P_q(n) = 5\binom{n+1}{2} + 2n,$$

so $e_2(q) = 0$.

Example 3.9. Let $R = k[x, y, z, t]/((x^2, z^4) \cap (x - y, z + t))$. Then R is a two-dimensional unmixed ring with depth one. The ideal $q = (x+t+y, z-y)$ is a parameter ideal with Hilbert-Samuel polynomial

$$P_q(n) = 9\binom{n+1}{2} + 2\binom{n}{1} - 1.$$

Hence, $e_2(q) = -1 < 0$. In this example, we have that $n(q) = 0$, that is, $P_q(0) \neq H_q(0)$ and $P_q(n) = H_q(n)$ for all $n \geq 1$. However, we do have that depth $gr_q(R) \geq 1$.

4. The higher Hilbert coefficients. In our first theorem of this section we use techniques similar to those of Marley to obtain a result reminiscent of Theorem 1 in [**7**].

Theorem 4.1. *Let* $(R, \mathfrak{m})$ *be a local Noetherian ring of dimension* d, *and suppose* q *is a parameter ideal for* R *satisfying* $\operatorname{depth} gr_q(R) \geq d-1$. *Then, for* $0 \leq i \leq d+1$ *and* $n \in \mathbf{Z}$,

$$(-1)^i \Delta^{d+1-i}(P_q(n) - H_q(n)) \geq 0.$$

Proof. We first note that it is enough to prove the result when $i = 0$. Indeed, suppose $g : \mathbf{Z} \to \mathbf{Z}$ satisfies $g(n) = 0$ for all n sufficiently large and $\Delta(g(n)) \geq 0$ for all n. Then we claim $g(n) \leq 0$ for all n. Let N be such that $g(n) = 0$ for all $n \geq N$. Then $\Delta(g(N-1)) = g(N) - g(N-1) \geq 0$ implies $g(N-1) \leq 0$. Inductively, one can show that $g(j) \leq 0$ for all j. In particular, if we set $g(n) = P_q(n) - H_q(n)$, and assume $(-1)^i \Delta^{d+1-i}(g(n)) \geq 0$ for all n, then $(-1)^i \Delta^{d+1-i-1}(g(n)) \leq 0$ gives the theorem for $i+1$. Hence, it is enough to prove

$$\Delta^{d+1}(P_q(n) - H_q(n)) \geq 0 \quad \text{for all } n.$$

We will use induction on the dimension d. Note the case $d = 1$ is proved in Corollary 2.4. Suppose $d > 1$. Let $a \in q \backslash \mathfrak{m}q$ be such that a^* is a $gr_q(R)$-regular element. Let $\overline{q} = q/(a)$ and $\overline{R} = R/(a)$. Then note that $\operatorname{depth}_{\overline{q}}(\overline{R}) \geq d-2$ and $\overline{q}$ is a parameter ideal for the $(d-1)$-dimensional ring $\overline{R}$. So, by induction,

$$\Delta^d(P_{\bar{q}}(n) - H_{\bar{q}}(n)) \geq 0 \quad \text{for all } n.$$

Now, as a^* is a non-zero-divisor in $gr_q(R)$, we have $H_{\bar{q}}(n) = H_q(n) - H_q(n-1)$ for all n. Similarly, $P_{\bar{q}}(n) = P_q(n) - P_q(n-1)$. Hence,

$$\begin{aligned}\Delta^{d+1}(P_q(n) - H_q(n)) &= \Delta^d(\Delta(P_q(n) - H_q(n)) \\ &= \Delta^d(P_{\bar{q}}(n+1) - H_{\bar{q}}(n+1)) \\ &\geq 0 \quad \text{for all } n.\end{aligned}$$

Thus,

$$\Delta^{d+1}(P_q(n) - H_q(n)) \geq 0 \quad \text{for all } n. \qquad \square$$

Corollary 4.2. *Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension d, and suppose q is a parameter ideal for R satisfying* $\operatorname{depth} gr_q(R) \geq d-1$. *Suppose $P_q(k) - H_q(k) = 0$ for some k. Then $P_q(n) - H_q(n) = 0$ for all $n \geq k$, i.e., $k > n(q)$.*

Proof. Letting $i = d$ in Theorem 4.1, we have $(-1)^d\Delta(P_q(n) - H_q(n)) \geq 0$ for all n. This gives $(-1)^d(P_q(n+1) - H_q(n+1)) \geq (-1)^d(P_q(n) - H_q(n))$ for all n. In particular, we have

$$0 = (-1)^d(P_q(k) - H_q(k)) \leq (-1)^d(P_q(n) - H_q(n)) \leq 0 \quad \text{for all } n \geq k,$$

where the last inequality holds because $P_q(N) - H_q(N) = 0$ for $N \gg 0$. Thus, $P_q(n) = H_q(n)$ for all $n \geq k$. $\square$

Remark 4.3. Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension d, and suppose q is a parameter ideal for R. For $0 \leq i \leq d-1$, we have the following:

(1) If $n(q) < i - d$, then $e_j(q) = 0$ for $j \geq i$.

(2) If $\operatorname{depth} gr_q(R) \geq d - 1$, the converse to (1) holds.

Proof. Note that (1) follows by using the fact that $P_q(j) = 0$ for $i - d < j < 0$. For (2), suppose $\operatorname{depth} gr_q(R) \geq d-1$ and $e_i(q) = 0$ for $j \geq i$. Then $P_q(i-d) = 0 = H_q(i-d)$ and, by Corollary 4.2, $n(q) < i - d$. $\square$

Question 4.4. Does the converse to part (1) of Remark 4.3 above hold in general?

Corollary 4.5. *Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension d, and suppose that q is a parameter ideal for R satisfying* $\operatorname{depth} gr_q(R) \geq d - 1$. *Then, for $1 \leq i \leq d$,*

(1) $e_i(q) \leq 0$.

(2) $(-1)^{j+1}(e_0(q) - e_1(q) + \cdots + (-1)^j e_j(q) - \lambda_R(R/q)) \geq 0$ *for* $j = 1, \ldots, d$.

Proof. Note that it is enough to prove (1) in the case $i = d$, as we can then use reduction by a superficial sequence to obtain $e_i(q) \leq 0$ for $i = 1, \ldots, d-1$. Letting $i = d+1$ in Theorem 4.1, we have

$$(-1)^{d+1}(P_q(n) - H_q(n)) \geq 0 \quad \text{for all } n. \tag{4.1}$$

If $n = 0$, $(-1)^{d+1}((-1)^d e_d(q) - H_q(0)) \geq 0$ implies $-e_d(q) \geq 0$; that is, $e_d(q) \leq 0$.

For (2), we will first prove the case $j = \dim R = d$. Indeed, letting $n = 1$ in equation (4.1), we see that

$$(-1)^{d+1}(e_0(q) - e_1(q) + \cdots + (-1)^d e_d - \lambda_R(R/q)) \geq 0.$$

Now, let $a_1, \ldots a_{d-j} \in q \backslash q^2$ be part of a minimal generating set for q such that $a_1^*, \ldots, a_{d-j}^*$ is a $gr_q(R)$-regular sequence. Then, setting $\overline{R} = R/(a_1, \ldots a_{d-j})$ and $\overline{q} = q/(a_1, \ldots a_{d-j})$, we have $\overline{q}$ is a parameter ideal in the j-dimensional ring $\overline{R}$, and $\operatorname{depth} gr_{\bar{q}}(\overline{R}) \geq j-1$. Finally, $\lambda_R(R/q) = \lambda_{\overline{R}}(\overline{R}/\overline{q})$ and since $a_1, \ldots a_{d-j}$ defines a superficial regular sequence in R, we have $e_i(\overline{q}) = e_i(q)$ for all $i = 0, \ldots, j$. It follows that

$$(-1)^{j+1}(e_0(q) - e_1(q) + \cdots + (-1)^j e_j(q) - \lambda_R(R/q)) \geq 0. \qquad \square$$

Corollary 4.6. *Let* $(R, \mathfrak{m})$ *be a local Noetherian ring of dimension* d*, and suppose that* q *is a parameter ideal for* R *satisfying* $\operatorname{depth} gr_q(R) \geq d-1$*. Suppose* $e_i(q) = 0$ *for some* $1 \leq i \leq d-1$*. Then* $e_j(q) = 0$ *for* $i \leq j \leq d$*.*

Proof. Note that it is enough to prove that $e_{i+1}(q) = 0$. Reducing by a superficial sequence if necessary, we may assume that $i = d-1$. Since $e_0(q) > 0$, we must have that $d > 1$, so by assumption, $\operatorname{depth} gr_q(R) > 0$. Let $a \in q$ be such that $a^* \in gr_q(R)$ is a non-zero-divisor. Then $e_{d-1}(\overline{q}) = e_{d-1}(q) = 0$ implies that $P_{\bar{q}}(0) = 0 = H_{\bar{q}}(0)$. Now, by Corollary 4.2, $n(\overline{q}) \leq -1$. As $n(\overline{q}) = n(q) + 1$, this gives $n(q) \leq -2$, and in particular, $(-1)^d e_d(q) = P_q(0) = H_q(0) = 0$. $\square$

Acknowledgments. Much of the work in this paper was done in preparation for the author's dissertation and would not have been possible without the guidance, encouragement, and input from my advisor, Tom Marley. The author is also grateful to Brian Harbourne and Luchezar Avramov for their feedback on this research.

REFERENCES

1. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, Cambridge, 1993.

2. L. Ghezzi, S. Goto, J. Hong, K. Ozeki, T.T. Phuong and W.V. Vasconcelos, *Negativity conjecture for the first Hilbert coefficient*, preprint.

3. L. Ghezzi, J. Hong and W.V. Vasconcelos, *The signature of the Chern coefficients of local rings*, Math. Res. Lett. **16** (2009), 279–289.

4. D. Grayson and M. Stillman, *Macaulay*2, *A software system for research in algebraic geometry*, available at http://www.math.uiuc.edu/Macaulay2.

5. S. Huckaba and T. Marley, *Hilbert coefficients and the depths of associated graded rings*, J. Lond. Math. Soc. **56** (1997), 64–76.

6. C. Huneke and I. Swanson, *Integral closure of ideals, rings, and modules*, Lond. Math. Soc. Lect. Note Series **336**, Cambridge University Press, Cambridge, 2006.

7. T. Marley, *The coefficients of the Hilbert polynomial and the reduction number of an ideal*, J. Lond. Math. Soc. **40** (1989), 1–8.

8. H. Matsumura. *Commutative ring theory*, Cambr. Stud. Adv. Math. **8**, Cambridge University Press, Cambridge, 1986.

9. M. Nagata, *Local rings*, Interscience, New York, 1962.

Department of Computer Science, Mathematics, and Physics, Missouri Western State University, 4525 Downs Drive, St. Joseph, MO 64068
Email address: lmccune@missouriwestern.edu