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A LOWER BOUND OF STANLEY DEPTH OF MONOMIAL IDEALS

RYOTA OKAZAKI

ABSTRACT. Let $S := \mathbf{k}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbf{k}$. In this paper, it is shown that Stanley depth of the monomial ideal of S, generated by m elements, is greater than or equal to $\max\{1, n - \lfloor m/2 \rfloor\}$.

1. Introduction. Let $S := \mathbf{k}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbf{k}$ with indeterminates $x_1, \dots, x_n$, and let M be a finitely generated $\mathbf{Z}^n$-graded S-module. We set $X := \{x_i \mid i = 1, \dots, n\}$. For a homogeneous element $u \in M$ and a subset $Z \subseteq X$, $u\mathbf{k}[Z]$ denotes the $\mathbf{k}$-subspace of M generated by all the homogeneous elements of the form uv, where v is a monomial in $\mathbf{k}[Z]$. The $\mathbf{k}$-subspace $u\mathbf{k}[Z]$ is said to be a *Stanley space of dimension* $|Z|$ if it is a free $\mathbf{k}[Z]$-module, where $|Z|$ denotes the cardinality of Z. A decomposition of M into its Stanley spaces

$$\mathcal{D} : M = \bigoplus_{i=1}^{r} u_i \mathbf{k}[Z_i]$$

as $\mathbf{Z}^n$-graded $\mathbf{k}$-vector spaces is called a *Stanley decomposition* of M, and the *Stanley depth* of $\mathcal{D}$, denoted by $\operatorname{sdepth} \mathcal{D}$, is $\min\{|Z_i| \mid i = 1, \dots, r\}$ by definition. The Stanley depth of M is defined to be the maximal value of Stanley depth of Stanley decompositions of M

$$\max\{\operatorname{sdepth} \mathcal{D} \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

and denoted by $\operatorname{sdepth} M$.

In his paper [**7**], Stanley posed a conjecture, and his conjecture reads as follows;

$$\operatorname{sdepth} M \geq \operatorname{depth} M$$

holds when $M = I$ or I/J for some monomial ideals I, J of S with $J \subseteq I$. Recently, Stanley depth of monomial ideals is studied by

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several authors (**1**–**3**, **6**]); in [**3**], Herzog, Vladoiu and Zheng gave some technique to compute Stanley depth and gave a lower bound of Stanley depth of monomial ideals (Proposition 2.1 in the next section). Apel [**1**] detected that a Borel type monomial ideal satisfies Stanley's conjecture (see [**3**] for the definition of a Borel type ideal) and all the generic monomial ideals in the sense of [**5**] also satisfy the conjecture; hence, complete intersection monomial ideals do in particular. For complete intersection monomial ideals, by Shen [**6**], their Stanley depths are completely determined (Theorem 2.2 in the next section).

For a monomial ideal I, let $G(I)$ denote the set of minimal monomial generators of I. The main result in this paper is the following: for a monomial ideal I, we have

$$\operatorname{sdepth} I \geq n - \left\lfloor \frac{|G(I)|}{2} \right\rfloor . \tag{1.1}$$

Recall that a monomial $v \in S$ is said to be *squarefree* if the exponent of each x_i in v is less than or equal to 1, and a monomial ideal I is said to be *squarefree* if it is generated by squarefree monomials. If I is squarefree, the inequality (1.1) is the question posed by Shen in [**6**], which is a motivation of the study. The author's result also improves Herzog-Vladoiu-Zheng's lower bound stated above. Furthermore, in the case $|G(I)| \leq 4$ and I is squarefree, where the inequality (1.1) is verified by Shen in [**6**], the author's proof is less constructive but more concise than his.

After the author finished the paper, he was told that Keller and Young had already solved Shen's problem [**4**]. However they showed the same inequality under the assumption that I is squarefree, and their proof is more combinatorial. So the author will give the main result and its proof in this paper.

2. Main results. For a monomial ideal, the following lower bound was given by Herzog, Vladoiu and Zhen:

Proposition 2.1 ([**3**, Proposition 3.4]). *Let I be a monomial ideal of S with $|G(I)| = m$. Then*

$$\operatorname{sdepth} I \geq \max\{1, n - m + 1\}.$$

As is stated in the Introduction, the Stanley depth of complete intersection ideals is completely computed by Shen.

Theorem 2.2 ([**6**, Theorem 2.4]). *Let I be a complete intersection monomial ideal with $|G(I)| = m$. Then*

$$\operatorname{sdepth} I = n - \left\lfloor \frac{m}{2} \right\rfloor.$$

The main theorem of this paper is:

Theorem 2.3. *For a monomial ideal I of S with $|G(I)| = m$, we have*

$$\operatorname{sdepth} I \geq \max\left\{1, n - \left\lfloor \frac{m}{2} \right\rfloor\right\}.$$

Let $\operatorname{mod}_{\mathbf{Z}}^{n} S$ denote the category whose objects are finitely generated $\mathbf{Z}^n$-graded S-modules and morphisms are degree-preserving S-homomorphisms, that is, S-homomorphisms $f : M \to N$ such that $f(M_{\mathbf{a}}) \subseteq N_{\mathbf{a}}$ for $\mathbf{a} \in \mathbf{Z}^n$. Clearly, the following holds.

Lemma 2.4. *Given an exact sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in $\operatorname{mod}_{\mathbf{Z}^n} S$*, we have*

$$\operatorname{sdepth} M \geq \min\{\operatorname{sdepth} L, \operatorname{sdepth} N\}.$$

Let $R := \mathbf{k}[x_1, \ldots, x_{n-1}]$. Note that by the natural surjective map $S \twoheadrightarrow R$, a $\mathbf{Z}^{n-1}$-graded R-module has a structure of $\mathbf{Z}^n$-graded S-modules. To prove Theorem 2.3, we shall verify that the following key lemma holds.

Lemma 2.5. *Let $v_1, \ldots, v_m$ be monomials in S. Assume x_n divides v_i for $i = 1, \ldots, r$ but not for $i = r+1, \ldots, m$, where $1 \leq r \leq m-1$. Let*

$\mathfrak{a}, \mathfrak{b}$ *be monomial ideals of* S *generated by* $v_1, \dots, v_r$ *and* $v_{r+1}, \dots, v_m$*, respectively, and set* $I := \mathfrak{a} + \mathfrak{b}$*,* $I' := \mathfrak{a} + x_n\mathfrak{b}$*. Then*

$$I/I' \cong \mathfrak{b} \cap R$$

as $\mathbf{Z}^n$*-graded* S*-modules, where the structure of* $\mathbf{Z}^n$*-graded* S*-modules of* $\mathfrak{b} \cap R$ *is given as above.*

Proof. The inclusion $\mathfrak{b} \subseteq I$ and $x_n\mathfrak{b} \subseteq I'$ induces the S-homomorphism

$$\varphi : \mathfrak{b}/x_n\mathfrak{b} \longrightarrow I/I'.$$

By the construction, its kernel is $(I' \cap \mathfrak{b})/x_n\mathfrak{b}$, and its cokernel is $I/(I' + \mathfrak{b})$. It is, however, clear that $I' \cap \mathfrak{b} \subseteq x_n\mathfrak{b}$ and $I = I' + \mathfrak{b}$, which indicates that φ is an isomorphism. Moreover, the composition of the inclusion $\mathfrak{b} \cap R \to \mathfrak{b}$ and the natural map $\mathfrak{b} \to \mathfrak{b}/x_n\mathfrak{b}$ gives an isomorphism as $\mathbf{Z}^n$-graded S-modules, which completes the proof. $\square$

For a monomial $v \in S$, let $\deg(v)$ denotes the multi-degree of v, and for $\mathbf{a} := (a_1, \dots, a_n) \in \mathbf{Z}^n$, set $|\mathbf{a}| := \sum_{i=1}^n a_i$ by abuse of notation.

Proof of Theorem 2.3. By Proposition 2.1, it suffices to show that $\operatorname{sdepth} I \geq n - \lfloor m/2 \rfloor$. Set $G(I) = \{v_1, \dots, v_m\}$ and $\varepsilon(I) := \sum_{i=1}^m |\mathbf{deg}(v_i)| (\geq m)$. We use induction on $\varepsilon(I)$. In the case $n = 0, 1$, there is nothing to do. Assume $n \geq 2$. The case $m \leq 2$ is a direct consequence of Proposition 2.1, and it suffices to consider only the case $m \geq 3$. For $i = 1, \dots, n$, we set

$$t_i(I) := |\{v_j \in G(I) \mid x_i \text{ divides } v_j\}|$$

($t_i(I)$ is called the *type* of x_i in [**6**]).

If $t_i(I) \leq 1$ for all i, then I is a complete intersection, and hence $\operatorname{sdepth} I = n - \lfloor m/2 \rfloor$ by Theorem 2.2. In particular, if $\varepsilon(I) = m$, then the assertion holds. Thus we may assume that $t_i(I) \geq 2$ for some i, and hence, without loss of generality, that $t_n(I) \geq 2$.

If $t_n(I) = m$, then each v_i can be divided by x_n. Set $v_i' := v_i/x_n$, and let I' be a monomial ideal of S (minimally) generated by $v_1', \dots, v_m'$. Since I and I' are isomorphic to each other up to degree shifting in

$\mathrm{mod}_{\mathbf{Z}^n} S$, it follows that $\operatorname{sdepth} I = \operatorname{sdepth} I'$, and moreover we have $\varepsilon(I') < \varepsilon(I)$. Therefore, by our inductive hypothesis we have

$$\operatorname{sdepth} I = \operatorname{sdepth} I' \geq n - \left\lfloor \frac{m}{2} \right\rfloor.$$

The remaining case is that $2 \leq t_n(I) \leq m-1$. We set $r := t_n(I)$. Without loss of generality, we may assume that x_n divides v_i for $i = 1, \ldots, r$ but not for $i = r+1, \ldots, m$. Let $\mathfrak{a}, \mathfrak{b}$ be a monomial ideal generated by $v_1, \ldots, v_r$ and $v_{r+1}, \ldots, v_m$ respectively, and hence $I = \mathfrak{a} + \mathfrak{b}$ hold. Set $I' := \mathfrak{a} + x_n\mathfrak{b}$, and consider the exact sequence

$$0 \longrightarrow I' \longrightarrow I \longrightarrow I/I' \longrightarrow 0.$$

It follows from Lemma 2.4 that

$$\operatorname{sdepth} I \geq \min\{\operatorname{sdepth} I', \operatorname{sdepth}(I/I')\}.$$

We set $G(I') := \{u_1, \ldots, u_{|G(I')|}\}$ (note that $|G(I')| \leq m$). Each minimal generator of I' can be divided by x_n; let I'' be the monomial ideal generated by $u_1/x_n, \ldots, u_{|G(I')|}/x_n$. By the same argument as in the case $t_n(I) = m$, we have $\operatorname{sdepth} I'' = \operatorname{sdepth} I'$. Since $\varepsilon(I'') \leq \varepsilon(I) - r \leq \varepsilon(I) - 2$ (recall that $r = t_n(I) \geq 2$), applying our inductive hypothesis yields

$$\operatorname{sdepth} I' = \operatorname{sdepth} I'' \geq n - \left\lfloor \frac{|G(I')|}{2} \right\rfloor \geq n - \left\lfloor \frac{m}{2} \right\rfloor.$$

As for I/I', we can apply Lemma 2.5, and it follows that

$$\operatorname{sdepth}(I/I') = \operatorname{sdepth}((v_{r+1}, \ldots, v_m) \cap \mathbf{k}[x_1, \ldots, x_{n-1}]).$$

Note that $(v_{r+1}, \ldots, v_m) \cap \mathbf{k}[x_1, \ldots, x_{n-1}]$ is minimally generated by $v_{r+1}, \ldots, v_m$ as an ideal of $\mathbf{k}[x_1, \ldots, x_{n-1}]$. Since $\sum_{i=r+1}^{m} |\mathbf{deg}(v_i)| < \varepsilon(I)$, applying our inductive hypothesis, we have

$$\begin{aligned} \operatorname{sdepth}((v_{r+1}, \ldots, v_m) \cap \mathbf{k}[x_1, \ldots, x_{n-1}]) &\geq n - 1 - \left\lfloor \frac{m-r}{2} \right\rfloor \\ &\geq n - 1 - \left\lfloor \frac{m-2}{2} \right\rfloor \\ &= n - 1 - \left(\left\lfloor \frac{m}{2} \right\rfloor - 1 \right) \\ &= n - \left\lfloor \frac{m}{2} \right\rfloor, \end{aligned}$$

since $r \geq 2$.

Summing up, we conclude that $\operatorname{sdepth} I \geq n - \lfloor m/2 \rfloor$ holds even if $2 \leq t_n(I) \leq m-1$, which completes the proof. $\square$

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DAI-2-COPO-RIBERA 305, NIWAI 1-4-8, SUMIYOSHIKU, OSAKA-CITY, OSAKA 558-0012, JAPAN
Email address: r-okazaki@ist.osaka-u.ac.jp, r_okazaki@tempo.ocn.ne.jp