JOURNAL OF COMMUTATIVE ALGEBRA
Volume 3, Number 1, Spring 2011

ARITHMETICAL RANK OF COHEN-MACAULAY SQUAREFREE MONOMIAL IDEALS OF HEIGHT TWO

KYOUKO KIMURA

ABSTRACT. In this paper, we prove that a squarefree monomial ideal of height 2 whose quotient ring is Cohen-Macaulay is a set-theoretic complete intersection.

1. Introduction. Let R be a polynomial ring over a field K. Let I be a squarefree monomial ideal of R and $G(I)$ the minimal set of monomial generators of I. The *arithmetical rank* of I is defined as the minimum number r of elements $a_1, \dots, a_r \in R$ such that

$$\sqrt{(a_1, \dots, a_r)} = \sqrt{I}. \tag{1.1}$$

We denote it by $\operatorname{ara} I$. When (1.1) holds, we say that $a_1, \dots, a_r$ *generate I up to radical*. By Krull's principal ideal theorem, we have $\operatorname{height} I \leq \operatorname{ara} I$. When equality holds, we say that I is a *set-theoretic complete intersection*. Moreover, Lyubeznik [**14**] proved that for a squarefree monomial ideal I, the projective dimension of R/I over R, denoted by $\operatorname{pd}_R R/I$ (or $\operatorname{pd} R/I$ if there is no confusion), provides a better lower bound for the arithmetical rank of I. Many authors among which Barile [**1**–**4**], Barile and Terai [**5, 6**], Ene, Olteanu and Terai [**10**], Kummini [**13**], Schmitt and Vogel [**16**], Terai and Yoshida with the author [**11, 12**], investigated when $\operatorname{ara} I = \operatorname{pd}_R R/I$ holds.

In this paper, we prove the following theorem:

Theorem 1.1 (see Theorem 4.1). *Let I be a squarefree monomial ideal of R of height 2. Suppose that R/I is Cohen-Macaulay. Then*

$$\operatorname{ara} I = \operatorname{pd}_R R/I = 2.$$

In particular, I is a set-theoretic complete intersection.

2010 AMS *Mathematics subject classification.* Primary 13F55, Secondary 13A15, 13H10.
Keywords and phrases. Arithmetical rank, Cohen-Macaulay, set-theoretic complete intersection, Alexander duality, cone.
This research was supported by JST, CREST.
Received by the editors on December 9, 2009, and in revised form on April 18, 2010.
DOI:10.1216/JCA-2011-3-1-31

In other words, the ideals as in Theorem 1.1 are generated by 2 elements up to radical. Note that the equality $\operatorname{ara} I = \operatorname{pd}_R R/I$ does not always hold for Cohen-Macaulay squarefree monomial ideals I of height 3 (when $\operatorname{char} K \neq 2$) as proved by Yan [**18**], Terai and Yoshida with the author [**12**].

We explain the organization of this paper. First in Section 2, we state the problem that motivated this paper (Problem 2.1), which corresponds to the Alexander dual of the results in Barile and Terai [**5**]. Partial answers to this problem are given in Section 3 (Propositions 3.1 and 3.2). In particular, Proposition 3.2 plays the key role in the proof of Theorem 1.1, which is given in Section 4.

The main result of Barile and Terai [**5**, Theorem 1], which is the paper that inspired the present work, required the assumption that K is algebraically closed. At the end of this paper, in Section 5, we give an improved proof of that result. Consequently, we can remove the assumption on K.

2. Preliminaries and the motivated problem. In this section, we state the problem which has motivated the present paper. First, we recall some definitions and properties of simplicial complexes and Stanley-Reisner ideals, in particular, Alexander duality. For more details, we refer to [**7**, Section 5], [**17**].

Let I be a squarefree monomial ideal of a polynomial ring R over a field K. The *graded Betti number* of R/I is defined by $\beta_{i,j}(R/I) = \dim_K [\operatorname{Tor}_i^R(R/I, K)]_j$. The *initial degree* and the (*Castelnuovo-Mumford*) *regularity* of I are defined by

$$\begin{aligned} \operatorname{indeg} I &= \min\{j : \beta_{1,j}(R/I) \neq 0\}, \\ \operatorname{reg} I &= \max\{j - i + 1 : \beta_{i,j}(R/I) \neq 0\}, \end{aligned}$$

respectively. In general, the inequality $\operatorname{reg} I \geq \operatorname{indeg} I$ holds. When $\operatorname{reg} I = \operatorname{indeg} I = k$, we say that I has a *k-linear resolution.*

Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of indeterminates over a field K. A *simplicial complex* Δ on the vertex set X is a collection of subsets of X with the properties (i) $\{x_i\} \in \Delta$ for all $x_i \in X$; (ii) $F \in \Delta$ and $G \subset F$ imply $G \in \Delta$. If Δ consists of all subsets of X, then Δ is called a *simplex.* An element of Δ is called a *face* of Δ. A maximal face of Δ with respect to inclusion is called a *facet* of Δ. The *dimension* of

Δ is defined by $\dim \Delta = \max\{|F| - 1 : F \in \Delta\}$, where $|F|$ denotes the cardinality of F. The *Alexander dual complex* Δ^* is defined by $\Delta^* = \{F \subset X : X \setminus F \notin \Delta\}$, which is also a simplicial complex. If $\dim \Delta < n-2$, then the vertex set of Δ^* coincides with X. When this is the case, $\Delta^{**} = \Delta$.

With a simplicial complex Δ on the vertex set $X = \{x_1, x_2, \ldots, x_n\}$, we associate a squarefree monomial ideal I_Δ of $K[X] = K[x_1, x_2, \ldots, x_n]$ as follows:

$$I_\Delta = \left(x_{i_1} \cdots x_{i_s} : 1 \le i_1 < \cdots < i_s \le n,\ \{x_{i_1}, \ldots, x_{i_s}\} \notin \Delta\right),$$

which is called the *Stanley-Reisner ideal* of Δ. The quotient ring $K[\Delta] = K[X]/I_\Delta$ is called the *Stanley-Reisner ring* of Δ. The minimal prime decomposition of I_Δ is given by

$$I_\Delta = \bigcap_{F \in \Delta : \text{facet}} P_F, \tag{2.1}$$

where $P_F = (x_i : x_i \in X \setminus F)$.

On the other hand, it is well known that for a squarefree monomial ideal I of $R = K[X]$ with $\operatorname{indeg} I \ge 2$, there exists a simplicial complex Δ on X such that $I = I_\Delta$. Assume that $\operatorname{height} I \ge 2$. Then since $\dim \Delta < n-2$, we can consider the ideal $I^* = I_{\Delta^*}$ of R, which is called the *Alexander dual ideal* of $I = I_\Delta$. Since $\Delta^{**} = \Delta$, we have $I^{**} = I$. The minimal set of monomial generators of $I^* = I_{\Delta^*}$ is given by

$$G(I^*) = G(I_{\Delta^*}) = \{m_{X \setminus F} : F \in \Delta \text{ is a facet of } \Delta\}, \tag{2.2}$$

where $m_{X\setminus F} = \prod_{x_i \in X \setminus F} x_i$. Then, as it can be easily seen, (2.1) and (2.2) imply that $\operatorname{indeg} I^* = \operatorname{height} I$. Moreover, Eagon and Reiner [**9**, Theorem 3] proved that I has a linear resolution if and only if R/I^* is Cohen-Macaulay.

Now we state the problem that has motivated the present paper.

Let Δ be a simplicial complex on the vertex set $X = \{x_1, x_2, \ldots, x_n\}$. Let x_0 be a new indeterminate and F a face of Δ. A *cone from* x_0 *over* F, denoted by $\mathrm{co}_{x_0} F$, is the simplex on the vertex set $F \cup \{x_0\}$. Then $\Delta' := \Delta \cup \mathrm{co}_{x_0} F$ is a simplicial complex on the vertex set $X' := X \cup \{x_0\}$. Barile and Terai [**5**] investigated some relations between the arithmetical ranks of I_Δ and $I_{\Delta'}$ ([**5**, Theorem 1]). Moreover, they proved that if $\operatorname{ara} I_\Delta = \operatorname{pd} K[\Delta]$ holds, then $\operatorname{ara} I_{\Delta'} = \operatorname{pd} K[\Delta']$ also holds ([**5**, Theorem 2]). As a corollary, they proved that if a squarefree monomial ideal $I \subset R$ has a 2-linear resolution, then $\operatorname{ara} I = \operatorname{pd}_R R/I$ holds. (This result was first proved by Morales [**15**] in a different way.)

We consider the following problem which corresponds to the Alexander dual of their results:

Problem 2.1. *Let Δ be a simplicial complex on the vertex set $X = \{x_1, x_2, \ldots, x_n\}$ with $\dim \Delta < n-2$. Let F be an arbitrary face of Δ^* and x_0 a new vertex. Set $X' = X \cup \{x_0\}$, $\Gamma = \Delta^*$, $\Gamma' = \Gamma \cup \operatorname{co}_{x_0} F$, and $\Delta' = (\Gamma')^*$.*

Are there any relations between the arithmetical ranks of I_Δ and $I_{\Delta'}$? In particular, if $\operatorname{ara} I_\Delta = \operatorname{pd} K[\Delta]$ holds, then does $\operatorname{ara} I_{\Delta'} = \operatorname{pd} K[\Delta']$ hold?

Set $R = K[X]$, $R' = K[X']$, $I = I_\Delta$, $I' = I_{\Delta'}$, and $G(I) = \{m_1, \ldots, m_\mu\}$. Then

$$I_\Gamma = I^* = P_{G_1} \cap \cdots \cap P_{G_\mu} \subset R,$$

where $G_1, \ldots, G_\mu$ are all facets of $\Gamma = \Delta^*$ and $m_j = \prod_{x_i \in P_{G_j}} x_i$. We may assume $F \subset G_1$ without loss of generality. Then

$$I_{\Gamma'} = P_{F \cup \{x_0\}} \cap (P_{G_1} R' + (x_0)) \cap \cdots \cap (P_{G_\mu} R' + (x_0)) \subset R'.$$

Hence

$$I' = (m_0, x_0 m_1, \ldots, x_0 m_\mu) R' = m_0 R' + x_0 I R',$$

where $m_0 = \prod_{x_i \in P_{F \cup \{x_0\}}} x_i$. Note that m_0 is divisible by m_1 since $X \setminus G_1 \subset X \setminus F$.

We first compare the projective dimension of $K[\Delta']$ with that of $K[\Delta]$.

Lemma 2.2. *Let Δ and Δ' be simplicial complexes as in Problem* 2.1. *Then*

$$\operatorname{pd} K[\Delta'] = \operatorname{pd} K[\Delta].$$

Proof. Let us consider the short exact sequence of R'-modules
(2.3)
$$0 \longrightarrow R'/m_0R' \cap x_0IR' \longrightarrow R'/m_0R' \oplus R'/x_0IR' \longrightarrow R'/I' \longrightarrow 0.$$

Note that $m_0R' \cap x_0IR' = x_0m_0R'$. Since $\mathrm{pd}_{R'} R'/x_0IR' = \mathrm{pd}_R R/I \geq \mathrm{height}\, I \geq 2$, $\mathrm{pd}_{R'} R'/x_0m_0R' = \mathrm{pd}_{R'} R'/m_0R' = 1$, the long exact sequence obtained by applying $\mathrm{Tor}^{R'}(-, K)$ to (2.3) yields

$$\mathrm{pd}\, K[\Delta'] = \mathrm{pd}_{R'} R'/I' = \mathrm{pd}_R R/I = \mathrm{pd}\, K[\Delta],$$

as desired. □

3. Partial answers to Problem 2.1. In this section, we give partial answers to Problem 2.1. Throughout this section, we use the same notation as in Problem 2.1.

First, we show a relation between the arithmetical ranks of I_Δ and $I_{\Delta'}$.

Proposition 3.1. *Let Δ and Δ' be simplicial complexes as in Problem* 2.1. *Then*
$$\mathrm{ara}\, I_{\Delta'} \leq \mathrm{ara}\, I_\Delta + 1.$$

In particular, if $\mathrm{ara}\, I_\Delta = \mathrm{pd}\, K[\Delta]$ *holds, then* $\mathrm{ara}\, I_{\Delta'}$ *coincides with either* $\mathrm{pd}\, K[\Delta']$ *or* $\mathrm{pd}\, K[\Delta'] + 1$.

Proof. Put $h = \mathrm{ara}\, I$ and let $q_1, \dots, q_h$ be elements of R which generate I up to radical. Then $x_0q_1, \dots, x_0q_h$ generate $(x_0m_1, \dots, x_0m_\mu)$ up to radical. This implies that $m_0, x_0q_1, \dots, x_0q_h$ generate I' up to radical. Therefore we have $\mathrm{ara}\, I' \leq h + 1$.

Then the second part of the claim immediately follows from Lemma 2.2 and the inequality $\mathrm{ara}\, I_{\Delta'} \geq \mathrm{pd}\, K[\Delta']$. □

Next, we give a partial answer to the second question of Problem 2.1.

Proposition 3.2. *Let Δ and Δ' be simplicial complexes as in Problem* 2.1. *Suppose that* $\mathrm{ara}\, I_\Delta = \mathrm{pd}\, K[\Delta] = 2$. *Then*

$$\mathrm{ara}\, I_{\Delta'} = \mathrm{pd}\, K[\Delta'] = 2.$$

In the study of the arithmetical rank, the technique based on linear algebraic consideration has been developed by Barile [**2**], Barile and Terai [**5**] (see also [**6**]). Our proof of this proposition also goes along this current.

Proof. By Lemma 2.2, we have $\operatorname{pd} K[\Delta'] = \operatorname{pd} K[\Delta] = 2$. Therefore it suffices to prove that $\operatorname{ara} I_{\Delta'} \leq 2$.

Let q_1, q_2 be elements of R which generate I up to radical. Note that $q_1, q_2 \in I$ because I is a squarefree monomial ideal. Since $m_i \in \sqrt{(q_1, q_2)}$, there exists some integer $\ell_i \geq 1$ such that $m_i^{\ell_i} \in (q_1, q_2)$. Then we can write

$$m_i^{\ell_i} = a_{i1} q_1 + a_{i2} q_2, \quad i = 1, \dots, \mu,$$

where $a_{i1}, a_{i2} \in R$. Set $A = (a_{ij})_{i=1,\dots,\mu; j=1,2}$. Then

$$\begin{pmatrix} m_1^{\ell_1} \\ \vdots \\ m_\mu^{\ell_\mu} \end{pmatrix} = A \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

Set

$$J' = (x_0 q_1 - a_{12} m_0, x_0 q_2 + a_{11} m_0) R'.$$

We prove that $\sqrt{J'} = I'$. Since $x_0 q_1 - a_{12} m_0, x_0 q_2 + a_{11} m_0 \in I'$, we have $\sqrt{J'} \subset I'$. We prove the opposite inclusion.

Since

$$A \begin{pmatrix} x_0 q_1 - a_{12} m_0 \\ x_0 q_2 + a_{11} m_0 \end{pmatrix} = \begin{pmatrix} x_0 m_1^{\ell_1} + f_1 m_0 \\ \vdots \\ x_0 m_\mu^{\ell_\mu} + f_\mu m_0 \end{pmatrix}, \text{ where } f_i = a_{11} a_{i2} - a_{12} a_{i1},$$

we have $x_0 m_i^{\ell_i} + f_i m_0 \in J'$ for $i = 1, \dots, \mu$. Note that $f_1 = a_{11} a_{12} - a_{12} a_{11} = 0$. Thus $x_0 m_1^{\ell_1} \in J'$, that is, $x_0 m_1 \in \sqrt{J'}$. Since m_1 divides m_0, multiplying $x_0 m_i^{\ell_i} + f_i m_0 \in J'$ by x_0 implies $x_0^2 m_i^{\ell_i} \in \sqrt{J'}$, that is, $x_0 m_i \in \sqrt{J'}$.

Here, recall that $q_1, q_2 \in I = (m_1, \dots, m_\mu)$. Thus $x_0 q_1, x_0 q_2 \in \sqrt{J'}$. Consequently, we have $a_{11} m_0, a_{12} m_0 \in \sqrt{J'}$. Since $m_1^{\ell_1} = a_{11} q_1 + a_{12} q_2$, we have

$$m_0 m_1^{\ell_1} = m_0 (a_{11} q_1 + a_{12} q_2) = (a_{11} m_0) q_1 + (a_{12} m_0) q_2 \in \sqrt{J'}.$$

This implies $m_0 \in \sqrt{J'}$ since m_0 is divisible by m_1. Therefore $\sqrt{J'} \supset I'$ holds, as required. $\square$

Example 3.3. Let Δ be the simplicial complex on the vertex set $\{x_1, x_2, x_3, x_4\}$ whose facets are $\{x_1, x_3\}, \{x_2, x_3\}, \{x_2, x_4\}$. Then

$$I = I_\Delta = (x_2, x_4) \cap (x_1, x_4) \cap (x_1, x_3) = (x_1x_2, x_1x_4, x_3x_4).$$

The Alexander dual complex Γ of Δ has facets $\{x_3, x_4\}$, $\{x_2, x_3\}$, $\{x_1, x_2\}$, that is, Γ is a line segment with 4 vertices. Take the face $F = \{x_4\} \in \Gamma$ and a new vertex $x_5 := x_0$. Then $\Gamma' = \Gamma \cup \mathrm{co}_{x_5}\{x_4\}$ is a line segment with 5 vertices and

$$I_{\Gamma'} = (x_1, x_2, x_3) \cap (x_1, x_2, x_5) \cap (x_1, x_4, x_5) \cap (x_3, x_4, x_5).$$

Thus $I' = I_{(\Gamma')^*}$ is generated by

$$x_1x_2x_3,\ x_1x_2x_5,\ x_1x_4x_5,\ x_3x_4x_5.$$

In this case, $m_0 = x_1x_2x_3$ and $m_1 = x_1x_2$. By the result of Schmitt and Vogel [**16**, page 249, Lemma], it is easy to see that the following two elements q_1, q_2 generate I up to radical:

$$q_1 = x_1x_4, \qquad q_2 = x_1x_2 + x_3x_4.$$

Then

$$m_1^2 = -x_2x_3q_1 + x_1x_2q_2. \tag{3.1}$$

By Proposition 3.2, the following two elements q_1', q_2' generate I' up to radical:

$$\begin{aligned} q_1' &= x_5q_1 - x_1x_2m_0 = x_1x_4x_5 - x_1^2x_2^2x_3, \\ q_2' &= x_5q_2 - x_2x_3m_0 = x_1x_2x_5 + x_3x_4x_5 - x_1x_2^2x_3^2. \end{aligned}$$

4. Proof of the main theorem. In this section, we prove the following theorem, which is the main result in this paper.

Theorem 4.1. *Let I be a squarefree monomial ideal of $R = K[X]$ of height 2. Suppose that R/I is Cohen-Macaulay. Then*

$$\operatorname{ara} I = \operatorname{pd}_R R/I = \operatorname{height} I = 2.$$

In particular, I is a set-theoretic complete intersection.

The Alexander dual of the ideals satisfying the assumptions of this theorem have a 2-linear resolution. To study these ideals, we recall the definition of generalized tree.

We say that a simplicial complex is a *generalized tree* if it can be obtained by the following recursive procedure: (i) a simplex is a generalized tree; (ii) if Δ is a generalized tree, then $\Delta \cup \operatorname{co}_{x_0} F$ is also a generalized tree for any $F \in \Delta$ and for any new vertex x_0. Then a Stanley-Reisner ideal I_Δ which has a 2-linear resolution is characterized by the following lemma.

Lemma 4.2 (See Barile and Terai [**5**, Lemma 2]). *Let Δ be a simplicial complex which is not a simplex. Then I_Δ has a 2-linear resolution if and only if Δ is a generalized tree.*

Now we prove Theorem 4.1. The proof is an application of Proposition 3.2.

Proof of Theorem 4.1. Since R/I is Cohen-Macaulay, we have $\operatorname{pd}_R R/I = \operatorname{height} I = 2$. First, we note that when $\mu(I) \leq \operatorname{pd}_R R/I + 1$, it is known that $\operatorname{ara} I = \operatorname{pd}_R R/I$ holds; see e.g., [**11**, Theorem 2.1]. Thus in our situation, $\operatorname{ara} I = \operatorname{pd}_R R/I = 2$ holds if $\mu(I) \leq 3$.

If $\operatorname{indeg} I = 1$, then I is of the form (x_1, m_2) by the assumptions on I. In this case, $\operatorname{ara} I = 2$ trivially holds.

Assume that $\operatorname{indeg} I \geq 2$. We proceed by induction on the number $|X|$ of variables. The minimum number $|X|$ in which there exists an ideal I satisfying our assumption is 3 and such an ideal is of the form

$$I = (x_1, x_2) \cap (x_1, x_3) \cap (x_2, x_3) = (x_1x_2, x_1x_3, x_2x_3).$$

Then, since $\mu(I) = 3$, we have $\operatorname{ara} I = \operatorname{pd}_R R/I = 2$.

Now assume $|X| > 3$. Since $I^* = I_\Gamma$ has a 2-linear resolution, Γ is a generalized tree by Lemma 4.2, and there exist a vertex $x \in X$, a generalized tree $\overline{\Gamma}$ on the vertex set $X \setminus \{x\}$ and a face $F \in \overline{\Gamma}$ such that $\Gamma = \overline{\Gamma} \cup \mathrm{co}_x F$ by definition of generalized tree. Note that $\overline{\Gamma}$ is not a simplex because $\operatorname{height} I_\Gamma = \operatorname{indeg} I \geq 2$. Then $\overline{J} := I_{\overline{\Gamma}}$ has a 2-linear resolution.

If $\operatorname{height} \overline{J} = 1$, then $\overline{J}$ is of the form $(x_1) \cap P_2$, and I^* is of the form

$$I^* = I_\Gamma = P_{F \cup \{x\}} \cap (x_1, x) \cap (P_2 R + (x)).$$

Therefore $\mu(I) \leq 3$.

Thus we may assume $\operatorname{height} \overline{J} \geq 2$. Then $\overline{I} := (\overline{J})^*$ satisfies the assumptions of Theorem 4.1. By the induction hypothesis, we have $\operatorname{ara} \overline{I} = \operatorname{pd}_R R/\overline{I} = 2$. Hence, we have $\operatorname{ara} I = \operatorname{pd}_R R/I = 2$ by Proposition 3.2. $\square$

The next example, which is a generalization of Example 3.3, presents a class of ideals which satisfy the assumptions of Theorem 4.1.

Example 4.3. Let us consider the squarefree monomial ideal I_n of $K[x_1, x_2, \ldots, x_n]$ $(n \geq 4)$ generated by the following $n-1$ elements:

$$m_i^{(n)} = \frac{x_1 \cdots x_n}{x_{n-i} x_{n-i+1}}, \quad i = 1, 2, \ldots, n-1.$$

That is, I_n is the Alexander dual ideal of the Stanley-Reisner ideal I_{Γ_n}, where Γ_n is the simplicial complex whose facets are $\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}$. The ideals I, I' in Example 3.3 are I_4, I_5, respectively.

Then the height of I_n is equal to 2, and the quotient ring is Cohen-Macaulay. Therefore by Theorem 4.1, we have $\operatorname{ara} I_n = 2$.

For $n = 4, 5$, two elements $q_1^{(n)}, q_2^{(n)}$ which generate I_n up to radical are given in Example 3.3, i.e.,

$$\begin{cases} q_1^{(4)} = m_2^{(4)}, \\ q_2^{(4)} = m_1^{(4)} + m_3^{(4)}, \end{cases} \qquad \begin{cases} q_1^{(5)} = x_5 q_1^{(4)} - x_1 x_2 m_1^{(5)}, \\ q_2^{(5)} = x_5 q_2^{(4)} - x_2 x_3 m_1^{(5)}. \end{cases}$$

In general, two elements $q_1^{(n)}, q_2^{(n)}$ which generate I_n up to radical are given by the following recursive formula:

$$\begin{cases} q_1^{(n+1)} = x_{n+1} q_1^{(n)} - x_{n-2}^{n-3} q_1^{(n-1)} m_1^{(n+1)}, \\ q_2^{(n+1)} = x_{n+1} q_2^{(n)} - x_{n-2}^{n-3} q_2^{(n-1)} m_1^{(n+1)}, \end{cases} \qquad n \geq 5. \tag{4.1}$$

We prove this by induction on n. Note that $I'_n = I_{n+1}$ with $F = \{x_n\}(\subset G_1 = \{x_{n-1}, x_n\})$ and $x_0 = x_{n+1}$ with respect to the notations of the proof of Proposition 3.2. Hence by the proof of Proposition 3.2, it suffices to check the following equality by induction on n under the hypothesis that $q_1^{(n)}, q_2^{(n)}$ generate I_n up to radical:

$$(m_1^{(n)})^{n-2} = -x_{n-2}^{n-3} q_2^{(n-1)} q_1^{(n)} + x_{n-2}^{n-3} q_1^{(n-1)} q_2^{(n)}, \quad n \geq 5. \tag{4.2}$$

When $n = 5$, since

$$\begin{aligned} &- q_2^{(4)} q_1^{(5)} + q_1^{(4)} q_2^{(5)} \\ &\qquad = -q_2^{(4)}(x_5 q_1^{(4)} - x_1 x_2 m_1^{(5)}) + q_1^{(4)}(x_5 q_2^{(4)} - x_2 x_3 m_1^{(5)}) \\ &\qquad = (x_1 x_2 q_2^{(4)} - x_2 x_3 q_1^{(4)}) m_1^{(5)} \\ &\qquad = (m_1^{(4)})^2 m_1^{(5)} \quad \text{by (3.1)}, \end{aligned}$$

and $x_3^2 (m_1^{(4)})^2 m_1^{(5)} = (m_1^{(5)})^3$, we have the desired equality. Similarly, for general n,

$$\begin{aligned} &- q_2^{(n-1)} q_1^{(n)} + q_1^{(n-1)} q_2^{(n)} \\ &\qquad = -q_2^{(n-1)}(x_n q_1^{(n-1)} - x_{n-3}^{n-4} q_1^{(n-2)} m_1^{(n)}) \\ &\qquad\quad + q_1^{(n-1)}(x_n q_2^{(n-1)} - x_{n-3}^{n-4} q_2^{(n-2)} m_1^{(n)}) \\ &\qquad = (x_{n-3}^{n-4} q_1^{(n-2)} q_2^{(n-1)} - x_{n-3}^{n-4} q_2^{(n-2)} q_1^{(n-1)}) m_1^{(n)} \\ &\qquad = (m_1^{(n-1)})^{n-3} m_1^{(n)} \quad \text{by the induction hypothesis} \end{aligned}$$

and $x_{n-2}^{n-3}(m_1^{(n-1)})^{n-3} m_1^{(n)} = (m_1^{(n)})^{n-2}$ yield the equation (4.2).

Another class of ideals which satisfies the assumptions of Theorem 4.1 was found in Barile [**1**, Section 3]. It is essentially the Alexander dual of the class of Ferrers ideals (see [**4, 8**]). In [**1**], Barile constructed 2 elements which generate the ideals up to radical in a different way.

5. Improved proof of the result by Barile and Terai. Let Δ be a simplicial complex on the vertex set X. Let F be a face of Δ and x_0 a new vertex. Set $\Delta' = \Delta \cup \mathrm{co}_{x_0} F$. Throughout this section, we will use these notations. Note that these are different from those of previous sections.

In the paper motivating the present one, Barile and Terai [**5**], the main result [**5**, Theorem 1] depends on the base field K. Precisely, it needs the assumption that K is algebraically closed. In this section, we give an improved proof which does not depend on the base field K.

Theorem 5.1 (cf. [**5**, Theorem 1]). *Let Δ be a simplicial complex on the vertex set $X = \{x_1, x_2, \dots, x_n\}$, F a face of Δ and x_0 a new vertex. Set $\Delta' = \Delta \cup \mathrm{co}_{x_0} F$. Then*

$$\operatorname{ara} I_{\Delta'} \leq \max\{\operatorname{ara} I_\Delta + 1, n - |F|\}.$$

As a consequence of our improvement, we can also omit the assumption on K for other results in [**5**]:

Theorem 5.2 (cf. [**5**, Theorem 2]). *Let Δ be a simplicial complex on the vertex set X, F a face of Δ, and x_0 a new vertex. Set $\Delta' = \Delta \cup \mathrm{co}_{x_0} F$. If* $\operatorname{ara} I_\Delta = \operatorname{pd} K[\Delta]$ *holds, then* $\operatorname{ara} I_{\Delta'} = \operatorname{pd} K[\Delta']$ *also holds.*

Corollary 5.3 (cf. [**5**, Corollary 3]). *Let I be a squarefree monomial ideal of $R = K[X]$. Suppose that I has a 2-linear resolution. Then*

$$\operatorname{ara} I = \operatorname{pd}_R R/I.$$

Corollary 5.3 was first proved by Morales [**15**, Theorems 8 and 9] in a different way, but he also assumed that K is algebraically closed.

Now, we prove Theorem 5.1. The proof is divided into two steps. We construct $\max\{\operatorname{ara} I_\Delta + 1, n - |F|\}$ elements which generate $I_{\Delta'}$ up to radical in the last step (Step 2). The first step (Step 1) is performed to transform elements which generate I_Δ up to radical so that the elements constructed in (Step 2) belong to $I_{\Delta'}$.

In our proof, (Step 1) is the same as in the paper by Barile and Terai (see also Barile [**2**, Theorem 1]). Thus we omit the details. Our improvement is in (Step 2). In Case 1 of (Step 2), the elements which generate $I_{\Delta'}$ up to radical are the same as in the paper by Barile and Terai. The difference is that we use the cofactor matrix instead of Cramer's rule which they used, and we do not use Hilbert's Nullstellensatz. In Case 2 of (Step 2), we give $\operatorname{ara} I_{\Delta'}$ elements generating $I_{\Delta'}$ up to radical; these are different from those given by Barile and Terai. This is our main improvement.

Proof of Theorem 5.1. (*Step* 1). First, we fix the notation. Set $R = K[X]$ and $R' = K[X']$ where $X' = X \cup \{x_0\}$. If $F = X$, then $I_\Delta = I_{\Delta'} = 0$ and the assertion is trivially true. Thus we assume $F \neq X$. Let G be a facet of Δ which contains F. We can assume that $G = \{x_{s+1}, \dots, x_n\}$ and $F = \{x_{t+1}, \dots, x_n\}$, where $s \leq t$. Then $I_{\Delta'} = I_\Delta R' + (x_0x_1, \dots, x_0x_t)R'$. We set $\operatorname{ara} I_\Delta = h$. Then we can rewrite the claim as

$$\operatorname{ara} I_{\Delta'} \leq \max\{h+1, t\}.$$

Assume that $q_1, \dots, q_h$ generate I_Δ up to radical. Since $I_\Delta \subset P_G = (x_1, \dots, x_s)$ and $q_i \in I_\Delta$, we can write

$$q_i = \sum_{j=1}^{s} a_{ij}x_j, \quad i = 1, 2, \dots, h,$$

where $a_{ij} \in R$. Since I_Δ is generated by monomials, we may assume that all monomials appearing $a_{ij}x_j$ belong to I_Δ. Let $\phi\colon R \to R$ be the ring homomorphism defined by $\phi(x_j) = x_j^2$, $j = 1, 2, \dots, n$. We set $\overline{q}_i = \phi(q_i)$ and $\overline{a}_{ij} = \phi(a_{ij})x_j$ for $i = 1, 2, \dots, h$; $j = 1, 2, \dots, s$. Then

$$\overline{q}_i = \sum_{j=1}^{s} \overline{a}_{ij}x_j,$$

and $\overline{a}_{ij} \in I_\Delta$. Moreover, $\overline{q}_1, \dots, \overline{q}_h$ also generate I_Δ up to radical; see [**5**, Proof of Theorem 1].

(*Step* 2). Now we find $\max\{h+1, t\}$ elements which generate $I_{\Delta'}$ up to radical. We distinguish between two cases.

Case 1. Suppose that $h+1>t$. We show that $\operatorname{ara} I_{\Delta'} \leq h+1$. We set $\overline{A}=(\overline{a}_{ij})_{i,j=1,\dots,t}$, where $\overline{a}_{ij}=0$ if $j>s$. Let $A_1=\overline{A}+x_0\mathrm{Id}_t$, where Id_t denotes the $t\times t$ identity matrix. Set

$$J_1=(\det A_1-x_0^t, \overline{q}_1+x_0x_1,\dots,\overline{q}_t+x_0x_t, \overline{q}_{t+1},\dots,\overline{q}_h)R'.$$

We prove that $\sqrt{J_1}=I_{\Delta'}$. Since $\overline{a}_{ij}\in I_\Delta$, we have $\det A_1-x_0^t\in I_\Delta R'$. Moreover since $\overline{q}_i\in I_\Delta$, $i=1,2,\dots,h$ and $x_0x_j\in I_{\Delta'}$, $j=1,2,\dots,t$, we have $\sqrt{J_1}\subset I_{\Delta'}$. We prove the opposite inclusion. To do this, it suffices to show that $\overline{q}_i\in\sqrt{J_1}$, $i=1,2,\dots,t$ and $x_0x_j\in\sqrt{J_1}$, $j=1,2,\dots,t$.

Let B_1 be the cofactor matrix of A_1. Then $B_1A_1=(\det A_1)\mathrm{Id}_t$. Since

$$\begin{pmatrix}\overline{q}_1+x_0x_1\\ \vdots\\ \overline{q}_t+x_0x_t\end{pmatrix}=A_1\begin{pmatrix}x_1\\ \vdots\\ x_t\end{pmatrix},$$

we have

$$B_1\begin{pmatrix}\overline{q}_1+x_0x_1\\ \vdots\\ \overline{q}_t+x_0x_t\end{pmatrix}=B_1A_1\begin{pmatrix}x_1\\ \vdots\\ x_t\end{pmatrix}=(\det A_1)\begin{pmatrix}x_1\\ \vdots\\ x_t\end{pmatrix}.$$

Then $(\det A_1)x_j\in J_1$ for $j=1,2,\dots,t$ since $\overline{q}_i+x_0x_i\in J_1$ for all $i=1,2,\dots,t$. Multiplying $\det A_1-x_0^t\in J_1$ by x_j, we have $x_0^tx_j\in J_1$. Hence $x_0x_j\in\sqrt{J_1}$ for $j=1,2,\dots,t$. Since $\overline{q}_i+x_0x_i\in J_1$, we have $\overline{q}_i\in\sqrt{J_1}$ for $i=1,2,\dots,t$, as required.

Case 2. Suppose that $h+1\leq t$. We show that $\operatorname{ara} I_{\Delta'}\leq t$. Note that in this case, $s\leq t-1$ because if $s=t$, then t is the height of the minimal prime P_G of I_Δ and Krull's principal ideal theorem shows that $\operatorname{ara} I_\Delta\geq t$. This contradicts $\operatorname{ara} I_\Delta=h\leq t-1$.

We set $\overline{A}'=(\overline{a}_{ij})_{i,j=1,\dots,t-1}$, where $\overline{a}_{ij}=0$ if $i>h$ or $j>s$. Let $A_2=\overline{A}'+x_0\mathrm{Id}_{t-1}$, where Id_{t-1} denotes the $(t-1)\times(t-1)$ identity matrix. Set

$$\begin{aligned}J_2=((\det A_2)(x_0+x_t)-x_0^t, \overline{q}_1+x_0x_1,\dots,&\\ \overline{q}_h+x_0x_h, x_0x_{h+1},\dots,x_0x_{t-1})R'.&\end{aligned}$$

We prove that $\sqrt{J_2} = I_{\Delta'}$. As $\overline{a}_{ij} \in I_\Delta$, similarly to Case 1, we have $\sqrt{J_2} \subset I_{\Delta'}$. We prove the opposite inclusion. Let B_2 be the cofactor matrix of A_2. Then $B_2A_2 = (\det A_2)\mathrm{Id}_{t-1}$. Since we have set $\overline{a}_{ij} = 0$ for $i > h$, we can write formally $x_0x_i = \overline{q}_i + x_0x_i$ for $i = h+1, \dots, t-1$. Using this notation, we have

$$\begin{pmatrix} \overline{q}_1 + x_0x_1 \\ \vdots \\ \overline{q}_{t-1} + x_0x_{t-1} \end{pmatrix} = A_2 \begin{pmatrix} x_1 \\ \vdots \\ x_{t-1} \end{pmatrix}.$$

Thus

$$B_2 \begin{pmatrix} \overline{q}_1 + x_0x_1 \\ \vdots \\ \overline{q}_{t-1} + x_0x_{t-1} \end{pmatrix} = B_2A_2 \begin{pmatrix} x_1 \\ \vdots \\ x_{t-1} \end{pmatrix} = (\det A_2) \begin{pmatrix} x_1 \\ \vdots \\ x_{t-1} \end{pmatrix}.$$

Then $(\det A_2)x_j \in J_2$ for $j = 1, 2, \dots, t-1$ since $\overline{q}_i + x_0x_i \in J_2$ for all $i = 1, 2, \dots, t-1$. Multiplying $(\det A_2)(x_0 + x_t) - x_0^t \in J_2$ by x_j, we have $x_0^tx_j \in J_2$ for $j = 1, 2, \dots, t-1$. Hence $x_0x_j \in \sqrt{J_2}$ for $j = 1, 2, \dots, t-1$. Since $\overline{q}_i + x_0x_i \in J_2$, we have $\overline{q}_i \in \sqrt{J_2}$ for $i = 1, 2, \dots, t-1$. In particular, $\overline{q}_i \in \sqrt{J_2}$ for $i = 1, 2, \dots, h$. Since $\sqrt{(\overline{q}_1, \dots, \overline{q}_h)} = I_\Delta$ and $\overline{a}_{ij} \in I_\Delta$, we have

$$\overline{a}_{ij} \in \sqrt{(\overline{q}_1, \dots, \overline{q}_h)} \subset \sqrt{J_2}, \quad \text{for all } i, j.$$

Therefore $(\det A_2)(x_0 + x_t) - x_0^t \in J_2$ implies $x_0^{t-1}(x_0 + x_t) - x_0^t \in \sqrt{J_2}$. Thus we have $x_0^{t-1}x_t \in \sqrt{J_2}$, that is $x_0x_t \in \sqrt{J_2}$. This completes the proof. □

Example 5.4. Let Δ be the simplicial complex on the vertex set $\{x_1, x_2, x_3, x_4\}$ whose facets are $\{x_1, x_2\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_3, x_4\}$. Then

$$\begin{aligned} I_\Delta &= (x_1, x_2) \cap (x_1, x_4) \cap (x_2, x_3) \cap (x_3, x_4) \\ &= (x_1x_3, x_2x_4). \end{aligned}$$

Thus I_Δ is a complete intersection. In particular, $h = \operatorname{ara} I_\Delta = 2$. Set $q_1 = x_1x_3$ and $q_2 = x_2x_4$. Take the face $F = \{x_4\} \in \Delta$, and let x_0 be a new vertex. Then $I_{\Delta'}$ is generated by the following 5 elements:

$$x_1x_3, x_2x_4, x_0x_1, x_0x_2, x_0x_3.$$

Then $t = 3$. We take the facet G as $\{x_3, x_4\}$. Then $P_G = (x_1, x_2)$. In this case, we have

$$\overline{q}_1 = x_1x_3^2 \cdot x_1, \qquad \overline{q}_2 = x_2x_4^2 \cdot x_2.$$

Since $h+1 = 3 = t$, we apply Case 2 of the proof of Theorem 5.1. Since

$$\overline{A}' = \begin{pmatrix} x_1x_3^2 & 0 \\ 0 & x_2x_4^2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} x_1x_3^2 + x_0 & 0 \\ 0 & x_2x_4^2 + x_0 \end{pmatrix},$$

the ideal $I_{\Delta'}$ is generated by the following 3 elements up to radical:

$$\begin{aligned} &(x_1x_3^2 + x_0)(x_2x_4^2 + x_0)(x_0 + x_3) - x_0^3 \\ &\qquad = x_0^2x_3 + x_0^2x_1x_3^2 + x_0^2x_2x_4^2 + x_0x_1x_2x_3^2x_4^2 \\ &\qquad\quad + x_0x_1x_3^3 + x_0x_2x_3x_4^2 + x_1x_2x_3^3x_4^2, \\ &x_1^2x_3^2 + x_0x_1, \quad x_2^2x_4^2 + x_0x_2. \end{aligned}$$

Acknowledgments. The author is grateful to Naoki Terai for drawing the author's attention to this problem. The author also thanks Ken-ichi Yoshida for giving her valuable suggestions. The author thanks the referee for many suggestions and comments.

REFERENCES

1. M. Barile, *On the number of equations defining certain varieties*, Manuscrip. Math. **91** (1996), 483–494.

2. ———, *Arithmetical ranks of Stanley-Reisner ideals via linear algebra*, Comm. Algebra **36** (2008), 4540–4556.

3. ———, *On the arithmetical rank of the edge ideals of forests*, Comm. Algebra **36** (2008), 4678–4703.

4. ———, *A note on the edge ideals of Ferrers graphs*, preprint, math.AC/0606353.

5. M. Barile and N. Terai, *Arithmetical ranks of Stanley-Reisner ideals of simplicial complexes with a cone*, Comm. Algebra, to appear, arXiv:0809.2194.

6. ———, *The Stanley-Reisner ideals of polygons as set-theoretic complete intersections*, preprint, arXiv:0909.1900.

7. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, revised ed., Cambridge University Press, Cambridge, 1998.

8. A. Corso and U. Nagel, *Monomial and toric ideals associated to Ferrers graphs*, Trans. Amer. Math. Soc. **361** (2009), 1371–1395.

9. J.A. Eagon and V. Reiner, *Resolutions of Stanley-Reisner rings and Alexander duality*, J. Pure Appl. Algebra **130** (1998), 265–275.

10. V. Ene, O. Olteanu and N. Terai, *Arithmetical rank of lexsegment edge ideals*, preprint, arXiv:0911.2884.

11. K. Kimura, N. Terai and K. Yoshida, *Arithmetical rank of squarefree monomial ideals of small arithmetic degree*, J. Algebraic Combin. **29** (2009), 389–404.

12. ———, *Arithmetical rank of monomial ideals of deviation two*, in *Combinatorial aspects of commutative algebra*, V. Ene and E. Miller, eds., Contemporary Math. **502**, American Mathematical Society, 2009.

13. M. Kummini, *Regularity, depth and arithmetic rank of bipartite edge ideals*, J. Algebraic Combin. **30** (2009), 429–445.

14. G. Lyubeznik, *On the local cohomology modules $H^i_{\mathfrak{a}}(R)$ for ideals $\mathfrak{a}$ generated by monomials in an R-sequence*, in *Complete intersections, Acireale*, S. Greco and R. Strano, eds., Lecture Notes Math. **1092**, Springer-Verlag, New York, 1984.

15. M. Morales, *Simplicial ideals, 2-linear ideals and arithmetical rank*, preprint, math.AC/0702668.

16. T. Schmitt and W. Vogel, *Note on set-theoretic intersections of subvarieties of projective space*, Math. Ann. **245** (1979), 247–253.

17. N. Terai, *Alexander duality in Stanley-Reisner rings*, in *Affine algebraic geometry*, in honor of Professor Masayoshi Miyanishi on the occasion of his retirement from Osaka University, T. Hibi, ed., Osaka University Press, 2007.

18. Z. Yan, *An étale analog of the Goresky-Macpherson formula for subspace arrangements*, J. Pure Appl. Algebra **146** (2000), 305–318.

Department of Mathematics, Faculty of Science, Shizuoka University, 836 Ohya, Suruga-ku, Shizuoka 422-8529, Japan
Email address: skkimur@ipc.shizuoka.ac.jp