

A CLASSIFICATION OF ONE-DIMENSIONAL LOCAL DOMAINS BASED ON THE INVARIANT $(c - \delta)r - \delta$

ANNA ONETO AND ELSA ZATINI

ABSTRACT. Let (R, \mathfrak{m}) be a one-dimensional, local, Noetherian domain, and let \overline{R} be the integral closure of R in its quotient field K . We assume that R is not regular, analytically irreducible and residually rational. The usual valuation $v : K \rightarrow \mathbf{Z} \cup \infty$ associated to \overline{R} defines the numerical semigroup $v(R) = \{v(a), a \in R, a \neq 0\} \subseteq \mathbf{N}$. The aim of the paper is to study the non-negative invariant $b := (c - \delta)r - \delta$, where c, δ, r denote the conductor, the length of \overline{R}/R and the Cohen-Macaulay type of R , respectively. In particular, the classification of the semigroups $v(R)$ for rings having $b \leq 2(r - 1)$ is realized. This method of classification might be successfully utilized with similar arguments but more boring computations in the cases $b \leq q(r - 1)$, for reasonably low values of q . The main tools are type sequences and the invariant k which estimates the number of elements in $v(R)$ belonging to the interval $[c - e, c)$, e being the multiplicity of R .

1. Introduction. Let (R, \mathfrak{m}) be a one-dimensional, local, Noetherian domain and let \overline{R} be the integral closure of R in its quotient field K . We assume that R is not regular and analytically irreducible, i.e., \overline{R} is a DVR with uniformizing parameter t and a finite R -module. We also suppose R to be residually rational, i.e., $R/\mathfrak{m} \simeq \overline{R}/t\overline{R}$. Called $v : K \rightarrow \mathbf{Z} \cup \infty$ the usual valuation associated to \overline{R} , the image $v(R) = \{v(a), a \in R, a \neq 0\} \subseteq \mathbf{N}$ is a numerical semigroup. Starting from the following classical invariants:

c , the *conductor* of R , the minimal $j \in v(R)$ such that $j + \mathbf{N} \subset v(R)$,
 $\delta := \ell_R(\overline{R}/R)$, the number of gaps of the semigroup $v(R)$ in \mathbf{N} ,
 $r := \ell_R((R : \mathfrak{m})/R)$, the *Cohen Macaulay type* of R , the new invariant

$$b := (c - \delta)r - \delta$$

The second author is the corresponding author.
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has been recently considered in the literature. The general problem of classifying rings according to the size of b has been examined by several authors. First, Brown and Herzog in [2] characterize all the one-dimensional reduced local rings having $b = 0$ or $b = 1$. Successively, in [3, 4, 6], Delfino, D'Anna and Micale consider the rings for which $b \leq r$. Partial answers in the case $b > r - 1$ are given in [5].

In [10, Section 4] we obtain some improvements of the quoted results. This is done by using the expression of the invariant b in terms of the type sequence $[r_1, \dots, r_n]$ (defined in (1.1)), where $n := c - \delta$ and r_1 equals the Cohen-Macaulay type r of R , namely:

$$b = \sum_{i=1}^n (r - r_i).$$

So, employing the properties of the type sequence, we get as a straightforward consequence of the preceding formula the well-known bounds

$$0 \leq b \leq (n - 1)(r - 1)$$

(for the positivity see [2, Theorem 1]; for the upper bound see [3, Proposition 2.1]). Also, we recover in an immediate way the two extremal cases:

$b = 0$, corresponding to the so-called rings of *maximal length*, i.e., the rings having maximal type sequence $[r, r, \dots, r]$;

$b = (n - 1)(r - 1)$, corresponding to the *almost Gorenstein* rings, i.e., the rings having minimal type sequence $[r, 1, \dots, 1]$.

Actually, for any integer $q \in \mathbf{N}$ it is natural to ask if it is possible to characterize the rings verifying

$$(q - 1)(r - 1) \leq b \leq q(r - 1).$$

In Section 3 we write explicitly all the possible values of $v(R)$ for $1 \leq q \leq 2$ (see Theorems (3.3), (3.4), (3.6)), but we outline that the method used here is absolutely general and analogous although more tedious computations might be repeated for greater values of q . To achieve our results, we utilize heavily the number

$$k := \ell_R(R/(\mathfrak{C} + xR)),$$

where $\mathfrak{C} := t^c \overline{R}$ denotes the *conductor ideal* of R in \overline{R} and x an element of R such that $v(x) = e(R)$, the *multiplicity*. In [5] it is established that $b = r - 1 \Rightarrow k = 1$ or 2 [5, Proposition 2.4], and that $b = r - 1$ and $k = 2 \Rightarrow r = e - 2$ [5, Corollary 2.13]. In [6] the lower bound $rk - e + 1 \leq b$ is found. Improvements of these results and several other inequalities relating the invariants k, b, r are now realized by means of the type sequence of R (see (1.4) and (2.1)). For this purpose we introduce in Section 1 a decomposition of the semigroup $v(R)$ as a disjoint union of subsets:

$$v(R) = \{0, e, 2e, \dots, pe, c, \longrightarrow\} \cup H_1 \cup \dots \cup H_{k-1},$$

where $H_i := \{y_i, y_i + e, \dots, y_i + l_i e\}$, $i = 1, \dots, k - 1$, $p, l_i \in \mathbf{N}$, and $\{y_i\}_{i=1, \dots, k-1}$ have distinct residues $(\bmod e)$ (see Setting 1.6). This allows us to obtain in Section 2 the useful formula (2.2.1):

$$b = X + Y + Z$$

where $X := (k - 1)(r - 1) \geq 0$, $Y := k - (e - r) \geq 0$, $Z := (r + 1)(p + \sum_{i=1}^{k-1} l_i) + k + h - pe - 1 \geq 0$. Obviously $X + Y = rk - e + 1$, and so the integer Z measures how far is b from the lower bound proved in [6].

The advantage of this formula is evident for low values of b . For instance, for rings having $b \in \{0, 1, 2\}$ we state in a quite simple way all the possible value sets (see Theorems (3.1), (3.8), (3.9)). Nevertheless, a such type of classification might be accomplished for greater values of b with similar arguments.

1. Preliminary results. We begin by giving the setting of the paper.

Setting 1.1. Let (R, \mathfrak{m}) be a one-dimensional local Noetherian domain with residue field κ and quotient field K . We assume throughout that R is not regular with normalization $\overline{R} \subset K$ a DVR and a finite R -module, i.e., R is analytically irreducible. Let $t \in \overline{R}$ be a uniformizing parameter for \overline{R} , so that $t\overline{R}$ is the maximal ideal of \overline{R} . We also suppose that the field κ is isomorphic to the residue field $\overline{R}/t\overline{R}$, i.e., R is residually rational. We denote the usual valuation on K associated

to \overline{R} by v ; that is, $v : K \rightarrow \mathbf{Z} \cup \infty$, and $v(t) = 1$. By [9, Proposition 1] in this setting it is possible to compute for a pair of fractional nonzero ideals $I \supseteq J$ the length of the R -module I/J by means of valuations:

$$(1.1.1) \quad \ell_R(I/J) = |v(I) \setminus v(J)|.$$

The set $v(R) := \{v(a) \mid a \in R, a \neq 0\} \subseteq \mathbf{N}$ is the *numerical semigroup* of R . Since the *conductor* $\mathfrak{C} := (R :_K \overline{R})$ is an ideal of both R and \overline{R} , there exists a positive integer c so that $\mathfrak{C} = t^c \overline{R}$, $\ell_R(\overline{R}/\mathfrak{C}) = c$ and $c \in v(R)$. Furthermore, denoting by $\delta := \ell_R(\overline{R}/R)$ the number of gaps of the semigroup $v(R)$ and $r := \ell_R((R : \mathfrak{m})/R)$ the *Cohen Macaulay type* of R , we define the invariant

$$b := (c - \delta)r - \delta.$$

We list the elements of $v(R)$ in order of size: $v(R) := \{s_i\}_{i \geq 0}$, where $s_0 = 0$ and $s_i < s_{i+1}$, for every $i \geq 0$. We put $e := s_1$ the *multiplicity* of R and $n = c - \delta$ the number such that $s_n = c$. For every $i \geq 0$, let R_i denote the ideal of elements whose values are bounded by s_i , that is,

$$R_i := \{a \in R \mid v(a) \geq s_i\}.$$

The ideals R_i give a strictly decreasing sequence

$$R = R_0 \supset R_1 = \mathfrak{m} \supset R_2 \supset \cdots \supset R_n = \mathfrak{C} \supset R_{n+1} \supset \cdots,$$

which induces the chain of duals:

$$R \subset (R : R_1) \subset \cdots \subset (R : R_n) = \overline{R} \subset (R : R_{n+1}) = t^{-1} \overline{R} \subset \cdots.$$

Put $r_i := \ell_R((R : R_i)/(R : R_{i-1}))$, $i \geq 1$; the finite sequence of integers

$$[r_1, \dots, r_n] \quad \text{is the } \textit{type sequence} \text{ of } R.$$

In particular $r_1 = r$, the Cohen-Macaulay type of R . Moreover it is known that:

- $1 \leq r_i \leq r$ for every $i \geq 1$, and $r_i = 1$ for every $i > n$,
- $\delta = \sum_1^n r_i$,
- $2\delta - c = \sum_1^n (r_i - 1) = \sum_1^\infty (r_i - 1)$ (see, e.g., [10, Proposition 2.7]).

We see now that type sequence is a suitable tool to study the behavior of the invariant b .

Proposition 1.2. *We have:*

- (1) $b = \sum_{i=1}^n (r - r_i)$.
- (2) $0 \leq b \leq (n-1)(r-1)$.

Proof. For (1) see [10, Section 4].

(2) We have: $\sum_{i=1}^n (r - r_i) = \sum_{i=2}^n (r - r_i) \leq (n-1)(r-1)$, because $r_1 = r$ and $r_i \geq 1$, for every $i \geq 1$. \square

Notation 1.3. Let R be as in (1.1). We set:

- $x \in \mathfrak{m}$ is an element such that $v(x) = e$; namely, $\ell_R(R/xR) = e$.
- For $a, b \in \mathbf{Z}$, $[a, b] = \{x \in \mathbf{Z} \mid a \leq x \leq b\}$.
- $i_0 \in [1, n]$ is such that $s_{i_0-1} = \min\{y \in v(R) \mid y \geq c - e\}$.
($i_0 = 1$ if and only if $c = e$).
- $B := [i_0, n]$ and $A := [1, n] \setminus B$. Note that $|A| \leq n-1$.
- $k := \ell_R(R/(\mathfrak{C} + xR))$ ($1 \leq k \leq e-1$).

Theorem 1.4. *The following facts hold.*

- (1) $k = |B| = \ell_R(\mathfrak{C} :_R \mathfrak{m}/\mathfrak{C}) \geq e - r > 0$.
- (2) $k \leq \sum_{i \in B} r_i \leq e - 1$. If $\sum_{i \in B} r_i = e - 1$, then $s_{i_0-1} = c - e$.

Proof. (1) and the inequality $\sum_{i \in B} r_i \leq e - 1$ of (2) are proved in [10, Lemma 4.2]. Since $r_i \geq 1$ for every i and $|B| = k$, the inequality $k \leq \sum_{i \in B} r_i$ is done.

Moreover, denoting by ω the canonical module of R (see [10] for the existence and the properties in our setting), we remark that

$$\sum_{i \in B} r_i = \ell_R(\overline{R}/(R : R_{i_0-1})) = |v(\omega R_{i_0-1})_{< c}|$$

and

$$v(\omega R_{i_0-1})_{< c} \subseteq [c - e, c - 2]$$

(see the proof of the quoted lemma). Thus, $\sum_{i \in B} r_i = e - 1 \Rightarrow v(\omega R_{i_0-1})_{<c} = [c - e, c - 2]$, and so s_{i_0-1} , the minimal element in $v(\omega R_{i_0-1})$, equals $c - e$. \square

The case $k = 1$ is completely known and recalled below for the convenience of the reader.

Proposition 1.5 [10, Lemma 4.4]. *The following facts are equivalent:*

- (1) $k = 1$.
- (2) $v(R) = \{0, e, \dots, pe, c \rightarrow\}$.
- (3) *The type sequence of R equals $[e - 1, \dots, e - 1, r_n]$.*

If R satisfies these equivalent conditions, then:

$$\delta = c - p - 1, \quad b = (p + 1)e - c \leq r - 1, \quad r = e - 1, \quad r_n = e - 1 - b.$$

By virtue of (1.1.1) we have $k = |v(R) \setminus v(\mathfrak{C} + xR)|$. This fact allows to write $v(R) = v(\mathfrak{C} + xR) \cup \{0, y_1, \dots, y_{k-1}\}$, obtaining the description of $v(R)$ as a disjoint union of the sets H_i given in the next setting. The construction is significant for $k > 1$.

Setting 1.6. Let $k > 1$. We set:

$$v(R) = \{0, e, 2e, \dots, pe, c, \longrightarrow\} \cup H_1 \cup \dots \cup H_{k-1},$$

where

- p is the integer such that $c - e \leq pe < c$, in other words, $pe + 2 \leq c \leq (p + 1)e$. ($p \geq 0$ and $p = 0$ if and only if $c = e$).
- $h := (p + 1)e - c$, ($0 \leq h \leq e - 2$).
- $H_i := \{y_i, y_i + e, \dots, y_i + l_i e\}$, $i = 1, \dots, k - 1, l_i \in \mathbf{N}$.
- The integers $y_i \in \mathbf{N}$ are such that $e < y_1 < y_2 < \dots < y_{k-1}$, $y_i \notin e\mathbf{Z}$, $\overline{y_i} \neq \overline{y_j} \pmod{e}$ for every $i, j \in \{1, \dots, k - 1\}$.
- The integers l_i , $i = 1, \dots, k - 1$, are defined by the relations:

$$y_i + l_i e < c \leq y_i + (l_i + 1)e.$$

- For $k = 2$ we shortly call $y := y_1$, $l := l_1$.

Example 1.7. If $v(R) = \langle 10, 11, 26 \rangle$, then: $v(R) = \{0, 10, 20, 30, 40, 50 \rightarrow\} \cup H_1 \cup \dots \cup H_7$ where $H_1 = \{11, 21, 31, 41\}$, $H_2 = \{22, 32, 42\}$, $H_3 = \{26, 36, 46\}$, $H_4 = \{33, 43\}$, $H_5 = \{37, 47\}$, $H_6 = \{44\}$, $H_7 = \{48\}$. According to notations previously introduced $y_1 = 11$, $y_2 = 22$, $y_3 = 26$, $y_4 = 33$, $y_5 = 37$, $y_6 = 44$, $y_7 = 48$ and $l_1 = 3$, $l_2 = l_3 = 2$, $l_4 = l_5 = 1$, $l_6 = l_7 = 0$. Moreover, $c = 50$, $p = 4$, $h = 0$.

Proposition 1.8. *Let $k > 1$, p , h , $\{l_i\}$ be the integers defined in (1.3) and (1.6). Then:*

- (1) $r \in \{e - k, \dots, e - 1\}$.
- (2) $0 \leq l_{k-1} \leq \dots \leq l_2 \leq l_1 \leq p - 1$.
- (3) $c - \delta = p + k + \sum_1^{k-1} l_i$, $\delta = (p + 1)(e - 1) - h - \sum_1^{k-1} (l_i + 1)$.

Proof. Assertion (1) follows immediately from (1.4.1).

(2) By definition of l_i and p , we have $(l_i + 1)e < y_i + l_i e < c \leq (p + 1)e$; then $l_i + 1 \leq p$, for every $i = 1, \dots, k - 1$. Now note that

$$y_i + l_i e < c \leq y_{i-1} + (l_{i-1} + 1)e \Rightarrow y_i - y_{i-1} < (l_{i-1} + 1 - l_i)e \Rightarrow l_i \leq l_{i-1}.$$

(3) Using the integers defined in (1.6) $c - \delta$ and δ can be expressed as:

$$\begin{aligned} c - \delta &= (p + 1) + (l_1 + 1) + \dots + (l_{k-1} + 1) = p + k + \sum_1^{k-1} l_i, \\ \delta &= c - (c - \delta) = (p + 1)e - h - \left(p + k + \sum_1^{k-1} l_i \right) \\ &= (p + 1)(e - 1) - h - \sum_1^{k-1} (l_i + 1). \quad \square \end{aligned}$$

It is natural to ask how the elements y_1, \dots, y_{k-1} introduced in (1.6) influence the Cohen Macaulay type of R . This will be analyzed in the following (1.9), (1.11), (1.12).

Proposition 1.9. *Let $k = \ell_R(R/(\mathfrak{C} + xR))$, and let $v(R)$ be as in (1.6). Further, let $x_1, \dots, x_{k-1} \in \mathfrak{m}$ be such that $v(x_i) = y_i$. The following facts are equivalent:*

- (1) $r = e - 1$, i.e., R is of maximal Cohen-Macaulay type.
- (2) $v(R) \setminus v(xR : \mathfrak{m}) = \{0\}$.
- (3) $y_1, \dots, y_{k-1} \in v(xR : \mathfrak{m})$.
- (4) $x_1, \dots, x_{k-1} \in (xR : \mathfrak{m})$.
- (5) $x_i x_j \in x\mathfrak{m}$ for every $i, j = 1, \dots, k - 1$.
- (6) $\ell_R(\mathfrak{m}/\mathfrak{m}^2) = e$, i.e., R is of maximal embedding dimension.

Proof. Since $e - r = \ell_R(R/xR) - \ell_R((xR : \mathfrak{m})/xR) = \ell_R(R/(xR : \mathfrak{m}))$, the equality $e - r = 1$ means $|v(R) \setminus v(xR : \mathfrak{m})| = 1$, and so $1 \Leftrightarrow 2$ is proved. In the same way we obtain that

- (*) $r = e - 1 \Leftrightarrow (xR : \mathfrak{m}) = \mathfrak{m} \Leftrightarrow \mathfrak{m}^2 = x\mathfrak{m}$. Moreover,
- (**) $v(x^{-1}\mathfrak{m}) \subseteq \mathbf{N} \Rightarrow x^{-1}\mathfrak{m}\mathfrak{C} \subseteq \mathfrak{C} \Rightarrow \mathfrak{m}\mathfrak{C} = x\mathfrak{C} \Rightarrow \mathfrak{C} \subseteq (xR : \mathfrak{m})$.

Considering the chain of ideals

$$R \supset \mathfrak{m} \supseteq \mathfrak{C} + (x, x_1, \dots, x_{k-1})R \supset \mathfrak{C} + (x, x_1, \dots, x_{k-2})R \supset \dots \supset \mathfrak{C} + xR,$$

we see that $\ell_R(R/(\mathfrak{C} + xR)) = k \Rightarrow \mathfrak{m} = \mathfrak{C} + (x, x_1, \dots, x_{k-1})R$, hence

- (***) $x_i\mathfrak{m} = (xx_i)R + (x_ix_j)R + x_i\mathfrak{C}$ for every $j = 1, \dots, k - 1$. By (*) we have immediately $1 \Leftrightarrow 6$ and $1 \Rightarrow 5$.

$5 \Rightarrow 4$. By the assumption $x_i x_j \in x\mathfrak{m}$, for all $i, j = 1, \dots, k - 1$ and by the obvious inclusion $x_i\mathfrak{C} \subseteq \mathfrak{m}\mathfrak{C} = x\mathfrak{C}$, from (***) we get $x_i\mathfrak{m} \subseteq xR$, then $x_i \in (xR : \mathfrak{m})$.

The implication $4 \Rightarrow 3$ is obvious.

Finally, $3 \Rightarrow 2$ holds by (**). \square

Remark 1.10. It is clear from (1.9) (see the equivalence 1-5) that the condition $y_i + y_j - e \in v(R)$ for every $i, j = 1, \dots, k - 1$, is necessary to have maximal Cohen Macaulay type. Unfortunately, it is not sufficient. For example, if $R = \kappa[[t^6, t^9 + t^{10}, t^{14}, t^{16}, t^{17}, t^{19}]]$, then $k = 2$, $y = 9$, $2y - e = 12 \in v(R)$, but $r = e - 2$. In this case (1.9.5) does not hold, because $(t^9 + t^{10})^2 \notin xR$.

Proposition 1.11. *Let $k = \ell_R(R/(\mathfrak{C} + xR))$, and let $v(R)$ be as in (1.6).*

- (1) $r = e - k$, $k \geq 2$, $\Leftrightarrow v(R) \setminus v(xR : \mathfrak{m}) = \{0, y_1, \dots, y_{k-1}\}$.
- (2) If $r < e - 1$, then
 - (a) $2y_1 < c + e$,
 - (b) $p \leq 2l_1 + 2$ and $p = 2l_1 + 2 \Rightarrow h > 0$.
- (3) If $r = e - k$, then
 - (a) $y_1 + y_j < c + e$, for every $j = 1, \dots, k - 1$.
 - (b) $p \leq l_1 + l_{k-1} + 2$ and $p = l_1 + l_{k-1} + 2 \Rightarrow h > 0$.
- (4) If $p \geq 3$ and i is such that $l_i = 0$, then $2y_i > c + e$.

Proof. (1) By means of (**) stated in the proof of (1.9), we have the inclusions $(\mathfrak{C} + xR) \subseteq (xR : \mathfrak{m}) \subseteq R$. Since $e - r = \ell_R(R/(xR : \mathfrak{m}))$ and $k = \ell_R(R/(\mathfrak{C} + xR))$, it follows that $e - r = k \Leftrightarrow (\mathfrak{C} + xR) = (xR : \mathfrak{m})$. To see (2.a), suppose $2y_1 \geq c + e$; then $y_i + y_j \geq c + e$ for every $i, j = 1, \dots, k - 1$. Let $x_i \in \mathfrak{m}$ be elements such that $v(x_i) = y_i$, and let $s \in \mathfrak{m}$. If $s \in (\mathfrak{C} + xR)$, then $x_i s \in \mathfrak{m}(\mathfrak{C} + xR) \subseteq xR$. If $s \notin (\mathfrak{C} + xR)$, then $v(s) = y_j$, for some j , $1 \leq j \leq k - 1$; hence, $v(x_i s) = y_i + y_j \geq c + e \Rightarrow x_i s \in x\mathfrak{C} \subset xR$. In both cases $x_i \in (xR : \mathfrak{m})$, and so $y_i \in v(xR : \mathfrak{m})$. Thus $v(R) \setminus v(xR : \mathfrak{m}) = \{0\}$ and $r = e - 1$ by (1.9), a contradiction.

To see (2.b), consider that by (1.3) and (1.6):

$$y_1 \geq c - (l_1 + 1)e = (p - l_1)e - h.$$

Combining this with the preceding (2.a), we obtain

$$(2p - 2l_1)e - 2h \leq 2y_1 < c + e = (p + 2)e - h.$$

Thus $(p - 2l_1 - 2)e < h$ and since $h \leq e - 2$, we see that $p \leq 2l_1 + 2$ and also that $p = 2l_1 + 2 \Rightarrow h > 0$.

To prove (3.a), it suffices to show that $y_1 + y_{k-1} < c + e$. Suppose $y_1 + y_{k-1} \geq c + e$; then $y_i + y_{k-1} \geq c + e$ for all i . Let $x_{k-1} \in \mathfrak{m}$ be an element such that $v(x_{k-1}) = y_{k-1}$. As in (2.a), we get $x_{k-1} \in (xR : \mathfrak{m})$, and so $y_{k-1} \in v(xR : \mathfrak{m})$, a contradiction, since the assumption $e - r = k$ means $v(R) \setminus v(xR : \mathfrak{m}) = \{0, y_1, \dots, y_{k-1}\}$ (see item 1).

We prove now (3b). As in (2.b),

$$y_j \geq c - (l_j + 1)e = (p - l_j)e - h, \quad \text{for } j = 1, \dots, k - 1,$$

and by (3.a)

$$(2p - l_j - l_1)e - 2h \leq y_1 + y_j < c + e = (p + 2)e - h.$$

Hence,

$$(p - l_j - l_1 - 2)e < h \leq e - 2, \quad \text{for every } j = 1, \dots, k - 1.$$

We conclude

$$p \leq l_1 + l_j + 2 \leq l_1 + l_{k-1} + 2$$

and also the last assertion.

For (4), note that $l_i = 0 \Rightarrow y_i + e \geq c$, and that $p \geq 3 \Rightarrow c > 3e$. Thus: $2y_i \geq 2c - 2e = c + (c - 2e) > c + e$, as desired. \square

We may describe the particular case $k = 2$ in a more precise way.

Proposition 1.12. *Assume $k = 2$. With setting (1.6) we have:*

- (1) $r = e - 1 \iff$ one of the following conditions is satisfied:
 - (a) $2y \geq c + e$;
 - (b) $2y = (2q + 1)e < c + e$, $q \geq 1$, $p \geq 2$ and $y \in v(xR : \mathfrak{m})$.
- (2) $r = e - 2 \iff 2y < c + e$ and if $2y = (2q + 1)e$, then $y \notin v(xR : \mathfrak{m})$.

Proof. First recall that by (1.4.1) one has $r \geq e - 2$. For implication \Rightarrow in (1), note that $y \in v(xR : \mathfrak{m})$, by (1.9), and so $2y - e \in v(\mathfrak{m})$. Then regarding the structure of $v(R)$, we have the claim. For the opposite implication, note that in case (a) for any $s \in \mathfrak{m}$ such that $v(s) = y$, $v(x^{-1}s^2) = 2y - e \geq c \Rightarrow x^{-1}s^2 \in \mathfrak{C} \Rightarrow s^2 \in x\mathfrak{m}$; now use again (1.9) to conclude.

(2) is immediate by (1). \square

2. Bounds for the invariant b . Starting from the preliminary result (1.2) we go on in studying the integer b . First (see (2.1)) we find lower and upper bounds using the properties of the type sequence, then (see (2.2)) we express b in terms of the integers k, p, l_i, h occurring in the decomposition of $v(R)$ as in (1.6). This description becomes quite

simple in the particular cases $k = 2, 3$ (see (2.3) and (2.4)). The last result of the present section (see (2.5)) furnishes information according to the range $(q-1)(r-1) < b \leq q(r-1)$ that will be basic in the next section.

Proposition 2.1. *With Notation 1.3, the following facts hold.*

- (1) $(e-r-1)(r-1) \leq rk - e + 1 \leq b - \sum_{i \in A} (r - r_i) \leq k(r-1)$.
- (2) $b = (k-1)(r-1) + \sum_{i \in A} (r - r_i) \Leftrightarrow \sum_{i \in B} r_i = e-1$ and $k = e-r$.
- (3) $b = k(r-1) + \sum_{i \in A} (r - r_i) \Leftrightarrow r_i = 1$ for every $i \in B$.
- (4) *The following conditions are equivalent:*
 - (a) $b = (e-r-1)(r-1)$.
 - (b) $b = (k-1)(r-1)$.
 - (c) $e-r = k$, $\sum_{i \in B} r_i = e-1$ and $r_i = r$ for every $i \in A$.

If these conditions hold, then $s_{i_0-1} = c - e$.

- (5) $b \geq (r-1)s$, where $s := |\{i \in [1, n] \mid r_i = 1\}|$.

Proof. (1) Write the invariant $b = \sum_{i=1}^n (r - r_i)$ in the following form:

$$(*) \quad b = \sum_{i \in B} (r - r_i) + \sum_{i \in A} (r - r_i) = rk - \sum_{i \in B} r_i + \sum_{i \in A} (r - r_i).$$

Using that $\sum_{i \in B} r_i \leq e-1$ (see 1.4.2), we obtain

$$(**) \quad rk - (e-1) \leq b - \sum_{i \in A} (r - r_i) \leq k(r-1).$$

Then, since $k \geq e-r$ by (1.4.1), the inequalities of (1) are clear.

(2) Supposing $b - \sum_{i \in A} (r - r_i) = (k-1)(r-1)$ we have by item (1) $(k-1)(r-1) \geq rk - e + 1$; hence, $k \leq e-r$ and since always $k \geq e-r$, it follows that $k = e-r$. From $(*)$ $\sum_{i \in B} r_i = rk - (k-1)(r-1) = k + r - 1 = e-1$. For the converse, it suffices to substitute $\sum_{i \in B} r_i = k + r - 1$ in $(*)$.

(3) Using $(*)$ we have $b - \sum_{i \in A} (r - r_i) = k(r-1) \Leftrightarrow \sum_{i \in B} (r - r_i) = k(r-1)$. Since $r - r_i \leq r-1$ for every i and $k = |B|$, the last fact is equivalent to say that $r_i = 1$ for every $i \in B$.

(4) $a \Rightarrow b$. By (1) we have immediately $\sum_{i \in A} (r - r_i) = 0$ and $(e - r - 1)(r - 1) = rk - e + 1 \Rightarrow e - r = k$, as desired.

$b \Rightarrow c$. By (1) we have $\sum_{i \in A} (r - r_i) \leq b - (rk - e + 1) = -k - r + e \leq 0$, then we can apply item (2) with $\sum_{i \in A} (r - r_i) = 0$.

$c \Rightarrow a$. Substitute in (*) the relations of (c).

The fact $s_{i_0-1} = c - e$ is immediate by (1.4.2).

By applying [10, Corollary 3.13.2], with $I = \mathfrak{C}$ we get (5). \square

Utilizing the description of the value set $v(R)$ introduced in (1.6), we obtain the next useful formula for the invariant b .

Theorem 2.2. *With Setting 1.6, assume $k > 1$. The following equalities hold:*

(1) $b = (r + 1) \sum_1^{k-1} (l_i + 1) - (p + 1)(e - r - 1) + h = X + Y + Z$ where

$$X := (k - 1)(r - 1) \geq 0,$$

$$Y := k - (e - r) \geq 0,$$

$$Z := (r + 1) \left(p + \sum_1^{k-1} l_i \right) + k + h - pe - 1 \geq \sum_{i \in A} (r - r_i) \geq 0.$$

$$(2) \ c = (p + 1 + \sum_1^{k-1} (l_i + 1))(r + 1) - b.$$

Proof. (1) To get the desired formula it suffices to substitute in the equality $b = (c - \delta)r - \delta$ the expressions of $c - \delta$ and δ given in (1.8.3). The positivity of Y is clear by (1.4.1). To prove the positivity of Z we use the second inequality of (2.1.1): $X + Y = kr - e + 1 \leq b - \sum_{i \in A} (r - r_i)$, and so we have the conclusion: $Z = b - (X + Y) \geq \sum_{i \in A} (r - r_i) \geq 0$.

(2) Since $b + c = (r + 1)(c - \delta)$, (2) follows easily. \square

Lemma 2.3. *Case $k = 2$. With Setting 1.6, assume $k = 2$.*

(1) *If $r = e - 1$, then $b = (l + 1)e + h \leq (l + 2)e - 2$. Further:*

$$b = (l + 2)e - 2 \Leftrightarrow h = e - 2.$$

(2) If $r = e - 2$, then:

$$b = (l + 1)(e - 1) + h - p - 1, \quad c = (p + l + 2)(e - 1) - b.$$

Further we have:

(a) $l + 1 \leq p \leq 2l + 2$ and $p = 2l + 2 \Rightarrow h > 0$.

(b) $(l + 1)(e - 3) \leq b \leq (l + 1)(e - 2) + e - 3$. In particular,

$$b = (l + 1)(e - 3) \Leftrightarrow p = 2l + 2, h = 1 \text{ or } p = 2l + 1, h = 0.$$

$$b = (l + 1)(e - 2) + e - 3 \Rightarrow p = l + 1, p > 1,$$

$$h = e - 2, \quad y = e + 1.$$

Proof. For $k = 2$, we write $v(R) = \{0, e, 2e, \dots, pe, c, \rightarrow\} \cup \{y, y + e, \dots, y + le\}$, with $r \in \{e - 2, e - 1\}$, $c - \delta = p + 2 + l$, $c = (p + 1)e - h$. (See (1.8), (1.6)). Then the expressions of b , in items (1), (2), come from (1.8.4) with $k = 2$ and $e - r = 1, 2$, respectively. To complete the proof of item (1), recall that $h \leq e - 2$.

The bounds for p in item (2) come from (1.8.2) and (1.11.3) and the value of c comes from (2.2.2).

Rewriting b in the form

$$b = (l + 1)(e - 2) + (l - p) + h,$$

and recalling that $l - p \leq -1$, $h \leq e - 2$, we obtain the upper bound for b .

Rewriting b in the form

$$b = (l + 1)(e - 3) + (2l + 2 - p) + (h - 1),$$

and using part a, we obtain the lower bound and also $b = (l + 1)(e - 3) \Leftrightarrow p = 2l + 2, h = 1$ or $p = 2l + 1, h = 0$.

Finally, note that $b = (l + 1)(e - 2) + e - 3 \Rightarrow h = (p - l - 1) + e - 2 \geq e - 2 \Rightarrow p = l + 1$, $h = e - 2 \Rightarrow c = pe + 2$ and since by definition of l , $y + le < c$, it follows that $y < e + 2$, hence $y = e + 1$ and $p > 1$. \square

Lemma 2.4. Case $k = 3$. With Setting 1.6, assume $k = 3$.

(1) If $r = e - 3$, then $b = (l_1 + l_2 + 2)(e - 2) + h - 2(p + 1)$. Moreover,

$$\begin{aligned} p < l_1 + l_2 + 2 &\implies b \geq (l_1 + l_2 + 2)(e - 4) + h, \quad h \geq 0. \\ p = l_1 + l_2 + 2 &\implies b = (l_1 + l_2 + 2)(e - 4) + h - 2, \quad h > 0. \end{aligned}$$

(2) If $r = e - 2$, then $b = (l_1 + l_2 + 2)(e - 1) + h - p - 1$ and $p \leq 2l_1 + 2$. Further, $p = 2l_1 + 2 \implies h > 0$.

(3) If $r = e - 1$, then $b = (l_1 + l_2 + 2)e + h$.

Proof. Recall that by (1.4) $e - r \leq 3$ and by (1.6)

$$v(R) = \{0, e, 2e, \dots, pe, c, \rightarrow\} \cup \{y_1, \dots, y_1 + l_1 e\} \cup \{y_2, \dots, y_2 + l_2 e\}.$$

Formula (2.2.1) with $k = 3$ becomes

$$b = (r + 1)(l_1 + l_2 + 2) - (p + 1)(e - r - 1) + h.$$

By substituting r with $e - 1, e - 2, e - 3$, we get the desired expressions for b in items (3), (2), (1), respectively. To complete the proof of (1) and (2), apply (1.11.3) and (1.11.2), respectively. \square

Proposition 2.5. *Let $q \in \mathbf{N}$ be such that $0 < b \leq q(r - 1)$. Then:*

(1) $r \geq e - q - 1$. In particular,

(a) *If $r = e - 1 - q$, then $b = q(r - 1)$, $q \leq e - 3$, and the equivalent conditions of (2.14) hold.*

(b) *If $r \geq e - q$, then $e - r \leq k \leq q$.*

(2) *If $(q - 1)(r - 1) < b \leq q(r - 1)$, we have:*

(c) $k - 1 \leq q \leq n - 1$.

(d) $(q - k - 1)(r - 1) < \sum_{i \in A} (r - r_i) \leq (q - k)(r - 1) + e - 1 - k$.

Proof. (1) First we deduce the inequalities

$$(\odot) \quad (e - r - 1)(r - 1) \leq b \leq q(r - 1),$$

by combining (2.1.1) with the assumption. Hence we get $r \geq e - q - 1$.

(a) If $r = e - 1 - q$, then relations (\odot) give $b = (e - r - 1)(r - 1)$, and so the conditions of (2.1.4) hold.

(b) Assertion $(**)$ in the proof of (2.1) insures that $kr - (e - 1) \leq b$. Hence assuming $r \geq e - q$, we have $rk \leq b + e - 1 \leq q(r - 1) + e - 1 \leq (q + 1)r - 1$; then $k \leq q$.

(2) Put $M := \sum_{i \in A} (r - r_i)$. We have to compare the two inequalities of (2.1.1)

$$(k - 1)(r - 1) + M + k - (e - r) \leq b \leq k(r - 1) + M$$

with the assumption

$$(q - 1)(r - 1) < b \leq q(r - 1).$$

We obtain the following:

$$\begin{aligned} (k - 1)(r - 1) + M + k - (e - r) &\leq q(r - 1), \text{ and also} \\ (q - 1)(r - 1) &< k(r - 1) + M. \end{aligned}$$

The first inequality gives $q \geq k - 1$ and $M \leq (q - k)(r - 1) + e - 1 - k$.

The second one says that $M > (q - k - 1)(r - 1)$. Moreover, combining the hypothesis with (2.1.2) $(q - 1)(r - 1) < b \leq (n - 1)(r - 1)$, and this implies $q \leq n - 1$, as desired. \square

3. Classification. Our aim is now to classify the value sets for one-dimensional local domains having

$$0 \leq b \leq 2(r - 1).$$

On this topic several results are presented in the literature. For semigroup rings $R = \kappa[[t^\alpha; \alpha \in S]]$, $S \subset \mathbf{N}$ a numerical semigroup, Brown and Herzog in [2, Corollary after Theorem 4] illustrate the case $b = 1$. This result can be extended to rings R as in Setting 1.1 (see (3.1)). Successively Delfino in [5, Corollaries 2.11, 2.14] gives a characterization of rings satisfying the condition $b < r - 1$ and exhibits all the possible value sets in the case $b \leq r$, under the additional assumption $r = e - 1$. See also Proposition 2.7 from [3] for a further generalization. An exhaustive description of the cases $0 < b < r - 1$ can be found in [10, Theorem 4.6].

In this section we assume Setting 1.1 and Notation 1.3. Moreover, $t.s.(R)$ will denote the type sequence of R , defined in (1.1).

First we recall in (3.1) and (3.2) the quoted known results, which now become an easy consequence of our preceding statements.

Theorem 3.1. *Case $b = 0$. The following conditions are equivalent:*

- (1) $b = 0$.
- (2) *Either R is Gorenstein, or $v(R) = \{0, e, \dots, pe, (p+1)e \rightarrow\}$.*
- (3) $t.s.(R) = [r, \dots, r]$.

Proof. By (2.1.1) $0 = b \geq (k-1)(r-1)$; hence either $r = 1$ or $k = 1$, and this last condition gives, by (1.5), $v(R) = \{0, e, \dots, pe, (p+1)e \rightarrow\}$, or equivalently, $t.s.(R) = [e-1, \dots, e-1] = [r, \dots, r]$. Hence $1 \implies 2 \iff 3$ are clear. Of course, in the Gorenstein case we have $t.s.(R) = [1, \dots, 1]$. Implication $3 \implies 1$ is immediate by (1.2.1). \square

Theorem 3.2 [10, Theorem 4.6.1]. *Case $0 < b < r-1$. The following facts are equivalent:*

- (1) $0 < b < r-1$.
- (2) $v(R) = \{0, e, \dots, pe, c \rightarrow\}$ with $pe+2 < c \leq (p+1)e$.
- (3) $t.s.(R) = [e-1, e-1, \dots, e-1, r_n], r_n > 1$.

If these conditions hold, then:

$$b < e-2, \quad r = e-1, \quad r_n = e-1-b, \quad k = 1, \quad c = (p+1)e-b.$$

Theorem 3.3. *Case $b = r-1$. If $b = r-1 > 0$, then either $r = e-1$ or $r = e-2$.*

1. Subcase $r = e-1$. *The following facts are equivalent:*

- (a) $b = r-1 > 0$ and $r = e-1$.
- (b) $v(R) = \{0, e, \dots, pe, pe+2 \rightarrow\}$, $e > 2$.
- (c) $t.s.(R) = [e-1, \dots, e-1, 1]$, $e > 2$.
- (d) $b = r-1 > 0$ and $k = 1$.

2. Subcase $r = e-2$. *The following facts are equivalent:*

- (e) $b = r - 1 > 0$ and $r = e - 2$.
- (f) either $v(R) = \{0, e, 2e - 1, 2e, 3e - 1 \rightarrow\}$, $e > 3$, or $v(R) = \{0, e, y, 2e \rightarrow\}$, with $2y < 3e$, $e > 3$.
- (g) either $t.s.(R) = [e - 2, e - 2, 1, e - 2]$, with $e > 3$, or $t.s.(R) = [e - 2, r_2, r_3]$, with $r_2 + r_3 = e - 1$, $e > 3$.
- (h) $b = r - 1 > 0$ and $k = 2$.

Proof. Applying (2.5.1) with $q = 1$, we obtain that $r \geq e - 2$. Further, if $b = r - 1$, then $r = e - 2 \Leftrightarrow k = 2$ by (2.5.1a) and (2.1.4); also, if $b = r - 1$, then $r = e - 1 \Leftrightarrow k = 1$ by (2.5.1b). This proves the first assertion and the equivalences $a \Leftrightarrow d$, $e \Leftrightarrow h$.

(1) First note that (a) implies $e > 2$; in fact, $e = 2$ would imply $r = 1, b = 0$.

$d \Rightarrow b$. Since $k = 1$, the equivalent conditions of (1.5) hold, and $v(R) = \{0, e, \dots, pe, c \rightarrow\}$, with $c = (p + 1)e - b = pe + 2$, $e > 2$.

$b \Rightarrow c$. If (b) holds, then by (1.5) $t.s.(R) = [e - 1, \dots, e - 1, r_n]$, with $r - r_n = b = r - 1$, hence $r_n = 1$, as in (c).

$c \Rightarrow a$. By (1.2.1), (c) implies $r = e - 1$ and $b = r - r_n = r - 1$, as in (a).

(2) First note that (a) implies $e > 3$; in fact, $e = 3$ would imply $r = 1$, $b = 0$.

$h \Rightarrow f$. Since $k = 2$ we use (2.3.2) recalling that $p \leq 2l + 2$:

$$e - 3 = b = (l + 1)(e - 1) + h - p - 1 \geq (l + 1)(e - 1) + h - 2l - 3.$$

Hence we get $l(e - 3) + h \leq 1$ and the following possibilities occur by (2.3.2c):

$$(l, p, h) = (0, 1, 0), \text{ or } (l, p, h) = (0, 2, 1), \text{ or } h = 0, e = 4, l = 1.$$

(i) If $(l, p, h) = (0, 1, 0)$, then $c = 2e$, $v(R) = \{0, e, y, 2e \rightarrow\}$ with $2y < 3e$, $e > 3$.

(ii) If $(l, p, h) = (0, 2, 1)$, then $v(R) = \{0, e, 2e, c \rightarrow\} \cup \{y\}$, with $c - \delta = 4$,

$$c = (p + 1)e - h = 3e - 1, \quad 2y < c + e = 4e - 1 \implies y \leq 2e - 1,$$

$c - e \in v(R) \Rightarrow y = 2e - 1$. Hence, $v(R) = \{0, e, 2e - 1, 2e, 3e - 1 \rightarrow\}$, $e > 3$.

(iii) If $h = 0$, $e = 4$, $l = 1$, then $e - 3 = b = (l + 1)(e - 1) + h - p - 1 \Rightarrow p = 4 = 2l + 2 \Rightarrow h > 0$, which is absurd. Hence, $h \Rightarrow f$ is proved.

$f \Rightarrow g$. Denoting $R_0 = \kappa[[t^d, d \in v(R)]]$ the monomial ring such that $v(R_0) = v(R) = \{0, e, 2e - 1, 2e, 3e - 1 \rightarrow\}$, we have $r(R_0) = e - 2$. Since $r(R) \leq r(R_0)$ and $r(R) \geq e - 2$ by (1.4.1), we conclude that $r(R) = e - 2$. The other invariants are easily derived from $v(R)$: $c - \delta = 4$, $\delta = 3e - 5$, $b = (c - \delta)r - \delta = e - 3$. By substituting in (2.1.1), we obtain $\sum_{h \in A} (r - r_h) = 0$; hence $r_2 = e - 2$ and $r_3 + r_4 = e - 1$, as desired. The same reasoning holds for $v(R) = \{0, e, y, 2e \rightarrow\}$.

To see $g \Rightarrow e$, it suffices to recall that $b = \sum_{h=1}^n (r - r_h)$, see (1.2.1). \square

Theorem 3.4. Case $r - 1 < b < 2(r - 1)$. We have $r - 1 < b < 2(r - 1)$ if and only if $v(R)$ is one of the following:

(1) $v(R) = \{0, e, \dots, pe, c \rightarrow\} \cup \{y\}$, with $y \notin e\mathbf{Z}$, and either $2y \geq c + e$, $pe + 5 \leq c \leq \min\{y + e, (p + 1)e\}$, $e \geq 5$, or $e = 2e'$, $y = 3e'$, $p = 2$, $4e' + 5 \leq c \leq 5e'$, $e \geq 10$, $y \in v(xR : \mathfrak{m})$.

(2) $v(R) = \{0, e, 2e, c \rightarrow\} \cup \{y\}$, with $y \notin e\mathbf{Z}$, $2y < c + e$ and:

if $2y \neq 3e$, then $2e + 3 \leq c \leq 3e - 2$, $e \geq 5$;

if $2y = 3e$, then $e = 2e'$, $4e' + 3 \leq c \leq 5e'$, $e \geq 6$, $y \notin v(xR : \mathfrak{m})$.

(3) $v(R) = \{0, e, y, c \rightarrow\}$, with $y \notin e\mathbf{Z}$, $e \geq 5$, $2y < c + e$, $e + 4 \leq c \leq 2e - 1$.

In each case $k = 2$; in case (1), $r = e - 1$ and $b \geq r + 1$; in cases (2) and (3), $r = e - 2$.

Proof. Assume $r - 1 < b < 2(r - 1)$.

Step 1. Claim: $k = 2$ and $e - 2 \leq r \leq e - 1$, $r > 2$.

We have $r > 2$, since $r = 2 \Rightarrow 1 < b < 2$, which is absurd. Further (2.1.1) gives $(k - 1)(r - 1) \leq b$, and so $k \leq 2$. But $k = 1$ would imply $b \leq r - 1$ by (1.5), then $k = 2$. We conclude using (1.4.1).

Now utilizing the notation in (1.6) we write:

$$(*) \quad \begin{cases} v(R) = \{0, e, 2e, \dots, pe, c, \rightarrow\} \cup \{y, y+e, \dots, y+le\}, \\ p \geq 1, \ y > e, \ y \notin e\mathbf{Z}, \\ y+le < c = (p+1)e - h \leq y + (l+1)e, \ l+1 \leq p. \end{cases}$$

Step 2. Claim: $l = 0$ and $e \geq 5$. Further, if $r = e - 2$, then $p \leq 2$.

(i) If $r = e - 1$, then, by (2.3.1) we know that $b = (l+1)e + h$, $l, h \geq 0$. Hence $b < 2(r-1) = 2e - 4 \Rightarrow (l+1)e + h < -4 \Rightarrow l = 0$, $h < e - 4$; further, we get : $c = (p+1)e - h \geq pe + 5$, $e \geq 5$ and $b = h + e \geq e = r + 1$.

(ii) If $r = e - 2$, we have $(l+1)(e-3) \leq b$ and $l+1 \leq p \leq 2l+2$ by (2.3.2). Then $b < 2(r-1) = 2(e-3) \Rightarrow l = 0$ and $p \leq 2$; also, the assumption $e-3 < b < 2e-6$ implies $e \geq 5$.

Step 3. When $r = e - 1$, recalling the relations proved in Step 2, we obtain $v(R) = \{0, e, \dots, pe, c \rightarrow\} \cup \{y\}$, with $e \geq 5$, $pe + 5 \leq c$, as in item (1). Recall that by definition of p and l we have $c \leq (p+1)e$ and $c \leq y + e$. Moreover, by (1.12.1) one of the following conditions is satisfied:

either (a) $2y \geq c + e$ or (b) $2y = (2q+1)e < c + e$, $p \geq 2$ and $y \in v(xR : \mathfrak{m})$. Further, as noted in (1.11.4), $p \geq 3$, $l = 0 \Rightarrow 2y > c + e$; hence, in case (b) we have $p = 2$ and consequently $(2q+1)e < c + e \leq 4e \Rightarrow q = 1$. This proves (1).

Step 4. When $r = e - 2$, we have by Step 2 that $l = 0$ and $p \leq 2$.

In the case $p = 2$ we get item 2. In fact from (2.3.2) we obtain $c = 4e - 4 - b$ and the bounds for c follow at once. The last assertion in item 2 comes from (1.12). Analogously, in the case $p = 1$ we get item 3. Notice that when $p = 1$ we cannot have $2y = 3e < c + e$, because $c + e \leq 3e - 1$.

To complete the proof, let $v(R)$ be as in items (1), (2), (3); we claim that $r - 1 < b < 2(r - 1)$. In every case $k = 2$; in case (1) $r = e - 1$ and in cases (2), (3) $r = e - 2$ by (1.12). The rest is a direct computation based on relation (2.2.2): $c = (p+2)(r+1) - b$. \square

Example 3.5. We supply an example for each case of the above proposition.

- Case (1) with $2y \geq c + e$. Let $R = \kappa[[t^5, t^{10}, t^{12}, t^{15}, \rightarrow]]$. Then: $y = 12, p = 2, c = 15, r = 4, b = 5$.
- Case (1) with $2y = 3e$. Let $R = \kappa[[t^{10}, t^{15}, t^{20}, t^{25}, \rightarrow]]$. Then: $y = 15, p = 2, c = 25, r = 9, b = 15$.
- Case (2). Let $R = \kappa[[t^{10}, t^{15} + t^{16}, t^{20}, t^{25}, \rightarrow]]$. As above, $y = 15, p = 2, c = 25$, but $r = 8$ by 1.9 since $(t^{15} + t^{16})^2 \notin x\mathfrak{m}$. Then $b = 11$.
- Case (3). Let $R = \kappa[[t^5, t^6, t^9, \rightarrow]]$. Here $y = 6, p = 1, c = 9, r = 3, b = 3$.

Theorem 3.6. Case $b = 2(r - 1)$. $b = 2(r - 1) > 0$ if and only if $v(R)$ is one of the following:

1. (a) $v(R) = \{0, e, e + 2, e + 4 \rightarrow\}, e \geq 4$.
 (b) $v(R) = \{0, e, 2e, 2e + 4 \rightarrow\} \cup \{y\}, e \geq 4, y \in v(xR : \mathfrak{m})$.
 (c) $v(R) = \{0, e, 2e, \dots, pe, pe + 4 \rightarrow\} \cup \{y\}, e \geq 4, y \geq (p - 1)e + 4, p \geq 3$.
2. (a) $v(R) = \{0, e, e + 1, e + 3 \rightarrow\}, e \geq 4$.
 (b) $v(R) = \{0, e, y, 2e, 2e + 2 \rightarrow\}, e \geq 5, 2e + 4 \leq 2y < 3e + 2, 2y \neq 3e$.
 (c) $v(R) = \{0, e, 2e, 3e - 1, 3e, 4e - 1, 4e, 5e - 1 \rightarrow\}, e \geq 4$.
 (d) $v(R) = \{0, e, 2e, y, 3e, y + e, 4e \rightarrow\}, e \geq 4, 2y < 5e$.
3. (a) $v(R) = \{0, e, y_1, y_2, 2e \rightarrow\}, e \geq 5, y_1 + y_2 < 3e$.
 (b) $v(R) = \{0, e, 2e - 2, 2e - 1, 2e, 3e - 2 \rightarrow\}, e \geq 5$.

Further: in case 1, $r = e - 1$ and $\ell_R(R/(\mathfrak{C} + xR)) = 2$; in case 2, $r = e - 2$ and $\ell_R(R/(\mathfrak{C} + xR)) = 2$; in case 3, $r = e - 3$ and $\ell_R(R/(\mathfrak{C} + xR)) = 3$.

Proof. Let, as above, $k = \ell_R(R/(\mathfrak{C} + xR))$. First we assume $b = 2(r - 1) > 0$ and we observe that by (2.1.1) $(k - 1)(r - 1) \leq b = 2(r - 1)$, then $k \leq 3$. Since $k = 1$ implies $b \leq r - 1$ by (1.5), one of the following cases occurs:

$$\left[\begin{array}{lll} \text{or} & k = 2 & \text{and} \quad r = e - 1 \\ \text{or} & k = 2 & \text{and} \quad r = e - 2 \\ \text{or} & k = 3 & \text{and} \quad r = e - 3. \end{array} \right.$$

In case $k = 2$ by Setting 1.6 we have:

$$(*) \quad \begin{cases} v(R) = \{0, e, 2e, \dots, pe, c, \rightarrow\} \cup \{y, y+e, \dots, y+le\}, \\ p \geq 1, y > e, y \notin e\mathbf{Z}, \\ y+le < c = (p+1)e - h \leq y + (l+1)e, l+1 \leq p. \end{cases}$$

Step 1. Assuming $r = e - 1$ and $k = 2$, we prove that $v(R)$ has the form described in item 1. By (2.3.1) and the assumption we have the equalities $b = (l+1)e + h = 2e - 4$; hence $(l-1)e + h = -4 \Rightarrow l = 0$, $h = e - 4$, $e \geq 4$, $c = (p+1)e - h = pe + 4$. Now, $l = 0 \Rightarrow y \geq c - e = (p-1)e + 4$, and so $v(R) = \{0, e, \dots, pe, pe+4 \rightarrow\} \cup \{y\}$, with $(p-1)e + 4 \leq y \leq pe + 2$, $e \geq 4$. For $p = 1$ we get (1.a). In fact, by (1.12.1) $2y \geq c + e = 2e + 4 \Rightarrow y \geq e + 2 \Rightarrow y = e + 2$. For $p = 2$ we get (1.b). For $p \geq 3$ we get (1.c).

Step 2. Assuming $r = e - 2$ and $k = 2$, we prove that $v(R)$ satisfies item 2. First, by (2.3.2) we have that $l+1 \leq p \leq 2l+2$ and also that

$$(**) \quad (l+1)(e-3) \leq (l+1)(e-1) + h - p - 1 = b.$$

Then $b = 2(e-3) > 0$ implies $(l+1)(e-3) \leq 2(e-3)$, i.e., $l \leq 1$.

Case $l = 0$, and consequently $1 \leq p \leq 2$.

(\cdot) If $l = 0$, $p = 1$, then by (**), $h = e - 3$; thus, $c = e + 3$, and (2.a) holds.

(\cdot) If $l = 0$, $p = 2$, then $h = e - 2$ and $c = 2e + 2$; hence, (2.b) holds.

Case $l = 1$. Now, relation (**) combined with the assumption $b = 2e - 6$ implies $h - p - 1 = -4$, $2 \leq p \leq 4$ and two possibilities occur:

(\cdot) $p = 4$, $h = 1$, $c = 5e - 1$. The relation $c \leq y + (l+1)e$ gives $y \geq 3e - 1$, the relation $2y < c + e$ gives $y \leq 3e - 1$. Hence (2.c) holds.

(\cdot) $p = 3$, $h = 0$, $c = 4e$; hence, (2.d) holds.

Step 3. Assuming $r = e - 3$ and $k = 3$, we prove that $v(R)$ has the form described in item 3. First, by Setting 1.6 and by (2.4.1) we have:

$$(\overline{*}) \quad \begin{cases} v(R) = \{0, e, \dots, pe, c \rightarrow\} \cup \{y_1, y_1 + e, \dots, y_1 + l_1 e\} \\ \quad \cup \{y_2, y_2 + e, \dots, y_2 + l_2 e\} \\ p \geq 1, y_2 > y_1 > e, y_i \notin e\mathbf{Z}, \\ y_i + l_i e < c = (p+1)e - h \leq y_i + (l_i + 1)e, l_i + 1 \leq p, \\ b = (l_1 + l_2 + 2)(e - 2) + h - 2(p + 1). \end{cases}$$

By (1.11.3), since $r = e - k$, then $p \leq l_1 + l_2 + 2$.

(\cdot) If $p < l_1 + l_2 + 2$, then substituting $b = 2(e - 4) > 0$ in $(\bar{*})$ we get $(l_1 + l_2)(e - 4) + h \leq 0$, $h \geq 0$. Hence, $h = l_1 = l_2 = 0$, $p = 1$, $c = 2e$, $y_1 + y_2 < c + e$ by (1.11.3), and so we have (3.a).

(\cdot) If $p = l_1 + l_2 + 2$, then analogously we get $(l_1 + l_2)(e - 4) + h - 2 = 0$, with $0 < h \leq 2$. The case $h = 1$ is impossible. In fact, $h = 1 \Rightarrow l_1 + l_2 = 1$ (in particular, by (1.8.2), $l_2 \leq l_1$; hence, $l_2 = 0$, $l_1 = 1$), $e = 5$, $p = 3$, $c = (p + 1)e - h = 19$. The relation of (1.6) $c \leq y_i + (l_i + 1)e$ gives $y_1 \geq 19 - 10 = 9$, $y_2 \geq 19 - 5 = 14$, but $y_1 + y_2 < c + e = 24$ by (1.11.3); the only possibility would be $y_1 = 9$, $y_2 = 14$. Absurd that $\overline{y_1} = \overline{y_2} \pmod{5}$. Hence, $h = 2$, $l_1 = l_2 = 0$, $p = 2$, $c = 3e - 2$ and

$$v(R) = \{0, e, 2e, 3e - 2, \rightarrow\} \cup \{y_1, y_2\}.$$

Since $l_1 = 0$, the bound $c \leq y_1 + e$ gives $y_1 \geq 2e - 2$. Recalling that by (1.11.3) $y_1 + y_2 < c + e$, we conclude $y_1 = 2e - 2$, $y_2 = 2e - 1$, as in (3.b).

Vice versa, we assume in the following $v(R)$ having the form described in items 1,2,3, and we prove that $b = 2(r - 1) > 0$.

For a $v(R)$ as in item 1 we see that $r = e - 1$ using (1.12). In fact, in case (1.a) we have $y = e + 2$, $2y = c + e$ and in case (1.c):

$$2y \geq 2(p - 1)e + 8 > c + e = (p + 1)e + 4.$$

In conclusion in each case of item 1 we have $\ell_R(R/(\mathfrak{C} + xR)) = 2$, $r = e - 1$, $l = 0$. Using (2.3.1) $b = e + h = 2e - 4 = 2(r - 1)$, as desired.

In case (2.a), $y = e + 1 \notin v(xR : \mathfrak{m})$, then $r = e - 2$ by (1.9). In case (2.b) by hypothesis $2y < c + e$ and $2y \neq 3e$, then $r = e - 2$ by (1.12). In case (2.c) we get by a direct calculation $v(xR_0 : \mathfrak{m}) \setminus v(R_0) = \{4e + 1, \dots, 5e - 2\}$; then $r = r(R_0) = e - 2$. In case (2.d) $2y < c + e$ and $2y \notin e\mathbb{Z}$; then $r = e - 2$ by (1.12). In conclusion, in each case of item 2 one has: $\ell_R(R/(\mathfrak{C} + xR)) = 2$, $r = e - 2$, and so by (2.3.2) $b = (l + 1)e - 1 + h - p - 1$. Putting in this formula

(\cdot) $l = 0$, $p = 1$, $h = e - 3$, in case (2.a),

(\cdot) $l = 0$, $p = 2$, $h = e - 2$, in case (2.b),

(\cdot) $l = 1$, $p = 4$, $h = 1$, in case (2.c),

(\cdot) $l = 1, p = 3, h = 0$, in case (2.d),

we get $b = 2e - 6 = 2(r - 1)$, as desired.

In both cases of item 3 we have $r = e - 3$. In fact, $y_1 + y_2 - e \notin v(R) \Rightarrow y_1, y_2 \notin v(xR : \mathfrak{m}) \Rightarrow e - r = 3$ by (1.11.1). Hence $\ell_R(R/(\mathfrak{C} + xR)) = 3$, $r = e - 3$, $l_1 = l_2 = 0$, and by (2.4.3) $b = 2(e - 2) + h - 2(p + 1)$. Putting in this formula

(\cdot) $h = 0, p = 1$ in case (3.a),

(\cdot) $h = p = 2$ in case (3.b),

we get $b = 2e - 8 = 2(r - 1)$, as desired. \square

With similar arguments one can evaluate the semigroups $v(R)$ of rings having $b > 2(r - 1)$. For instance, if $2(r - 1) < b \leq 3(r - 1)$ there are few possible cases and the classification is tedious but easy. Now, for each $q \geq 3$ we construct a family of rings of multiplicity e and Cohen Macaulay type $r = e - 1$ having $b = q(r - 1)$ or $(q - 1)(r - 1) < b < q(r - 1)$.

Example 3.7. Let $q \geq 3$. Following notations of Setting 1.6 we consider $v(R) = \{0, e, 2e, \dots, pe, c \rightarrow\} \cup \{y, y + e, \dots, y + le\}$, with $e > p$, $p = 2q$, $l = q - 2$. In this case $k = 2$. Using (1.12) we see that $r = e - 1$, because $y + (q - 1)e \geq c > 2qe \Rightarrow y > (q + 1)e \Rightarrow 2y > 2(q + 1)e \geq c + e$. Then by (2.3.1) $b = (q - 1)e + h$, with $0 \leq h \leq e - 2$. Now, with an additional hypothesis on the conductor, we are in goal. In fact:

1) Assuming $c = pe + p$, we have $h = (p + 1)e - c = -p + e = -2q + e$; then $b = (q - 1)e + (-2q + e) = q(e - 2) = q(r - 1)$.

2) Assuming $c > pe + p$, i.e., $e - h > 2q$, we have $(q - 1)(e - 2) < (q - 1)e \leq b = (q - 1)e + h = q(e - 2) + 2q - e + h < q(e - 2)$; hence $(q - 1)(r - 1) < b < q(r - 1)$.

As a further application of the previous results we describe exhaustively the cases $b = 1$ and $b = 2$ (see next (3.8), (3.9); for $b = 1$ see also [2, Section 4]). With regard to the formula

$$b = \sum_{i=1}^n (r - r_i)$$

it becomes natural to consider the invariant b as a measure of how far is the type sequence $[r_1, \dots, r_n]$ from the maximal one $[r, \dots, r]$. For instance, for $b = 1$ one expects a type sequence of the form $[r, \dots, r-1, \dots, r]$, for $b = 2$ $[r, \dots, r-1, \dots, r-1, \dots, r]$ or $[r, \dots, r-2, \dots, r]$, and so on. Surprisingly, after finding by a direct computation all the possible value sets and the corresponding type sequences, we discover that very few choices are possible. For $b = 1$ (respectively $b = 2$) either $e \leq 4$ (respectively $e \leq 5$) or $t.s.(R) = [e-1, \dots, e-1, e-1-b]$.

Corollary 3.8. *Case $b = 1$. Here $t.s.$ stands for $t.s.(R)$. $b = 1$ if and only if $v(R)$ is one of the following:*

$$\begin{aligned} v(R) &= \{0, 4, 7, 8, 11 \rightarrow\}, \quad \text{with } t.s.[2, 2, 1, 2]; \\ v(R) &= \{0, 4, 5, 8, \rightarrow\}, \quad \text{with } t.s.[2, 1, 2]; \\ v(R) &= \{0, e, \dots, pe, (p+1)e-1, \rightarrow\}, \quad e \geq 3, \\ &\quad \text{with } t.s.[e-1, \dots, e-1, e-2]. \end{aligned}$$

Proof. First recall that $b > 0 \Rightarrow r > 1$ by (1.2.1). Let, as in (2.2.1), $b = X + Y + Z$, where $X := (k-1)(r-1) \geq 0$, $Y := k - (e-r) \geq 0$, and $Z := (r+1)(p + \sum_1^{k-1} l_i) + k + h - pe - 1 \geq 0$. Assuming $b = 1$, we have to consider the choices:

	X	Y	Z
a)	1	0	0
b)	0	1	0
c)	0	0	1

In a) $k = r = 2$, $2 - (e-2) = 0 \Rightarrow e = 4$. By (3.3.2) with $e = 4$ we find:

$$\begin{aligned} v(R) &= \{0, 4, 7, 8, 11 \rightarrow\}, \\ v(R) &= \{0, 4, 5, 8, \rightarrow\}. \end{aligned}$$

In b) $k = 1$, $1 - (e-r) = 1$, which is absurd. In c) $k = 1$, $1 - (e-r) = 0 \Rightarrow r = e-1$, $e \geq 3$, $Z = ep+1+h-pe-1 = 1 \Rightarrow h = 1 \Rightarrow c = (p+1)e-1$. By (1.5) we find:

$$v(R) = \{0, e, \dots, pe, (p+1)e-1, \rightarrow\}, \quad e \geq 3. \quad \square$$

Corollary 3.9. Case $b = 2$. As above, $t.s.$ stands for $t.s.(R)$. $b = 2$ if and only if $v(R)$ is one of the following:

$$\begin{aligned}
v(R) &= \{0, 4, 5, 7, \rightarrow\}, \text{ with } t.s.[2, 1, 1]; \\
v(R) &= \{0, 4, 8, 11, 12, 15, 16, 19, \rightarrow\}, \text{ with } t.s.[2, 2, 2, 1, 2, 1, 2]; \\
v(R) &= \{0, 4, 8, 9, 12, 13, 16, \rightarrow\}, \text{ with } t.s.[2, 2, 1, 2, 1, 2]; \\
v(R) &= \{0, 5, 9, 10, 14, \rightarrow\}, \text{ with } t.s.[3, 3, 1, 3]; \\
v(R) &= \{0, 5, 6, 10, \rightarrow\}, \text{ with } t.s.[3, 1, 3]; \\
v(R) &= \{0, 5, 7, 10, \rightarrow\}, \text{ with } t.s.[3, 2, 2]; \\
v(R) &= \{0, 5, 6, 7, 10, \rightarrow\}, \text{ with } t.s.[2, 1, 1, 2]; \\
v(R) &= \{0, 5, 6, 8, 10, \rightarrow\}, \text{ with } t.s.[2, 2, 1, 1]; \\
v(R) &= \{0, 5, 8, 9, 10, 13, \rightarrow\}, \text{ with } t.s.[2, 2, 1, 1, 2]; \\
v(R) &= \{0, e, \dots, pe, (p+1)e-2, \rightarrow\}, \quad e \geq 4, \\
&\quad \text{with } t.s.[e-1, \dots, e-1, e-3].
\end{aligned}$$

Proof. As in the preceding proof, assuming $b = 2$, we have to consider the following choices:

	X	Y	Z
a)	0	1	1
b)	1	0	1
c)	1	1	0
d)	2	0	0
e)	0	2	0
f)	0	0	2

First recall that $k = 1 \Rightarrow r = e - 1$ by (1.5), and so $X = 0$ (with $r > 0$) $\Rightarrow k - (e - r) = Y = 0$ and cases a) and e) are impossible.

In b) $X = 1 \Rightarrow k = r = 2$, $2 - (e - 2) = Y = 0 \Rightarrow e = 4$; hence, $b = 2(r - 1)$ and we can apply (3.6.2) with $e = 4$. We find:

$$\begin{aligned}
v(R) &= \{0, 4, 5, 7, \rightarrow\}, \\
v(R) &= \{0, 4, 8, 11, 12, 15, 16, 19, \rightarrow\}, \\
v(R) &= \{0, 4, 8, 9, 12, 13, 16, \rightarrow\}.
\end{aligned}$$

In c) $X = 1 \Rightarrow k = r = 2$, $2 - (e - 2) = Y = 1 \Rightarrow e = 3$, $Z = 3(p + l) + 2 + h - 3p - 1 = 0 \Rightarrow 3l + h + 1 = 0$, which is absurd.

In d) the condition $X = (k - 1)(r - 1) = 2$ implies two possibilities:

d_1) $k = 2, r = 3, 2 - (e - 3) = 0 \Rightarrow e = 5$. We are in case $b = r - 1, r = e - 2$. By (3.3.2) with $e = 5$ we find:

$$v(R) = \{0, 5, 9, 10, 14, \rightarrow\},$$

$$v(R) = \{0, 5, 6, 10, \rightarrow\},$$

$$v(R) = \{0, 5, 7, 10, \rightarrow\}.$$

d_2) $k = 3, r = 2, e = 5$. We are in case $b = 2(r - 1), r = e - 3$, and so by (3.6.3) with $e = 5$ we find:

$$v(R) = \{0, 5, 6, 7, 10, \rightarrow\},$$

$$v(R) = \{0, 5, 6, 8, 10, \rightarrow\},$$

$$v(R) = \{0, 5, 8, 9, 10, 13, \rightarrow\}.$$

In f) $k = 1, r = e - 1, Z = ep + 1 + h - pe - 1 = 2 \Rightarrow h = 2 \Rightarrow c = (p + 1)e - 2$. By (1.5) we find:

$$v(R) = \{0, e, \dots, pe, (p + 1)e - 2, \rightarrow\}, \quad e \geq 4. \quad \square$$

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DITPEM, UNIVERSITÀ DI GENOVA, P.LE KENNEDY, PAD. D – I 16129 GENOVA, ITALY

Email address: oneto@diptem.unige.it

DIMA, UNIVERSITÀ DI GENOVA, VIA DODECANESO 35–I 16146 GENOVA, ITALY

Email address: zatini@dima.unige.it