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# ON THE SUPPORT OF LOCAL COHOMOLOGY MODULES AND FILTER REGULAR SEQUENCES

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ABSTRACT. Let $R$ be a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of $R$, and $M$ a finitely generated $R$-module. In this paper, for non-negative integers $n$ and $j$, we examine the question of whether the support of $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ must be closed in Zariski topology, where $H^n_{\mathfrak{a}}(M)$ is the $n$-th local cohomology module of $M$ with respect to $\mathfrak{a}$. Several results giving an affirmative answer to this question are given.

**1. Introduction.** The fourth of Huneke's four problems in local cohomology [**5**] is to determine when the set of associated primes of a local cohomology module $H^n_{\mathfrak{a}}(M)$ is finite for an ideal $\mathfrak{a}$ of a commutative Noetherian ring $R$ and finitely generated $R$-module $M$. The answer to this problem is known in several cases (cf. [**1, 2, 7, 11**–**14**]). Examples of Singh [**16**], Katzman [**8**], and later Singh and Swanson [**17**] provided some local cohomology modules with infinite set of associated primes. So it is natural to ask whether the sets of primes minimal in the support of such local cohomology modules are always finite. This is equivalent to asking the following question.

**Question 1.1.** *Let $R$ be a Noetherian ring, $M$ a finitely generated $R$-module, $\mathfrak{a}$ an ideal of $R$, and $n$ a non-negative integer. Is the support of $H^n_{\mathfrak{a}}(M)$ Zariski-closed subset of* $\operatorname{Spec} R$*?*

Question 1.1 has an important role in the study of cohomological dimension and understanding the local-global properties of local cohomology. Clearly, if the set of associated primes of a given local cohomology module is finite, then its support is closed. Recently, Huneke,

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Katz and Marley in [**6**] provided some partial answers for Question 1.1. In fact, in [**6**], the above question in the case that the ideal $\mathfrak{a}$ is generated by $n$ elements is studied and so, the top local cohomology modules are considered. For instance, they showed that if $H^n_{\mathfrak{a}}(R) = 0$ for all $n > 2$, then, for a finitely generated $R$-module $M$, the support of $H^2_{\mathfrak{a}}(M)$ is closed. Also, by using the ideas of Hellus [**4**], they showed that, for a finitely generated $R$-module $M$, if $\mathrm{Supp}_R H^3_{\mathfrak{a}}(M)$ is closed for every three-generated ideal $\mathfrak{a}$ of $R$, then, for all non-negative integers $n$, $\mathrm{Supp}_R H^n_{\mathfrak{a}}(M)$ is closed for every $n$-generated ideal $\mathfrak{a}$ of $R$. In addition, over a Cohen-Macaulay ring $R$, they proved that the following conditions are equivalent:

(a) For all positive integers $n$, $\mathrm{Supp}_R H^n_{\mathfrak{a}}(R)$ is closed for every ideal $\mathfrak{a}$ of $R$.

(b) $\mathrm{Supp}_R H^2_{\mathfrak{a}}(R)$ is closed for every three-generated ideal $\mathfrak{a}$ of $R$, $\mathrm{Supp}_R H^3_{\mathfrak{a}}(R)$ is closed for every four-generated ideal $\mathfrak{a}$ of $R$, and $\mathrm{Supp}_R H^4_{\mathfrak{a}}(R)$ is closed for every five-generated ideal $\mathfrak{a}$ of $R$.

Since $\mathrm{Ass}_R H^n_{\mathfrak{a}}(M) = \mathrm{Ass}_R \mathrm{Hom}_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$ and $R$ is Noetherian, $\mathrm{Supp}_R H^n_{\mathfrak{a}}(M) = \mathrm{Supp}_R \mathrm{Hom}_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$. This provides some motivations for studying the following question.

**Question 1.2.** *Let $R$ be a Noetherian ring, $M$ a finitely generated $R$-module, $\mathfrak{a}$, $\mathfrak{b}$ ideals of $R$, and $n, j$ non-negative integers. Is the support of $\mathrm{Ext}^j_R(R/\mathfrak{b}, H^n_{\mathfrak{a}}(M))$ a Zariski-closed subset of $\mathrm{Spec} R$?*

In this short note, by using a natural generalization of regular sequence which is called *filter regular sequence*, we study Question 1.2 in several cases. For example, without any restriction on the ring $R$, we show that the following conditions are equivalent:

(a) For all positive integers $n$, $\mathrm{Supp}_R H^n_{\mathfrak{a}}(M)$ is closed for every ideal $\mathfrak{a}$.

(b) For $i = 2, 3, 4$, $\mathrm{Supp}_R \mathrm{Hom}_R(R/(x_1, \dots, x_{i+1}), H^i_{(x_1, \dots, x_i)}(M))$ is closed, for every sequence $x_1, \dots, x_{i+1}$ of elements of $R$ such that $x_1, \dots, x_i$ is an $(x_1, \dots, x_{i+1})$-filter regular sequence on $M$.

(c) $\mathrm{Supp}_R H^2_{\mathfrak{a}}(M)$ is closed for every three-generated ideal $\mathfrak{a}$ of $R$, $\mathrm{Supp}_R H^3_{\mathfrak{a}}(M)$ is closed for every four-generated ideal $\mathfrak{a}$ of $R$, and $\mathrm{Supp}_R H^4_{\mathfrak{a}}(M)$ is closed for every five-generated ideal $\mathfrak{a}$ of $R$.

This result is an improved form of [**6**, Corollary 5.6] which we mentioned earlier. Also, we prove, among the other things, that if $n$ is a positive integer such that $H^n_{\mathfrak{a}}(M) = 0$, then the following conditions are equivalent:

(a) $\mathrm{Supp}_R H^{n+1}_{\mathfrak{a}}(M)$ is closed.

(b) $\mathrm{Supp}_R \mathrm{Ext}^1_R(R/\mathfrak{a}, H^n_{(x_1,\dots,x_n)}(M))$ is closed, for every $\mathfrak{a}$-filter regular sequence $x_1,\dots,x_n$ on $M$.

(c) $\mathrm{Supp}_R \mathrm{Ext}^1_R(R/\mathfrak{a}, H^n_{(x_1,\dots,x_n)}(M))$ is closed, for some $\mathfrak{a}$-filter regular sequence $x_1,\dots,x_n$ on $M$.

Throughout this paper, $R$ will denote a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of $R$, and $M$ a finitely generated $R$-module. Also, we use $\mathbf{N}_0$ to denote the set of non-negative integers. Our terminology follows the textbook [**3**] on local cohomology.

**2. Support of local cohomology modules.** The concept of filter regular sequence plays an important role in this paper. A sequence $x_1,\dots,x_n$ of elements of the ideal $\mathfrak{a}$ of $R$ is said to be an $\mathfrak{a}$-filter regular sequence on $M$, if

$$\mathrm{Supp}_R\left(\frac{(x_1,\dots,x_{i-1})M :_M x_i}{(x_1,\dots,x_{i-1})M}\right) \subseteq V(\mathfrak{a})$$

for all $i = 1,\dots,n$, where $V(\mathfrak{a})$ denotes the set of prime ideals of $R$ containing $\mathfrak{a}$. The concept of an $\mathfrak{a}$-filter regular sequence on $M$ is a generalization of the one of a filter regular sequence which has been studied in [**9, 15, 18**] and has led to some interesting results. Note that both concepts coincide if $\mathfrak{a}$ is the maximal ideal in local ring. Also note that $x_1,\dots,x_n$ is a weak $M$-sequence if and only if it is an $R$-filter regular sequence on $M$. It is easy to see that the analogue of [**18**, Appendix 2 (ii)] holds true whenever $R$ is Noetherian, $M$ is finitely generated and $\mathfrak{m}$ replaced by $\mathfrak{a}$; so that, if $x_1,\dots,x_n$ is an $\mathfrak{a}$-filter regular sequence on $M$, then there is an element $y \in \mathfrak{a}$ such that $x_1,\dots,x_n,y$ is an $\mathfrak{a}$-filter regular sequence on $M$. Thus, for a positive integer $n$, there exists an $\mathfrak{a}$-filter regular sequence on $M$ of length $n$.

**Proposition 2.1** (See [**9**, Proposition 2.1].) *Let $x_1,\dots,x_n$ $(n > 0)$ be an $\mathfrak{a}$-filter regular sequence on $M$. Then there are the following isomorphisms*

$$H^i_{\mathfrak{a}}(M) \cong \begin{cases} H^i_{(x_1,\dots,x_n)}(M) & \text{for } 0 \leqslant i < n, \\ H^{i-n}_{\mathfrak{a}}(H^n_{(x_1,\dots,x_n)}(M)) & \text{for } n \leqslant i. \end{cases}$$

By using the ideas of Hellus [**4**], in conjunction with a weak version of the Dedekind-Mertens lemma, Huneke, Katz and Marley showed that, for a non-negative integer $n$, if $\mathfrak{a} = (x_1, \dots, x_{n+k})$ is an ideal of $R$ such that $k \in \mathbf{N}_0$ and $n \geqslant k+4$, then there exists an ideal $\mathfrak{b} = (y_1, \dots, y_{n+k-1}) \subseteq \mathfrak{a}$ such that $H^n_{\mathfrak{a}}(M) \cong H^{n-1}_{\mathfrak{b}}(M)$. So, by employing Proposition 5.1 in [**6**] several times, one can show that $H^n_{\mathfrak{a}}(M) \cong H^4_{\mathfrak{c}}(M)$ for some ideal $\mathfrak{c}$ generated by $k+4$ elements. In the following proposition, by using the concept of filter regular sequences, we present a new version of [**6**, Proposition 5.1] which we need in this paper.

**Theorem 2.2.** *Let $R$ be a commutative Noetherian ring. Let $\mathfrak{a}$ be an ideal of $R$ and $n \geqslant 5$. Then there exists a five-generated ideal $\mathfrak{b} \subseteq \mathfrak{a}$ of $R$ such that $H^n_{\mathfrak{a}}(M) \cong H^4_{\mathfrak{b}}(M)$.*

*Proof.* Let $x_1, \dots, x_{n+1} \in \mathfrak{a}$ be an $\mathfrak{a}$-filter regular sequence on $M$. (Note that the existence of such a sequence is explained in the beginning of this section.) Then, in view of Proposition 2.1, $H^n_{\mathfrak{a}}(M) \cong H^n_{(x_1,\dots,x_{n+1})}(M)$. Now, by [**6**, Proposition 5.1], $H^n_{(x_1,\dots,x_{n+1})}(M) \cong H^4_{\mathfrak{b}}(M)$, for some five-generated ideal $\mathfrak{b}$ of $R$ with $\mathfrak{b} \subseteq (x_1, \dots, x_{n+1}) \subseteq \mathfrak{a}$, and the result follows. □

The following exact sequence of local cohomology modules is included here for the reader's convenience.

**Lemma 2.3** (See [**10**, Lemma 2.2]). *Let $\mathfrak{a}$ be an ideal of $R$. Then, for any non-negative integer $n$ and any $\mathfrak{a}$-filter regular sequence $x_1, \dots, x_{n+1} \in \mathfrak{a}$ on $M$, there exists an exact sequence*

$$\begin{aligned} 0 \longrightarrow H^n_{\mathfrak{a}}(M) \longrightarrow H^n_{(x_1,\dots,x_n)}(M) \longrightarrow (H^n_{(x_1,\dots,x_n)}(M))_{x_{n+1}} \\ \longrightarrow H^{n+1}_{(x_1,\dots,x_{n+1})}(M) \longrightarrow 0. \end{aligned}$$

**Proposition 2.4.** *Let $\mathfrak{a}$ be an ideal of $R$. Then, for any non-negative integer $n$ and any $\mathfrak{a}$-filter regular sequence $x_1, \dots, x_n \in \mathfrak{a}$ on $M$,*

$$\mathrm{Supp}_R H^n_{\mathfrak{a}}(M) = \mathrm{Supp}_R \mathrm{Hom}_R(R/\mathfrak{a}, H^n_{(x_1,\dots,x_n)}(M)).$$

*Proof.* Let $n \in \mathbf{N}_0$ and $x_1, \ldots, x_{n+1} \in \mathfrak{a}$ be an $\mathfrak{a}$-filter regular sequence on $M$. Then, by Lemma 2.3, there is an exact sequence

$$0 \longrightarrow H^n_{\mathfrak{a}}(M) \longrightarrow H^n_{(x_1,\ldots,x_n)}(M) \longrightarrow (H^n_{(x_1,\ldots,x_n)}(M))_{x_{n+1}}.$$

Since multiplication by $x_{n+1}$ is an automorphism on $(H^n_{(x_1,..,x_n)}(M))_{x_{n+1}}$ and $x_{n+1} \in \mathfrak{a}$, by applying the functor $\mathrm{Hom}_R(R/\mathfrak{a}, -)$ on the above exact sequence, one can obtain the isomorphism

$$\mathrm{Hom}_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M)) \cong \mathrm{Hom}_R(R/\mathfrak{a}, H^n_{(x_1,\ldots,x_n)}(M)).$$

Now, since $\mathrm{Ass}_R H^n_{\mathfrak{a}}(M) = \mathrm{Ass}_R \mathrm{Hom}_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$ and $R$ is Noetherian,

$$\begin{aligned}\mathrm{Supp}_R H^n_{\mathfrak{a}}(M) &= \mathrm{Supp}_R \mathrm{Hom}_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M)) \\ &= \mathrm{Supp}_R \mathrm{Hom}_R(R/\mathfrak{a}, H^n_{(x_1,\ldots,x_n)}(M)). \qquad \square\end{aligned}$$

**Corollary 2.5.** *Let $\mathfrak{a}$ be an ideal of $R$. Then there exists a sequence $y_1, y_2, y_3$ of elements of $\mathfrak{a}$ such that*

$$\mathrm{Supp}_R H^4_{\mathfrak{a}}(M) = \mathrm{Supp}_R \mathrm{Hom}_R(R/\mathfrak{a}, H^3_{(y_1,y_2,y_3)}(M)).$$

*Proof.* The result is immediate from Proposition 2.4 and [**6**, Corollary 5.2]. $\square$

The following theorem is an improved form of [**6**, Corollary 5.6].

**Theorem 2.6.** *Let $R$ be a commutative Noetherian ring. Let $M$ be a finitely generated $R$-module. The following conditions are equivalent*:

(a) *For all positive integers $n$, $\mathrm{Supp}_R H^n_{\mathfrak{a}}(M)$ is closed for every ideal $\mathfrak{a}$ of $R$.*

(b) *For all positive integers $n$,*

$$\mathrm{Supp}_R \mathrm{Hom}_R(R/(x_1, \ldots, x_{n+1}), H^n_{(x_1,\ldots,x_n)}(M))$$

*is closed, for every sequence* $x_1, \dots, x_{n+1}$ *of elements of* $R$ *such that* $x_1, \dots, x_n$ *is an* $(x_1, \dots, x_{n+1})$*-filter regular sequence on* $M$.

(c) *For* $n = 2, 3, 4$, $\mathrm{Supp}_R \mathrm{Hom}_R(R/(x_1, \dots, x_{n+1}), H^n_{(x_1,\dots,x_n)}(M))$ *is closed, for every sequence* $x_1, \dots, x_{n+1}$ *of elements of* $R$ *such that* $x_1, \dots, x_n$ *is an* $(x_1, \dots, x_{n+1})$*-filter regular sequence on* $M$

(d) $\mathrm{Supp}_R H^2_{\mathfrak{a}}(M)$ *is closed for every three-generated ideal* $\mathfrak{a}$ *of* $R$, $\mathrm{Supp}_R H^3_{\mathfrak{a}}(M)$ *is closed for every four-generated ideal* $\mathfrak{a}$ *of* $R$, *and* $\mathrm{Supp}_R H^4_{\mathfrak{a}}(M)$ *is closed for every five-generated ideal* $\mathfrak{a}$ *of* $R$.

*Proof.* $(a) \Rightarrow (b)$. Apply Proposition 2.4 with $\mathfrak{a} = (x_1, \dots, x_{n+1})$.

$(b) \Rightarrow (c)$. It is clear.

$(c) \Rightarrow (d)$. In view of Proposition 2.1, for any non-negative integer $n$ and any ideal $\mathfrak{a}$, $H^n_{\mathfrak{a}}(M) \cong H^n_{(x_1,\dots,x_{n+1})}(M)$ for every $\mathfrak{a}$-filter regular sequence $x_1, \dots, x_{n+1}$ on $M$. So the result is immediate from Proposition 2.4.

$(d) \Rightarrow (a)$. Apply Theorem 2.2 and Proposition 2.1. Also note that, for every ideal $\mathfrak{a}$ of $R$, $\mathrm{Supp}_R H^1_{\mathfrak{a}}(M)$ is closed. $\square$

In connection with Questions 1.1 and 1.2, we have the following:

**Theorem 2.7** (cf. [**6**, Corollary 5.3]). *Suppose that* $R$ *is a commutative Noetherian ring. If* $\mathrm{Supp}_R \mathrm{Hom}_R(R/\mathfrak{b}, H^3_{\mathfrak{c}}(M))$ *is closed for every three-generated ideal* $\mathfrak{c}$ *of* $R$ *and every five-generated ideal* $\mathfrak{b}$ *of* $R$ *with* $\mathfrak{c} \subseteq \mathfrak{b}$, *then, for all integers* $n$ *with* $n \geqslant 3$, $\mathrm{Supp}_R H^n_{\mathfrak{a}}(M)$ *is closed for every ideal* $\mathfrak{a}$ *of* $R$.

*Proof.* Let $\mathfrak{a}$ be an ideal of $R$ and $n \geqslant 3$. Then, by Theorem 2.2, $H^n_{\mathfrak{a}}(M) \cong H^4_{\mathfrak{b}}(M)$ for some five-generated ideal $\mathfrak{b}$ of $R$ with $\mathfrak{b} \subseteq \mathfrak{a}$. We can therefore deduce from Corollary 2.5 that there exist $y_1, y_2, y_3 \in \mathfrak{b}$ such that

$$\mathrm{Supp}_R H^n_{\mathfrak{a}}(M) = \mathrm{Supp}_R \mathrm{Hom}_R(R/\mathfrak{b}, H^3_{(y_1,y_2,y_3)}(M)).$$

Hence, $\mathrm{Supp}_R H^n_{\mathfrak{a}}(M)$ is closed. $\square$

It is well-known that the first non-zero local cohomology has finitely many associated primes (cf. [**2**, Proposition 2.2]) and hence has closed support. In the following theorem, for a fixed non-negative integer $n$ with $H^n_{\mathfrak{a}}(M) = 0$, we study the support of $H^{n+1}_{\mathfrak{a}}(M)$.

**Theorem 2.8.** *Let $R$ be a commutative Noetherian ring. Let $\mathfrak{a}$ be an ideal of $R$ and $n$ a non-negative integer such that $H^n_{\mathfrak{a}}(M) = 0$. Then the following conditions are equivalent*:

(a) $\mathrm{Supp}_R H^{n+1}_{\mathfrak{a}}(M)$ *is closed.*

(b) $\mathrm{Supp}_R \mathrm{Ext}^1_R(R/\mathfrak{a}, H^n_{(x_1,\dots,x_n)}(M))$ *is closed, for every $\mathfrak{a}$-filter regular sequence $x_1, \dots, x_n$ on $M$.*

(c) $\mathrm{Supp}_R \mathrm{Ext}^1_R(R/\mathfrak{a}, H^n_{(x_1,\dots,x_n)}(M))$ *is closed, for some $\mathfrak{a}$-filter regular sequence $x_1, \dots, x_n$ on $M$.*

*Proof.* Let $x_1, \dots, x_{n+1} \in \mathfrak{a}$ be an $\mathfrak{a}$-filter regular sequence on $M$. Then, in view of Lemma 2.3, there is an exact sequence

$$0 \to H^n_{(x_1,\dots,x_n)}(M) \to (H^n_{(x_1,\dots,x_n)}(M))_{x_{n+1}} \to H^{n+1}_{(x_1,\dots,x_{n+1})}(M) \to 0.$$

Since the multiplication by $x_{n+1}$ provides an automorphism on $(H^n_{(x_1,\dots,x_n)}(M))_{x_{n+1}}$ and $x_{n+1} \in \mathfrak{a}$, by applying the functor $\mathrm{Hom}_R(R/\mathfrak{a}, -)$ on the above exact sequence, we have that $\mathrm{Hom}_R(R/\mathfrak{a}, H^{n+1}_{(x_1,\dots,x_{n+1})}(M)) \cong \mathrm{Ext}^1_R(R/\mathfrak{a}, H^n_{(x_1,\dots,x_n)}(M))$. The result now follows from Proposition 2.4. $\square$

Recall that an $R$-module $N$ is $\mathfrak{a}$-cofinite if the support of $N$ is contained in $V(\mathfrak{a})$ and $\mathrm{Ext}^j_R(R/\mathfrak{a}, N)$ is finitely generated for all $j \in \mathbf{N}_0$.

**Theorem 2.9.** *Let $R$ be a commutative Noetherian ring. Let $\mathfrak{a}$ be an ideal of $R$ and $n$ be a positive integer such that*

(a) $H^i_{\mathfrak{a}}(M)$ *is $\mathfrak{a}$-cofinite for all $i < n$, and*

(b) $\mathrm{Ass}_R \mathrm{Ext}^2_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$ *is finite.*

*Then $H^{n+1}_{\mathfrak{a}}(M)$ has only finitely many associated primes and so $\mathrm{Supp}_R H^{n+1}_{\mathfrak{a}}(M)$ is closed.*

*Proof.* Let $x_1, \dots, x_{n+1} \in \mathfrak{a}$ be an $\mathfrak{a}$-filter regular sequence on $M$. Then, by Lemma 2.3, there exists an exact sequence

$$\begin{aligned} 0 \longrightarrow H^n_{\mathfrak{a}}(M) \longrightarrow H^n_{(x_1,\dots,x_n)}(M) &\xrightarrow{f} (H^n_{(x_1,\dots,x_n)}(M))_{x_{n+1}} \\ &\longrightarrow H^{n+1}_{(x_1,\dots,x_{n+1})}(M) \longrightarrow 0. \end{aligned}$$

Put $L := \operatorname{Im} f$. Now, by breaking the above exact sequence in two exact sequences and applying the functor $\operatorname{Hom}_R(R/\mathfrak{a}, -)$ on them, one can obtain the isomorphism

$$\operatorname{Hom}_R(R/\mathfrak{a}, H^{n+1}_{(x_1,\dots,x_{n+1})}(M)) \cong \operatorname{Ext}^1_R(R/\mathfrak{a}, L)$$

and the exact sequence
(†)
$$\operatorname{Ext}^1_R(R/\mathfrak{a}, H^n_{(x_1,\dots,x_n)}(M)) \xrightarrow{\varphi} \operatorname{Ext}^1_R(R/\mathfrak{a}, L) \xrightarrow{\psi} \operatorname{Ext}^2_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M)).$$

On the other hand, in light of [**9**, Lemma 3.12], it follows from assumption (a) that $\operatorname{Ext}^1_R(R/\mathfrak{a}, H^n_{(x_1,\dots,x_n)}(M))$ is finitely generated. Hence $\operatorname{Im}\varphi$ is also finitely generated. Moreover

$$\operatorname{Ass}_R \operatorname{Im}\psi \subseteq \operatorname{Ass}_R \operatorname{Ext}^2_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M)$$

is finite. Therefore, in view of the exact sequence (†), we have that $\operatorname{Ass}_R \operatorname{Ext}^1_R(R/\mathfrak{a}, L) \subseteq \operatorname{Ass}_R \operatorname{Im}\varphi \cup \operatorname{Ass}_R \operatorname{Im}\psi$ is finite. Thus

$$\operatorname{Ass}_R H^{n+1}_{\mathfrak{a}}(M) = \operatorname{Ass}_R \operatorname{Hom}_R(R/\mathfrak{a}, H^{n+1}_{(x_1,\dots,x_{n+1})}(M))$$

is finite. □

Next, let us recall the $\mathfrak{a}$-finiteness dimension of $M$: (see [**3**, Definition 9.1.3]):

$$f_{\mathfrak{a}}(M) := \min\{j \in \mathbf{N}_0 \mid H^j_{\mathfrak{a}}(M) \text{ is not finitely generated}\}.$$

Using this notation we have the following corollary.

**Corollary 2.10.** *Let $\mathfrak{a}$ be an ideal of $R$ and $f := f_{\mathfrak{a}}(M)$. Suppose that*

$$\operatorname{Ass}_R \operatorname{Ext}^2_R(R/\mathfrak{a}, H^f_{\mathfrak{a}}(M))$$

*is finite. Then $\operatorname{Ass}_R H^{f+1}_{\mathfrak{a}}(M)$ is finite and so $\operatorname{Supp}_R H^{f+1}_{\mathfrak{a}}(M)$ is closed.*

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