

THE JOIN-MEET IDEAL OF A FINITE LATTICE

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Dedicated to Professor Jürgen Herzog on the occasion of his 70th birthday

ABSTRACT. Radical binomial ideals associated with finite lattices are studied. Gröbner basis theory turns out to be an efficient tool in this investigation.

1. Introduction. Let L be a finite lattice and $K[L]$ the polynomial ring over a field K whose variables are the elements of L . Let I_L be the *join-meet ideal* of L , that is, the ideal of $K[L]$ which is generated by all the binomials of the form $f = ab - (a \wedge b)(a \vee b)$, where $a, b \in L$ are incomparable elements. Of course one may ask whether algebraic properties of I_L are related to the combinatorial properties of L . I_L is a prime ideal if and only if L is distributive as was shown in [9], and if L is distributive the Gröbner bases of I_L with respect to various monomial orders have been studied; see, for instance, [1, 8, 9, 13]. In the same hypothesis on L , the toric ring $K[L]/I_L$ is well understood; see [6, 9–11].

Almost nothing is known about the join-meet ideal I_L when L is not distributive. In the present paper we focus on the join-meet ideals of modular and non-distributive lattices. For basic properties of lattices, like distributivity and modularity, we refer the reader to the well-known monographs [2, 14].

It was conjectured in [8] that, given a modular lattice L , for any monomial order $<$ on $K[L]$, the initial ideal $\text{in}_<(I_L)$ is not squarefree, unless L is distributive. We give a proof of this conjecture in Section 1. This result shows, in particular, that for deciding whether a join-meet ideal I_L of a modular and non-distributive lattice L is radical, one

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cannot use the known statement that a polynomial ideal is radical if it has a squarefree initial ideal. Moreover, easy examples show that, even if the lattice L is rather close to a distributive lattice, the ideal I_L might not be radical; see Example 3.1. A general characterization of radical join-meet ideals associated with modular non-distributive lattices seems to be difficult. However, in Section 3, we find a class of modular non-distributive lattices L whose join-meet ideal I_L is radical. To prove this property, we intensively use the Gröbner basis theory.

For radical join-meet ideals, in Section 2, we describe the minimal prime ideals. This description is used later, in Section 4, to obtain a complete characterization of the minimal primes of the radical join-meet ideals studied in Section 3.

1. The squarefree conjecture. Let L be a finite lattice and $K[L]$ the polynomial ring over a field K whose variables are the elements of L . A binomial of $K[L]$ of the form $f = ab - (a \wedge b)(a \vee b)$, where $a, b \in L$ are incomparable, is called a *basic binomial*. In some recent papers, the basic binomials are called Hibi relations.

Definition 1.1. The *join-meet ideal* of L is the ideal of $K[L]$ generated by the basic binomials, that is,

$$I_L = (ab - (a \wedge b)(a \vee b) : a, b \in L, a, b \text{ incomparable}) \subset K[L].$$

The join-meet ideal of a lattice was introduced in [9]. For fundamental notions on lattices we refer to [2, 14].

The main result of this section answers positively a conjecture made in [8]. We first need a preparatory result on modular and non-distributive lattices which might be known, but we include its proof since we could not find any reference.

Lemma 1.2. *Let L be a modular non-distributive lattice. Then L has a diamond sublattice L' such that $\text{rank max } L' - \text{rank min } L' = 2$.*

Proof. Let δ be a diamond of L labeled as in Figure 1 (i) of minimal rank, that is, $\text{rank } e - \text{rank } a$ is minimal.

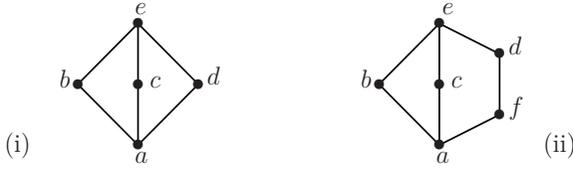


FIGURE 1.

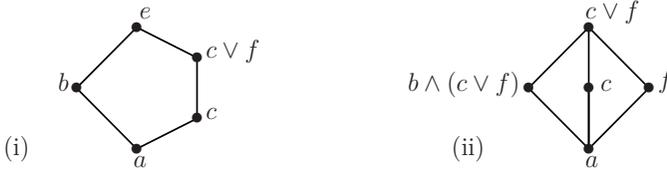


FIGURE 2.

We show that $\text{rank } a - \text{rank } e = 2$. Let us assume that $\text{rank } e > \text{rank } a + 2$. By duality, we may assume, for instance, that $\text{rank } d > \text{rank } a + 1$, that is, there exists an $f \in L$ such that $a < f < d$. Then we get the lattice displayed in Figure 1 (ii) where $c \wedge f = c \wedge d = a$ and $c \vee f \leq c \vee d = e$. If $c \vee f = e$, then L has a pentagon sublattice (with the elements a, c, f, d, e), which is impossible since L is modular. Therefore, we must have $c \vee f < e$.

We now look at the lattice with the elements $a, b, c, c \vee f$, and e . Here we have $b \vee (c \vee f) = (b \vee c) \vee f = e \vee f = e$ and $b \wedge (c \vee f) \geq a$. If $b \wedge (c \vee f) = a$, we again get a pentagon sublattice of L ; see Figure 2 (i). Since L is modular, we must have $b \wedge (c \vee f) > a$. We look at the lattice with elements $a, c, f, b \wedge (c \vee f)$ and $c \vee f$, see Figure 2 (ii). The following relations hold:

$$c \wedge (b \wedge (c \vee f)) = (c \wedge b) \wedge (c \vee f) = a,$$

and

$$c \vee (b \wedge (c \vee f)) = (c \vee b) \wedge (c \vee f) = e \wedge (c \vee f) = c \vee f,$$

the first equality in the latter relation being true by modularity. Moreover, we have

$$f \wedge (b \wedge (c \vee f)) = (f \wedge b) \wedge (c \vee f) = a \wedge (c \vee f) = a,$$

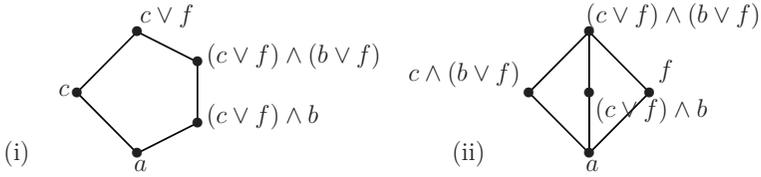


FIGURE 3.

and

$$f \vee (b \wedge (c \vee f)) = (f \vee b) \wedge (c \vee f),$$

again by modularity, and, thus,

$$f \vee (b \wedge (c \vee f)) \leq c \vee f.$$

If $f \vee (b \wedge (c \vee f)) = c \vee f$, then we get a diamond sublattice of L as in Figure 2 (ii) of smaller rank than δ , which is impossible by our assumption. Hence, we must have

$$(f \vee b) \wedge (c \vee f) < c \vee f.$$

Let us now consider the lattice with the elements $a, c, (c \vee f) \wedge b, f \vee (b \wedge (c \vee f)) = (c \vee f) \wedge (b \vee f)$ and $c \vee f$. The following equalities hold:

$$((c \vee f) \wedge b) \wedge c = (c \vee f) \wedge (b \wedge c) = a,$$

and, by modularity,

$$c \vee (b \wedge (c \vee f)) = (c \vee b) \wedge (c \vee f) = c \vee f.$$

Next, we have:

$$c \vee (f \vee (b \wedge (c \vee f))) = (c \vee f) \vee (b \wedge (c \vee f)) = c \vee f.$$

Therefore, if $c \wedge ((c \vee f) \wedge (b \vee f)) = c \wedge (b \vee f) = a$, then L has a pentagon sublattice; see Figure 3 (i). Hence, we must have

$$c \wedge (b \vee f) > a.$$

Finally, we look at the lattice with the elements $a, c \wedge (b \vee f), (c \vee f) \wedge b, f$ and $(c \vee f) \wedge (b \vee f)$. The following equalities hold:

$$f \wedge (c \wedge (b \vee f)) = (c \wedge f) \wedge (b \vee f) = a \wedge (b \vee f) = a,$$

and

$$f \vee (c \wedge (b \vee f)) = (f \vee c) \wedge (b \vee f) \text{ (by modularity).}$$

Next,

$$f \wedge (b \wedge (c \vee f)) = (f \wedge b) \wedge (c \vee f) = a \wedge (c \vee f) = a,$$

and

$$f \vee (b \wedge (c \vee f)) = (f \vee b) \wedge (c \vee f) \text{ (by modularity).}$$

We also have:

$$(c \wedge (b \vee f)) \wedge ((c \vee f) \wedge b) = (b \wedge c) \wedge (b \vee f) \wedge (c \vee f) = a$$

and, by applying modularity,

$$\begin{aligned} (c \wedge (b \vee f)) \vee ((c \vee f) \wedge b) &= ((c \wedge (b \vee f)) \vee b) \wedge (c \vee f) \\ &= ((b \vee c) \wedge (b \vee f)) \wedge (c \vee f) \\ &= e \wedge (b \vee f) \wedge (c \vee f) \\ &= (b \vee f) \wedge (c \vee f). \end{aligned}$$

Consequently, we have got another diamond sublattice of L (see Figure 3 (ii)) with a smaller rank than δ , again a contradiction. \square

In the proof of the next theorem we use some arguments which are taken from the proof of [8, Theorem 1.1], but we include them for the convenience of the reader.

Theorem 1.3. *Let L be a modular non-distributive lattice. Then, for any monomial order $<$ on $K[L]$, the initial ideal $\text{in}_{<}(I_L)$ is not squarefree.*

Proof. By Lemma 1.2, L has a sublattice L' with $a = \min L'$, $e = \max L'$ such that $\text{rank } e - \text{rank } a = 2$. Let $b_1, b_2, \dots, b_k \in L$,

$k \geq 3$, be the elements of L , distinct from a, e , such that, for any $1 \leq i < j \leq n$, $b_i \vee b_j = e$ and $b_i \wedge b_j = a$. Therefore, we have the following relations in I_L : $b_i b_j - ae$ for $1 \leq i < j \leq k$ since, obviously, b_i and b_j are incomparable for any $i \neq j$.

Let $<$ be an arbitrary monomial order on $K[L]$. We may assume that, with respect to this order, we have $b_1 > \dots > b_k$. We are going to show that $\text{in}_<(I_L)$ is not squarefree. We have to analyze the following two cases.

Case 1. Assume that $ae < b_i b_j$ for any $1 \leq i < j \leq k$. Let $b = b_k$, and consider the binomial $f = ab^2e - a^2e^2$ which, by the proof of [8, Theorem 1.1], belongs to I_L . Let us assume that $\text{in}_<(I_L)$ is squarefree. Then, since $f \in I_L$, we must have $abe \in \text{in}_<(I_L)$; hence, following the arguments of the proof of [8, Theorem 1.1], there exists a binomial $g = abe - u \in I_L$ where $u = \ell mn$ with $\ell, m, n \in L$, all of them in the interval $[a, e]$ of L , and, in addition, with $\text{in}_<(g) = abe$. Also, from the arguments of the cited proof, it follows that at least two of the variables ℓ, m, n are distinct. Indeed, let

$$(1.1) \quad g = \sum_{i=1}^N x_i(v_i - w_i),$$

where each x_i is a variable and $v_i - w_i$ is a basic binomial of I_L such that $x_1 v_1 = abe$, $x_i w_i = x_{i+1} v_{i+1}$ for $1 \leq i < N$, and $x_N w_N = u$. Then each variable that appears in the binomial $x_i(v_i - w_i)$ must belong to the interval $[a, e]$ of L . This is true since, for any basic binomial $v - w$, one has $\text{supp}(v) \subset [a, e]$ if and only if $\text{supp}(w) \subset [a, e]$. In particular, $x_N w_N = u$ is of the form $u = \ell mn$ with $\ell, m, n \in [a, e]$ and, by (1.1), at least two of ℓ, m, n are distinct. Moreover, by (1.1), it also follows that $\text{rank } a + \text{rank } b + \text{rank } e = \text{rank } \ell + \text{rank } m + \text{rank } n$. Since in L' , we have $\text{rank } e - \text{rank } a = 2$ and it follows that

$$(1.2) \quad \text{rank } \ell + \text{rank } m + \text{rank } n = 3\text{rank } a + 3.$$

Of course, we may assume that $\text{rank } \ell \geq \text{rank } m \geq \text{rank } n$. Let us suppose that $\text{rank } n > \text{rank } a$. Then, by using equation (1.2), we obtain $\text{rank } \ell = \text{rank } m = \text{rank } n = \text{rank } a + 1$; hence, $\ell, m, n \in \{b_1, b_2, \dots, b_k\}$. It follows that $g = abe - b_i b_j b_p$ for some $i, j, p \in \{1, 2, \dots, k\}$ with at least two of them distinct. Let us assume that $i \neq j$. Then,

since $ae < b_i b_j$ and $b \leq b_p$, we get a contradiction to the fact that $\text{in}_<(g) = abe$.

Now let $\text{rank } n = \text{rank } a$. This implies that $\text{rank } \ell + \text{rank } m = 2\text{rank } a + 3$, which leads to the conclusion that $\text{rank } \ell = \text{rank } a + 2$ and $\text{rank } m = \text{rank } a + 1$. Therefore, we get $n = a, \ell = e$ and $m = b_i$ for some $1 \leq i \leq k$. We then have $g = abe - ab_i e$, which is impossible since obviously $abe \leq ab_i e$ by the choice of b .

Hence, in Case 1, $\text{in}_<(I_L)$ is not squarefree.

Case 2. There exist $1 \leq i < j \leq k$ such that $ae > b_i b_j$. Let bd be the smallest monomial among all the monomials $b_i b_j, 1 \leq i < j \leq k$. In particular, it follows that $ae > bd$. We first claim that $b^2 d - bd^2 \in I_L$. Indeed, one may easily check the following identity:

$$b^2 d - bd^2 = (b - d)(bd - ae) - b(cd - ae) + d(bc - ae),$$

where c is an arbitrary variable in $\{b_1, \dots, b_k\} \setminus \{b, d\}$. Let us assume that $\text{in}_<(I_L)$ is squarefree. Then we have $bd \in \text{in}_<(I_L)$. This implies that there exists a binomial $g = bd - \ell m \in I_L$ with $bd \in \text{in}_<(I_L)$. Since $ae > bd$, we cannot have $\ell m = ae$. Therefore, $g = bd - b_i b_j$ for some $1 \leq i < j \leq n$, which is again impossible by our choice of the monomial bd . \square

2. Radical join-meet ideals of finite lattices. In this section we describe the associated primes of a radical join-meet ideal of a finite lattice.

Proposition 2.1. *Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K , and let $I \subset S$ be a binomial ideal, that is, an ideal which is generated by differences of two monomials. If I is a radical ideal, then:*

- (a) $I : (\prod_{i=1}^n x_i)^\infty = I : \prod_{i=1}^n x_i$.
- (b) $I : \prod_{i=1}^n x_i$ is a prime ideal.

Proof. (a) Let $\text{Min}^*(I)$ be the set of all prime ideals of I which contain no variable. Then

$$\begin{aligned} I : \prod_{i=1}^n x_i &= \bigcap_{P \in \text{Min}(I)} \left(P : \prod_{i=1}^n x_i \right) = \bigcap_{P \in \text{Min}^*(I)} P \\ &= \bigcap_{P \in \text{Min}(I)} \left(P : \left(\prod_{i=1}^n x_i \right)^\infty \right) = I : \left(\prod_{i=1}^n x_i \right)^\infty. \end{aligned}$$

(b) By [3, 12], $I : \prod_{i=1}^n x_i$ is a lattice ideal, let us say $I_{\mathcal{L}}$ where $\mathcal{L} \subset \mathbf{Z}^n$ is a lattice. By [3, Theorem 2.1], it is enough to show that \mathcal{L} is saturated, in other words, if $\mathbf{x}^{m\mathbf{a}} - \mathbf{x}^{m\mathbf{b}} \in I : \prod_{i=1}^n x_i$ for some positive integer m , then $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \in I : \prod_{i=1}^n x_i$.

The proof depends on the characteristic of the field. Let us first assume that $\text{char } K = 0$. Since, by the proof of (a), we have $I : \prod_{i=1}^n x_i = \bigcap_{P \in \text{Min}^*(I)} P$, we get $\mathbf{x}^{m\mathbf{a}} - \mathbf{x}^{m\mathbf{b}} \in P$ for any prime ideal $P \in \text{Min}^*(I)$. Since P does not contain any variable, it follows that the polynomial $g = \mathbf{x}^{(m-1)\mathbf{a}} + \dots + \mathbf{x}^{(m-1)\mathbf{b}} \notin P$ since $g(1, \dots, 1) = m \neq 0$; hence, $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \in P$ for any $P \in \text{Min}^*(I)$. Therefore, we obtain $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \in I : \prod_{i=1}^n x_i$.

A similar proof works in positive characteristic. Indeed, let $p > 0$ be the characteristic of the field, and let $m = p^t q$ for some non-negative integer t and some positive integer q such that $(p, q) = 1$. Then

$$\mathbf{x}^{m\mathbf{a}} - \mathbf{x}^{m\mathbf{b}} = (\mathbf{x}^{q\mathbf{a}} - \mathbf{x}^{q\mathbf{b}})^{p^t} = (\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}})^{p^t} (\mathbf{x}^{(q-1)\mathbf{a}} + \dots + \mathbf{x}^{(q-1)\mathbf{b}})^{p^t} \in P$$

for all $P \in \text{Min}^*(I)$. Let $h = (\mathbf{x}^{(q-1)\mathbf{a}} + \dots + \mathbf{x}^{(q-1)\mathbf{b}})^{p^t} = \mathbf{x}^{(q-1)\mathbf{a}p^t} + \dots + \mathbf{x}^{(q-1)\mathbf{b}p^t}$. Then $h(1, \dots, 1) = q \neq 0$. It follows by using the same argument as in the zero characteristic that $(\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}})^{p^t} \in P$, and thus $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \in P$ for every $P \in \text{Min}^*(I)$. This implies that $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \in I : \prod_{i=1}^n x_i$. \square

Now we are going to characterize the associated primes of a radical join-meet ideal of a finite lattice. We first need the following.

Definition 2.2. Let L be a lattice and A a subset of L . A is called *admissible* if it is empty or it is non-empty and has the following property: for any basic binomial $ab - cd$ of I_L , if $a \in A$ or $b \in A$, then $c \in A$ or $d \in A$.

In other words, the set A is admissible if and only if, for any basic binomial, either A “covers” both monomials of the binomial or none of them. Of course, the empty set and L are admissible sets for I_L .

Remark 2.3. Let A be an admissible set for I_L . We set $L_A = L \setminus A$. Then L_A is a sublattice of L with respect to the order induced from

L . Indeed, let $a, b \in L_A$ be two incomparable elements. Since A is admissible, it follows that $a \vee b$ and $a \wedge b$ do not belong to A

Proposition 2.4. *Let I_L be a radical ideal. Then, for any admissible set, the ideal I_{L_A} is radical.*

Proof. Assume that there exists an $A \subset L$ such that I_{L_A} is not radical; hence, there exists a polynomial $f \in K[\{a : a \in L \setminus A\}]$ such that $f \in \sqrt{I_{L_A}} \setminus I_{L_A}$. Then, obviously, $f \in \sqrt{I_L}$. We claim that $f \notin I_L$ which shows that I_L is not radical, a contradiction. Let us assume that $f \in I_L$. Then we may write

$$f = \sum_{a, b \notin A} h_{ab}(ab - (a \wedge b)(a \vee b)) + \sum_{a \in A \text{ or } b \in A} h_{ab}(ab - (a \wedge b)(a \vee b))$$

for some polynomials $h_{ab} \in K[L]$. We map to zero all the variables of A . In this way, since A is admissible, it follows that the second sum in the above formula vanishes while, in the first sum, all the basic binomials survive. Therefore, $f \in I_{L_A}$, a contradiction. \square

Remark 2.5. We are going to see in Example 3.7 that the radical property does not pass from a lattice to any of its proper sublattices.

For an admissible set $A \subset L$, we set

$$P_A(L) = I_{L_A} : \prod_{a \notin A} a + (a : a \in A).$$

If I_L is a radical ideal, then I_{L_A} is a radical ideal by Proposition 2.4, and, by Proposition 2.1, it follows that $I_{L_A} : \prod_{a \notin A} a$ is prime. Thus $P_A(L)$ is a prime ideal for any admissible set A if I_L is a radical ideal. Obviously, $P_A(L) \supset I_L$ for any admissible set A .

Theorem 2.6. *Let L be a lattice such that I_L is a radical ideal. Then*

$$I_L = \bigcap_{\substack{A \subset L \\ A \text{ admissible}}} P_A(L).$$

Proof. It is enough to show that any minimal prime ideal of I_L is of the form $P_A(L)$ for some admissible set $A \subset L$.

Let P be a minimal prime of I_L and $A = \{a : a \in P\}$. If $A = \emptyset$, that is, P does not contain any variable, then $P \supset I_L : \prod_{a \in L} a \supset I_L$. Since, by Proposition 2.1, $I_L : \prod_{a \in L} a$ is a prime ideal, we obtain $P = P_\emptyset(L)$.

Now let A be nonempty. We claim that A is admissible. Indeed, let $ab - cd$ be a basic binomial such that $a \in A$. It follows that $cd \in P$, which implies that $c \in A$ or $d \in A$. We show that $P = P_A(L)$. Indeed, since $P \supset I_L$ and $P \supset (a : a \in A)$, we also have $P \supset I_L + (a : a \in A) = I_{L_A} + (a : a \in A)$. It follows that $P \supset (I_{L_A} + (a : a \in A)) : \prod_{a \notin A} a = P_A(L)$. Since P is minimal over I , we must have $P = P_A(L)$. \square

Proposition 2.7. *Let I_L be radical. Then, for two admissible sets $A, B \subset L$, we have $P_A(L) \subsetneq P_B(L)$ if and only if*

$$A \subsetneq B \quad \text{and} \quad I_{L_A} : \prod_{a \notin A} a \subset I_{L_B} : \prod_{b \notin B} b + (b : b \in B \setminus A).$$

Proof. Let $A \subset B$. Then $P_A(L) \subset P_B(L)$ if and only if

$$\begin{aligned} I_{L_A} : \prod_{a \notin A} a = P_A(L)/(a : a \in A) &\subset P_B(L)/(a : a \in A) \\ &= I_{L_B} : \prod_{b \notin B} b + (b : b \in B \setminus A). \quad \square \end{aligned}$$

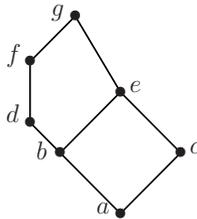


FIGURE 4.

The following example illustrates Theorem 2.6 and Proposition 2.7.

Example 2.8. Let Q be the lattice of Figure 4. The Gröbner basis of I_Q with respect to the lexicographic order induced by $a > b > \dots > g$ is $\{ae - bc, ag - cf, bg - ef, cd - cf, de - ef\}$. Thus, $\text{in}_<(I_Q)$ is squarefree, which implies that I_Q is a radical ideal and we may apply Theorem 2.6 and Proposition 2.7 to determine the minimal primes of I_Q .

One easily sees that

$$I_Q : \prod_{x \in Q} x \supset J = (ae - bc, ag - cf, bg - ef, d - f) \supset I_Q.$$

But $K[a, b, c, d, e, f, g]/J \cong K[a, b, c, d, e, g]/(ae - bc, ag - cd, bg - de)$, and the latter quotient ring is a domain. Therefore, J is a prime ideal. Moreover, $I_Q : \prod_{x \in Q} x = J = P_\emptyset(Q)$. The other minimal primes of I are (a, b, c, e) and (c, e, g) , that is, $I = J \cap (a, b, c, e) \cap (c, e, g)$. Note that, for instance, the set $A = \{g, d, f\}$ is an admissible set, but the corresponding prime ideal $P_A(Q)$ is not a minimal prime of I_Q since $P_A(Q) \supsetneq P_\emptyset(Q)$.

3. Join-meet ideals of modular non-distributive lattices.

It is well known that, given an ideal I of a polynomial ring S over a field, if $\text{in}_<(I)$ is radical for some monomial order $<$ on S , then the ideal I is radical as well; see [7, Proposition 3.3.7] or [4, Lemma 6.51] for an alternative proof. This also gives a procedure to show that a polynomial ideal is radical. However, there are radical polynomial ideals whose initial ideals are always non-radical. For such ideals one has to use other kind of arguments to prove the radical property.

In this section we mainly study a class of modular non-distributive lattices whose join-meet ideals are radical. Before beginning our study, let us look at the next example.

Example 3.1. Let N be the lattice of rank 4 of Figure 5. This is rather a simple example of a modular non-distributive lattice. We “included” only one diamond into a distributive lattice with eight elements. However, as we are going to show, the join-meet ideal of lattice N is not radical.

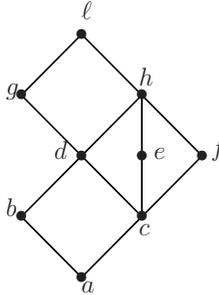


FIGURE 5. Lattice N .

We claim that $alg(d-f)^2 \in I_N$, which implies that $(alg(d-f))^2 \in I_N$; therefore, $alg(d-f) \in \sqrt{I_N}$. Indeed, one may easily see that

$$\begin{aligned} alg(d-f)^2 &= algd^2 - 2algdf + algf^2 \\ &\equiv ag^2hd - ag^2hf - agf(gh - lf) \\ &\equiv ag^2h(d-f) - agfl(d-f) \\ &\equiv ag^2h(d-f) - al^2c(d-f) \pmod{I_N}. \end{aligned}$$

On the other hand, $ah(d-f) \in I_N$ and $lc(d-f) \in I_N$. One may easily check this. For instance, for the first membership, we may use the following identity:

$$ah(d-f) = b(de - ch) + (f-d)(be - ah) - b(ef - ch).$$

Thus, $alg(d-f) \in \sqrt{I_N}$. The Gröbner basis of I_N with respect to reverse lexicographic order contains, apart from the basic binomials of N , the following binomials: $cel - cfl$, $cdl - cfl$, $ceh - cfh$, $afh - afh$, $adh - afh$, $cf^2\ell - c^2h\ell$, $ad^2\ell - ach\ell$, $cf^2h - c^2h^2$, $af^2h - ach^2$. Thus, $\text{in}_<(algd - algf) \notin \text{in}_<(I_N)$, which implies that $alg(d-f) \notin I_N$.

Let us still look a little at Example 3.1. Here we have obtained a non-radical ideal by "including" the vertex e (to get the diamond with the vertices c, d, e, f, h) in the distributive lattice with the vertices $a, b, c, d, f, g, h, \ell$. On the other hand, as one may easily check, the sublattice of N with the vertices c, d, e, f, g, h, ℓ has a radical join-meet ideal. Moreover, as Figure 7 and Example 3.7 show, if we "complete"

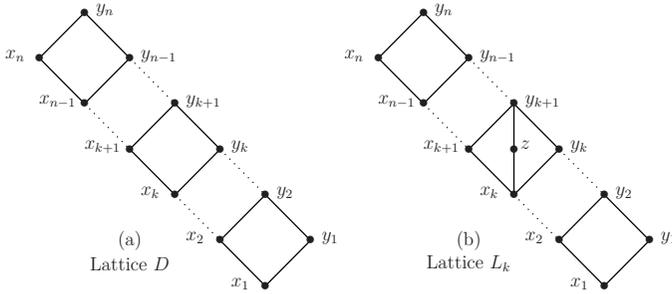


FIGURE 6.

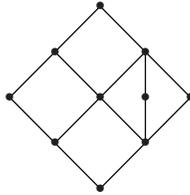


FIGURE 7.

the square with vertices b, d, g in lattice N , then we again get a radical ideal. These quite easy examples show that the join-meet ideals of non-distributive lattices have a wild behavior.

The following question arises. May we find, however, a class of distributive lattices such that by “including” just one small diamond one may get a radical joint-meet ideal for the new lattice? We are going to answer this question in the next theorem.

Let D be the distributive lattice of the divisors of $2 \cdot 3^n$ for some integer $n \geq 1$ with the elements labeled as in Figure 6 (a). For every $1 \leq k \leq n - 1$, we denote by L_k the lattice of Figure 6 (b).

Before stating our first preparatory result, we need to introduce some notation. For $1 \leq k \leq n - 1$, let

$$\begin{aligned}
 p_k &= x_{k+1}z - y_kz; & r_k &= y_k^2z - y_kz^2; \\
 g_i &= x_iy_{k+1} - y_iz, & \text{for } 1 \leq i < k;
 \end{aligned}$$

$$\begin{aligned}
h_j &= x_k y_j - x_j z, \text{ for } k+1 \leq j \leq n; \\
f_{ij} &= x_j y_i - x_i y_j, \text{ for } 1 \leq i < j \leq n, j \neq k+1, i \neq k; \\
f_{i,k+1} &= x_{k+1} y_i - y_i z, \text{ for } 1 \leq i \leq k; \\
f_{kj} &= x_j y_k - x_j z, \text{ for } j > k+1; \\
p_{ij} &= x_i x_{k+1} y_j - x_i y_j z, \text{ for } 1 \leq i < k < k+1 < j \leq n, \\
t_{ij} &= x_i y_k y_j - x_i y_j z, \text{ for } 1 \leq i < k < k+1 < j \leq n, \text{ and} \\
q_{ik} &= y_i y_k z - y_i z^2, \text{ for } 1 \leq i < k.
\end{aligned}$$

Lemma 3.2. *The set*

$$\begin{aligned}
\mathcal{G} = & \{p_k, r_k\} \cup \{g_i, q_{ik} : 1 \leq i < k\} \cup \{h_j : k+1 \leq j \leq n\} \\
& \cup \{f_{ij} : 1 \leq i < j \leq n\} \cup \{p_{ij}, t_{ij} : 1 \leq i < k < k+1 < j \leq n\}
\end{aligned}$$

is a Gröbner basis of $I = I_{L_k}$ with respect to the reverse lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n > z$. In particular, it follows that $\text{in}_<(I)$ is generated by the following set of monomials:

$$\begin{aligned}
\mathcal{M} = & \{x_j y_i : 1 \leq i < j \leq n\} \\
& \cup \{x_i y_{k+1} : 1 \leq i < k\} \cup \{x_k y_j : k+1 \leq j \leq n\} \\
& \cup \{x_i x_{k+1} y_j, x_i y_k y_j : 1 \leq i < k < k+1 < j \leq n\} \\
& \cup \{y_i y_k z : 1 \leq i < k\} \cup \{x_{k+1} z, y_k^2 z\}.
\end{aligned}$$

Proof. We first note that \mathcal{G} is a generating set of I and that next one applies Buchberger's criterion, that is, one checks that all the S -polynomials of the pairs $(f, g) \in \mathcal{G} \times \mathcal{G}$ reduce to zero modulo \mathcal{G} . Note that, for many pairs $(f, g) \in \mathcal{G} \times \mathcal{G}$, the checks are superfluous since the initial monomials $\text{in}_<(f)$ and $\text{in}_<(g)$ are relatively prime. Moreover, in order to eliminate many checks, one may use the following known fact. If f, g are two polynomials with $\text{in}_<(f)$ and $\text{in}_<(g)$ relatively prime, then, for any monomials u, v the S -polynomial $S(uf, vg)$ reduces to zero modulo uf and vg . \square

Theorem 3.3. *For every $1 \leq k \leq n-1$, the join-meet ideal I_{L_k} is radical.*

The proof of this theorem has several steps which are shown in the following lemmas, but the basic idea of the proof is very simple. We

actually show that one may decompose I as an intersection of two radical ideals, namely, $I = (I, x_{k+1} - y_k) \cap (I, z)$; hence, I itself is a radical ideal.

Lemma 3.4. *Let $1 \leq k \leq n - 1$ and $I = I_{L_k}$. Then $I = (I, x_{k+1} - y_k) \cap (I, z)$.*

Proof. The inclusion $I \subset (I, x_{k+1} - y_k) \cap (I, z)$ is obvious. For obtaining the equality, we show that

$$(3.1) \quad \text{in}_{<}(I, x_{k+1} - y_k) \cap \text{in}_{<}(I, z) \subset \text{in}_{<}(I).$$

This will imply that $\text{in}_{<}((I, x_{k+1} - y_k) \cap (I, z)) \subset \text{in}_{<}(I)$; thus,

$$\text{in}_{<}(I) = \text{in}_{<}((I, x_{k+1} - y_k) \cap (I, z)),$$

which leads to the desired statement.

We know the generators of $\text{in}_{<}(I)$ from Lemma 3.2. We now compute the Gröbner bases of (I, z) and $(I, x_{k+1} - y_k)$ with respect to the reverse lexicographic order induced by $x_1 > \dots > x_n > y_1 > \dots > y_n > z$. By using the Gröbner basis of I , one easily sees that (I, z) is generated by the binomials $f_{ij} = x_j y_i - x_i y_j$ where $1 \leq i < j \leq n$ and $j \neq k + 1, i \neq k$, and by the following set of monomials: $\{z\} \cup \{x_i y_{k+1} : 1 \leq i < k\} \cup \{x_k y_j : k + 1 \leq j \leq n\} \cup \{x_{k+1} y_i : 1 \leq i \leq k\} \cup \{x_j y_k : j > k + 1\} \cup \{x_i x_{k+1} y_j, x_i y_k y_j : 1 \leq i < k < k + 1 < j \leq n\}$. By using Buchberger's criterion, one immediately checks that the above set of generators of (I, z) is a Gröbner basis of (I, z) . Consequently,

$$G(\text{in}_{<}(I, z)) = (G(\text{in}_{<}(I) \setminus \{x_{k+1} z, y_k z^2, y_i y_k z : 1 \leq i < k\}) \cup \{z\}),$$

which implies that

$$\text{in}_{<}(I, z) = (\text{in}_{<}(I), z).$$

Here we used the notation $G(J)$ for the minimal set of monomial generators of the monomial ideal J .

By using the Gröbner basis of I it follows that the ideal $(I, x_{k+1} - y_k)$ is generated by the binomials $x_{k+1} - y_k, g_i, 1 \leq i < k, h_j, k + 1 \leq j \leq n, f_{ij}, 1 \leq i < j \leq n, j \neq k + 1, i \neq k, f'_{i,k+1} = y_i y_k - y_i z = f_{i,k+1} -$

$y_i(x_{k+1}-z)$, $1 \leq i \leq k$, f_{kj} , $j > k+1$, r_k and $p'_{ij} = t_{ij} = x_i y_j y_k - x_i y_j z$, $1 \leq i < k < k+1 < j \leq n$, since $q_{ik} = z f'_{i,k+1}$. Buchberger's criterion applied to this set of generators shows that they form a Gröbner basis of $(I, x_{k+1} - y_k)$. Moreover, we obtain

$$\begin{aligned} G(\text{in}_{<}(I, x_{k+1} - y_k)) &= (G(\text{in}_{<}(I) \setminus (\{x_{k+1}z, x_{k+1}y_i : 1 \leq i \leq k\} \\ &\cup \{y_i y_k z : 1 \leq i < k\})) \\ &\cup \{x_{k+1}, y_i y_k : 1 \leq i \leq k\}); \end{aligned}$$

therefore, we get the following equality:

$$(3.3) \quad \text{in}_{<}(I, x_{k+1} - y_k) = (\text{in}_{<}(I), x_{k+1}, y_1 y_k, \dots, y_{k-1} y_k, y_k^2).$$

By using relations (3.2) and (3.3), we get

$$\begin{aligned} \text{in}_{<}(I, x_{k+1} - y_k) \cap \text{in}_{<}(I, z) \\ = (\text{in}_{<}(I), x_{k+1}z, y_1 y_k z, \dots, y_k^2 z) \subset \text{in}_{<}(I). \quad \square \end{aligned}$$

From the above proof, we may also derive the following:

Corollary 3.5. *(I, z) is a radical ideal.*

Proof. By (3.2), we have $\text{in}_{<}(I, z) = (\text{in}_{<}(I), z)$. Since $\text{in}_{<}(I)$ has only one non-squarefree generator, namely, $y_k^2 z$ which is “killed” by z , it follows that $\text{in}_{<}(I, z)$ is square free and, consequently, (I, z) is a radical ideal. \square

The last step in the proof of Theorem 3.3 is shown in the following.

Lemma 3.6. *The ideal $(I, x_{k+1} - y_k)$ is radical.*

Proof. We show that $(I, x_{k+1} - y_k)$ has a squarefree initial ideal with respect to the lexicographic order induced by $z > x_1 > \dots > x_n > y_1 > \dots > y_n$. We recall from the proof of Lemma 3.4 that $(I, x_{k+1} - y_k)$ is generated by $x_{k+1} - y_k, g_i, 1 \leq i < k, h_j, k+1 \leq j \leq n, f_{ij}, 1 \leq i < j \leq$

$n, j \neq k+1, i \neq k, f'_{i,k+1} = y_i y_k - y_i z, 1 \leq i \leq k, f_{kj}, j > k+1, r_k,$ and $p'_{ij} = t_{ij} = x_i y_j y_k - x_i y_j z, 1 \leq i < k < k+1 < j \leq n.$ In this generating set, the generators r_k and p'_{ij} are redundant. Indeed, $r_k = z f'_{k,k+1}$ and $p'_{ij} = (y_k - z) f_{ij} - x_j f'_{i,k+1}$ for any $1 \leq i < k < k+1 < j \leq n.$ Moreover, for every $1 \leq i < k,$ we may replace the generator g_i by $g'_i = x_i y_{k+1} - y_i y_k = f'_{i,k+1} - g_i.$ Finally, for $j > k+1,$ we may replace the generator f_{kj} by $x_k y_j - x_j y_k = f_{kj} - h_j.$ Therefore, $(I, x_{k+1} - y_k)$ is generated by the following binomials: $x_{k+1} - y_k, g'_i = x_i y_{k+1} - y_i y_k$ for $1 \leq i < k, h_j = z x_j - x_k y_j$ for $k+1 \leq j \leq n, f'_{i,k+1} = z y_i - y_i y_k$ for $1 \leq i \leq k,$ and $f_{ij} = x_i y_j - x_j y_i$ for $1 \leq i < j \leq n$ with $j \neq k+1.$ By trivial calculations, one may check that this set of generators is a Gröbner basis of $(I, x_{k+1} - y_k)$ with respect to the lexicographic order induced by $z > x_1 > \dots > x_n > y_1 > \dots > y_n.$ Since all these generators have squarefree initial monomials, it follows that the initial ideal of $(I, x_{k+1} - y_k)$ is squarefree and, thus, $(I, x_{k+1} - y_k)$ is a radical ideal. \square

We end this section with a few comments. Going back to Example 3.1, by applying Theorem 3.3, we see that every proper sublattice N' of N has a radical join-meet ideal although I_N is not radical. The following example shows that the radical property does not pass from a lattice to any of its proper sublattices.

Example 3.7. Let R be the lattice of Figure 7.

One may check with Singular [5] that I_R is a radical ideal. However, the ideal I_N attached to its proper sublattice N is not radical, as we have seen in Example 3.1.

4. The minimal primes of the join-meet ideal of $L_k.$ In this section we apply the results of Section 2 to determine explicitly the minimal primes of the ideals I_{L_k} for $1 \leq k \leq n - 1.$ We recall that we denoted by D the distributive lattice displayed in Figure 6 (a), and by L_k the lattice displayed in Figure 6 (b). We denote by D_k the sublattice of D with the elements $x_i, y_i, 1 \leq i \leq k,$ and by D'_k the sublattice of D with the elements $x_i, y_i, k+1 \leq i \leq n.$

Before stating the main theorem of this section, we need to prove a preparatory result.

Lemma 4.1. *For any $1 \leq k \leq n - 1$, the ideal $(I_D, x_{k+1} - y_k)$ is prime.*

Proof. It is enough to show that $(I_D, x_2 - y_1)$ is a prime ideal since, by an appropriate change of variables, we may map the ideal $(I_D, x_2 - y_1)$ into $(I_D, x_{k+1} - y_k)$.

Let $f_{ij} = x_i y_j - x_j y_i$, $1 \leq i < j \leq n$, be the generators of I_D . By [8, Theorem 2.2], $\{f_{ij} : 1 \leq i < j \leq n\}$ is a Gröbner basis of I_D with respect to any monomial order. Actually, if $\text{in}_{<} f_{ij}$ and $\text{in}_{<} f_{k\ell}$ are not relatively prime, then the S -polynomial of the pair $(f_{ij}, f_{k\ell})$ may be expressed as

$$(4.1) \quad S(f_{ij}, f_{k\ell}) = z f_{pq}$$

for some variable $z \in K[D]$ and $1 \leq p < q \leq n$.

Let $<$ be an arbitrary monomial order on $K[D]$. For any $1 \leq i < j \leq n$, we denote by g_{ij} the reduction of f_{ij} modulo $x_2 - y_1$. More precisely, g_{ij} is obtained from f_{ij} by replacing x_2 by y_1 if $x_2 > y_1$ or y_1 by x_2 if $y_1 > x_2$. Since $\{f_{ij} : 1 \leq i < j \leq n\}$ is a Gröbner basis of I_D with respect to $<$, it follows that the set $\mathcal{G} = \{g_{ij} : 1 \leq i < j \leq n\} \cup \{x_2 - y_1\}$ is a Gröbner basis of $(I_D, x_2 - y_1)$ with respect to $<$. This is essentially due to equation (4.1). In particular, \mathcal{G} is a Gröbner basis of $(I_D, x_2 - y_1)$ with respect to the lexicographic order induced by $x_1 > \dots > x_n > y_1 > \dots > y_n$. In this case, it follows that the initial ideal of $(I_D, x_2 - y_1)$ is generated by the following squarefree monomials: $x_2, x_i y_j$ for $i, j \neq 2$, $x_1 y_2$, and $x_j y_2$ for $2 < j \leq n$. This shows that $(I_D, x_2 - y_1)$ is a radical ideal. On the other hand, by applying [15, Lemma 12.1], it follows that all the variables are regular on $(I_D, x_2 - y_1)$, which implies that $(I_D, x_2 - y_1) : \prod_{1 \leq i \leq n} x_i \prod_{1 \leq j \leq n} y_j = (I_D, x_2 - y_1)$. Finally, by applying Proposition 2.1, we get the desired conclusion. \square

Theorem 4.2. *Let $1 \leq k \leq n - 1$ and $I = I_{L_k}$ be the join-meet ideal of lattice L_k . The minimal primes of I are the following:*

$$\begin{aligned} P &= (I, z - x_{k+1}, z - y_k), \\ P_1 &= (z, x_1, \dots, x_n), \\ P'_1 &= (z, y_1, \dots, y_n), \end{aligned}$$

$$\begin{aligned}
 P_2 &= (z, x_1, \dots, x_k, y_1, \dots, y_k) + I_{D'_k}, \\
 P'_2 &= (z, x_{k+1}, \dots, x_n, y_{k+1}, \dots, y_n) + I_{D_k}, \\
 P_3 &= (x_1, \dots, x_n, y_1, \dots, y_k), \\
 P'_3 &= (y_1, \dots, y_n, x_{k+1}, \dots, x_n).
 \end{aligned}$$

Proof. By Theorem 2.6, since I is a radical ideal, we know that any minimal prime of I is of the form $P_A(L_k)$ where A is an admissible set of I .

Let $P = P_\emptyset(L_k)$. Then $P = I : (z \prod_{1 \leq i \leq n} x_i \prod_{1 \leq j \leq n} y_j)$. We obviously have

$$(4.2) \quad P \supset (I, z - x_{k+1}, x - y_k) \supset I.$$

On the other hand,

$$\frac{K[L_k]}{(I, z - x_{k+1}, x - y_k)} \cong \frac{K[D]}{(I_D, x_{k+1} - y_k)}.$$

Since, by Lemma 4.1, $(I_D, x_{k+1} - y_k)$ is a prime ideal, it follows that $(I, z - x_{k+1}, x - y_k)$ is a prime ideal as well. Therefore, since P is a minimal prime of I , by using (4.2), we must have $P = (I, z - x_{k+1}, x - y_k)$.

Now we look at the minimal primes which correspond to non-empty admissible sets. Let A be such an admissible set, and assume first that $z \in A$. If $y_\ell \notin A$ for every $1 \leq \ell \leq n$, then, by using the basic binomials $z y_i - x_i y_{k+1}$ for $i \leq k$ and $x_k y_j - x_j y_k$ for $j \geq k + 1$, it follows that $P \supset (z, x_1, \dots, x_n) \supset I$; hence, we get $P_A(L_k) = (z, x_1, \dots, x_n) = P_1$. Since the dual lattice of L_k obviously has the same relation ideal, it follows that P'_1 is the minimal prime which correspond to the admissible set A which contains z and does not contain any of the variables x_i , $i = 1, \dots, n$.

Now we consider an admissible set A which contains z and has the property that there exist $1 \leq i, j \leq n$ such that $x_i, y_j \notin A$. If $i \neq j$, then, since $x_i y_j - x_j y_i$ is a basic binomial, it follows that $x_j, y_i \notin A$. Therefore, we may assume that there exists $1 \leq i \leq n$ such that $x_i, y_i \notin A$. Let us suppose that $x_k, y_k \notin A$. From the

relations $x_j z - x_k y_j$, we get $y_j \in A$ for $j \geq k + 1$ and, next, from the relations $x_j y_k - x_k y_j$, we get $x_j \in A$ for $j \geq k + 1$. Thus, in this case, $P_A(L_k) \supset P'_2 \supset I$. But P'_2 is obviously a prime ideal, therefore, $P_A(L) = P'_2$. The dual situation corresponds to $x_{k+1}, y_{k+1} \notin A$ and, in this case, one gets $P_A(L_k) = P_2$. It remains to consider $x_k, y_k, x_{k+1}, y_{k+1} \in A$. Then it follows that $P_A(L_k) \supsetneq P$, which implies that $P_A(L_k)$ is not a minimal prime.

We still need to identify the minimal primes which correspond to non-empty admissible sets A which do not contain z . Let A be such that $z \notin A$ and $P_A(L_k)$ is a minimal prime of I . Since $z y_k - x_k y_{k+1}$, $z x_{k+1} - x_k y_{k+1}$, $y_k x_{k+1} - x_k y_{k+1} \in I \subset P_A(L_k)$, we get $z(y_k - x_{k+1}) \in P_A(L_k)$; hence, $y_k - x_{k+1} \in P_A(L)$ and $x_{k+1}(z - y_k) \in P_A(L)$. If $x_{k+1} \notin P_A(L)$, it follows that $z - x_{k+1} \in P_A(L)$. But this further implies that $P_A(L_k) \supsetneq P$; hence, $P_A(L_k)$ is not a minimal prime. Consequently, $x_{k+1} \in A$, and, next, $y_k \in P_A(L_k)$. By again using the basic binomial $y_k x_{k+1} - x_k y_{k+1}$, we obtain $x_k \in A$ or $y_{k+1} \in A$.

We analyze the following cases.

Case 1. $x_k \in A$ and $y_{k+1} \notin A$. By using the relations $x_j z - x_k y_j$ for $j > k + 1$, we get $x_j \in A$ for $j > k + 1$. Similarly, by using the basic binomials $y_{k+1} x_i - y_i x_{k+1}$ for $i < k$, we get $x_i \in A$ for all $i < k$. Therefore, we have $x_i \in A$ for all $i = 1, \dots, n$. By using the basic binomials $z y_i - x_i y_{k+1}$ for $i < k$, we also get $y_i \in A$. Then we have actually proved that $P_A(L_k) \supset (x_1, \dots, x_n, y_1, \dots, y_k) = P_3 \supset I$. Since $P_A(L_k)$ is a minimal prime of I , we must have $P_A(L_k) = P_3$.

Case 2. $x_k \notin A$ and $y_{k+1} \in A$. This is the dual of the above case and leads to the conclusion that $P_A(L_k) = P'_3$.

Case 3. Let $x_k, y_{k+1} \in A$. From the relations $z y_i - x_i y_{k+1}$ for $i < k$, and $x_j z - x_k y_j$ for $j > k$, we obtain $y_i \in A$ for $i < k$, and $x_j \in A$ for $j > k$. If there exists an $i < k$ such that $x_i \notin A$, by using the relations $x_i y_j - x_j y_i$ for $j > k + 1$, we get $y_j \in A$ for all $j > k + 1$. In this case, it follows that $A \supset \{y_1, \dots, y_n, x_k, \dots, x_n\}$ and $P_A(L_k) \supsetneq P'_3$; hence, $P_A(L_k)$ is not a minimal prime, a contradiction. In other words, Case 3 does not hold, and this completes the proof. \square

Corollary 4.3. *The join-meet ideal I_{L_k} is not unmixed and $\dim(K[L_k]/I_{L_k}) = n$.*

Proof. It is known (see [9]), that if \mathcal{D} is a distributive lattice, then $\dim(K[\mathcal{D}]/I_{\mathcal{D}})$ is equal to the number of the join irreducible elements of \mathcal{D} plus 1. Therefore, we get

$$\begin{aligned}\dim(K[L_k]/P) &= n = \dim(K[L_k]/P_1) = \dim(K[L_k]/P'_1), \\ \dim(K[L_k]/P_2) &= n - k, \dim(K[L_k]/P'_2) = k, \\ \dim(K[L_k]/P_3) &= n - k + 1, \dim(K[L_k]/P'_3) = k + 1.\end{aligned}$$

The above equalities yield the desired statements. \square

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