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TORIC RINGS AND IDEALS OF NESTED CONFIGURATIONS

HIDEFUMI OHSUGI AND TAKAYUKI HIBI

ABSTRACT. The toric ring together with the toric ideal arising from a nested configuration is studied, with particular attention given to the algebraic study of normality of the toric ring as well as the Gröbner bases of the toric ideal. One of the combinatorial applications of these algebraic findings leads to insights on smooth 3×3 transportation polytopes.

Introduction. Toric rings and toric ideals play a central role in combinatorial and computational aspects of commutative algebra. In [**1**], from a viewpoint of algebraic statistics, the concept of nested configurations was introduced. In the present paper, the toric ring together with the toric ideal arising from a nested configuration will be studied in detail.

Let $K[\mathbf{t}] = K[t_1, \dots, t_d]$ denote the polynomial ring in d variables over a field K. A (*point*) *configuration* of $K[\mathbf{t}]$ is a finite set $A = \{\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n}\}$ of monomials belonging to $K[\mathbf{t}]$ for which there exists a vector $\mathbf{w} \in \mathbf{R}^d$ such that $\mathbf{w} \cdot \mathbf{a}_i = 1$ for all $1 \leq i \leq n$. We will associate each configuration A of $K[\mathbf{t}]$ with the homogeneous semigroup ring $K[A]$, called the *toric ring* of A, which is the subalgebra of $K[\mathbf{t}]$ generated by the monomials belonging to A. The toric ring $K[A]$ is called *normal* if $K[A]$ is integrally closed in its field of fractions. It is known that $K[A]$ is normal if and only if $\mathbf{Z}_{\geq 0}\{\mathbf{a}_1, \dots, \mathbf{a}_n\} = \mathbf{Z}\{\mathbf{a}_1, \dots, \mathbf{a}_n\} \cap \mathbf{Q}_{\geq 0}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. See, e.g., [**9**, Proposition 13.5]. In addition, $K[A]$ is called *very ample* if

$$(\mathbf{Z}\{\mathbf{a}_1, \dots, \mathbf{a}_n\} \cap \mathbf{Q}_{\geq 0}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}) \setminus \mathbf{Z}_{\geq 0}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$$

is a finite set. In particular, $K[A]$ is very ample if $K[A]$ is normal.

Let $K[\mathbf{x}] = K[x_1, \dots, x_n]$ denote the polynomial ring over K in n variables with each $\deg(x_i) = 1$. The *toric ideal* I_A of A is the kernel

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of the surjective homomorphism $\pi : K[\mathbf{x}] \to K[A]$ defined by setting $\pi(x_i) = \mathbf{t}^{\mathbf{a}_i}$ for each $1 \leq i \leq n$. It is known (e.g., [**9**, Section 4]) that the toric ideal I_A is generated by those homogeneous binomials $u - v$, where u and v are monomials of $K[\mathbf{x}]$, with $\pi(u) = \pi(v)$. Fix a monomial order $<$ on $K[\mathbf{x}]$. The *initial monomial* $\text{in}_<(f)$ of $0 \neq f \in I_A$ with respect to $<$ is the biggest monomial appearing in f with respect to $<$. The *initial ideal* of I_A with respect to $<$ is the ideal $\text{in}_<(I_A)$ of $K[\mathbf{x}]$ generated by all initial monomials $\text{in}_<(f)$ with $0 \neq f \in I_A$. An initial ideal $\text{in}_<(I_A)$ is called *quadratic* (respectively *squarefree*) if $\text{in}_<(I_A)$ is generated by quadratic (respectively squarefree) monomials. Let $\mathcal{G}$ be a finite subset of I_A, and write $\text{in}_<(\mathcal{G})$ for the ideal $\langle \text{in}_<(g) \mid g \in \mathcal{G} \rangle$ of $K[X]$. A finite set $\mathcal{G}$ of I_A is said to be a *Gröbner basis* of I_A with respect to $<$ if $\text{in}_<(\mathcal{G}) = \text{in}_<(I_A)$. It is known that a Gröbner basis of I_A with respect to $<$ always exists.

Moreover, if $\mathcal{G}$ is a Gröbner basis of I_A, then I_A is generated by $\mathcal{G}$. A Gröbner basis $\mathcal{G}$ of I_A is called *quadratic* if $\text{in}_<(\mathcal{G})$ is quadratic. We are interested in two implications below:

$$\begin{aligned} I_A \text{ has a squarefree initial ideal} &\Longrightarrow K[A] \text{ is normal} \\ &\Longrightarrow K[A] \text{ is very ample;} \\ I_A \text{ has a quadratic Gröbner basis} &\Longrightarrow K[A] \text{ is Koszul} \\ &\Longrightarrow I_A \text{ is generated by quadratic binomials.} \end{aligned}$$

It is known that each converse is false in general. See, e.g., [**6, 7**].

For the sake of simplicity, let $A = \{\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}\}$ be a configuration of $K[\mathbf{t}]$ with the following properties:

- $|\mathbf{a}_j| = r$ for each $1 \leq j \leq n$;
- t_i divides the monomial $\mathbf{t}^{\mathbf{a}_1} \cdots \mathbf{t}^{\mathbf{a}_n}$ for each $1 \leq i \leq d$.

(Note that any configuration is isomorphic to such a configuration.) Assume that, for each $1 \leq i \leq d$, a configuration $B_i = \{m_1^{(i)}, \ldots, m_{\lambda_i}^{(i)}\}$ of a polynomial ring $K[\mathbf{u}^{(i)}] = K[u_1^{(i)}, \ldots, u_{\mu_i}^{(i)}]$ in μ_i variables over K is given. Then the *nested configuration* [**1**] arising from A and $B_1, \ldots, B_d$ is the configuration

$$\begin{aligned} &A(B_1, \ldots, B_d) \\ &\quad := \left\{ m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)} \mid t_{i_1} \cdots t_{i_r} \in A,\ 1 \leq j_k \leq \lambda_{i_k} \text{ for } 1 \leq k \leq r \right\} \end{aligned}$$

of the polynomial ring $K[\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(d)}]$ in $\sum_{i=1}^{d} \mu_i$ variables over K. Here, $t_{i_1} \cdots t_{i_r} \in A$ is not necessarily squarefree. If $A = \{t_1 t_2\}$, then $K[A(B_1, B_2)]$ is the Segre product of $K[B_1]$ and $K[B_2]$. Moreover, if $A = \{t_1^m\}$, then $K[A(B_1)]$ is the mth Veronese subring of $K[B_1]$.

Example 0.1. Let $A = \{t_1^2, t_1 t_2\}$, $B_1 = \{u_1^2, u_1 u_2, u_2^2\}$ and $B_2 = \{v_1^2 v_2, v_1 v_2^2\}$. Then, the nested configuration $A(B_1, B_2)$ consists of the monomials

$$u_1^4, u_1^3 u_2, u_1^2 u_2^2, u_1 u_2^3, u_2^4, \qquad u_1^2 v_1^2 v_2, u_1 u_2 v_1^2 v_2,$$
$$u_2^2 v_1^2 v_2, u_1^2 v_1 v_2^2, u_1 u_2 v_1 v_2^2, u_2^2 v_1 v_2^2.$$

Then, the matrices

$$M_A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_{B_1} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad M_{B_2} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

$$M_{A(B_1,B_2)} = \left(\begin{array}{ccccc|cccccc} 4 & 3 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \end{array}\right)$$

correspond to the configurations A, B_1, B_2 and $A(B_1, B_2)$, respectively.

One of the fundamental facts of the nested configuration is

Theorem 0.2 [**1**]. *If each of the toric ideals I_A, $I_{B_1}, \ldots, I_{B_d}$ possesses a quadratic Gröbner basis, then the toric ideal $I_{A(B_1,\ldots,B_d)}$ possesses a quadratic Gröbner basis.*

In Section 1, we study the normality of the toric ring arising from a nested configuration. Our first main result is Theorem 1.2: if each of $K[A], K[B_1], \ldots, K[B_d]$ is normal, then $K[A(B_1, \ldots, B_d)]$ is also normal. In general–see Example 1.3–the converse does not hold. However, Corollary 1.9 guarantees that, when A consists of squarefree monomials, each of $K[A], K[B_1], \ldots, K[B_d]$ is normal if and only if $K[A(B_1, \ldots, B_d)]$ is normal.

In Section 2, we study Gröbner bases of the toric ideal arising from a nested configuration. A natural generalization of Theorem 0.2 will be obtained. In fact, Theorem 2.5 together with Theorem 2.6 guarantees that if each of $I_A, I_{B_1}, \ldots, I_{B_d}$ possesses a Gröbner basis consisting of binomials of degree at most p, then $I_{A(B_1,\ldots,B_d)}$ possesses a Gröbner basis consisting of binomials of degree at most $\max(2,p)$. Moreover, if each of $I_A, I_{B_1}, \ldots, I_{B_d}$ possesses a squarefree initial ideal, then $I_{A(B_1,\ldots,B_d)}$ possesses a squarefree initial ideal.

In Section 3, as one of the combinatorial applications of our algebraic theory of nested configurations, we discuss the toric ideal of a multiple of the Birkhoff polytope $\mathcal{B}_3$. Here $\mathcal{B}_3$ is the convex hull of

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\sigma_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_5 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

in $\mathbf{R}^{3\times 3}$. The toric ideal of $\mathcal{B}_3$ is the toric ideal of the configuration

$$B_1 = \{u_{11}u_{22}u_{33}, u_{12}u_{23}u_{31}, u_{13}u_{21}u_{32}, u_{11}u_{23}u_{32}, u_{12}u_{21}u_{33}, u_{13}u_{22}u_{31}\}$$

of polynomial ring $K[u_{11}, \ldots, u_{33}]$, and it is a principal ideal generated by $z_1z_2z_3 - z_4z_5z_6$. Given an integer $m \geq 1$, the m multiple of $\mathcal{B}_3$ is defined by $m\mathcal{B}_3 = \{m\alpha \mid \alpha \in \mathcal{B}_3\}$. Since it is well known (due to Birkhoff) that

$$m\mathcal{B}_3 \cap \mathbf{Z}^{3\times 3} = \{\sigma_{i_1} + \cdots + \sigma_{i_m} \mid 1 \leq i_1, \ldots, i_m \leq 6\},$$

the toric ideal of $m\mathcal{B}_3$ is the toric ideal of the nested configuration $A(B_1)$ where $A = \{t_1^m\}$. In [**2**], it is stated that Piechnik and Haase proved that the toric ideal of the multiple $2n\mathcal{B}_3$ possesses a squarefree quadratic initial ideal for $n > 1$. This fact is directly obtained by Theorem 2.6 since the toric ideal of the multiple $2\mathcal{B}_3$ possesses a squarefree quadratic initial ideal. Similarly, since the toric ideal of the multiple $3\mathcal{B}_3$ possesses a squarefree quadratic initial ideal, Theorem 2.6 guarantees that the toric ideal of the multiple $3n\mathcal{B}_3$ possesses a squarefree quadratic initial ideal for $n > 1$. However, since there are infinitely many prime numbers, it is difficult to show the existence of a squarefree quadratic initial ideal of the toric ideal of $m\mathcal{B}_3$ for all $m > 1$ in this way. In

Theorem 3.4, using another monomial order, we will prove that the toric ideal of the multiple $m\mathcal{B}_3$ possesses a quadratic Gröbner basis for all $m > 1$.

In Section 4, we give a summary of our algebraic theory of nested configurations.

1. Normality of toric rings of nested configurations. The purpose of this section is to study normality of $K[A(B_1, \dots, B_d)]$.

Lemma 1.1 [**3**]. *The toric ring $K[A]$ is normal if and only if*

$$\left\{ \frac{M_1}{M_2} \,\middle|\, M_1, M_2 \in K[A] \text{ are monomials and } \left(\frac{M_1}{M_2}\right)^m \in K[A] \text{ for some } 0 < m \in \mathbf{Z} \right\}$$

is a subset of $K[A]$.

Theorem 1.2. *If $K[A]$, $K[B_1], \dots, K[B_d]$ are normal, then $K[A(B_1, \dots, B_d)]$ is normal.*

Proof. Suppose that $K[A]$, $K[B_1], \dots, K[B_d]$ are normal and that $K[A(B_1, \dots, B_d)]$ is not normal. Thanks to Lemma 1.1, there exist monomials M_1, M_2, M_3 belonging to $K[A(B_1, \dots, B_d)]$ such that $M_1/M_2 \notin K[A(B_1, \dots, B_d)]$ and that $(M_1/M_2)^n = M_3$ for some integer $n > 1$.

Let $\psi : K[A(B_1, \dots, B_d)] \to K[A]$ be the surjective homomorphism defined by $\psi(m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)}) = t_{i_1} \cdots t_{i_r} \in A$. Then $\psi(M_1), \psi(M_2) \in K[A]$ and

$$(\psi(M_1)/\psi(M_2))^n = \psi(M_3) \in K[A].$$

Since $K[A]$ is normal, we have $\psi(M_1)/\psi(M_2) \in K[A]$. Thus, $\psi(M_1)/\psi(M_2) = \mathbf{t}^{\mathbf{a}_{i_1}} \cdots \mathbf{t}^{\mathbf{a}_{i_p}}$ for some $1 \leq i_1, \dots, i_p \leq n$.

Let $\rho_k : K[A(B_1, \dots, B_d)] \to K[B_k]$ be the homomorphism defined by

$$\rho_k(u_j^{(i)}) = \begin{cases} u_j^{(i)} & \text{if } i = k \\ 1 & \text{otherwise.} \end{cases}$$

Then $\rho_k(M_1), \rho_k(M_2) \in K[B_k]$ and $(\rho_k(M_1)/\rho_k(M_2))^n = \rho_k(M_3) \in K[B_k]$. Since $K[B_k]$ is normal, $\rho_k(M_1)/\rho_k(M_2) \in K[B_k]$. Thus, $\rho_k(M_1)/\rho_k(M_2) = m_{j_1}^{(k)} \cdots m_{j_{q_k}}^{(k)}$ for some $1 \leq j_1, \dots, j_{q_k} \leq \lambda_k$. Since B_k is a configuration, it follows that $\rho_k(M_1) = m_{u_1}^{(k)} \cdots m_{u_{q_k+r_k}}^{(k)}$ and $\rho_k(M_2) = m_{v_1}^{(k)} \cdots m_{v_{r_k}}^{(k)}$. Then $\psi(M_1) = t_1^{q_1+r_1} \cdots t_d^{q_d+r_d}$ and $\psi(M_2) = t_1^{r_1} \cdots t_d^{r_d}$. Thus, we have

$$\frac{\psi(M_1)}{\psi(M_2)} = \mathbf{t}^{\mathbf{a}_{i_1}} \cdots \mathbf{t}^{\mathbf{a}_{i_p}} = t_1^{q_1} \cdots t_d^{q_d}.$$

Hence, $M_1/M_2 \in K[A(B_1, \dots, B_d)]$, and this is a contradiction. □

The converse of Theorem 1.2 is false in general.

Example 1.3. Let $A = \{t_1^2\}$ and $B_1 = \{v, uv, u^3v, u^4v\}$. Then $K[B_1]$ is not normal. However, $I_{A(B_1)}$ has a squarefree quadratic initial ideal and hence $K[A(B_1)] = K[\{u^i v^2 \mid i = 0, 1, \dots, 8\}]$ is normal.

Theorem 1.2 did not hold when we replaced "normal" with "very ample."

Example 1.4. Let $A = \{t_1, t_2\}$, $B_1 = \{v, uv, u^3v, u^4v\}$ and $B_2 = \{w\}$. Then $K[A]$ and $K[B_2]$ are polynomial rings. On the other hand, $K[B_1]$ is very ample, but not normal. However, $K[A(B_1, B_2)] = K[v, uv, u^3v, u^4v, w]$ is not very ample. In fact, the monomial u^2vw^α does not belong to $K[A(B_1, B_2)]$ for all $\alpha \in \mathbf{Z}_{\geq 0}$.

Let P_A denote the convex hull of $\{\mathbf{a} \in \mathbf{Z}_{\geq 0}^d \mid \mathbf{t}^\mathbf{a} \in A\}$. For a subset $B \subset A$, $K[B]$ is called the *combinatorial pure subring* ([**4, 5**]) of $K[A]$ if there exists a face F of P_A such that $\{\mathbf{b} \in \mathbf{Z}_{\geq 0}^d \mid \mathbf{t}^\mathbf{b} \in B\} = \{\mathbf{a} \in \mathbf{Z}_{\geq 0}^d \mid \mathbf{t}^\mathbf{a} \in A\} \cap F$. For example, if $B = A \cap K[t_{i_1}, \dots, t_{i_s}]$ for some $1 \leq i_1 < \cdots < i_s \leq d$, then $K[B]$ is a combinatorial pure subring of $K[A]$. (This is the original definition of a combinatorial pure subring in [**5**].)

Lemma 1.5. *The toric ring $K[A(B_1, \dots, B_d)]$ has a combinatorial pure subring which is isomorphic to $K[A]$.*

Proof. For each $i = 1, 2, \dots, d$, let σ_i be an arbitrary monomial of B_i which corresponds to a vertex of P_{B_i}. It follows that $K[A(\{\sigma_1\}, \dots, \{\sigma_d\})]$ is a combinatorial pure subring of $K[A(B_1, \dots, B_d)]$. Then $K[A(\{\sigma_1\}, \dots, \{\sigma_d\})] \simeq K[A]$. □

It is known [**8**, Lemma 1] that every combinatorial pure subring of a normal (respectively, very ample) semigroup ring is normal (respectively, very ample). Thus we have the following.

Theorem 1.6. *If $K[A(B_1, \dots, B_d)]$ is normal* (*respectively, very ample*)*, then $K[A]$ is normal* (*respectively, very ample*)*.*

Lemma 1.7. *Let $m = \max(i \mid t_1^i t_2^{a_2} \cdots t_d^{a_d} \in A) \geq 1$. Then $K[A(B_1, \dots, B_d)]$ has a combinatorial pure subring which is isomorphic to $K[A'(B_1)]$ where $A' = \{t_1^m\}$. In particular, if $m = 1$, then we have $K[A'(B_1)] = K[B_1]$.*

Proof. Let $t_1^m t_2^{a_2} \cdots t_d^{a_d}$ be the largest monomial of A with respect to a lexicographic order $t_1 > \cdots > t_d$. Let $A = \{\mathbf{t}^{\mathbf{a}_1} = t_1^m t_2^{a_2} \cdots t_d^{a_d}, \mathbf{t}^{\mathbf{a}_2}, \dots, \mathbf{t}^{\mathbf{a}_n}\}$. Thanks to [**9**, Proposition 1.11], there exists a nonnegative integer vector $\mathbf{v}$ such that $\mathbf{v} \cdot \mathbf{a}_1 > \mathbf{v} \cdot \mathbf{a}_i$ for all $2 \leq i \leq n$. Then $(m, a_2, \dots, a_d)$ is a $\mathbf{v}$-vertex of P_A. Hence, $K[A(B_1, \dots, B_d)]$ has a combinatorial pure subring $K[A''(B_1, \dots, B_d)]$ with $A'' = \{t_1^m t_2^{a_2} \cdots t_d^{a_d}\}$. For each $i = 2, \dots, d$, let σ_i be an arbitrary monomial of B_i which corresponds to a vertex of P_{B_i}. It follows that $K[A''(B_1, \{\sigma_2\}, \dots, \{\sigma_d\})]$ is a combinatorial pure subring of $K[A''(B_1, \dots, B_d)]$. Then $K[A''(B_1, \{\sigma_2\}, \dots, \{\sigma_d\})] \simeq K[A'(B_1)]$ where $A' = \{t_1^m\}$. □

Thanks to Lemma 1.7, we have the following.

Theorem 1.8. *If A has no monomial divided by t_i^2 and if $K[A(B_1, \dots, B_d)]$ is normal* (*respectively, very ample*)*, then $K[B_i]$ is normal* (*respectively, very ample*)*.*

Corollary 1.9. *Suppose that a configuration A consists of squarefree monomials. Then $K[A]$, $K[B_1], \dots, K[B_d]$ are normal if and only if $K[A(B_1, \dots, B_d)]$ is normal.*

2. Gröbner bases of toric ideals of nested configurations. In this section, using the technique (sorting operator) in the proof of [**9**, Theorem 14.2], we study Gröbner bases of the toric ideal of a nested configuration. The present section has three subsections:

- Gröbner bases for polynomial ring case, i.e., each $K[B_i]$ is a polynomial ring;
- Gröbner bases for general case;
- Generators.

First, we introduce the sorting operator used in [**9**]:

Example 2.1 [**9**, Theorem 14.2]. Fix positive integers r and $s_1, \ldots, s_d$. Let

$$A = \{t_1^{i_1} \cdots t_d^{i_d} \mid i_1 + \cdots + i_d = r,\ 0 \le i_1 \le s_1, \ldots, 0 \le i_d \le s_d\}.$$

We define a natural bijection between the element of A and weakly increasing strings of length r over the alphabet $\{1, 2, \ldots, d\}$ having at most s_j occurrences of the letter j which maps the monomial $t_1^{i_1} \cdots t_d^{i_d} \in A$ to the weakly increasing string

$$u_1 u_2 \cdots u_r = \underbrace{11\cdots1}_{i_1 \text{ times}}\underbrace{22\cdots2}_{i_2 \text{ times}}\underbrace{33\cdots3}_{i_3 \text{ times}}\cdots\underbrace{dd\cdots d}_{i_d \text{ times}}.$$

We write $x_{u_1 u_2 \cdots u_r}$ for the corresponding variable in $K[\mathbf{x}]$. Let $\mathrm{sort}\,(\cdot)$ denote the operator which takes any string over the alphabet $\{1, 2, \ldots, d\}$ and sorts it into weakly increasing order. It is known [**9**, Theorem 14.2] that there exists a monomial order $<$ on $K[\mathbf{x}]$ such that

$$\{x_{u_1 u_2 \cdots u_r} x_{v_1 v_2 \cdots v_r} - x_{w_1 w_3 \cdots w_{2r-1}} x_{w_2 w_4 \cdots w_{2r}} \mid w_1 w_2 w_3 \cdots w_{2r} = \mathrm{sort}\,(u_1 v_1 u_2 v_2 \cdots u_r v_r)\}$$

is a quadratic Gröbner basis of I_A with respect to $<$ and $\mathrm{in}_<(I_A)$ is squarefree. For example, $x_{12}x_{33} - x_{13}x_{23}$ belongs to the Gröbner basis since we have $1233 = \mathrm{sort}(1323)$.

Let, as before, $A = \{\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}\}$ and $B_i = \{m_1^{(i)}, \ldots, m_{\lambda_i}^{(i)}\}$ for $1 \le i \le d$. Let $K[\mathbf{x}]$ be a polynomial ring with the set of variables

$$\left\{ x^{(k)}_{(i_1,j_1)\cdots(i_r,j_r)} \;\middle|\; \begin{array}{c} 1 \le i_1 \le \cdots \le i_r \le d,\ 1 \le k \le n \\ t_{i_1} \cdots t_{i_r} = \mathbf{t}^{\mathbf{a}_k} \in A \\ m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)} \in A(B_1, \ldots, B_d) \end{array} \right\},$$

and let $K[\mathbf{y}] = K[y_1, \ldots, y_n]$ and $K\left[\mathbf{z}^{(i)}\right] = K\left[z_1^{(i)}, \ldots, z_{\lambda_i}^{(i)}\right]$ $(i = 1, 2, \ldots, d)$ be polynomial rings. The toric ideal I_A is the kernel of the homomorphism $\pi_0 : K[\mathbf{y}] \to K[\mathbf{t}]$ defined by setting $\pi_0(y_k) = \mathbf{t}^{\mathbf{a}_k}$. The toric ideal I_{B_i} is the kernel of the homomorphism $\pi_i : K[\mathbf{z}^{(i)}] \to K[\mathbf{u}^{(i)}]$ defined by setting $\pi_i(z_j^{(i)}) = m_j^{(i)}$. The toric ideal $I_{A(B_1,\ldots,B_d)}$ is the kernel of the homomorphism $\pi : K[\mathbf{x}] \to K[\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(d)}]$ defined by setting $\pi(x^{(k)}_{(i_1,j_1)\cdots(i_r,j_r)}) = m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)}$.

Lemma 2.2. *Let $p_1 = x^{(k)}_{(i_1,j_1)\cdots(i_r,j_r)} x^{(k)}_{(i_{r+1},j_{r+1})\cdots(i_{2r},j_{2r})}$ be a quadratic monomial in $K[\mathbf{x}]$, and let* $\operatorname{sort}(\cdot)$ *be the sorting operator over the alphabet*

$$\{(1,1), (1,2), \ldots, (1,\lambda_1), (2,1), \ldots, (d,\lambda_d)\}$$

with respect to the ordering

$$(1,1) \succ (1,2) \succ \cdots \succ (1,\lambda_1) \succ (2,1) \succ \cdots \succ (d,\lambda_d).$$

Then, $p_2 = x^{(k)}_{(i'_1,j'_1)(i'_3,j'_3)\cdots(i'_{2r-1},j'_{2r-1})} x^{(k)}_{(i'_2,j'_2)(i'_4,j'_4)\cdots(i'_{2r},j'_{2r})}$ where

$$(i'_1,j'_1)\cdots(i'_{2r},j'_{2r}) = \operatorname{sort}\left((i_1,j_1)\cdots(i_{2r},j_{2r})\right)$$

is a monomial belonging to $K[\mathbf{x}]$ and, in particular, we have $p_1 - p_2 \in I_{A(B_1,\ldots,B_d)}$.

Proof. Suppose that $x^{(k)}_{(i'_1,j'_1)(i'_3,j'_3)\cdots(i'_{2r-1},j'_{2r-1})}$ is not a variable in $K[\mathbf{x}]$. Then we have $t_{i'_1} t_{i'_3} \cdots t_{i'_{2r-1}} \neq \mathbf{t}^{\mathbf{a}_k}$, and hence there exist integers $1 \leq i \leq d$ and α such that t_i^{α} divides $\mathbf{t}^{\mathbf{a}_k}$ and does not divide $t_{i'_1} t_{i'_3} \cdots t_{i'_{2r-1}}$. Since $i'_1 \leq \cdots \leq i'_{2r}$, it then follows that $t_i^{2\alpha}$ does not divide $t_{i'_1} t_{i'_2} \cdots t_{i'_{2r}}$. Thanks to $(i'_1,j'_1)\cdots(i'_{2r},j'_{2r}) = \operatorname{sort}\left((i_1,j_1)\cdots(i_{2r},j_{2r})\right)$, we have $t_{i_1} t_{i_2} \cdots t_{i_{2r}} = t_{i'_1} t_{i'_2} \cdots t_{i'_{2r}}$. Hence $t_i^{2\alpha}$ does not divide $t_{i_1} t_{i_2} \cdots t_{i_{2r}}$. It follows that t_i^{α} does not divide either $t_{i_1} t_{i_2} \cdots t_{i_r}$ or $t_{i_{r+1}} t_{i_{r+2}} \cdots t_{i_{2r}}$. Thus, either $t_{i_1} t_{i_2} \cdots t_{i_r}$ or $t_{i_{r+1}} t_{i_{r+2}} \cdots t_{i_{2r}}$ is not equal to $\mathbf{t}^{\mathbf{a}_k}$. This contradicts that p_1 is a monomial of $K[\mathbf{x}]$.

On the other hand, by virtue of $(i'_1, j'_1) \cdots (i'_{2r}, j'_{2r}) = \mathrm{sort}\,((i_1, j_1) \cdots (i_{2r}, j_{2r}))$, we have $\pi(p_1) = \pi(p_2)$, and hence $p_1 - p_2 \in I_{A(B_1, \ldots, B_d)}$ as desired. □

Lemma 2.3. *Let $y_{k_1} \cdots y_{k_p} - y_{k'_1} \cdots y_{k'_p}$ be a binomial in I_A, and let*

$$\prod_{\ell=1}^{p} x^{(k_\ell)}_{(i_{(\ell-1)r+1}, j_{(\ell-1)r+1}) \cdots (i_{\ell r}, j_{\ell r})}$$

be a monomial in $K[\mathbf{x}]$. Then, there exists a binomial

$$\prod_{\ell=1}^{p} x^{(k_\ell)}_{(i_{(\ell-1)r+1}, j_{(\ell-1)r+1}) \cdots (i_{\ell r}, j_{\ell r})} - \prod_{\ell=1}^{p} x^{(k'_\ell)}_{(i'_{(\ell-1)r+1}, j'_{(\ell-1)r+1}) \cdots (i'_{\ell r}, j'_{\ell r})} \in I_{A(B_1, \ldots, B_d)},$$

where $\mathrm{sort}\,((i_1, j_1) \cdots (i_{pr}, j_{pr})) = \mathrm{sort}\,((i'_1, j'_1) \cdots (i'_{pr}, j'_{pr}))$.

Proof. Let $\pi_0(y'_{k_\ell}) = t_{i'_{(\ell-1)r+1}} \cdots t_{i'_{\ell r}}$ for each $1 \leq \ell \leq p$. Since $y_{k_1} \cdots y_{k_p} - y_{k'_1} \cdots y_{k'_p}$ belongs to I_A, we have $\prod_{\ell=1}^{pr} t_{i_\ell} = \prod_{\ell=1}^{pr} t_{i'_\ell}$. Hence there exist $j'_1, \ldots, j'_{pr}$ such that

$$\mathrm{sort}\,((i_1, j_1) \cdots (i_{pr}, j_{pr})) = \mathrm{sort}\,((i'_1, j'_1) \cdots (i'_{pr}, j'_{pr})).$$

It then follows that

$$\prod_{\ell=1}^{p} x^{(k_\ell)}_{(i_{(\ell-1)r+1}, j_{(\ell-1)r+1}) \cdots (i_{\ell r}, j_{\ell r})} - \prod_{\ell=1}^{p} x^{(k'_\ell)}_{(i'_{(\ell-1)r+1}, j'_{(\ell-1)r+1}) \cdots (i'_{\ell r}, j'_{\ell r})} \in I_{A(B_1, \ldots, B_d)},$$

as desired. □

Fix a monomial order $<_i$ on $K[\mathbf{z}^{(i)}]$ for each $1 \leq i \leq d$. Let $\mathcal{G}_i$ be a Gröbner basis of I_{B_i} with respect to $<_i$. For each $M \in A(B_1, \ldots, B_d)$, the expression $M = m^{(i_1)}_{j_1} \cdots m^{(i_r)}_{j_r}$ is called *standard* if

$$\prod_{\substack{i_\ell = j, \\ 1 \leq \ell \leq r}} z^{(i_\ell)}_{j_\ell}$$

is a standard monomial with respect to $\mathcal{G}_j$ for all $1 \leq j \leq d$. In order to study the relation among I_A, I_{B_i} and $I_{A(B_1,\dots,B_d)}$, we define homomorphisms

$$\varphi_0 : K[\mathbf{x}] \longrightarrow K[\mathbf{y}], \qquad \varphi_0\left(x^{(k)}_{(i_1,j_1)\cdots(i_r,j_r)}\right) = y_k,$$
$$\varphi_j : K[\mathbf{x}] \longrightarrow K[\mathbf{z}^{(j)}], \quad \varphi_j\left(x^{(k)}_{(i_1,j_1)\cdots(i_r,j_r)}\right) = \prod_{i_\ell=j,\quad 1\leq\ell\leq r} z^{(i_\ell)}_{j_\ell},$$

where $m^{(i_1)}_{j_1}\cdots m^{(i_r)}_{j_r}$ is the standard expression defined above.

Lemma 2.4 [**1**]. *Let f be a binomial in $K[\mathbf{x}]$. Then $f \in I_{A(B_1,\dots,B_d)}$ if and only if $\varphi_i(f) \in I_{B_i}$ for all $1 \leq i \leq d$. Moreover, if f belongs to $I_{A(B_1,\dots,B_d)}$, then we have $\varphi_0(f) \in I_A$.*

2.1. Polynomial ring case. First, we study the case when all of $K[B_i]$ are polynomial rings.

Theorem 2.5. *Let $\mathcal{G}_0$ be a Gröbner basis of I_A with respect to a monomial order $<_0$. If each B_i is a set of variables, then the toric ideal $I_{A(B_1,\dots,B_d)}$ possesses a Gröbner basis consisting of the following binomials:*

(1) $$\underline{\prod_{\ell=1}^{p} x^{(k_\ell)}_{(i_{(\ell-1)r+1},j_{(\ell-1)r+1})\cdots(i_{\ell r},j_{\ell r})}} - \prod_{\ell=1}^{p} x^{(k'_\ell)}_{(i'_{(\ell-1)r+1},j'_{(\ell-1)r+1})\cdots(i'_{\ell r},j'_{\ell r})}$$

where $\underline{y_{k_1}\cdots y_{k_p}} - y_{k'_1}\cdots y_{k'_p} \in \mathcal{G}_0$ *and*

$$\operatorname{sort}\left((i_1,j_1)\cdots(i_{pr},j_{pr})\right) = \operatorname{sort}\left((i'_1,j'_1)\cdots(i'_{pr},j'_{pr})\right).$$

(2)

$$\underline{x^{(k)}_{(i_1,j_1)\cdots(i_r,j_r)} x^{(k)}_{(i_{r+1},j_{r+1})\cdots(i_{2r},j_{2r})}} - x^{(k)}_{(i'_1,j'_1)(i'_3,j'_3)\cdots(i'_{2r-1},j'_{2r-1})} x^{(k)}_{(i'_2,j'_2)(i'_4,j'_4)\cdots(i'_{2r},j'_{2r})}$$

where $\operatorname{sort}\left((i_1,j_1)\cdots(i_{2r},j_{2r})\right) = (i'_1,j'_1)\cdots(i'_{2r},j'_{2r})$ *with respect to the ordering* $(1,1) \succ (1,2) \succ \cdots \succ (1,\lambda_1) \succ (2,1) \succ \cdots \succ (d,\lambda_d)$.

(3) $\underline{x^{(k)}_{(i_1,j_1)\cdots(i_\ell,j_\ell)\cdots(i_r,j_r)} x^{(k')}_{(i'_1,j'_1)\cdots(i'_{\ell'},j'_{\ell'})\cdots(i'_r,j'_r)}} - x^{(k)}_{(i_1,j_1)\cdots(i'_{\ell'},j'_{\ell'})\cdots(i_r,j_r)} x^{(k')}_{(i'_1,j'_1)\cdots(i_\ell,j_\ell)\cdots(i'_r,j'_r)}$

where $k < k'$, $i_\ell = i'_{\ell'}$ *and* $j_\ell > j'_{\ell'}$.

The initial monomial of each binomial is the first (*underlined*) *monomial and, in particular, the initial monomial of each binomial in* (2) *and* (3) *is squarefree. Moreover, the initial monomial of each binomial in* (1) *is squarefree* (*respectively, quadratic*) *if the corresponding monomial* $y_{k_1} \cdots y_{k_p}$ *is squarefree* (*respectively, quadratic*).

Proof. Let $\mathcal{G}$ denote the set of binomials above. Thanks to Lemmas 2.2 and 2.3, it is easy to see that $\mathcal{G}$ is a (finite) subset of $I_{A(B_1,\ldots,B_d)}$.

Claim 1. *There exists a monomial order such that the initial monomial of each binomial in* $\mathcal{G}$ *is the underlined monomial.*

By virtue of [**9**, Theorem 3.12], it is enough to show that the reduction modulo $\mathcal{G}$ is Noetherian. Suppose that there exists a sequence of reductions modulo $\mathcal{G}$ which does not terminate. Let v be a monomial in $K[\mathbf{x}]$, and assume $v \xrightarrow{g} v'$ with $g \in \mathcal{G}$. Then we have

$$\begin{cases} \varphi_0(v) >_0 \varphi_0(v') & \text{if } g \text{ in (1)}, \\ \varphi_0(v) = \varphi_0(v') & \text{otherwise.} \end{cases}$$

Hence, the number of binomials in (1) appearing in the sequence is finite. Thus we may assume that the binomials in (1) do not appear in the sequence. Let v be a monomial in $K[\mathbf{x}]$, and assume $v \xrightarrow{g} v'$ where $g \in \mathcal{G}$ belongs to either (2) or (3). Since g belongs to either (2) or (3), v and v' are of the form $v = \prod_{\ell=1}^{p} x^{(k_\ell)}_{(i_{(\ell-1)r+1}, j_{(\ell-1)r+1})\cdots(i_{\ell r}, j_{\ell r})}$, $v' = \prod_{\ell=1}^{p} x^{(k_\ell)}_{(i'_{(\ell-1)r+1}, j'_{(\ell-1)r+1})\cdots(i'_{\ell r}, j'_{\ell r})}$. Let

$$\text{Inversion}\,(v) = \left\{ (\xi, \xi') \;\middle|\; \begin{array}{r} \ell(r-1)+1 \le \xi \le \ell r \\ \ell'(r-1)+1 \le \xi' \le \ell' r \\ i_\xi = i_{\xi'},\ j_\xi > j_{\xi'} \\ k_\ell < k_{\ell'} \end{array} \right\},$$

$$\text{Inversion}\,(v') = \left\{ (\xi, \xi') \;\middle|\; \begin{array}{r} \ell(r-1)+1 \le \xi \le \ell r \\ \ell'(r-1)+1 \le \xi' \le \ell' r \\ i'_\xi = i'_{\xi'},\ j'_\xi > j'_{\xi'} \\ k_\ell < k_{\ell'} \end{array} \right\}.$$

Then the cardinality of these sets satisfies $\sharp|\text{Inversion}\,(v)| \ge \sharp|\text{Inversion}\,(v')|$ where equality holds if and only if g belongs to (2). Hence, the number of binomials in (3) appearing in the sequence is finite. Thus, we may assume that the binomials in (3) do not appear in the sequence. However, any sequence of reductions modulo the set of binomials in (2) corresponds to the sort of the indices and hence it terminates. This is a contradiction.

Claim 2. *The set $\mathcal{G}$ is a Gröbner basis of $I_{A(B_1,\dots,B_d)}$.*

Suppose that $\mathcal{G}$ is not a Gröbner basis of $I_{A(B_1,\dots,B_d)}$. Thanks to Lemmas 2.2 and 2.3, there exists a binomial $f = p_1 - p_2 \in I_{A(B_1,\dots,B_d)}$ such that neither p_1 nor p_2 is divisible by the initial monomial of any binomial in $\mathcal{G}$. By virtue of Lemma 2.4, we have $\varphi_0(f) = \varphi_0(p_1) - \varphi_0(p_2) \in I_A$. If $\varphi_0(p_1) - \varphi_0(p_2) \ne 0$, then there exists a binomial $g \in \mathcal{G}_0$ such that the initial monomial of g divides either $\varphi_0(p_1)$ or $\varphi_0(p_2)$. This contradicts that neither p_1 nor p_2 is divisible by the initial monomial of any binomial in (1). Hence, we have $\varphi_0(p_1) = \varphi_0(p_2)$. Thus, f is of the form

$$f = \prod_{\ell=1}^{p} x^{(k_\ell)}_{(i_{(\ell-1)r+1}, j_{(\ell-1)r+1})\cdots(i_{\ell r}, j_{\ell r})} - \prod_{\ell=1}^{p} x^{(k_\ell)}_{(i'_{(\ell-1)r+1}, j'_{(\ell-1)r+1})\cdots(i'_{\ell r}, j'_{\ell r})}.$$

Since neither p_1 nor p_2 is divisible by the initial monomial of any binomial in either (2) or (3), it follows that $p_1 = p_2$ and hence $f = 0$. $\square$

2.2. General case. We now study the general case.

Theorem 2.6. *Let $\mathcal{G}_0$ be a Gröbner basis of I_A, and let $\mathcal{G}_i$ be a Gröbner basis of I_{B_i} with respect to $<_i$. Then the toric ideal $I_{A(B_1,\dots,B_d)}$ possesses a Gröbner basis consisting of the binomials* (1), (2) *and* (3) *appearing in Theorem* 2.5 *together with the following binomials*:

(4) $\underline{\prod_{\ell=1}^{p} x^{(k_\ell)}_{M_\ell(i,j_{\ell,1})\cdots(i,j_{\ell,q_\ell})M'_\ell}} - \prod_{\ell=1}^{p} x^{(k_\ell)}_{M_\ell(i,j'_{\ell,1})\cdots(i,j'_{\ell,q_\ell})M'_\ell}$ *where the binomial* $0 \neq \underline{\prod_{\ell=1}^{p} z^{(i)}_{j_{\ell,1}} \cdots z^{(i)}_{j_{\ell,q_\ell}}} - \prod_{\ell=1}^{p} z^{(i)}_{j'_{\ell,1}} \cdots z^{(i)}_{j'_{\ell,q_\ell}}$ *belongs to* $\mathcal{G}_i$.

The initial monomial of each binomial is the first (underlined) monomial and, in particular, the initial monomial of each binomial above is squarefree (respectively, quadratic) if the corresponding monomial $\prod_{\ell=1}^{p} z^{(i)}_{j_{\ell,1}} \cdots z^{(i)}_{j_{\ell,q_\ell}}$ *is squarefree (respectively, quadratic).*

Proof. Let $\mathcal{G}$ denote the set of binomials above. Thanks to Lemmas 2.2, 2.3 and 2.4, $\mathcal{G}$ is a (finite) subset of $I_{A(B_1,\ldots,B_d)}$.

Claim 1. *There exists a monomial order such that the initial monomial of each binomial in* $\mathcal{G}$ *is the underlined monomial.*

By virtue of [**9**, Theorem 3.12], it is enough to show that the reduction modulo $\mathcal{G}$ is Noetherian. Suppose that there exists a sequence of reductions modulo $\mathcal{G}$ which does not terminate. Let v be a monomial in $K[\mathbf{x}]$, and assume $v \xrightarrow{g} v'$ with $g \in \mathcal{G}$. Then we have

$$\begin{cases} \varphi_j(v) >_j \varphi_j(v') & \text{if } g \text{ is in (4) and arising from } \mathcal{G}_j, \\ \varphi_j(v) = \varphi_j(v') & \text{otherwise.} \end{cases}$$

Hence, the number of binomials in (4) appearing in the sequence is finite. Thus, we may assume that the binomials in (4) do not appear in the sequence. However, as we proved in the proof of Theorem 2.5, there exists no sequence of reductions modulo the set of binomials in (1), (2) and (3) which does not terminate. This is a contradiction.

Claim 2. *The set* $\mathcal{G}$ *is a Gröbner basis of* $I_{A(B_1,\ldots,B_d)}$.

Suppose that $\mathcal{G}$ is not a Gröbner basis of $I_{A(B_1,\ldots,B_d)}$. Thanks to Lemmas 2.2, 2.3 and 2.4, there exists a binomial $f = p_1 - p_2 \in I_{A(B_1,\ldots,B_d)}$ such that neither p_1 nor p_2 is divisible by the initial monomial of any binomial in $\mathcal{G}$. By virtue of Lemma 2.4, we have $\varphi_i(f) = \varphi_i(p_1) - \varphi_i(p_2) \in I_{B_i}$ for all $1 \leq i \leq d$. If $\varphi_i(p_1) - \varphi_i(p_2) \neq 0$ for some i, then there exists a binomial $g' \in \mathcal{G}_i$ such that the initial monomial of g' divides either $\varphi_i(p_1)$ or $\varphi_i(p_2)$. This contradicts that neither p_1 nor p_2 is divisible by the initial monomial of any binomial in (4). Hence we have $\varphi_i(p_1) = \varphi_i(p_2)$ for all i. Moreover, thanks to the argument in the proof of Theorem 2.5, we have $\varphi_0(p_1) = \varphi_0(p_2)$.

Thus, f is of the form

$$f = \prod_{\ell=1}^{p} x^{(k_\ell)}_{(i_{(\ell-1)r+1}, j_{(\ell-1)r+1}) \cdots (i_{\ell r}, j_{\ell r})} - \prod_{\ell=1}^{p} x^{(k_\ell)}_{(i'_{(\ell-1)r+1}, j'_{(\ell-1)r+1}) \cdots (i'_{\ell r}, j'_{\ell r})},$$

where $\operatorname{sort}((i_1, j_1) \cdots (i_{pr}, j_{pr})) = \operatorname{sort}((i'_1, j'_1) \cdots (i'_{pr}, j'_{pr}))$. Since neither p_1 nor p_2 is divisible by the initial monomial of any binomial in either (2) or (3), it follows that $p_1 = p_2$, and hence $f = 0$. $\square$

If $\mathcal{G}_i$ possesses a binomial of degree 3, then we need the following binomials:

(a) $x^{(k_1)}_{M_1(i,j_1)M'_1} x^{(k_2)}_{M_2(i,j_2)M'_2} x^{(k_3)}_{M_3(i,j_3)M'_3} - x^{(k_1)}_{M_1(i,j'_1)M'_1} x^{(k_2)}_{M_2(i,j'_2)M'_2} x^{(k_3)}_{M_3(i,j'_3)M'_3}$ where $z^{(i)}_{j_1} z^{(i)}_{j_2} z^{(i)}_{j_3} - z^{(i)}_{j'_1} z^{(i)}_{j'_2} z^{(i)}_{j'_3} \in \mathcal{G}_i$.

(b) $x^{(k_1)}_{M_1(i,j_1)(i,j_2)M'_1} x^{(k_2)}_{M_2(i,j_3)M'_2} - x^{(k_1)}_{M_1(i,j'_1)(i,j'_2)M'_1} x^{(k_2)}_{M_2(i,j'_3)M'_2}$ where $z^{(i)}_{j_1} z^{(i)}_{j_2} z^{(i)}_{j_3} - z^{(i)}_{j'_1} z^{(i)}_{j'_2} z^{(i)}_{j'_3} \in \mathcal{G}_i$.

We do not need (b) if A has no monomial divided by t_i^2. In general, we have

$$\deg\left(\prod_{\ell=1}^{p} z^{(i)}_{j_{\ell,1}} \cdots z^{(i)}_{j_{\ell,q_\ell}}\right) = \sum_{\ell=1}^{p} q_\ell \geq p = \deg\left(\prod_{\ell=1}^{p} x^{(k_\ell)}_{M_\ell(i,j_{\ell,1}) \cdots (i,j_{\ell,q_\ell})M'_\ell}\right).$$

The binomials of type (a) are not always needed for a minimal Gröbner basis even if $\mathcal{G}_i$ has a cubic binomial. In such a case, $I_{A(B_1,\ldots,B_d)}$ may have a quadratic Gröbner basis. In Section 3, we will show an example.

2.3. Generators. Thanks to a part of the argument in the proof of Theorem 2.6, we have the following.

Proposition 2.7. *Let $\mathcal{H}_0$ be a set of binomial generators of I_A, and let $\mathcal{H}_i$ be a set of binomial generators of I_{B_i}. Then, the toric ideal $I_{A(B_1,\ldots,B_d)}$ is generated by the following binomials:*

(1) $\displaystyle\prod_{\ell=1}^{p} x^{(k_\ell)}_{(i_{(\ell-1)r+1},j_{(\ell-1)r+1})\cdots(i_{\ell r},j_{\ell r})} - \prod_{\ell=1}^{p} x^{(k'_\ell)}_{(i'_{(\ell-1)r+1},j'_{(\ell-1)r+1})\cdots(i'_{\ell r},j'_{\ell r})}$
where $y_{k_1}\cdots y_{k_p} - y_{k'_1}\cdots y_{k'_p} \in \mathcal{H}_0$ *and*

$$\operatorname{sort}\left((i_1,j_1)\cdots(i_{pr},j_{pr})\right) = \operatorname{sort}\left((i'_1,j'_1)\cdots(i'_{pr},j'_{pr})\right).$$

(2)

$$x^{(k)}_{(i_1,j_1)\cdots(i_r,j_r)} x^{(k)}_{(i_{r+1},j_{r+1})\cdots(i_{2r},j_{2r})} - x^{(k)}_{(i'_1,j'_1)(i'_3,j'_3)\cdots(i'_{2r-1},j'_{2r-1})} x^{(k)}_{(i'_2,j'_2)(i'_4,j'_4)\cdots(i'_{2r},j'_{2r})}$$

where $\operatorname{sort}((i_1,j_1)\cdots(i_{2r},j_{2r})) = (i'_1,j'_1)\cdots(i'_{2r},j'_{2r})$ *with respect to the ordering* $(1,1) \succ (1,2) \succ \cdots \succ (1,\lambda_1) \succ (2,1) \succ \cdots \succ (d,\lambda_d)$.

(3) $x^{(k)}_{(i_1,j_1)\cdots(i_\ell,j_\ell)\cdots(i_r,j_r)} x^{(k')}_{(i'_1,j'_1)\cdots(i'_{\ell'},j'_{\ell'})\cdots(i'_r,j'_r)} - x^{(k)}_{(i_1,j_1)\cdots(i'_{\ell'},j'_{\ell'})\cdots(i_r,j_r)} x^{(k')}_{(i'_1,j'_1)\cdots(i_\ell,j_\ell)\cdots(i'_r,j'_r)}$ *where* $k < k'$, $i_\ell = i'_{\ell'}$ *and* $j_\ell > j'_{\ell'}$.

(4) $\prod_{\ell=1}^{p} x^{(k_\ell)}_{M_\ell(i,j_{\ell,1})\cdots(i,j_{\ell,q_\ell})M'_\ell} - \prod_{\ell=1}^{p} x^{(k_\ell)}_{M_\ell(i,j'_{\ell,1})\cdots(i,j'_{\ell,q_\ell})M'_\ell}$ *where the binomial* $0 \neq \displaystyle\prod_{\ell=1}^{p} z^{(i)}_{j_{\ell,1}}\cdots z^{(i)}_{j_{\ell,q_\ell}} - \prod_{\ell=1}^{p} z^{(i)}_{j'_{\ell,1}}\cdots z^{(i)}_{j'_{\ell,q_\ell}}$ *belongs to* $\mathcal{H}_i$.

3. Toric ideals of multiples of the Birkhoff polytope. Let $\mathbf{c} = (c_1,c_2,c_3) \in \mathbf{Z}^3_{\geq 0}$, and let $\mathbf{r} = (r_1,r_2,r_3) \in \mathbf{Z}^3_{\geq 0}$ be vectors with $c_1+c_2+c_3 = r_1+r_2+r_3$. Then 3×3 *transportation polytope* $T_{\mathbf{rc}}$ is the set of all non-negative 3×3 matrices $A = (a_{ij})$ satisfying

$$\sum_{i=1}^{3} a_{ik} = c_k \text{ and } \sum_{j=1}^{3} a_{\ell j} = r_\ell$$

for $1 \leq k,\ell \leq 3$. It is known that this is a bounded convex polytope of dimension 4 whose vertices are lattice points in $\mathbf{R}^{3\times 3}$. The toric ideal of $T_{\mathbf{rc}}$ is the toric ideal of the configuration $\{\mathbf{t}^\alpha \mid \alpha \in T_{\mathbf{rc}} \cap \mathbf{Z}^{3\times 3}\}$.

Example 3.1. Let $\mathbf{c} = \mathbf{r} = (1,1,1)$. Then the transportation polytope $\mathcal{B}_3 := T_{\mathbf{rc}}$ is called the *Birkhoff polytope.* The lattice points in $\mathcal{B}_3$ are

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\sigma_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_5 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The toric ideal of $\mathcal{B}_3$ is the toric ideal of the configuration

$$\{u_{11}u_{22}u_{33}, u_{12}u_{23}u_{31}, u_{13}u_{21}u_{32}, u_{11}u_{23}u_{32}, u_{12}u_{21}u_{33}, u_{13}u_{22}u_{31}\},$$

and it is a principal ideal generated by $z_1z_2z_3 - z_4z_5z_6$.

The following is proved by Haase-Paffenholz [**2**]:

• The toric ideal of the 3×3 transportation polytope is generated by quadratic binomials except for $\mathcal{B}_3$.

• The toric ideal of 3×3 transportation polytope possesses a quadratic squarefree initial ideal if it is not a multiple of $\mathcal{B}_3$.

Thus, it is natural to ask whether the toric ideal of a multiple of $\mathcal{B}_3$ possesses a quadratic Gröbner basis except for $\mathcal{B}_3$. The following fact is due to Birkhoff:

• Every non-negative integer $p \times p$ matrix with equal row and column sums can be written as a sum of permutation matrices.

Hence, in particular, we have

$$n\mathcal{B}_3 \cap \mathbf{Z}^{3\times3} = \{\sigma_{i_1} + \cdots + \sigma_{i_n} \mid 1 \le i_1, \ldots, i_n \le 6\}.$$

Thus, in order to study the toric ideal of n multiples of $\mathcal{B}_3$, we consider the following:

Example 3.2. Let $A = \{t_1^n\}$, and suppose that B_1 satisfies $\sharp|B_1| = 6$ and $I_{B_1} = \langle z_1z_2z_3 - z_4z_5z_6\rangle$. If $n = 1$, then $A(B_1) = B_1$ and $\{x_1x_2x_3 - x_4x_5x_6\}$ is the reduced Gröbner basis of $I_{A(B_1)}$ with respect to any monomial order. If $n > 1$, then, by virtue of Theorem 2.6, $I_{A(B_1)}$ has a Gröbner basis consisting of the following binomials:

(a) $x_{1M_1}x_{2M_2}x_{3M_3} - x_{4M_1}x_{5M_2}x_{6M_3}$,

(b) $x_{j_1j_2M_1}x_{j_3M_2} - x_{j_4j_5M_1}x_{j_6M_2}$, where $\{j_1, j_2, j_3\} = \{1, 2, 3\}$ and $\{j_4, j_5, j_6\} = \{4, 5, 6\}$,

(c) $x_{j_1\cdots j_n}x_{j_{n+1}\cdots j_{2n}} - x_{j'_1j'_3\cdots j'_{2n-1}}x_{j'_2j'_4\cdots j'_{2n}}$, where $\mathrm{sort}\,(j_1\cdots j_{2n}) = j'_1\cdots j'_{2n}$.

Since the Gröbner basis in Example 3.2 is not quadratic, we have to consider another monomial order to find a quadratic Gröbner basis.

Remark 3.3. In [**2**], it is stated that Piechnik and Haase proved that the toric ideal of the multiple $2n\mathcal{B}_3$ possesses a squarefree quadratic initial ideal for $n > 1$. This fact is directly obtained by Theorem 2.6 since the toric ideal of the multiple $2\mathcal{B}_3$ possesses a squarefree quadratic initial ideal. Similarly, since the toric ideal of the multiple $3\mathcal{B}_3$ possesses a squarefree quadratic initial ideal, Theorem 2.6 guarantees that the toric ideal of the multiple $3n\mathcal{B}_3$ possesses a squarefree quadratic initial ideal for $n > 1$. However, since there are infinitely many prime numbers, it is difficult to show the existence of a squarefree quadratic initial ideal of the toric ideal of $m\mathcal{B}_3$ for all $m > 1$ in this way.

Theorem 3.4. *Let $A = \{t_1^n\}$ with $n > 1$, and suppose that B_1 satisfies $\sharp|B_1| = 6$ and $I_{B_1} = \langle z_1z_2z_3 - z_4z_5z_6\rangle$. Then, $I_{A(B_1)}$ has a quadratic Gröbner basis consisting of the following binomials*:

(i) $\underline{x_{j_1j_2M_1}x_{j_3M_2}} - x_{j_4j_5M_1}x_{j_6M_2}$ *where* $\{j_1, j_2, j_3\} = \{1, 2, 3\}$ *and* $\{j_4, j_5, j_6\} = \{4, 5, 6\}$,

(ii) $\underline{x_{j_1\cdots j_n}x_{j_{n+1}\cdots j_{2n}}} - x_{1\cdots 1j'_1\cdots j'_\alpha}x_{1\cdots 1j'_{\alpha+1}\cdots j'_{2\alpha}}$ *where* $\mathrm{sort}\,(j_1\cdots j_{2n}) = 1\cdots 1j'_1\cdots j'_{2\alpha}$ *and* $j'_2 > 1$.

Proof. Let $\mathcal{G}$ denote the set of binomials above. Since $A = \{t_1^n\}$, each binomial in (ii) and (iii) belongs to $I_{A(B_1)}$. In addition, thanks to Lemma 2.3, each binomial in (i) belongs to $I_{A(B_1)}$. Hence $\mathcal{G}$ is a (finite) subset of $I_{A(B_1)}$.

Claim 1. *There exists a monomial order such that the initial monomial of each binomial in $\mathcal{G}$ is the underlined monomial.*

By virtue of [**9**, Theorem 3.12], it is enough to show that the reduction modulo $\mathcal{G}$ is Noetherian. Suppose that there exists a sequence of reductions modulo $\mathcal{G}$ which does not terminate. Let v be a monomial in $K[\mathbf{x}]$ and assume $v \xrightarrow{g} v'$ with $g \in \mathcal{G}$. Then we have

$$\begin{cases} \varphi_1(v) >_1 \varphi_1(v') & \text{if } g \text{ in (i)},\\ \varphi_1(v) = \varphi_1(v') & \text{if } g \text{ in (ii)}.\end{cases}$$

Hence, the number of binomials in (i) appearing in the sequence is finite. Thus we may assume that the binomials in (i) do not appear in the sequence. Let $v = \prod_{\ell=1}^{q} x_{i_{(\ell-1)r+1}\cdots i_{\ell r}}$, $v' = \prod_{\ell=1}^{q} x_{i'_{(\ell-1)r+1}\cdots i'_{\ell r}}$, and let m_ℓ (respectively, m'_ℓ) denote the number of 1's appearing in $i_{(\ell-1)r+1}\cdots i_{\ell r}$ (respectively, $i'_{(\ell-1)r+1}\cdots i'_{\ell r}$). Then, we have

$$\sum_{1\le \ell_1<\ell_2\le q} |m_{\ell_1} - m_{\ell_2}| \ge \sum_{1\le \ell_1<\ell_2\le q} |m'_{\ell_1} - m'_{\ell_2}|$$

if $g \in \mathcal{G}$ belongs to (ii). (The equality holds if and only if $g = \underline{x_{j_1\cdots j_n} x_{j_{n+1}\cdots j_{2n}}} - x_{1\cdots 1 j'_1\cdots j'_\alpha} x_{1\cdots 1 j'_{\alpha+1}\cdots j'_{2\alpha}}$ satisfies that the difference between the number of 1's in $j_1\cdots j_n$ and that in $j_{n+1}\cdots j_{2n}$ is at most one.) Hence, we may assume that 1's in the indices are stable. Then, since the inversion number is strictly decreasing in the sequence of reductions modulo binomials in (ii), the sequence is finite.

Claim 2. *The set $\mathcal{G}$ is a Gröbner basis of $I_{A(B_1)}$.*

Suppose that $\mathcal{G}$ is not a Gröbner basis of $I_{A(B_1)}$. Then there exists a binomial $0 \neq g = p_1 - p_2 \in I_{A(B_1)}$ such that neither p_1 nor p_2 is divisible by the initial monomial of any binomial in $\mathcal{G}$. Let $p_1 = \prod_{\ell=1}^{p} x_{i_{(\ell-1)r+1}\cdots i_{\ell r}}$, $p_2 = \prod_{\ell=1}^{p} x_{i'_{(\ell-1)r+1}\cdots i'_{\ell r}}$. By Lemma 2.4, we have $\varphi_1(p_1) - \varphi_1(p_2) = \prod_{\xi=1}^{pr} z_{i_\xi} - \prod_{\xi=1}^{pr} z_{i'_\xi} \in \langle z_1z_2z_3 - z_4z_5z_6\rangle$.

Suppose that $\prod_{\xi=1}^{pr} z_{i_\xi} - \prod_{\xi=1}^{pr} z_{i'_\xi} \neq 0$. We may assume that $\prod_{\xi=1}^{pr} z_{i_\xi}$ is divided by $z_1z_2z_3$. Since p_1 is not divided by the initial monomial of any binomial in (i), p_1 is divided by a cubic monomial $x_{1M_1}x_{2M_2}x_{3M_3}$ where $2,3 \notin M_1$, $1,3 \notin M_2$ and $1,2 \notin M_3$. Note that $M_i \neq \varnothing$ by $n > 1$. Since p_1 is not divided by the initial monomial of any binomial in (ii), the number of 1's in iM_i is different by at most one. Since 1 appears in neither $2M_2$ nor $3M_3$, we have $1 \notin M_1$. Thus,

$M_1 \subset \{4,5,6\}$. Then p_1 is divided by the initial monomial of the binomial $g = x_{1M_1}x_{2M_2} - x_{12M_1'}x_{M_2'}$ where sort $(1M_12M_2) = 12M_1'M_2'$ and g belongs to (ii).

Suppose that $\prod_{\xi=1}^{pr} z_{i_\xi} - \prod_{\xi=1}^{pr} z_{i'_\xi} = 0$. Since neither p_1 nor p_2 is divisible by the initial monomial of any binomial in (ii), there exists $0 \leq p' \leq p$ and $0 \leq \beta \leq r$ such that

$$p_1 = p_2 = \prod_{\ell=1}^{p'} x_{\zeta_{(\ell-1)r+1}\cdots\zeta_{\ell r}} \prod_{\ell=p'+1}^{p} x_{\theta_{(\ell-1)r+1}\cdots\theta_{\ell r}}$$

where $\zeta_{(\ell-1)r+\eta} = 1$ for all $1 \leq \eta \leq \beta$, $\theta_{(\ell-1)r+\eta} = 1$ for all $1 \leq \eta \leq \beta - 1$ and $\zeta_{\beta+1} \leq \cdots \leq \zeta_r \leq \zeta_{r+\beta+1} \leq \cdots \leq \zeta_{2r} \leq \cdots \leq \zeta_{(p'-1)r+\beta+1} \leq \cdots \leq \zeta_{p'r} \leq \theta_{p'r+\beta} \leq \cdots \leq \theta_{(p'+1)r} \leq \theta_{(p'+1)r+\beta} \leq \cdots \leq \theta_{(p'+2)r} \leq \cdots \leq \theta_{(p-1)r+\beta} \leq \cdots \leq \theta_{pr}$. Hence, $g = p_1 - p_2 = 0$, and this is a contradiction.

Thus, there exists no binomial $0 \neq g = p_1 - p_2 \in I_{A(B_1)}$ such that neither p_1 nor p_2 is divisible by the initial monomial of any binomial in $\mathcal{G}$, and hence $\mathcal{G}$ is a Gröbner basis of $I_{A(B_1)}$ as desired. □

4. Observation. Finally, we conclude this paper with a summary of our algebraic theory of nested configurations. For a configuration A, let $\mathcal{G}_<$ denote the reduced Gröbner basis of I_A with respect to a monomial order $<$. Let

$$\lambda(A) := \min_{<} \left(\max \left(\deg(g) \mid g \in \mathcal{G}_< \right) \right).$$

(If $I_A = (0)$, then we set $\lambda(A) = 0$.) Thanks to the results in Section 2, if $\lambda(A(B_1, \ldots, B_d)) \neq 0$, then

$$\max(2, \lambda(A)) \leq \lambda(A(B_1, \ldots, B_d)) \leq \max(2, \lambda(A), \lambda(B_1), \ldots, \lambda(B_d)).$$

Moreover, if $\lambda(A(B_1, \ldots, B_d)) \neq 0$ and A consists of squarefree monomials, then

$$\lambda(A(B_1, \ldots, B_d)) = \max(2, \lambda(A), \lambda(B_1), \ldots, \lambda(B_d)).$$

Let $n \geq 2$ be an integer, and let X be the one of the following algebraic properties:

(1) The toric ring is normal;

(2) The toric ideal has a squarefree initial ideal;

(3) The toric ideal has a quadratic initial ideal;

(4) The toric ideal has a squarefree quadratic initial ideal;

(5) The toric ideal has an initial ideal of degree $\leq n$;

(6) The toric ideal is generated by quadratic binomials;

(7) The toric ideal is generated by binomials of degree $\leq n$.

Then we have

$$A, B_1, \dots, B_d \text{ have the property } X \\ \Longrightarrow A(B_1, \dots, B_d) \text{ has the property } X.$$

$$A(B_1, \dots, B_d) \text{ has the property } X \Longrightarrow A \text{ has the property } X.$$

Moreover, if A consists of squarefree monomials, then we have

$$A, B_1, \dots, B_d \text{ have the property } X \Longleftrightarrow \\ A(B_1, \dots, B_d) \text{ has the property } X.$$

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Department of Mathematics, College of Science, Rikkyo University, Toshima-ku, Tokyo 171-8501, Japan
Email address: ohsugi@rikkyo.ac.jp

Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan
Email address: hibi@math.sci.osaka-u.ac.jp