J. Math. Kyoto Univ. (JMKYAZ)
49-4 (2009), 645–689

Construction of a Galois action on modular forms for an arbitrary unitary group

By

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Abstract

In this paper we will construct a certain action of $\mathrm{Aut}(\mathbb{C})$ on holomorphic modular forms (of any weights) with respect to an arbitrary unitary group, which is compatible with Hecke operators. We can write this action explicitly and simply. The image of the action in general is a holomorphic modular form for another unitary group.

0. Introduction

In Section 25 of [13], or essentially in Theorem 1.5 of [11], G. Shimura proved the existence of a certain Galois action on holomorphic modular forms for any symplectic group $\mathrm{Sp}(l, F)$, where F is a totally real algebraic number field of finite degree. In this case, a holomorphic modular form f on $\mathfrak{H}_l^{\mathbf{a}}$ (the Hilbert-Siegel domain) has a Fourier expansion of the following form:

$$f((z_v)_{v\in\mathbf{a}}) = \sum_h c_h \exp\left(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}} \mathrm{tr}(h_v z_v)\right), \tag{0.1}$$

where $\mathbf{a}$ denotes the set of all archimedean primes of F, and h runs over the elements in a certain lattice in symmetric matrices of degree l with coefficients in F. Note that each c_h is a constant. It is shown that, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, there exists a holomorphic modular form f^σ whose Fourier expansion is given by

$$f^\sigma((z_v)_{v\in\mathbf{a}}) = \sum_h c_h^\sigma \exp\left(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}} \mathrm{tr}(h_v z_v)\right). \tag{0.2}$$

This Galois action is also compatible with Hecke operators.

In this paper we will construct a similar Galois action on holomorphic modular forms *of any weights* with respect to an arbitrary unitary group over

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2000 *Mathematics Subject Classification*(*s*). Primary 11F30, 11F55, 11G18.
Received December 10, 2004
Revised June 26, 2009

any CM-field K. The same action was essentially constructed in [4] and [1], but the action was not explicitly described in those papers. In this paper we will write it explicitly and simply, which enables us to consider the precise arithmeticity for holomorphic modular forms. The method of the proof in this paper is completely different from those of [4] or [1].

From now on let us describe the Galois action concretely. For any m-dimensional skew-hermitian matrix T with coefficients in K, we denote by $\mathrm{U}(T,\Psi)$ the unitary group with respect to T. (Here we take for T, without loss of generality, a certain normal form by a suitable choice of basis, and Ψ denotes a CM-type of K depending on T.) In this case, ($\mathbb{C}$-valued) modular forms have weights in $\sum_{v\in\mathbf{a}}\mathbb{Z}\cdot v$, where $\mathbf{a}$ denotes the set of all archimedean primes of K.

Let q be the dimension of a maximal isotropic subspace with respect to T. Then we can define an embedding $\varepsilon_0=\varepsilon_0(T,\Psi)$ of $\mathrm{Sp}(q,F)$ into $\mathrm{U}(T,\Psi)$, where F is the maximal real subfield of K. For any modular form f for $\mathrm{U}(T,\Psi)$, we denote by $f|\varepsilon_0$ the modular form for $\mathrm{Sp}(q,F)$ which is the pull-back of f by ε_0. We denote by $K_{\mathbf{h}}$ and $\mathrm{U}(T,\Psi)_{\mathbf{h}}$, the non-archimedean components of K_A (the adele ring of K) and $\mathrm{U}(T,\Psi)_A$ (the adelization of $\mathrm{U}(T,\Psi)$) respectively. Then we have the following theorem (Theorem 5.1).

Main Theorem. *Take any modular form f of weight k for $\mathrm{U}(T,\Psi)$ and any $\sigma\in\mathrm{Aut}(\mathbb{C})$. Then there exists a modular form $f^{(\sigma;T,\Psi;\underline{a})}$ of weight k^σ for another unitary group $\mathrm{U}(\tilde{T},\Psi\sigma)$, where $\underline{a}\in(K_A^\times)^{m-2q+1}$ is determined by σ and T at the beginning of Section* 5, *and $\tilde{T}$ is a skew-hermitian matrix of dimension m determined by σ, T and $\underline{a}$ as in Theorem* 5.1, *which satisfies one of the following properties.*
(i) *In case $q>0$, we have*

$$(f^{(\sigma;T,\Psi;\underline{a})}|_{k^\sigma}\tilde{\alpha})|\varepsilon_0=\{(f|_k\alpha)|\varepsilon_0\}^\sigma$$

for any $\tilde{\alpha}\in\mathrm{U}(\tilde{T},\Psi\sigma)$ and any $\alpha\in\mathrm{U}(T,\Psi)$ satisfying the relation $(*)$ below. Here the action of σ on the right hand side is as defined in (0.2). *The relation between α and $\tilde{\alpha}$ is*

$$(*)\qquad \alpha_{\mathbf{h}}\in C_{\mathbf{h}}\cdot B(\sigma;T,\Psi;\underline{a})\tilde{\alpha}_{\mathbf{h}}B(\sigma;T,\Psi;\underline{a})^{-1},$$

where $\alpha_{\mathbf{h}}$ and $\tilde{\alpha}_{\mathbf{h}}$ denote the non-archimedean components of α and $\tilde{\alpha}$, $B(\sigma;T,\Psi;\underline{a})\in\mathrm{GL}(m,K_{\mathbf{h}})$, and $C_{\mathbf{h}}$ is some open compact subgroup of $\mathrm{U}(T,\Psi)_{\mathbf{h}}$ depending only on f.
(ii) *In case $q=0$, we have*

$$\left(f^{(\sigma;T,\Psi;\underline{a})}|_{k^\sigma}\tilde{\alpha}\right)(\mathbf{0})=\{(f|_k\alpha)(\mathbf{0})\}^\sigma,$$

for any $\tilde{\alpha}\in\mathrm{U}(\tilde{T},\Psi\sigma)$ and any $\alpha\in\mathrm{U}(T,\Psi)$ satisfying $(*)$ above. Here the symbol $\mathbf{0}$ in each side denotes a certain fixed point in the corresponding symmetric domain.

This is a natural generalization of the result in [15], which needs Fourier-Jacobi expansions of modular forms.

We can prove that this action of $(\sigma; T, \Psi; \underline{a})$ is compatible with Hecke operators, that is,

$$(\mathbf{f}|\mathfrak{T}(\mathfrak{a}))^{(\sigma;T,\Psi;\underline{a})} = \mathbf{f}^{(\sigma;T,\Psi;\underline{a})}|\mathfrak{T}(\mathfrak{a}), \tag{0.3}$$

where $\mathbf{f}$ is an adelized modular form for $\mathrm{U}(T, \Psi)$ and $\mathfrak{T}(\mathfrak{a})$ is a Hecke operator. Note that $\mathfrak{T}(\mathfrak{a})$ in the left hand side is a Hecke operator for $\mathrm{U}(T, \Psi)$, while $\mathfrak{T}(\mathfrak{a})$ in the right hand side is the one for $\mathrm{U}(\tilde{T}, \Psi\sigma)$. The equation (0.3) implies that by this Galois action of $(\sigma; T, \Psi; \underline{a})$ a Hecke common eigenform of eigenvalues $\{\lambda(\mathfrak{a})\}_{\mathfrak{a}}$ is moved to a Hecke common eigenform of eigenvalues $\{\lambda(\mathfrak{a})^{\sigma}\}_{\mathfrak{a}}$. Using this fact, we can prove that Hecke eigenvalues of a common eigen cusp form with respect to any unitary groups are contained in a CM-field. See Theorem 6.6.

We conjecture the following.

Conjecture. *Let $0 \neq \mathbf{f}, \mathbf{g}_1$ and $\mathbf{g}_2$ be cusp forms for $\mathrm{U}(T, \Psi)$ which are common eigenforms of $\{\mathfrak{T}(\mathfrak{a})\}_{\mathfrak{a}}$ having the same eigenvalue for every $\mathfrak{a}$. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have $\langle \mathbf{f}^{(\sigma;T,\Psi;\underline{a}),-}, \mathbf{f}^{(\sigma;T,\Psi;\underline{a})}\rangle \neq 0$ and*

$$\frac{\langle \mathbf{g}_1^{(\sigma;T,\Psi;\underline{a}),-}, \mathbf{g}_2^{(\sigma;T,\Psi;\underline{a})}\rangle}{\langle \mathbf{f}^{(\sigma;T,\Psi;\underline{a}),-}, \mathbf{f}^{(\sigma;T,\Psi;\underline{a})}\rangle} = \left\{ \frac{\langle \mathbf{g}_1, \mathbf{g}_2\rangle}{\langle \mathbf{f}, \mathbf{f}\rangle} \right\}^{\sigma},$$

where $\mathbf{f}^{(\sigma;T,\Psi;\underline{a}),-}(\tilde{x}) = \mathbf{f}^{(\rho\sigma\rho;T,\Psi;\underline{a}^{\rho})}(\tilde{x}(B(\sigma;T,\Psi;\underline{a})^{-1}B(\rho\sigma\rho;T,\Psi;\underline{a}^{\rho})))$ for each $\tilde{x} \in \mathrm{U}(\tilde{T}, \Psi\sigma)_A$ and ρ denotes the complex conjugation.

The author believes that this conjecture is the first step of a more precise research of special values of L-functions. If this conjecture is proved, we will be able to show that special values belong to a specified algebraic number field of finite degree.

The technique of the proof of Main Theorem is essentially the same as that of [15]. However, to define the Galois action, the main theorem of [15] needs Fourier-Jacobi expansions of modular forms. In this paper we need no such expansions. This is an important and essential progress. Moreover, the author hopes that, by finding a substitute of $B(\sigma; T, \Psi; \underline{a})$, we can construct such a Galois action concretely on modular forms for any classical group.

In Section 1 we will define holomorphic modular forms for an arbitrary unitary group. In Section 2, we define equivariant embeddings of groups and symmetric domains, and consider pull-backs of modular forms. In Section 3, we will review canonical models for symplectic and unitary groups. In Section 4, we will define the embeddings of canonical models and show that their inverse rational maps are regular if we choose suitable congruence subgroups. The Main Theorem will be proved in Section 5, using the result of Section 4. In Section 6 we will prove this Galois action is compatible with Hecke operators.

Notation. For a set A, we denote by $A^{n_1}_{n_2}$ the set of all $n_1 \times n_2$-matrices with entries in A, and denote A^n_1 simply by A^n. We write zero matrix of size $n_1 \times n_2$ as $0^{n_1}_{n_2}$, and 1_n means the identity matrix of degree n. The transpose of a matrix X is denoted by ${}^tX$. We denote as usual by $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ the ring of rational integers, the set of all positive rational integers, the field of rational numbers, real numbers, and complex numbers, respectively. If K is an algebraic number field, K_{ab} denotes the maximal abelian extension of K, and we denote by K_A (resp. $K_A^\times$) the adele ring (resp. the idele group) of K. The archimedean component of K_A (resp. $K_A^\times$) is denoted by K_∞ (resp. $K_\infty^\times$). By class field theory, every element x of $K_A^\times$ defines an element of $\mathrm{Gal}(K_{ab}/K)$. We denote this by $[x, K]$. We denote by $\mathcal{O}_K$ and $\mathcal{O}_K^\times$ the ring of algebraic integers of K and its unit group. For each finite prime $\mathfrak{p}$ of K, we denote the $\mathfrak{p}$-completion of K and its maximal compact subring by $K_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$. In the same way, $\mathbb{Q}_p$ and $\mathbb{Z}_p$ denote the p-completion of $\mathbb{Q}$ and $\mathbb{Z}$ for each rational prime number p. By a variety, we understand a Zariski open subset of an absolutely irreducible projective variety.

1. Modular forms with respect to an arbitrary unitary group

Let F be a totally real algebraic number field of finite degree and K be its CM-extension (namely, a totally imaginary quadratic extension of F). Such a field K is called a CM-field. As is well known, the non-trivial element of $\mathrm{Gal}(K/F)$ is given by the complex conjugation for any embedding of K into $\mathbb{C}$. We denote this by ρ. Let $\mathbf{a}$ be the set of all archimedean primes of F, which can be identified with those of K. We denote by $|\mathbf{a}|$ the number of elements in $\mathbf{a}$, which is equal to $[F : \mathbb{Q}]$. For each $v \in \mathbf{a}$, there are exactly two embeddings of K into $\mathbb{C}$ which lie above v. By a CM-type of K, we mean a set $\Psi = (\Psi_v)_{v\in\mathbf{a}}$ where each Ψ_v is an embedding of K into $\mathbb{C}$ which lies above v.

Given a set X, we denote by $X^\mathbf{a}$ the set of all indexed elements $(x_v)_{v\in\mathbf{a}}$ with $x_v \in X$. We can view a CM-type $\Psi = (\Psi_v)_{v\in\mathbf{a}}$ of K as an embedding of K into $\mathbb{C}^\mathbf{a}$ such that $b^\Psi = (b^{\Psi_v})_{v\in\mathbf{a}}$ for $b \in K$. Through Ψ, we can regard K as a dense subset of $\mathbb{C}^\mathbf{a}$. For $v \in \mathbf{a}$ and $b \in F$, we denote by b_v the image of b under the embedding $v : F \hookrightarrow \mathbb{R}$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $v \in \mathbf{a}$, we denote by $v\sigma$ the element of $\mathbf{a}$ such that $b_{v\sigma} = (b_v)^\sigma$.

For a positive integer m, take a non-degenerate skew-hermitian matrix $T \in K_m^m$, namely, $\det(T) \neq 0$ and ${}^tT^\rho = -T$. We view T as a skew-hermitian form on K_m^1 by $(x_1, x_2) \mapsto x_1 T\, {}^t x_2^\rho$ and denote by q the dimension of a maximal isotropic subspace of K_m^1 with respect to T. Take a CM-type $\Psi = (\Psi_v)_{v\in\mathbf{a}}$ of K so that each hermitian matrix $-\sqrt{-1}T^{\Psi_v}$ has signature (r_v, s_v) $(r_v + s_v = m)$ with $r_v \geq s_v$. The choice of Ψ is unique if and only if $r_v \neq s_v$ for each $v \in \mathbf{a}$. By the Hasse principle, we can take a suitable basis of K_m^1 such that T is expressed in the following form satisfying the conditions (1)–(3).

$$T = \begin{pmatrix} \tau 1_q & & & & & \\ & t_1 & & & & \\ & & t_2 & & & \\ & & & \ddots & & \\ & & & & t_{m-2q} & \\ & & & & & \tau^\rho 1_q \end{pmatrix}, \tag{1.1}$$

(1) $\tau, t_j \in K^\times$ and $\tau^\rho = -\tau,\ t_j^\rho = -t_j\ (1 \leq j \leq m-2q)$.
(2) $\mathrm{Im}(\tau^{\Psi_v}) > 0$ for each $v \in \mathbf{a}$.
(3) $\mathrm{Im}(t_j^{\Psi_v}) > 0$ if $1 \leq j \leq r_v - q$ and $\mathrm{Im}(t_j^{\Psi_v}) < 0$ if $r_v - q + 1 \leq j \leq m - 2q$ for each $v \in \mathbf{a}$.

We call such a T "normal" skew-hermitian matrix with respect to Ψ. We introduce this notion for technical convenience. For T as in (1.1) and $1 \leq j \leq m-2q$, we denote by $\Psi(T,j) = (\Psi(T,j)_v)_{v\in\mathbf{a}}$, the CM-type of K such that $\mathrm{Im}(t_j^{\Psi(T,j)_v}) > 0$ for each $v \in \mathbf{a}$. Clearly, we have $\Psi(T,j) = \Psi$ if $j \leq \frac{m}{2} - q$.

Note that, for each $v \in \mathbf{a}$, a "normal" skew-hermitian matrix T with respect to Ψ can be written as

$$T = \begin{pmatrix} T_{1,v} & \\ & T_{2,v} \end{pmatrix} \tag{1.2}$$

with diagonal matrices $T_{1,v}$ and $T_{2,v}$ of degree r_v and s_v which satisfy $-\sqrt{-1}T_{1,v}^{\Psi_v} > 0$ and $-\sqrt{-1}T_{2,v}^{\Psi_v} < 0$. Here and henceforth, the symbol $X > 0$ (resp. $X < 0$) indicates that X is positive definite (resp. negative definite). In case $r_v = s_v = \frac{m}{2}$ for any $v \in \mathbf{a}$, we have $q = \frac{m}{2}$ if $\det(T) \in N_{K/F}(K^\times)$ and $q = \frac{m}{2} - 1$ if $\det(T) \notin N_{K/F}(K^\times)$. In case $r_v > s_v$ for some $v \in \mathbf{a}$, the minimum of $\{s_v\}_{v\in\mathbf{a}}$ is equal to q.

Let $T \in K_m^m$ be a "normal" skew-hermitian matrix with respect to a CM-type $\Psi = (\Psi_v)_{v\in\mathbf{a}}$. Then we can define the algebraic groups corresponding to T and Ψ as follows.

$$\begin{aligned}
\mathrm{GU}(T,\Psi) &= \left\{\alpha \in \mathrm{GL}(m,K) \,\middle|\, \alpha T {}^t\alpha^\rho = \nu(\alpha)T \text{ with } \nu(\alpha) \in F^\times \right\}, \\
\mathrm{U}(T,\Psi) &= \left\{\alpha \in \mathrm{GL}(m,K) \,\middle|\, \alpha T {}^t\alpha^\rho = T \right\}, \\
\mathrm{U}_1(T,\Psi) &= \left\{\alpha \in \mathrm{GL}(m,K) \,\middle|\, \alpha T {}^t\alpha^\rho = T,\ \det(\alpha) = 1 \right\}.
\end{aligned}$$

As is well known, the algebraic group $\mathrm{U}_1(T,\Psi)$ has the strong approximation property.

For each $v \in \mathbf{a}$, we can define the v-components of these algebraic groups as follows.

$$\begin{aligned}
\mathrm{GU}(T,\Psi)_v &= \left\{\alpha \in \mathrm{GL}(m,\mathbb{C}) \,\middle|\, \alpha T^{\Psi_v} \overline{{}^t\alpha} = \nu(\alpha)T^{\Psi_v} \text{ with } \nu(\alpha) \in \mathbb{R}^\times \right\}, \\
\mathrm{U}(T,\Psi)_v &= \left\{\alpha \in \mathrm{GL}(m,\mathbb{C}) \,\middle|\, \alpha T^{\Psi_v} \overline{{}^t\alpha} = T^{\Psi_v} \right\}, \\
\mathrm{U}_1(T,\Psi)_v &= \left\{\alpha \in \mathrm{GL}(m,\mathbb{C}) \,\middle|\, \alpha T^{\Psi_v} \overline{{}^t\alpha} = T^{\Psi_v},\ \det(\alpha) = 1 \right\},
\end{aligned}$$

where the bars mean complex conjugates. We also define

$$\mathrm{GU}(T,\Psi)_{v+} = \{\alpha \in \mathrm{GU}(T,\Psi)_v \,|\, \nu(\alpha) > 0\}.$$

Note that $\mathrm{GU}(T,\Psi)_{v+} = \mathrm{GU}(T,\Psi)_v$ if $r_v > s_v$.

For each $v \in \mathbf{a}$, we can define the corresponding symmetric domain $\mathfrak{D}_v = \mathfrak{D}(T,\Psi)_v$ by

$$\mathfrak{D}(T,\Psi)_v = \left\{\mathfrak{z}_v \in \mathbb{C}_{s_v}^{r_v} \,\middle|\, -\sqrt{-1}\left((T_{2,v}^{\Psi_v})^{-1} + \overline{{}^t\mathfrak{z}_v}(T_{1,v}^{\Psi_v})^{-1}\mathfrak{z}_v\right) > 0 \right\},$$

where $T_{1,v}$ and $T_{2,v}$ are as in (1.2). For any $\mathfrak{z}_v \in \mathfrak{D}(T,\Psi)_v$ and any $\alpha = \begin{pmatrix} A_\alpha & B_\alpha \\ C_\alpha & D_\alpha \end{pmatrix} \in \mathrm{GU}(T,\Psi)_{v+}$ (where $A_\alpha \in \mathbb{C}^{r_v}_{r_v}$, $B_\alpha \in \mathbb{C}^{r_v}_{s_v}$, $C_\alpha \in \mathbb{C}^{s_v}_{r_v}$, $D_\alpha \in \mathbb{C}^{s_v}_{s_v}$), put

$$\alpha(\mathfrak{z}_v) = (A_\alpha \mathfrak{z}_v + B_\alpha)(C_\alpha \mathfrak{z}_v + D_\alpha)^{-1}.$$

Then the group $\mathrm{GU}(T,\Psi)_{v+}$ acts on $\mathfrak{D}(T,\Psi)_v$ as a group of holomorphic automorphisms by $\mathfrak{z}_v \to \alpha(\mathfrak{z}_v)$. The automorphic factors are given by

$$\begin{aligned} \mu_v(\alpha,\mathfrak{z}_v) &= C_\alpha \mathfrak{z}_v + D_\alpha, \\ \lambda_v(\alpha,\mathfrak{z}_v) &= \overline{A_\alpha} - \overline{B_\alpha} T^{\Psi_v}_{2,v} {}^t\mathfrak{z}_v (T^{\Psi_v}_{1,v})^{-1}. \end{aligned}$$

We have

$$\begin{aligned} \mu_v(\beta\alpha,\mathfrak{z}_v) &= \mu_v(\beta,\alpha(\mathfrak{z}_v))\mu_v(\alpha,\mathfrak{z}_v), \\ \lambda_v(\beta\alpha,\mathfrak{z}_v) &= \lambda_v(\beta,\alpha(\mathfrak{z}_v))\lambda_v(\alpha,\mathfrak{z}_v), \\ \det(\alpha)\det(\lambda_v(\alpha,\mathfrak{z}_v)) &= \nu(\alpha)^{r_v}\det(\mu_v(\alpha,\mathfrak{z}_v)), \end{aligned}$$

for any $\alpha, \beta \in \mathrm{GU}(T,\Psi)_{v+}$ and any $\mathfrak{z}_v \in \mathfrak{D}(T,\Psi)_v$. Clearly, $\det(\mu_v(\alpha,\mathfrak{z}_v)) \neq 0$ for any $\alpha \in \mathrm{GU}(T,\Psi)_{v+}$ and $\mathfrak{z}_v \in \mathfrak{D}(T,\Psi)_v$.

For $\mathfrak{z}_v \in \mathfrak{D}(T,\Psi)_v$, set

$$\begin{aligned} \eta_v(\mathfrak{z}_v) &= -\sqrt{-1}\left((T^{\Psi_v}_{2,v})^{-1} + \overline{{}^t\mathfrak{z}_v}(T^{\Psi_v}_{1,v})^{-1}\mathfrak{z}_v\right), \\ \kappa_v(\mathfrak{z}_v) &= \sqrt{-1}\left((T^{\Psi_v}_{1,v})^{-1} + (T^{\Psi_v}_{1,v})^{-1}\overline{\mathfrak{z}_v} T^{\Psi_v}_{2,v} {}^t\mathfrak{z}_v (T^{\Psi_v}_{1,v})^{-1}\right). \end{aligned}$$

Then we have

$$\begin{aligned} \overline{{}^t\mu_v(\alpha,\mathfrak{z}_v)}\eta_v(\alpha(\mathfrak{z}_v))\mu_v(\alpha,\mathfrak{z}_v) &= \nu(\alpha)\eta_v(\mathfrak{z}_v), \\ \overline{{}^t\lambda_v(\alpha,\mathfrak{z}_v)}\kappa_v(\alpha(\mathfrak{z}_v))\lambda_v(\alpha,\mathfrak{z}_v) &= \nu(\alpha)\kappa_v(\mathfrak{z}_v), \end{aligned} \tag{1.3}$$

for any $\alpha \in \mathrm{GU}(T,\Psi)_{v+}$ and $\mathfrak{z}_v \in \mathfrak{D}(T,\Psi)_v$.

Set

$$\begin{aligned} \mathrm{GU}(T,\Psi)_{\mathbf{a}} &= \prod_{v\in\mathbf{a}} \mathrm{GU}(T,\Psi)_v, \\ \mathrm{GU}(T,\Psi)_{\mathbf{a}+} &= \prod_{v\in\mathbf{a}} \mathrm{GU}(T,\Psi)_{v+}, \\ \mathrm{U}(T,\Psi)_{\mathbf{a}} &= \prod_{v\in\mathbf{a}} \mathrm{U}(T,\Psi)_v, \\ \mathrm{U}_1(T,\Psi)_{\mathbf{a}} &= \prod_{v\in\mathbf{a}} \mathrm{U}_1(T,\Psi)_v, \\ \mathfrak{D}(T,\Psi) &= \prod_{v\in\mathbf{a}} \mathfrak{D}(T,\Psi)_v, \end{aligned}$$

and define the action of $\mathrm{GU}(T,\Psi)_{\mathbf{a}+}$ on $\mathfrak{D}(T,\Psi)$ componentwise.

Put

$$\mathrm{GU}(T,\Psi)_+ = \{\alpha \in \mathrm{GU}(T,\Psi)\,|\,\nu(\alpha) >> 0\},$$

where $>> 0$ means totally positive. Note that $\mathrm{GU}(T,\Psi)_+ \neq \mathrm{GU}(T,\Psi)$ only if $r_v = s_v = \frac{m}{2}$ for some $v \in \mathbf{a}$. We define an embedding of $\mathrm{GU}(T,\Psi)_+$ into $\mathrm{GU}(T,\Psi)_{\mathbf{a}+}$ by $\alpha \to (\alpha^{\Psi_v})_{v\in\mathbf{a}}$ and also define an action of $\mathrm{GU}(T,\Psi)_+$ on $\mathfrak{D}(T,\Psi)$ by

$$\alpha((\mathfrak{z}_v)_{v\in\mathbf{a}}) = \left(\alpha^{\Psi_v}(\mathfrak{z}_v)\right)_{v\in\mathbf{a}},$$

where $\alpha \in \mathrm{GU}(T,\Psi)_+$ and $\mathfrak{z} = (\mathfrak{z}_v)_{v\in\mathbf{a}} \in \mathfrak{D}(T,\Psi)$. We write

$$\begin{aligned}
\mu_v(\alpha,\mathfrak{z}) &= \mu_v(\alpha^{\Psi_v},\mathfrak{z}_v),\\
\lambda_v(\alpha,\mathfrak{z}) &= \lambda_v(\alpha^{\Psi_v},\mathfrak{z}_v),\\
\eta_v(\mathfrak{z}) &= \eta_v(\mathfrak{z}_v),\\
\kappa_v(\mathfrak{z}) &= \kappa_v(\mathfrak{z}_v),
\end{aligned}$$

for any $\alpha \in \mathrm{GU}(T,\Psi)_+$, $\mathfrak{z} = (\mathfrak{z}_v)_{v\in\mathbf{a}} \in \mathfrak{D}(T,\Psi)$ and $v \in \mathbf{a}$. We denote by $\mathbf{0}$ the point $(0^{r_v}_{s_v})_{v\in\mathbf{a}} \in \mathfrak{D}(T,\Psi)$.

Now let us define a congruence subgroup of $\mathrm{GU}(T,\Psi)_+$. Let $\mathcal{O}_K$ be the ring of integers in K. For any integral ideal $\mathfrak{a}$ of $\mathcal{O}_K$, put

$$\Gamma_{\mathfrak{a}} = \{\alpha \in \mathrm{U}_1(T,\Psi) \cap \mathrm{SL}(m,\mathcal{O}_K)\,|\,\alpha - 1_m \in (\mathfrak{a})^m_m\}.$$

By a congruence subgroup of $\mathrm{GU}(T,\Psi)_+$, we understand a subgroup Γ of $\mathrm{GU}(T,\Psi)_+$ which contains $\Gamma_{\mathfrak{a}}$ for some integral ideal $\mathfrak{a}$ of $\mathcal{O}_K$ and $K^\times\Gamma_{\mathfrak{a}}$ is a subgroup of $K^\times\Gamma$ of finite index. Any element (except a scalar matrix) of a congruence subgroup Γ has no fixed points in $\mathfrak{D}(T,\Psi)$ if and only if the group $K^\times\Gamma/K^\times$ is torsion free. As is well known, $K^\times\Gamma_{\mathfrak{a}}/K^\times$ is torsion free if $\mathfrak{a}$ is sufficiently small.

Set $k = (k_v)_{v\in\mathbf{a}} \in \mathbb{Z}^{\mathbf{a}}$. For $\alpha = (\alpha_v)_{v\in\mathbf{a}} \in \mathrm{GU}(T,\Psi)_{\mathbf{a}+}$ and a $\mathbb{C}$-valued function f on $\mathfrak{D}(T,\Psi)$, We define a $\mathbb{C}$-valued function $f|_k\alpha$ on $\mathfrak{D}(T,\Psi)$ by

$$(f|_k\alpha)(\mathfrak{z}) = f(\alpha(\mathfrak{z}))\prod_{v\in\mathbf{a}}\det(\mu_v(\alpha_v,\mathfrak{z}_v))^{-k_v},$$

where $\mathfrak{z} = (\mathfrak{z}_v)_{v\in\mathbf{a}} \in \mathfrak{D}(T,\Psi)$. If f is holomorphic on $\mathfrak{D}(T,\Psi)$, so is $f|_k\alpha$. For $\alpha \in \mathrm{GU}(T,\Psi)_+$, we define

$$(f|_k\alpha)(\mathfrak{z}) = f(\alpha(\mathfrak{z}))\prod_{v\in\mathbf{a}}\det(\mu_v(\alpha,\mathfrak{z}))^{-k_v}.$$

For any congruence subgroup Γ of $\mathrm{GU}(T,\Psi)_+$, we denote by $\mathcal{M}_k(T,\Psi)(\Gamma)$ the set of all holomorphic functions on $\mathfrak{D}(T,\Psi)$ such that $f|_k\gamma = f$ for any $\gamma \in \Gamma$. An element of $\mathcal{M}_k(T,\Psi)(\Gamma)$ is called a holomorphic modular form of weight k with respect to Γ. (In case $m = 2$, $q = 1$ and $F = \mathbb{Q}$, we need holomorphy at every cusp to define a holomorphic modular form. From now on, however, we will not treat this case since it is very easy.) We denote by $\mathcal{M}_k(T,\Psi)$ the union of $\mathcal{M}_k(T,\Psi)(\Gamma)$ for all congruence subgroups Γ of $\mathrm{GU}(T,\Psi)_+$. Put

$$\mathcal{A}_k(T,\Psi) = \bigcup_{e\in\mathbb{Z}^{\mathbf{a}}} \left\{ f_1 f_2^{-1} \,|\, f_1 \in \mathcal{M}_{k+e}(T,\Psi),\ 0 \not\equiv f_2 \in \mathcal{M}_e(T,\Psi) \right\},$$
$$\mathcal{A}_k(T,\Psi)(\Gamma) = \{ f \in \mathcal{A}_k(T,\Psi) \,|\, f|_k\gamma = f \text{ for any } \gamma \in \Gamma \}.$$

We write simply $\mathcal{M}_k(T,\Psi)(\Gamma)$, $\mathcal{M}_k(T,\Psi)$, $\mathcal{A}_k(T,\Psi)(\Gamma)$, $\mathcal{A}_k(T,\Psi)$, by $\mathcal{M}_k(\Gamma)$, $\mathcal{M}_k$, $\mathcal{A}_k(\Gamma)$, $\mathcal{A}_k$, respectively if there is no fear of confusion.

Hereafter we identify $\mathbb{Z}^{\mathbf{a}}$ with the free module $\sum_{v\in\mathbf{a}} \mathbb{Z}\cdot v$ by putting $(k_v)_{v\in\mathbf{a}} = \sum_{v\in\mathbf{a}} k_v v$. Also put $\mathbf{1} = (1)_{v\in\mathbf{a}} = \sum_{v\in\mathbf{a}} v$. We can define the action of $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathbb{Z}^{\mathbf{a}}$ by $\left(\sum_{v\in\mathbf{a}} k_v v\right)^{\sigma} = \sum_{v\in\mathbf{a}} k_v(v\sigma)$. For any $k \in \mathbb{Z}^{\mathbf{a}}$, we denote by $F(k)$ the algebraic number field corresponding to $\left\{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \,|\, k^{\sigma} = k\right\}$. Then the field $F(k)$ is contained in the Galois closure of F over $\mathbb{Q}$.

2. Some embeddings of groups and symmetric domains

In order to use Shimura's many results for symplectic cases, we will define two kinds of embeddings of groups and symmetric domains in this section. First let us review symplectic groups and corresponding symmetric domains. Let $F, \mathbf{a}$ be as in Section 1. For any positive integer l, put

$$\mathrm{GSp}(l,F) = \left\{ \gamma \in \mathrm{GL}(2l,F) \,\middle|\, {}^t\gamma \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} \gamma = \nu(\gamma) \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} \text{ with } \nu(\gamma) \in F^{\times} \right\},$$
$$\mathrm{Sp}(l,F) = \left\{ \gamma \in \mathrm{GL}(2l,F) \,\middle|\, {}^t\gamma \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} \gamma = \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} \right\}.$$

As is well known, we have $\det(\gamma) = 1$ for any $\gamma \in \mathrm{Sp}(l,F)$. Set

$$\mathrm{GSp}(l,F)_+ = \{ \gamma \in \mathrm{GSp}(l,F) \,|\, \nu(\gamma) >> 0 \},$$

where $>> 0$ means totally positive, and set

$$\mathfrak{H}_l^{\mathbf{a}} = \left\{ z = (z_v)_{v\in\mathbf{a}} \in (\mathbb{C}_l^l)^{\mathbf{a}} \,\middle|\, {}^t z_v = z_v,\ \mathrm{Im}(z_v) > 0 \text{ for each } v \in \mathbf{a} \right\},$$

where > 0 means positive definite. Then $\mathrm{GSp}(l,F)_+$ acts on $\mathfrak{H}_l^{\mathbf{a}}$ as $\alpha((z_v)_{v\in\mathbf{a}}) = \left((a_v z_v + b_v)(c_v z_v + d_v)^{-1}\right)_{v\in\mathbf{a}}$ with $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GSp}(l,F)_+$ and $a,b,c,d \in F_l^l$. The automorphic factor is defined by

$$\mu_v^{(l)}(\alpha,(z_v)_{v\in\mathbf{a}}) = c_v z_v + d_v$$

for each $v \in \mathbf{a}$. We define congruence subgroups of $\mathrm{GSp}(l,F)_+$ as in [13]. For any $k = (k_v)_{v\in\mathbf{a}} \in \mathbb{Z}^{\mathbf{a}}$ and any congruence subgroup Γ of $\mathrm{GSp}(l,F)_+$, we denote by $\mathcal{M}_k^{(l)}(\Gamma)$ the space of holomorphic functions f on $\mathfrak{H}_l^{\mathbf{a}}$ which satisfy $(f|_k\gamma) = f$ for any $\gamma \in \Gamma$ (and are holomorphic at every cusp if $l = 1$ and $F = \mathbb{Q}$). Here $f|_k\gamma$ denotes the holomorphic function on $\mathfrak{H}_l^{\mathbf{a}}$ defined by $(f|_k\gamma)(z) = f(\gamma(z))\prod_{v\in\mathbf{a}}\det(\mu_v^{(l)}(\gamma,z))^{-k_v}$. Let $\mathcal{M}_k^{(l)}$ denote the union of $\mathcal{M}_k^{(l)}(\Gamma)$ for all congruence subgroups Γ of $\mathrm{GSp}(l,F)_+$. Moreover, set

$$\mathcal{A}_k^{(l)} = \bigcup_{e\in\mathbb{Z}^{\mathbf{a}}}\left\{f_1f_2^{-1}\,\middle|\,f_1 \in \mathcal{M}_{k+e}^{(l)},\ 0 \not\equiv f_2 \in \mathcal{M}_e^{(l)}\right\}.$$

We also need to recall the Galois action on modular forms for symplectic groups, which is constructed in [11]. As is well known, any $f \in \mathcal{M}_k^{(l)}$ has a Fourier expansion of the form

$$f((z_v)_{v\in\mathbf{a}}) = \sum_{h\in L} c_h \exp\left(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}}\mathrm{tr}(h_vz_v)\right), \tag{2.1}$$

where L is a certain lattice in the space of symmetric matrices of degree l with coefficients in F. Note that $c_h \neq 0$ only if the real symmetric matrix h_v is semi-positive definite for each $v \in \mathbf{a}$. For $f \in \mathcal{M}_k^{(l)}$ as (2.1) and any $\sigma \in \mathrm{Aut}(\mathbb{C})$, there exists $f^\sigma \in \mathcal{M}_{k^\sigma}^{(l)}$ whose Fourier expansion is given by

$$f^\sigma((z_v)_{v\in\mathbf{a}}) = \sum_{h\in L} c_h^\sigma \exp\left(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}}\mathrm{tr}(h_vz_v)\right). \tag{2.2}$$

This fact is proved in [11]. This Galois action is also constructed on vector-valued modular forms in §25 of [13]. For any subfield Ω of $\mathbb{C}$, we denote by $\mathcal{M}_k^{(l)}(\Omega)$ the set of all $f \in \mathcal{M}_k^{(l)}$ whose Fourier coefficients c_h are all contained in Ω. Set

$$\mathcal{A}_k^{(l)}(\Omega) = \bigcup_{e\in\mathbb{Z}^{\mathbf{a}}}\left\{f_1f_2^{-1}\,\middle|\,f_1 \in \mathcal{M}_{k+e}^{(l)}(\Omega),\ 0 \not\equiv f_2 \in \mathcal{M}_e^{(l)}(\Omega)\right\}.$$

Now let us define the first embedding. Take T, Ψ and $m,\ q$ as in section 1. For $z = (z_v)_{v\in\mathbf{a}} \in \mathfrak{H}_q^{\mathbf{a}}$, put

$$\varepsilon_0(T,\Psi)(z) = \left(\begin{array}{cc} 0^q_{s_v-q} & (z_v - \frac{\tau^{\Psi_v}}{2}\cdot 1_q)\cdot(z_v + \frac{\tau^{\Psi_v}}{2}\cdot 1_q)^{-1} \\ 0^{r_v-q}_{s_v-q} & 0^{r_v-q}_q \end{array}\right)_{v\in\mathbf{a}},$$

where $r_v,\ s_v$ are as in section 1. Then $\varepsilon_0(T,\Psi)$ gives a holomorphic embedding of $\mathfrak{H}_q^{\mathbf{a}}$ into $\mathfrak{D}(T,\Psi)$. This is compatible with the injection $I_0(T,\Psi)$ from $\mathrm{Sp}(q,F)$ into $\mathrm{U}_1(T,\Psi)$ defined by

$$I_0(T,\Psi)\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$$
$$= \begin{pmatrix} 1_q & 0 & -\frac{\tau}{2}\cdot 1_q \\ 0 & 1_{m-2q} & 0 \\ 1_q & 0 & \frac{\tau}{2}\cdot 1_q \end{pmatrix}\begin{pmatrix} \alpha_1 & 0 & \alpha_2 \\ 0 & 1_{m-2q} & 0 \\ \alpha_3 & 0 & \alpha_4 \end{pmatrix}\begin{pmatrix} 1_q & 0 & -\frac{\tau}{2}\cdot 1_q \\ 0 & 1_{m-2q} & 0 \\ 1_q & 0 & \frac{\tau}{2}\cdot 1_q \end{pmatrix}^{-1},$$

where $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in \mathrm{Sp}(q, F)$ with $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in F_q^q$. We denote $I_0(T, \Psi)$, $\varepsilon_0(T, \Psi)$ by I_0, ε_0 if there is no fear of confusion. We have

$$I_0(T, \Psi)(\alpha)\,(\varepsilon_0(T, \Psi)(z)) = \varepsilon_0(T, \Psi)\,(\alpha(z))$$

for any $\alpha \in \mathrm{Sp}(q, F)$ and $z \in \mathfrak{H}_q^{\mathbf{a}}$. The automorphic factors satisfy

$$\begin{aligned} &\mu_v(I_0(T, \Psi)(\alpha), \varepsilon_0(T, \Psi)(z)) \\ &\quad = \begin{pmatrix} 1_{s_v - q} & 0 \\ 0 & (\tau^{\Psi_v})^{-1}\{\alpha(z)\}_v + \frac{1}{2} \cdot 1_q \end{pmatrix} \begin{pmatrix} 1_{s_v - q} & 0 \\ 0 & \mu_v^{(q)}(\alpha, z) \end{pmatrix} \\ &\quad\quad \times \begin{pmatrix} 1_{s_v - q} & 0 \\ 0 & (\tau^{\Psi_v})^{-1} z_v + \frac{1}{2} \cdot 1_q \end{pmatrix}^{-1} \end{aligned}$$

for any $\alpha \in \mathrm{Sp}(q, F)$, $z = (z_v)_{v \in \mathbf{a}} \in \mathfrak{H}_q^{\mathbf{a}}$ and each $v \in \mathbf{a}$. Hence we can define pull-backs of modular forms. For $k = (k_v)_{v \in \mathbf{a}} \in \mathbb{Z}^{\mathbf{a}}$ and $f \in \mathcal{M}_k(T, \Psi)$, define a function $f|\varepsilon_0 = f|\varepsilon_0(T, \Psi)$ on $\mathfrak{H}_q^{\mathbf{a}}$ as

$$(f|\varepsilon_0)(z) = f(\varepsilon_0(z)) \prod_{v \in \mathbf{a}} \det\left((\tau^{\Psi_v})^{-1} z_v + \frac{1}{2} \cdot 1_q \right)^{-k_v}.$$

Then we clearly have $f|\varepsilon_0 \in \mathcal{M}_k^{(q)}$. (In case $F = \mathbb{Q}$ and $q = 1$, the holomorphy of $f|\varepsilon_0$ at every cusp can be proved by using Lemma 2.1 of [15].)

Next we will define an embedding of $\mathfrak{D}(T, \Psi)$ into $\mathfrak{H}_m^{\mathbf{a}}$, which is compatible with that of $\mathrm{GU}(T, \Psi)$ into $\mathrm{GSp}(m, F)$. For $\alpha \in \mathrm{GU}(T, \Psi)$, put

$$I(T, \Psi)(\alpha) = \begin{pmatrix} 1_m & T \\ 1_m & T^\rho \end{pmatrix}^{-1} \begin{pmatrix} \alpha^\rho & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1_m & T \\ 1_m & T^\rho \end{pmatrix}.$$

Then we have $I(T, \Psi)(\alpha) \in \mathrm{GSp}(m, F)$ and $\nu(I(T, \Psi)(\alpha)) = \nu(\alpha)$. Hence the image of $\mathrm{GU}(T, \Psi)_+$ by $I(T, \Psi)$ is contained in $\mathrm{GSp}(m, F)_+$. We write simply $I(T, \Psi)$ by I if there is no fear of confusion. For $\mathfrak{z} = (\mathfrak{z}_v)_{v \in \mathbf{a}} \in \mathfrak{D}(T, \Psi)$ and each $v \in \mathbf{a}$, put

$$\omega_v(\mathfrak{z}) = \begin{pmatrix} 1_{r_v} & \mathfrak{z}_v \\ -T_{2,v}^{\Psi_v\,t}\mathfrak{z}_v (T_{1,v}^{\Psi_v})^{-1} & 1_{s_v} \end{pmatrix}.$$

Then $\det(\omega_v(\mathfrak{z})) \neq 0$ for any $\mathfrak{z} \in \mathfrak{D}(T, \Psi)$ and each $v \in \mathbf{a}$. We can define an embedding $\varepsilon = \varepsilon(T, \Psi)$ of $\mathfrak{D}(T, \Psi)$ into $\mathfrak{H}_m^{\mathbf{a}}$ by

$$\varepsilon(T, \Psi)(\mathfrak{z}) = \left(\omega_v(\mathfrak{z}) \begin{pmatrix} 1_{r_v} & 0 \\ 0 & -1_{s_v} \end{pmatrix} \omega_v(\mathfrak{z})^{-1} T^{\Psi_v} \right)_{v \in \mathbf{a}}.$$

Then the map $\varepsilon = \varepsilon(T, \Psi)$ is a holomorphic injection from $\mathfrak{D}(T, \Psi)$ into $\mathfrak{H}_m^{\mathbf{a}}$ and the set $\varepsilon(T, \Psi)(\mathfrak{D}(T, \Psi))$ is an analytic set in $\mathfrak{H}_m^{\mathbf{a}}$ (hence it is closed in $\mathfrak{H}_m^{\mathbf{a}}$). Moreover, it can easily be verified that the Jacobian of $\varepsilon(T, \Psi)$ is non-zero at each $\mathfrak{z} \in \mathfrak{D}(T, \Psi)$. We have

$$\varepsilon(T, \Psi)(\alpha(\mathfrak{z})) = I(T, \Psi)(\alpha)\,(\varepsilon(T, \Psi)(\mathfrak{z}))$$

for any $\alpha \in \mathrm{GU}(T,\Psi)_+$ and $\mathfrak{z} \in \mathfrak{D}(T,\Psi)$.

We need another embedding of $\mathfrak{D}(T,\Psi)$ into $\mathfrak{H}_m^{\mathbf{a}}$ to use many results on modular forms with respect to symplectic groups. Take $\delta \in K^\times$ so that $\delta^\rho = -\delta$ and put

$$C(T,\delta) = \left(\begin{array}{ccc|ccc} \frac{1}{2}1_q & 0 & \frac{1}{2}1_q & 0 & 0 & 0 \\ 0 & 1_{m-2q} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\tau\delta}{2}1_q & 0 & -\frac{\tau\delta}{2}1_q \\ \hline 0 & 0 & 0 & 1_q & 0 & 1_q \\ 0 & 0 & 0 & 0 & 1_{m-2q} & 0 \\ -(\tau\delta)^{-1}1_q & 0 & (\tau\delta)^{-1}1_q & 0 & 0 & 0 \end{array}\right).$$

Then $C(T,\delta) \in \mathrm{Sp}(m,F)$. Define an embedding $I_\delta(T,\Psi)$ (resp. $\varepsilon_\delta(T,\Psi)$) of $\mathrm{GU}(T,\Psi)$ into $\mathrm{GSp}(m,F)$ (resp. $\mathfrak{D}(T,\Psi)$ into $\mathfrak{H}_m^{\mathbf{a}}$) by

$$\begin{aligned} I_\delta(T,\Psi)(\alpha) &= C(T,\delta)I(T,\Psi)(\alpha)C(T,\delta)^{-1}, \\ \varepsilon_\delta(T,\Psi)(\mathfrak{z}) &= C(T,\delta)\left(\varepsilon(T,\Psi)(\mathfrak{z})\right). \end{aligned}$$

Obviously, we have

$$\begin{aligned} I_\delta(T,\Psi)(\alpha)\left(\varepsilon_\delta(T,\Psi)(\mathfrak{z})\right) &= \varepsilon_\delta(T,\Psi)\left(\alpha(\mathfrak{z})\right), \\ \nu(I_\delta(T,\Psi)(\alpha)) &= \nu(\alpha), \end{aligned}$$

for any $\alpha \in \mathrm{GU}(T,\Psi)_+$ and $\mathfrak{z} \in \mathfrak{D}(T,\Psi)$. For each $v \in \mathbf{a}$, $\alpha \in \mathrm{GU}(T,\Psi)_+$ and $\mathfrak{z} \in \mathfrak{D}(T,\Psi)$, the automorphic factors have the following relation.

(2.3)
$$\begin{aligned} &\mu_v^{(m)}\left(I_\delta(T,\Psi)(\alpha), \varepsilon_\delta(T,\Psi)(\mathfrak{z})\right) \\ &\quad = \mu_v^{(m)}\left(C(T,\delta)^{-1}, \varepsilon_\delta(T,\Psi)(\alpha(\mathfrak{z}))\right)^{-1}(T^{\Psi_v})^{-1}\omega_v(\alpha(\mathfrak{z})) \\ &\quad\quad \times \begin{pmatrix} \lambda_v(\alpha,\mathfrak{z}) & 0 \\ 0 & \mu_v(\alpha,\mathfrak{z}) \end{pmatrix} \omega_v(\mathfrak{z})^{-1}T^{\Psi_v}\mu_v^{(m)}\left(C(T,\delta)^{-1}, \varepsilon_\delta(T,\Psi)(\mathfrak{z})\right). \end{aligned}$$

We write simply $I_\delta(T,\Psi)$, $\varepsilon_\delta(T,\Psi)$ by I_δ, ε_δ respectivly if there is no fear of confusion. We easily obtain the following lemma from the property of $\varepsilon(T,\Psi)$.

Lemma 2.1. *The Jacobian of $\varepsilon_\delta(T,\Psi)$ is non-zero at each $\mathfrak{z} \in \mathfrak{D}(T,\Psi)$ and the set $\varepsilon_\delta(T,\Psi)\left(\mathfrak{D}(T,\Psi)\right)$ is an analytic set (hence closed) in $\mathfrak{H}_m^{\mathbf{a}}$.*

Now we can consider pull-backs of modular forms on $\mathfrak{H}_m^{\mathbf{a}}$. For any $f \in \mathcal{M}_k^{(m)}$ (with $k \in \mathbb{Z}^{\mathbf{a}}$), define a function $f|\varepsilon_\delta = f|\varepsilon_\delta(T,\Psi)$ on $\mathfrak{D}(T,\Psi)$ by

$$(f|\varepsilon_\delta)(\mathfrak{z}) = f(\varepsilon_\delta(\mathfrak{z}))\prod_{v\in\mathbf{a}}\left\{\left(-\frac{\delta^{\Psi_v}}{2}\right)^q \det\left(\omega_v(\mathfrak{z})\mu_v^{(m)}(C(T,\delta)^{-1}, \varepsilon_\delta(\mathfrak{z}))^{-1}\right)\right\}^{-k_v}.$$

Then we can easily obtain $f|\varepsilon_\delta(T,\Psi) \in \mathcal{M}_{2k}(T,\Psi)$ by using (2.3).

To conclude this section, let us write down $I_\delta \circ I_0 = I_\delta(T,\Psi) \circ I_0(T,\Psi)$ and $\varepsilon_\delta \circ \varepsilon_0 = \varepsilon_\delta(T,\Psi) \circ \varepsilon_0(T,\Psi)$ explicitly. For any $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(q,F)$ (with $a,b,c,d \in F_q^q$), we have

$$I_\delta \circ I_0(\alpha) = \left(\begin{array}{ccc|ccc} a & & & b & & \\ & 1_{m-2q} & & & 0 & \\ & & a & & & -\delta^2 b \\ \hline c & & & d & & \\ & 0 & & & 1_{m-2q} & \\ & & -\delta^{-2} c & & & d \end{array}\right).$$

On the other hand, for any $z = (z_v)_{v\in\mathbf{a}} \in \mathfrak{H}_q^{\mathbf{a}}$, we have

$$\varepsilon_\delta \circ \varepsilon_0(z) = \begin{pmatrix} z_v & & & & \\ & t_1^{\Psi(T,1)_v} & & & \\ & & \ddots & & \\ & & & t_{m-2q}^{\Psi(T,m-2q)_v} & \\ & & & & (-\delta^2)_v z_v \end{pmatrix}_{v\in\mathbf{a}}. \tag{2.4}$$

Moreover, for $z \in \mathfrak{H}_q^{\mathbf{a}}$ and $\alpha \in \mathrm{Sp}(q,F)$, we obtain

$$\mu_v^{(m)}(I_\delta \circ I_0(\alpha), \varepsilon_\delta \circ \varepsilon_0(z)) = \begin{pmatrix} \mu_v^{(q)}(\alpha,z) & & \\ & 1_{m-2q} & \\ & & \mu_v^{(q)}(\alpha,z) \end{pmatrix}$$

for each $v \in \mathbf{a}$. By a computation, we can verify

$$((f|\varepsilon_\delta)|\varepsilon_0)(z) = f(\varepsilon_\delta(\varepsilon_0(z))) \tag{2.5}$$

for $z \in \mathfrak{H}_q^{\mathbf{a}}$.

3. Canonical models and the conjugations

In [7], Shimura constructed canonical models with respect to symplectic groups, and in [5], Miyake constructed them with respect to unitary groups. We will recall and study them more precisely in this section, to use the results in later sections.

First let us consider the adelization of $\mathrm{GU}(T,\Psi)$, that is,

$$\mathrm{GU}(T,\Psi)_A = \left\{ x \in \mathrm{GL}(m,K_A) \,\middle|\, xT^t x^\rho = \nu(x)T \quad \text{with } \nu(x) \in F_A^\times \right\}.$$

Note that $x_\mathfrak{p}$, the $\mathfrak{p}$-component of x, belongs to $\mathrm{GL}(m,\mathcal{O}_\mathfrak{p})$ for almost all non-archimedean primes $\mathfrak{p}$ of K. Moreover, put

$$\mathrm{GU}(T,\Psi)_{A+} = \{ x \in \mathrm{GU}(T,\Psi)_A \,|\, \nu(x)_v > 0 \quad \text{for any } v \in \mathbf{a} \},$$

where $\nu(x)_v$ denotes the v-component of $\nu(x)$. We also put

$$\begin{aligned} \mathrm{U}(T,\Psi)_A &= \{x \in \mathrm{GU}(T,\Psi)_A \,|\, \nu(x) = 1\}\,, \\ \mathrm{U}_1(T,\Psi)_A &= \{x \in \mathrm{GU}(T,\Psi)_A \,|\, \nu(x) = \det(x) = 1\}\,. \end{aligned}$$

We denote by $\mathrm{GU}(T,\Psi)_{\mathbf{h}}$, $\mathrm{U}(T,\Psi)_{\mathbf{h}}$ and $\mathrm{U}_1(T,\Psi)_{\mathbf{h}}$, the non-archimedean components of $\mathrm{GU}(T,\Psi)_A$, $\mathrm{U}(T,\Psi)_A$ and $\mathrm{U}_1(T,\Psi)_A$, respectively, and view $\mathrm{GU}(T,\Psi)_{\mathbf{a}}$, $\mathrm{GU}(T,\Psi)_{\mathbf{a}+}$, $\mathrm{U}(T,\Psi)_{\mathbf{a}}$ and $\mathrm{U}_1(T,\Psi)_{\mathbf{a}}$, as the archimedean components of $\mathrm{GU}(T,\Psi)_A$, $\mathrm{GU}(T,\Psi)_{A+}$, $\mathrm{U}(T,\Psi)_A$ and $\mathrm{U}_1(T,\Psi)_A$, respectively. For each $x \in \mathrm{GU}(T,\Psi)_A$, we denote by $x_{\mathbf{a}}$ (resp. $x_{\mathbf{h}}$) the archimedean part (resp. non-archimedean part) of x. We regard $\mathrm{GU}(T,\Psi)$, $\mathrm{GU}(T,\Psi)_+$, $\mathrm{U}(T,\Psi)$ and $\mathrm{U}_1(T,\Psi)$, as subgroups of $\mathrm{GU}(T,\Psi)_A$, $\mathrm{GU}(T,\Psi)_{A+}$, $\mathrm{U}(T,\Psi)_A$ and $\mathrm{U}_1(T,\Psi)_A$, through diagonal embeddings. As is well known, the algebraic group $\mathrm{U}_1(T,\Psi)$ has the strong approximation property.

Let $\mathcal{Z}(T,\Psi)$ be the set of all subgroups X of $\mathrm{GU}(T,\Psi)_{A+}$ which are written as $X = \mathrm{GU}(T,\Psi)_{\mathbf{a}+} \times X_{\mathbf{h}}$ with open compact subgroups $X_{\mathbf{h}}$ of $\mathrm{GU}(T,\Psi)_{\mathbf{h}}$. For any $X \in \mathcal{Z}(T,\Psi)$, take $\Gamma_X = X \cap \mathrm{GU}(T,\Psi)$. Then Γ_X is a congruence subgroup of $\mathrm{GU}(T,\Psi)_+$. In [5] it is shown that, for each $X \in \mathcal{Z}(T,\Psi)$, there exists a variety (more precisely, a Zariski open subset of a projective variety) V_X defined over $\overline{\mathbb{Q}}$, and a holomorphic map $\varphi_X : \mathfrak{D}(T,\Psi) \to V_X$ so that φ_X defines a biregular isomorphism of $\Gamma_X \setminus \mathfrak{D}(T,\Psi)$ onto V_X. This is the so-called canonical model. It is also known that any $f \in \mathcal{A}_0(T,\Psi)$ can be written as $f = p \circ \varphi_X$ with some $X \in \mathcal{Z}(T,\Psi)$ and a rational function p on V_X. Put $\Phi_X(f) = p$ for such f, p and X.

Canonical models with respect to a symplectic group are constructed in [7]. For a positive integer l, put

$$\mathrm{GSp}(l,F)_A = \left\{ x \in \mathrm{GL}(2l,F_A) \;\middle|\; \begin{array}{r} {}^t x \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} x = \nu(x) \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix} \\ \text{with } \nu(x) \in F_A^\times \end{array} \right\},$$

$$\begin{aligned} \mathrm{GSp}(l,F)_{A+} &= \{x \in \mathrm{GSp}(l,F)_A \,|\, \nu(x)_v > 0 \quad \text{for any } v \in \mathbf{a}\}\,, \\ \mathrm{Sp}(l,F)_A &= \{x \in \mathrm{GSp}(l,F)_A \,|\, \nu(x) = 1\}\,. \end{aligned}$$

We denote by $\mathrm{GSp}(l,F)_{\mathbf{a}}$ (resp. $\mathrm{GSp}(l,F)_{\mathbf{h}}$) and $\mathrm{Sp}(l,F)_{\mathbf{a}}$ (resp. $\mathrm{Sp}(l,F)_{\mathbf{h}}$), the archimedean components (resp. non-archimedean components) of $\mathrm{GSp}(l,F)_A$ and $\mathrm{Sp}(l,F)_A$. The archimedean component of $\mathrm{GSp}(l,F)_{A+}$ is denoted by $\mathrm{GSp}(l,F)_{\mathbf{a}+}$, which is the connected component of $\mathrm{GSp}(l,F)_{\mathbf{a}}$ containing the identity. We regard $\mathrm{GSp}(l,F)$, $\mathrm{GSp}(l,F)_+$ and $\mathrm{Sp}(l,F)$ as subgroups of $\mathrm{GSp}(l,F)_A$, $\mathrm{GSp}(l,F)_{A+}$ and $\mathrm{Sp}(l,F)_A$ through diagonal embeddings.

The embeddings $I_0(T,\Psi) : \mathrm{Sp}(q,F) \hookrightarrow \mathrm{U}_1(T,\Psi)$ and $I_\delta(T,\Psi) : \mathrm{GU}(T,\Psi) \hookrightarrow \mathrm{GSp}(m,F)$ can naturally be extended to those of $\mathrm{Sp}(q,F)_A \hookrightarrow \mathrm{U}_1(T,\Psi)_A$ and $\mathrm{GU}(T,\Psi)_A \hookrightarrow \mathrm{GSp}(m,F)_A$ respectively, since they can be viewed as homomorphisms of algebraic groups.

Consider the closed subgroup $\mathcal{G}_+^{(l)}$ of $\mathrm{GSp}(l,F)_{A+}$ defined by

$$\mathcal{G}_+^{(l)} = \left\{ x \in \mathrm{GSp}(l,F)_A \;\middle|\; \begin{array}{l} \nu(x) \in F^\times F_{\infty+}^\times \mathbb{Q}_A^\times, \\ \nu(x)_v > 0 \quad \text{for any } v \in \mathbf{a} \end{array} \right\},$$

where $F_{\infty+}^{\times}$ denotes the connected component of $F_{\infty}^{\times}$ containing the identity. Note that $\mathcal{G}_{+}^{(l)} \supset \mathrm{GSp}(l,F)_{+}$. Let $\mathcal{Z}^{(l)}$ be the set of all subgroups Y of $\mathrm{GSp}(l,F)_{A+}$ which are written as $Y = \mathrm{GSp}(l,F)_{\mathbf{a}+} \times Y_{\mathbf{h}}$ with open compact subgroups $Y_{\mathbf{h}}$ of $\mathrm{GSp}(l,F)_{\mathbf{h}}$. For any $Y \in \mathcal{Z}^{(l)}$, put $\Gamma_Y^{(l)} = Y \cap \mathrm{GSp}(l,F)$. Then $\Gamma_Y^{(l)}$ is a congruence subgroup of $\mathrm{GSp}(l,F)_{+}$. For each $Y \in \mathcal{Z}^{(l)}$, there exists a variety $V_Y^{(l)}$ defined over $\mathbb{Q}_{ab}$, and a holomorphic map $\varphi_Y^{(l)} : \mathfrak{H}_l^{\mathbf{a}} \to V_Y^{(l)}$ so that $\varphi_Y^{(l)}$ defines a biregular isomorphism of $\Gamma_Y^{(l)} \setminus \mathfrak{H}_l^{\mathbf{a}}$ onto $V_Y^{(l)}$. Moreover, any $f \in \mathcal{A}_0^{(l)}$ can be written as $f = p \circ \varphi_Y^{(l)}$ with some $Y \in \mathcal{Z}^{(l)}$ and a rational function p on $V_Y^{(l)}$. Put $\Phi_Y^{(l)}(f) = p$ for such f, p and Y.

For any $\sigma \in \mathrm{Gal}(\mathbb{Q}_{ab}/\mathbb{Q})$, we define $\chi(\sigma) \in \prod_p \mathbb{Z}_p^{\times} \subset \mathbb{Q}_A^{\times}$ by the formula $[\chi(\sigma)^{-1}, \mathbb{Q}] = \sigma$. (Then $\chi(\sigma)$ is uniquely determined.) Take $x \in \mathcal{G}_{+}^{(l)}$ and $Y, Y' \in \mathcal{Z}^{(l)}$ so that $Y \supset xY'x^{-1}$. Then there exists a morphism $J_{YY'}^{(l)}(x)$ (defined over $\mathbb{Q}_{ab}$) of $V_{Y'}^{(l)}$ to $(V_Y^{(l)})^{\sigma(x)}$, where $\sigma(x) \in \mathrm{Gal}(\mathbb{Q}_{ab}/\mathbb{Q})$ which satisfies $\nu(x) \in F^{\times} F_{\infty+}^{\times} \chi(\sigma(x))$. We also have $J_{YY'}^{(l)}(x_1)^{\sigma(x_2)} \circ J_{Y'Y''}^{(l)}(x_2) = J_{YY''}^{(l)}(x_1 x_2)$, if the both components of the left hand side are defined.

The Galois action defined in (2.2) and canonical models have the following relation. For any non-zero $f_1, f_2 \in \mathcal{M}_k^{(l)}$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$(f_1^{\sigma}/f_2^{\sigma}) = \Phi_Y^{(l)}(f_1/f_2)^{\sigma} \circ J_{Y\tilde{Y}}^{(l)}\left(\begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}\right) \circ \varphi_{\tilde{Y}}^{(l)}, \tag{3.1}$$

for any $Y \in \mathcal{Z}^{(l)}$ so that f_1/f_2 can be written as $(f_1/f_2) = \Phi_Y^{(l)}(f_1/f_2) \circ \varphi_Y^{(l)}$, where

$$\tilde{Y} = \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}^{-1} Y \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix} \quad \in \mathcal{Z}^{(l)}.$$

This fact is proved in [11].

In the rest of this section, we will study $J_{Y\tilde{Y}}^{(l)}\left(\begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}\right)$ more concretely. Before doing that, we must recall the reflex of a CM-type and the conjugation of abelian varieties of CM-type.

For a CM-field K, its CM-type $\Psi = (\Psi_v)_{v \in \mathbf{a}}$, and any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we can define another CM-type $\Psi\sigma = \{\Psi_v \sigma \,|\, v \in \mathbf{a}\}$ of K. We denote by K_{Ψ}^{*} (or simply K^{*} if there is no fear of confusion) the algebraic number field corresponding to the subgroup $\{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \,|\, \Psi\sigma = \Psi\}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of finite index. As is well known, K_{Ψ}^{*} is a CM-field contained in the Galois closure of K. Viewing Ψ as a union of $|\mathbf{a}|$ different right $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-cosets in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we define a CM-type Ψ^{*} of K_{Ψ}^{*} as follows:

$$\mathrm{Gal}(\overline{\mathbb{Q}}/K_{\Psi}^{*})\Psi^{*} = \left(\mathrm{Gal}(\overline{\mathbb{Q}}/K)\Psi\right)^{-1}.$$

We call Ψ^{*} "the reflex of Ψ" and the couple (K_{Ψ}^{*}, Ψ^{*}) "the reflex of (K, Ψ)". From the definition, we have $(K_{\Psi}^{*})^{\sigma} = K_{\Psi\sigma}^{*}$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (or $\in \mathrm{Aut}(\mathbb{C})$). By N'_{Ψ}, we denote the group homomorphism of $K_{\Psi}^{*\times}$ to $K^{\times}$ defined by $x \to \prod_{\psi^* \in \Psi^*} x^{\psi^*}$. It is a morphism of algebraic groups if we view $K_\Psi^{*\times}$ and $K^\times$ as algebraic groups defined over $\mathbb{Q}$, and so it can naturally be extended to the homomorphism of $(K_\Psi^*)_A^\times$ to $K_A^\times$.

For a CM-type Ψ and any $\sigma \in \mathrm{Aut}(\mathbb{C})$, a certain idele class $g_\Psi(\sigma) \in K_A^\times / K^\times K_\infty^\times$ is defined in Chapter 7 of [3] (or essentially in [2]). Take an abelian variety (A, ι) of type (K, Ψ) with an $\mathcal{O}_K$-lattice L in K and a complex analytic isomorphism Θ of $\mathbb{C}^{\mathbf{a}}/L^\Psi$ onto A. (See, [13].) We denote by A_{tor} the subgroup consistofing of all torsion elements of A, which coincides with the image of K/L by $\Theta \circ \Psi$. Next take $(A, \iota)^\sigma$. Then it is an abelian variety of type $(K, \Psi\sigma)$ and we have the following commutative diagram

$$\begin{array}{ccc} K/L & \xrightarrow{\Theta\circ\Psi} & A_{\mathrm{tor}} \\ {\scriptstyle \times a}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\ K/aL & \xrightarrow{\Theta_a\circ(\Psi\sigma)} & A_{\mathrm{tor}}^\sigma \end{array}$$

with some $a \in K_A^\times$ and complex analytic isomorphism Θ_a of $\mathbb{C}^{\mathbf{a}}/(aL)^{\Psi\sigma}$ onto A^σ. The coset $aK^\times K_\infty^\times$ is uniquely determined by (K, Ψ) and σ, not depending on A or L. We denote this coset by $g_\Psi(\sigma)$. For $a \in g_\Psi(\sigma)$, we have $aa^\rho \in \chi(\sigma) F^\times F_\infty^\times$. We define $\iota(\sigma, a) \in F^\times$ by $\frac{\chi(\sigma)}{aa^\rho} \in \iota(\sigma, a) F_\infty^\times$. If σ is trivial on K_Ψ^*, we have $g_\Psi(\sigma) = N'_\Psi(b) K^\times K_\infty^\times$ with $b \in (K_\Psi^*)_A^\times$ such that $[b^{-1}, K_\Psi^*] = \sigma|_{K^*_{\Psi ab}}$; this fact is a main theorem of complex multiplication theory of [13]. Note that $g_\Psi(\sigma_1) g_{\Psi\sigma_1}(\sigma_2) = g_\Psi(\sigma_1\sigma_2)$. The following lemma is an immediate consequence of Theorem 3.1 of Chapter 7 in [3].

Lemma 3.1. *Take a CM-type $\Psi = (\Psi_v)_{v \in \mathbf{a}}$ and $t \in K^\times$ so that $t^\rho = -t$ and $\mathrm{Im}(t^{\Psi_v}) > 0$ for each $v \in \mathbf{a}$. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $a \in g_\Psi(\sigma)$, we have $\mathrm{Im}\left((\iota(\sigma, a)t)^{\Psi_v\sigma}\right) > 0$ for each $v \in \mathbf{a}$.*

The morphism $J^{(l)}_{Y\tilde{Y}}\left(\begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}\right)$ moves so-called "CM-points" in $\mathfrak{H}_l^{\mathbf{a}}$ according to the following proposition.

Proposition 3.2. *For a CM-type Ψ of K, take $z = \tau^\Psi \in \mathfrak{H}_l^{\mathbf{a}}$ with $\tau \in K_l^l$ (such that ${}^t\tau = \tau$). For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, choose $Y, \tilde{Y} \in \mathcal{Z}^{(l)}$ so that*

$$\tilde{Y} = \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}^{-1} Y \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}.$$

Take $\tilde{z} \in \mathfrak{H}_l^{\mathbf{a}}$ as

$$\varphi^{(l)}_{\tilde{Y}}(\tilde{z}) = \left[J^{(l)}_{\tilde{Y}Y}\left(\begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix}^{-1}\right)\left(\varphi^{(l)}_Y(z)\right)\right]^\sigma.$$

Then we have

$$\varphi^{(l)}_{\tilde{Y}}(\tilde{z}) = \varphi^{(l)}_{\tilde{Y}}\left(\left((\alpha_1\tau + \alpha_2)(\alpha_3\tau + \alpha_4)^{-1}\right)^{\Psi\sigma}\right),$$

where $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in \mathrm{GSP}(l, F) \quad (\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in F_l^l)$ *so that*

$$\begin{aligned} \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in (\tilde{Y} \cap \mathrm{Sp}(l, F)_A) \mathrm{GSp}(l, F)_{\mathbf{a}} & \begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma) 1_l \end{pmatrix}^{-1} \\ & \times \begin{pmatrix} (a\tau - a^\rho \tau^\rho)(\tau - \tau^\rho)^{-1} & -(a - a^\rho)\tau^\rho(\tau - \tau^\rho)^{-1}\tau \\ (a - a^\rho)(\tau - \tau^\rho)^{-1} & (\tau - \tau^\rho)^{-1}(a^\rho \tau - a\tau^\rho) \end{pmatrix} \\ & \cap \mathrm{GSp}(l, F) \end{aligned}$$

for $a \in g_\Psi(\sigma)$.

This is Proposition 3.2 of [15].

We can naturally consider the embedding $I_{l_1,l_2,\ldots,l_r}$ of $\prod_{j=1}^r \mathrm{Sp}(l_j, F)$ into $\mathrm{SP}(l_1 + \cdots + l_r, F)$ defined by

$$I_{l_1,l_2,\ldots,l_r}\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix}\right) = \left(\begin{array}{ccc|ccc} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_r & & & b_r \\ \hline c_1 & & & d_1 & & \\ & \ddots & & & \ddots & \\ & & c_r & & & d_r \end{array}\right),$$

where $a_j, b_j, c_j, d_j \in F_{l_j}^{l_j}$ for each $1 \le j \le r$. Obviously, this is compatible with the embedding $\varepsilon_{l_1,l_2,\ldots,l_r}$ of $\prod_{j=1}^r \mathfrak{H}_{l_j}^{\mathbf{a}}$ into $\mathfrak{H}_{l_1+\cdots+l_r}^{\mathbf{a}}$ defined by

$$\varepsilon_{l_1,l_2,\ldots,l_r}\left((z_{1,v})_{v\in\mathbf{a}}, \ldots, (z_{r,v})_{v\in\mathbf{a}}\right) = \begin{pmatrix} z_{1,v} & & \\ & \ddots & \\ & & z_{r,v} \end{pmatrix}_{v\in\mathbf{a}}.$$

That is,

$$\begin{aligned} I_{l_1,l_2,\ldots,l_r}(\alpha_1, \alpha_2, \ldots, \alpha_r) & \left(\varepsilon_{l_1,l_2,\ldots,l_r}(z_1, z_2, \ldots, z_r)\right) \\ & = \varepsilon_{l_1,l_2,\ldots,l_r}(\alpha_1(z_1), \alpha_2(z_2), \ldots \alpha_r(z_r)), \end{aligned}$$

for any $\alpha_j \in \mathrm{Sp}(l_j, F)$ and $z_j \in \mathfrak{H}_{l_j}^{\mathbf{a}}$ $(1 \le j \le r)$. Moreover, it is also clear that

$$\begin{aligned} \mu_v^{(l_1+\cdots+l_r)} & \left(I_{l_1,l_2,\ldots,l_r}(\alpha_1, \ldots, \alpha_r), \varepsilon_{l_1,\ldots,l_r}(z_1, \ldots, z_r)\right) \\ & = \begin{pmatrix} \mu_v^{(l_1)}(\alpha_1, z_1) & & \\ & \ddots & \\ & & \mu_v^{(l_r)}(\alpha_r, z_r) \end{pmatrix}, \end{aligned}$$

for each $v \in \mathbf{a}$. We have the following lemmas corresponding to this embedding.

Lemma 3.3. *For any $k \in \mathbb{Z}^{\mathbf{a}}$, take $f \in \mathcal{M}_k^{(l_1+\cdots+l_r)}$. Then $f \circ \varepsilon_{l_1,\ldots,l_r}$ can be expressed as*

$$(f \circ \varepsilon_{l_1,\ldots,l_r})(z_1,\ldots,z_r) = \sum_{i=1}^{t} \left(\prod_{j=1}^{r} f_{j,i}(z_j) \right), \tag{3.2}$$

with $f_{j,i} \in \mathcal{M}_k^{(l_j)}$. In case $f \in \mathcal{M}_k^{(l_1+\cdots+l_r)}(\overline{\mathbb{Q}})$, we can take $f_{j,i} \in \mathcal{M}_k^{(l_j)}(\overline{\mathbb{Q}})$. Moreover, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$(f^{\sigma} \circ \varepsilon_{l_1,\ldots,l_r})(z_1,\ldots,z_r) = \sum_{i=1}^{t} \left(\prod_{j=1}^{r} f_{j,i}^{\sigma}(z_j) \right).$$

Remark. The choice of $\{f_{j,i}\}$ is not unique.

Lemma 3.4. *For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, take $Y_j, \tilde{Y}_j \in \mathcal{Z}^{(l_j)}$ $(1 \le j \le r)$ and $Y, \tilde{Y} \in \mathcal{Z}^{(l_1+\cdots+l_r)}$ satisfying* (1)–(3).

(1) $\tilde{Y}_j = \begin{pmatrix} 1_{l_j} & 0 \\ 0 & \chi(\sigma)1_{l_j} \end{pmatrix}^{-1} Y_j \begin{pmatrix} 1_{l_j} & 0 \\ 0 & \chi(\sigma)1_{l_j} \end{pmatrix}$.

(2) $\tilde{Y} = \begin{pmatrix} 1_{l_1+\cdots+l_r} & 0 \\ 0 & \chi(\sigma)1_{l_1+\cdots+l_r} \end{pmatrix}^{-1} Y \begin{pmatrix} 1_{l_1+\cdots+l_r} & 0 \\ 0 & \chi(\sigma)1_{l_1+\cdots+l_r} \end{pmatrix}$.

(3) *There exists an r-tuple $\{\Gamma_j\}_{j=1}^{r}$ of congruence subgroups Γ_j $(1 \le j \le r)$ of $\mathrm{Sp}(l_j, F)$ which satisfies $\mathcal{O}_F^{\times}\Gamma_j \supset \mathcal{O}_F^{\times}\Gamma_{Y_j}^{(l_j)}$ for each $1 \le j \le r$ and $I_{l_1,\ldots,l_r}(\Gamma_1,\ldots,\Gamma_r) \subset \Gamma_Y^{(l_1+\cdots+l_r)} = Y \cap \mathrm{GSp}(l_1+\cdots+l_r, F)$.*

For each $1 \le j \le r$, take $z_j, \tilde{z}_j \in \mathfrak{H}_{l_j}^{\mathbf{a}}$ so that

$$\varphi_{\tilde{Y}_j}^{(l_j)}(\tilde{z}_j) = \left[J_{\tilde{Y}_j Y_j}^{(l_j)} \left(\begin{pmatrix} 1_{l_j} & 0 \\ 0 & \chi(\sigma)1_{l_j} \end{pmatrix}^{-1} \right) \left(\varphi_{Y_j}^{(l_j)}(z_j) \right) \right]^{\sigma}.$$

Then we have

$$\begin{aligned} &\varphi_{\tilde{Y}}^{(l_1+\cdots+l_r)}\left(\varepsilon_{l_1,\ldots,l_r}(\tilde{z}_1,\ldots,\tilde{z}_r)\right) \\ &\quad = \Bigg[J_{\tilde{Y}Y}^{(l_1+\cdots+l_r)} \left(\begin{pmatrix} 1_{l_1+\cdots+l_r} & 0 \\ 0 & \chi(\sigma)1_{l_1+\cdots+l_r} \end{pmatrix}^{-1} \right) \\ &\qquad\qquad \left(\varphi_Y^{(l_1+\cdots+l_r)}\left(\varepsilon_{l_1,\ldots,l_r}(z_1,\ldots,z_r)\right) \right) \Bigg]^{\sigma}. \end{aligned}$$

Proofs of Lemmas 3.3 *and* 3.4.

We only have to prove these lemmas when $r = 2$. First let us prove Lemma 3.3. For any $f \in \mathcal{M}_k^{(l_1+l_2)}(\Gamma)$ (with a congruence subgroup Γ of $\mathrm{Sp}(l_1+l_2, F)$), the function $f \circ \varepsilon_{l_1,l_2}(z_1,z_2)$ is a holomorphic function on $(z_1,z_2) \in \mathfrak{H}_{l_1}^{\mathbf{a}} \times \mathfrak{H}_{l_2}^{\mathbf{a}}$. Take congruence subgroups Γ_1, Γ_2 of $\mathrm{Sp}(l_1,F), \mathrm{Sp}(l_2,F)$ so that

$I_{l_1,l_2}(\Gamma_1,\Gamma_2)\subset\Gamma$. If we fix $z_2\in\mathfrak{H}^{\mathbf{a}}_{l_2}$, and view $f\circ\varepsilon_{l_1,l_2}(z_1,z_2)$ as a holomorphic function on $z_1\in\mathfrak{H}^{\mathbf{a}}_{l_1}$, it is contained in $\mathcal{M}_k^{(l_1)}(\Gamma_1)$. Hence we can write

$$f\circ\varepsilon_{l_1,l_2}(z_1,z_2)=\sum_{i=1}^{t}f_{1,i}(z_1)f_{2,i}(z_2), \tag{3.3}$$

where $\{f_{1,i}\}_{i=1}^t$ is a basis of $\mathcal{M}_k^{(l_1)}(\Gamma_1)$ (over $\mathbb{C}$) and each $f_{2,i}$ is a holomorphic function in $\mathfrak{H}^{\mathbf{a}}_{l_2}$. As is well known, we can take $\{f_{1,i}\}_{i=1}^t\subset\mathcal{M}_k^{(l_1)}(\Gamma_1,\overline{\mathbb{Q}})$. Note that each $f_{2,i}$ is uniquely determined if we fix $\{f_{1,i}\}_{i=1}^t$. Therefore we have $f_{2,i}\in\mathcal{M}_k^{(l_2)}(\Gamma_2)$ for each $1\le i\le t$.

Next let us consider the Fourier expansions of these modular forms. Put

$$f((z_v)_{v\in\mathbf{a}})=\sum_h c(f;h)\exp\left(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}}\mathrm{tr}(h_vz_v)\right),$$

$$f_{1,i}((z_{1,v})_{v\in\mathbf{a}})=\sum_{h_1} c(f_{1,i};h_1)\exp\left(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}}\mathrm{tr}(h_{1,v}z_{1,v})\right),$$

$$f_{2,i}((z_{2,v})_{v\in\mathbf{a}})=\sum_{h_2} c(f_{2,i};h_2)\exp\left(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}}\mathrm{tr}(h_{2,v}z_{2,v})\right),$$

for each $1\le i\le t$. Moreover, we can express $f\circ\varepsilon_{l_1,l_2}$ as

$$\begin{aligned}(f\circ\varepsilon_{l_1,l_2})&((z_{1,v})_{v\in\mathbf{a}},(z_{2,v})_{v\in\mathbf{a}})\\ &=\sum_{h_1,h_2}c(f;h_1,h_2)\exp\left(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}}(\mathrm{tr}(h_{1,v}z_{1,v})+\mathrm{tr}(h_{2,v}z_{2,v}))\right),\end{aligned}$$

where h_1 (resp. h_2) runs through a lattice in the space of symmetric matrices of degree l_1 (resp. l_2) with coefficients in F. Note that the constant $c(f;h_1,h_2)$ is uniquely determined by f,h_1 and h_2. We easily obtain

$$c(f;h_1,h_2)=\sum_{i=1}^{t}c(f_{1,i};h_1)c(f_{2,i};h_2). \tag{3.4}$$

Considering f on $\varepsilon_{l_1,l_2}(\mathfrak{H}^{\mathbf{a}}_{l_1}\times\mathfrak{H}^{\mathbf{a}}_{l_2})$, we can get

$$c(f;h_1,h_2)=\sum_h c(f;h) \tag{3.5}$$

where h runs through the matrices of the form $\begin{pmatrix} h_1 & h_3\\ {}^th_3 & h_2\end{pmatrix}$ with $h_3\in F^{l_1}_{l_2}$. Note that this is a finite sum. Hence we have

$$c(f^\sigma;h_1,h_2)=\sum_h c(f;h)^\sigma=c(f;h_1,h_2)^\sigma$$

for any $\sigma \in \mathrm{Aut}(\mathbb{C})$. Combining this with (3.4), we have

$$(f^\sigma \circ \varepsilon_{l_1,l_2})(z_1, z_2) = \sum_{i=1}^{t} f_{1,i}^\sigma(z_1) f_{2,i}^\sigma(z_2).$$

In case $f \in \mathcal{M}_k^{(l_1+l_2)}(\overline{\mathbb{Q}})$ and $\{f_{1,i}\}_{i=1}^t \subset \mathcal{M}_k^{(l_1)}(\overline{\mathbb{Q}})$, this equation implies $\{f_{2,i}\}_{i=1}^t \subset \mathcal{M}_k^{(l_2)}(\overline{\mathbb{Q}})$ since the choice of $\{f_{2,i}\}_{i=1}^t$ is unique for a fixed $\{f_{1,i}\}_{i=1}^t$.

Take $Y \in \mathcal{Z}^{(l_1+l_2)}$, $Y_1 \in \mathcal{Z}^{(l_1)}$, $Y_2 \in \mathcal{Z}^{(l_2)}$, and congruence subgroups Γ_1, Γ_2 of $\mathrm{Sp}(l_1, F)$, $\mathrm{Sp}(l_2, F)$ which satisfy $\mathcal{O}_F^\times \Gamma_1 \supset \mathcal{O}_F^\times \Gamma_{Y_1}^{(l_1)}$, $\mathcal{O}_F^\times \Gamma_2 \supset \mathcal{O}_F^\times \Gamma_{Y_2}^{(l_2)}$ and $I_{l_1,l_2}(\Gamma_1, \Gamma_2) \subset \Gamma_Y^{(l_1+l_2)}$, we then obtain a holomorphic map $E_{(Y_1,Y_2)}^Y : V_{Y_1}^{(l_1)} \times V_{Y_2}^{(l_2)} \to V_Y^{(l_1+l_2)}$ which makes the following diagram

$$\begin{tikzcd}
\mathfrak{H}_{l_1}^{\mathbf{a}} \times \mathfrak{H}_{l_2}^{\mathbf{a}} \arrow[r, "\varepsilon_{l_1,l_2}"] \arrow[d, "\varphi_{Y_1}^{(l_1)} \times \varphi_{Y_2}^{(l_2)}"] & \mathfrak{H}_{l_1+l_2}^{\mathbf{a}} \arrow[d, "\varphi_Y^{(l_1+l_2)}"] \\
V_{Y_1}^{(l_1)} \times V_{Y_2}^{(l_2)} \arrow[r, "E_{(Y_1,Y_2)}^Y"'] & V_Y^{(l_1+l_2)}
\end{tikzcd}$$

commutative. The equation (3.3) implies that $E_{(Y_1,Y_2)}^Y$ is a rational map.

Take $\tilde{Y}_1, \tilde{Y}_2$ and $\tilde{Y}$ as in (1) and (2) above. Then we can also consider the rational map $E_{(\tilde{Y}_1,\tilde{Y}_2)}^{\tilde{Y}} : V_{\tilde{Y}_1}^{(l_1)} \times V_{\tilde{Y}_2}^{(l_2)} \to V_{\tilde{Y}}^{(l_1+l_2)}$. Let us prove

(3.6)
$$\begin{aligned}
&(E_{(\tilde{Y}_1,\tilde{Y}_2)}^{\tilde{Y}})^{\sigma^{-1}} \circ \left[J_{\tilde{Y}_1 Y_1}^{(l_1)} \left(\begin{pmatrix} 1_{l_1} & 0 \\ 0 & \chi(\sigma) 1_{l_1} \end{pmatrix}^{-1} \right) \times J_{\tilde{Y}_2 Y_2}^{(l_2)} \left(\begin{pmatrix} 1_{l_2} & 0 \\ 0 & \chi(\sigma) 1_{l_2} \end{pmatrix}^{-1} \right) \right] \\
&\quad = J_{\tilde{Y} Y}^{(l_1+l_2)} \left(\begin{pmatrix} 1_{l_1+l_2} & 0 \\ 0 & \chi(\sigma) 1_{l_1+l_2} \end{pmatrix}^{-1} \right) \circ E_{(Y_1,Y_2)}^Y
\end{aligned}$$

as rational maps : $V_{Y_1}^{(l_1)} \times V_{Y_2}^{(l_2)} \to (V_{\tilde{Y}}^{(l_1+l_2)})^{\sigma^{-1}}$. Using Proposition 3.2, we can easily verify that (3.6) is true at $(\varphi_{Y_1}^{(l_1)}(\tau_1^\Psi), \varphi_{Y_2}^{(l_2)}(\tau_2^\Psi))$ for any $\tau_1 \in K_{l_1}^{l_1}$, $\tau_2 \in K_{l_2}^{l_2}$ and CM-type Ψ of K so that $\tau_1^\Psi \in \mathfrak{H}_{l_1}^{\mathbf{a}}$ and $\tau_2^\Psi \in \mathfrak{H}_{l_2}^{\mathbf{a}}$. Since the set of all such points is dense in $V_{Y_1}^{(l_1)} \times V_{Y_2}^{(l_2)}$, we can obtain (3.6), which is equivalent to Lemma 3.4. □

In Theorem 26.10 in [13], or essentially in [11], the action of a certain extended Galois group on arithmetic modular forms is constructed as follows.

Proposition 3.5. *Put*

$$\mathfrak{G}^{(l)} = \left\{ (x, \sigma) \in \mathcal{G}_+^{(l)} \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \,|\, \sigma(x) = \sigma|_{\mathbb{Q}_{ab}} \right\}.$$

Then any element (x, σ) of the group $\mathfrak{G}^{(l)}$ gives a ring-automorphism of the graded algebra $\sum_{k \in \mathbb{Z}^{\mathbf{a}}} \mathcal{A}_k^{(l)}(\overline{\mathbb{Q}})$, written $f \to f^{(x,\sigma)}$, which satisfies the following *properties.*

(1) $(f_1 f_2)^{(x,\sigma)} = f_1^{(x,\sigma)} f_2^{(x,\sigma)}$.

(2) $(f^{(x_1,\sigma_1)})^{(x_2,\sigma_2)} = f^{(x_1 x_2, \sigma_1\sigma_2)}$.

(3) $f^{(\alpha,1)} = f|_k\alpha$ *if* $\alpha \in \mathrm{GSp}(l,F)_+$ *and* $f \in \mathcal{A}_k^{(l)}(\overline{\mathbb{Q}})$.

(4) $f^{\left(\begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)1_l \end{pmatrix},\sigma\right)} = f^{\sigma}$.

(5) $\mathcal{A}_k^{(l)}(\overline{\mathbb{Q}})^{(x,\sigma)} = \mathcal{A}_{k^\sigma}^{(l)}(\overline{\mathbb{Q}})$ *and* $\mathcal{M}_k^{(l)}(\overline{\mathbb{Q}})^{(x,\sigma)} = \mathcal{M}_{k^\sigma}^{(l)}(\overline{\mathbb{Q}})$.

(6) *If* $f \in \mathcal{A}_0^{(l)}(\overline{\mathbb{Q}})$, *express* f *as* $f = \Phi_Y^{(l)}(f) \circ \varphi_Y^{(l)}$ *with* $\overline{\mathbb{Q}}$*-rational rational function* $\Phi_Y^{(l)}(f)$ *on* $V_Y^{(l)}$ *(where* $Y \in \mathcal{Z}^{(l)}$*). Then* $f^{(x,\sigma)} = \Phi_Y^{(l)}(f)^\sigma \circ J_{Y\tilde{Y}}^{(l)}(x) \circ \varphi_{\tilde{Y}}^{(l)}$ *with* $\tilde{Y} = x^{-1}Yx$.

This action is extended to vector-valued modular forms in 10.2 of [14].

4. The embeddings of canonical models

We defined an embedding $\varepsilon_\delta(T,\Psi)$ of $\mathfrak{D}(T,\Psi)$ into $\mathfrak{H}_m^{\mathbf{a}}$ in Section 2. We will consider the embeddings of canonical models corresponding to $\varepsilon_\delta(T,\Psi)$ precisely in this section. The content of this section is essentially the same as that of Section 4 of [15].

For $X \in \mathcal{Z}(T,\Psi)$ and $Y \in \mathcal{Z}^{(m)}$ which satisfy $I_\delta(T,\Psi)(X) \subset Y$, there exists a rational map $E_{YX} : V_X \to V_Y^{(m)}$ so that the diagram

$$\begin{array}{ccc} \mathfrak{D}(T,\Psi) & \xrightarrow{\ \varepsilon_\delta(T,\Psi)\ } & \mathfrak{H}_m^{\mathbf{a}} \\ {\scriptstyle \varphi_X}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_Y^{(m)}} \\ V_X & \xrightarrow[\ E_{YX}\ ]{} & V_Y^{(m)} \end{array}$$

is commutative. In case $K^\times\Gamma_X/K^\times$ is torsion free, the rational map E_{YX} is regular on whole V_X since φ_X is locally biholomorphic then.

For $\mathbf{0} \in \mathfrak{D}(T,\Psi)$ defined in section 1 and any $\alpha \in \mathrm{GU}(T,\Psi)_+$, consider $\varphi_X(\alpha(\mathbf{0})) \in V_X$. Then $\varphi_X(\alpha(\mathbf{0}))$ is $\overline{\mathbb{Q}}$-rational since the point $\alpha(\mathbf{0})$ is a so-called "isolated fixed point" in the sense of [5]. In the same way, $\varepsilon_\delta(T,\Psi)(\alpha(\mathbf{0})) = I_\delta(T,\Psi)(\alpha)(\varepsilon_\delta(T,\Psi)(\mathbf{0})) \in \mathfrak{H}_m^{\mathbf{a}}$ is a so-called CM-point. Hence the point $\varphi_Y^{(m)}(\varepsilon_\delta(T,\Psi)(\alpha(\mathbf{0}))) = E_{YX}(\varphi_X(\alpha(\mathbf{0}))) \in V_Y^{(m)}$ is also $\overline{\mathbb{Q}}$-rational. This implies that E_{YX} is defined over $\overline{\mathbb{Q}}$, since the set $\{\varphi_X(\alpha(\mathbf{0})) \,|\, \alpha \in \mathrm{GU}(T,\Psi)_+\}$ is dense in V_X.

The purpose of this section is to prove the following theorem, which is essentially same as Theorem 4.1 of [15]. This theorem will be used to construct a regular morphism between canonical models for different unitary groups in (5.7).

Theorem 4.1. *For any integral ideal* $\mathfrak{a}$ *of* $\mathcal{O}_K$ *such that* $\mathfrak{a}^\rho = \mathfrak{a}$*, take* $X = X(\mathfrak{a}) \in \mathcal{Z}(T,\Psi)$ *of the following form*

$$X(\mathfrak{a}) = \left\{ x \in \mathrm{GU}(T,\Psi)_{A+} \;\middle|\; \begin{array}{l} x_\mathfrak{p} \in \mathrm{GL}(m,\mathcal{O}_\mathfrak{p}) \text{ and} \\ x_\mathfrak{p} - 1_m \in (\mathfrak{a}\mathcal{O}_\mathfrak{p})_m^m \\ \text{for any finite prime } \mathfrak{p} \text{ of } K \end{array} \right\},$$

where $x_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-component of x. In the same way, take $Y(\mathfrak{a}) \in \mathcal{Z}^{(m)}$ as

$$Y(\mathfrak{a}) = \left\{ y \in \mathrm{GSp}(m,F)_{A+} \;\middle|\; \begin{array}{l} D(T,\delta)^{-1} y_{\mathfrak{p}} D(T,\delta) \in \mathrm{GL}(2m,\mathcal{O}_{\mathfrak{p}}) \textit{ and} \\ D(T,\delta)^{-1} y_{\mathfrak{p}} D(T,\delta) - 1_{2m} \in (\mathfrak{a}\mathcal{O}_{\mathfrak{p}})_{2m}^{2m} \\ \textit{for any finite prime } \mathfrak{p} \textit{ of } K \end{array} \right\},$$

where $D(T,\delta) = C(T,\delta) \begin{pmatrix} 1_m & T \\ 1_m & T^{\rho} \end{pmatrix}^{-1}$ and $y_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-component of y. (Then $I_\delta(T,\Psi)^{-1}(Y(\mathfrak{a})) = X(\mathfrak{a})$.) Assume that $K^\times \Gamma_{X(\mathfrak{a})}/K^\times$ and $F^\times \Gamma^{(m)}_{Y(\mathfrak{a})}/F^\times$ are torsion free. Then there exists $\hat{Y} \in \mathcal{Z}^{(m)}$ satisfying the following properties (1)–(3).
(1) *$\hat{Y} \subset Y(\mathfrak{a})$ and $I_\delta(T,\Psi)^{-1}(\hat{Y}) = X(\mathfrak{a})$.*
(2) *$E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})}) = \overline{E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})}$ and it is a non-singular subvariety of $V^{(m)}_{\hat{Y}}$ (where the bar means the Zariski closure in $V^{(m)}_{\hat{Y}}$).*
(3) *$E_{\hat{Y}X(\mathfrak{a})}$ is a (set theorically) injective map on $V_{X(\mathfrak{a})}$, regular on whole $V_{X(\mathfrak{a})}$, and its inverse rational map $E^{-1}_{\hat{Y}X(\mathfrak{a})} : E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})}) \to V_{X(\mathfrak{a})}$ is regular on whole $E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})$.*

Remark 1. Any $X' \in \mathcal{Z}(T,\Psi)$ contains some $X(\mathfrak{a})$.

Remark 2. Any $\hat{Y}' \in \mathcal{Z}^{(m)}$ such that $I_\delta(T,\Psi)(X(\mathfrak{a})) \subset \hat{Y}' \subset \hat{Y}$, also has the properties (1)–(3).

Proof. In this proof, any varieties and rational maps are defined over $\overline{\mathbb{Q}}$. The word "generic" means generic over $\overline{\mathbb{Q}}$. As is well known, any algebraic set defined over $\overline{\mathbb{Q}}$ is a union of finitely many varieties defined over $\overline{\mathbb{Q}}$.

First of all, from the definition of $I_\delta = I_\delta(T,\Psi)$, we easily have

$$I_\delta(T,\Psi)(x) = D(T,\delta) \begin{pmatrix} x^{\rho} & 0 \\ 0 & x \end{pmatrix} D(T,\delta)^{-1} \qquad (x \in \mathrm{GU}(T,\Psi)_A).$$

For any integral ideal $\mathfrak{b}$ of $\mathcal{O}_K$ such that $\mathfrak{b} \subset \mathfrak{a}$ and $\mathfrak{b}^{\rho} = \mathfrak{b}$, take $Y(\mathfrak{a},\mathfrak{b}) \in \mathcal{Z}^{(m)}$ as

$$Y(\mathfrak{a},\mathfrak{b}) = \left\{ y \in Y(\mathfrak{a}) \;\middle|\; \begin{array}{l} D(T,\delta)^{-1} y_{\mathfrak{p}} D(T,\delta) - \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \in (\mathfrak{b}\mathcal{O}_{\mathfrak{p}})_{2m}^{2m} \\ \text{with some } b_1, b_2 \in (\mathcal{O}_{\mathfrak{p}})_m^m \\ \text{for any finite prime } \mathfrak{p} \text{ of } K \end{array} \right\}.$$

Clearly we have $Y(\mathfrak{a},\mathfrak{b}) \subset Y(\mathfrak{a})$ and $I_\delta(T,\Psi)^{-1}(Y(\mathfrak{a},\mathfrak{b})) = X(\mathfrak{a})$. Moreover, we can obtain

$$I_\delta(T,\Psi)(X(\mathfrak{a})) = \bigcap_{\mathfrak{b}} Y(\mathfrak{a},\mathfrak{b}), \tag{4.1}$$

$$I_\delta(T,\Psi)(\Gamma_{X(\mathfrak{a})}) = \bigcap_{\mathfrak{b}} \Gamma^{(m)}_{Y(\mathfrak{a},\mathfrak{b})}, \tag{4.2}$$

where $\mathfrak{b}$ runs through all integral ideals of $\mathcal{O}_K$ such that $\mathfrak{b}^\rho = \mathfrak{b}$ and $\mathfrak{b} \subset \mathfrak{a}$. For each $P \in V^{(m)}_{Y(\mathfrak{a},\mathfrak{b})}$, the set $E^{-1}_{Y(\mathfrak{a},\mathfrak{b})X(\mathfrak{a})}(P) = \varphi_{X(\mathfrak{a})} \circ \left(\varepsilon_\delta^{-1}((\varphi^{(m)}_{Y(\mathfrak{a},\mathfrak{b})})^{-1}(P))\right)$ is at most countable, and clearly an algebraic set in $V_{X(\mathfrak{a})}$. Hence $E^{-1}_{Y(\mathfrak{a},\mathfrak{b})X(\mathfrak{a})}(P)$ is a finite set.

Assume $Q_1, Q_2 \in V_{X(\mathfrak{a})}$, $Q_1 \neq Q_2$ and $E_{Y(\mathfrak{a},\mathfrak{b})X(\mathfrak{a})}(Q_1) = E_{Y(\mathfrak{a},\mathfrak{b})X(\mathfrak{a})}(Q_2)$. Take $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{D}(T, \Psi)$ so that $\varphi_{X(\mathfrak{a})}(\mathfrak{z}_1) = Q_1$, $\varphi_{X(\mathfrak{a})}(\mathfrak{z}_2) = Q_2$ and fix them. Then we have $\Gamma_{X(\mathfrak{a})}(\mathfrak{z}_1) \cap \Gamma_{X(\mathfrak{a})}(\mathfrak{z}_2) = \phi$ and $\Gamma^{(m)}_{Y(\mathfrak{a},\mathfrak{b})}(\varepsilon_\delta(\mathfrak{z}_1)) = \Gamma^{(m)}_{Y(\mathfrak{a},\mathfrak{b})}(\varepsilon_\delta(\mathfrak{z}_2))$. Since $F^\times \Gamma^{(m)}_{Y(\mathfrak{a},\mathfrak{b})}/F^\times$ is torsion free, any (non-scalar) element of $\Gamma^{(m)}_{Y(\mathfrak{a},\mathfrak{b})}$ has no fixed points in $\mathfrak{H}^{\mathbf{a}}_m$. Hence there exists unique $\gamma \in \Gamma^{(m)}_{Y(\mathfrak{a},\mathfrak{b})}$ which satisfies $\gamma(\varepsilon_\delta(\mathfrak{z}_1)) = \varepsilon_\delta(\mathfrak{z}_2)$ up to scalar multiples. Clearly we have $\gamma \notin I_\delta(\Gamma_{X(\mathfrak{a})})$. Because of (4.2), we can choose an integral ideal $\mathfrak{b}' \subset \mathfrak{b}$ so that $\mathfrak{b}'^\rho = \mathfrak{b}'$ and $\gamma \notin \Gamma^{(m)}_{Y(\mathfrak{a},\mathfrak{b}')}$. Then we have $F^\times\gamma \cap F^\times\Gamma^{(m)}_{Y(\mathfrak{a},\mathfrak{b}')} = \phi$ since any scalar element in $\Gamma^{(m)}_{Y(\mathfrak{a})}$ is contained in $I_\delta(\Gamma_{X(\mathfrak{a})})$ $(\subset \Gamma^{(m)}_{Y(\mathfrak{a},\mathfrak{b}')})$. This implies $\Gamma^{(m)}_{Y(\mathfrak{a},\mathfrak{b}')}(\varepsilon_\delta(\mathfrak{z}_1)) \cap \Gamma^{(m)}_{Y(\mathfrak{a},\mathfrak{b}')}(\varepsilon_\delta(\mathfrak{z}_2)) = \phi$ and hence we obtain $E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(Q_1) \neq E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(Q_2)$. Using this method repeatedly, we can obtain the following lemma, since $E^{-1}_{Y(\mathfrak{a},\mathfrak{b})X(\mathfrak{a})}(P)$ is finite for each $P \in V^{(m)}_{Y(\mathfrak{a},\mathfrak{b})}$.

Lemma 4.2. *For any $Q \in V_{X(\mathfrak{a})}$, there exists some integral ideal $\mathfrak{b}$ of $\mathcal{O}_K$ (such that $\mathfrak{b} \subset \mathfrak{a}$ and $\mathfrak{b}^\rho = \mathfrak{b}$) which satisfies $E^{-1}_{Y(\mathfrak{a},\mathfrak{b})X(\mathfrak{a})}(E_{Y(\mathfrak{a},\mathfrak{b})X(\mathfrak{a})}(Q)) = \{Q\}$.*

For $X(\mathfrak{a}) \in \mathcal{Z}(T, \Psi)$ and $Y(\mathfrak{a},\mathfrak{b}) \in \mathcal{Z}^{(m)}$ as above, put

$$W(X(\mathfrak{a}), Y(\mathfrak{a},\mathfrak{b})) = \overline{\left\{ Q \in V_{X(\mathfrak{a})} \;\middle|\; \begin{array}{l} \text{There exists } Q' \in V_{X(\mathfrak{a})} \\ \text{such that } Q' \neq Q \text{ and} \\ E_{Y(\mathfrak{a},\mathfrak{b})X(\mathfrak{a})}(Q) = E_{Y(\mathfrak{a},\mathfrak{b})X(\mathfrak{a})}(Q') \end{array} \right\}},$$

where the bar means the ($\overline{\mathbb{Q}}$-)Zariski closure in $V_{X(\mathfrak{a})}$. Set

$$W(X(\mathfrak{a}), Y(\mathfrak{a},\mathfrak{b})) = \bigcup_{j=1}^{r} W_j,$$

where W_j $(1 \leq j \leq r)$ are subvarieties of $V_{X(\mathfrak{a})}$ defined over $\overline{\mathbb{Q}}$, and none of them are contained in the other. We can assume that

$$\dim W(X(\mathfrak{a}), Y(\mathfrak{a},\mathfrak{b})) = \dim W_1 = \cdots = \dim W_s > \dim W_{s+1} \geq \cdots \geq \dim W_r,$$

with $1 \leq s \leq r$. For $1 \leq j \leq s$, let Q_j be generic points of W_j (over $\overline{\mathbb{Q}}$).

Using Lemma 4.2 (s times repeatedly), we can find an integral ideal $\mathfrak{b}' \subset \mathfrak{b}$ so that ($\mathfrak{b}'^\rho = \mathfrak{b}'$ and) $E^{-1}_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(Q_j)) = \{Q_j\}$ $(1 \leq j \leq s)$ hold. Then the Zariski closure $\overline{E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(W_j)}$ of $E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(W_j)$ $(1 \leq j \leq s)$ are subvarieties of $V^{(m)}_{Y(\mathfrak{a},\mathfrak{b}')}$ whose generic points (over $\overline{\mathbb{Q}}$) are $E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(Q_j)$. Note that, for $1 \leq j \leq s$, $\dim \overline{E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(W_j)} = \dim(W_j)$, and $\overline{E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(W_1)}$,

$\ldots, \overline{E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(W_s)}$ are all different varieties. Since each $E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}|_{W_j}$ $(1 \le j \le s)$ is generically injective, we can define the inverse rational map $(E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}|_{W_j})^{-1} : \overline{E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(W_j)} \to W_j$ (for each $1 \le j \le s$), which is regular on some non-empty ($\overline{\mathbb{Q}}$-)Zariski open subset C_j of $\overline{E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(W_j)}$. (This does not mean $E^{-1}_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}$ is regular on C_j in general.) For each $1 \le j \le s$, define the subset U_j of W_j as

$$U_j = W_j \cap E^{-1}_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}\left[C_j - \bigcup_{\substack{1 \le i \le r \\ i \ne j}} \left(\overline{E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(W_i)} \cap \overline{E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(W_j)}\right)\right].$$

The dimension of each $\overline{E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(W_i)} \cap \overline{E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(W_j)}$ is smaller than $\dim(W_j)$, since we have $\overline{E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(W_i)} \ne \overline{E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(W_j)}$ if $1 \le i \le s$ and $\dim(W_i) < \dim(W_j)$ if $s+1 \le i \le r$. Therefore the set in the square blacket is a non-empty ($\overline{\mathbb{Q}}$-)Zariski open subset of $\overline{E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(W_j)}$ and this implies that U_j is a non-empty ($\overline{\mathbb{Q}}$-)Zariski open subset of W_j. Note that, for any $Q \in U_j$, we have $E^{-1}_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(Q)) = \{Q\}$.

In the same way as the definition of $W(X(\mathfrak{a}), Y(\mathfrak{a},\mathfrak{b}))$, we take

$$W(X(\mathfrak{a}), Y(\mathfrak{a},\mathfrak{b}')) = \overline{\left\{ Q \in V_{X(\mathfrak{a})} \;\middle|\; \begin{array}{l} \text{There exists } Q' \in V_{X(\mathfrak{a})} \\ \text{such that } Q' \ne Q \text{ and} \\ E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(Q) = E_{Y(\mathfrak{a},\mathfrak{b}')X(\mathfrak{a})}(Q') \end{array} \right\}}.$$

Then clearly we have $W(X(\mathfrak{a}), Y(\mathfrak{a},\mathfrak{b}')) \subset W(X(\mathfrak{a}), Y(\mathfrak{a},\mathfrak{b})) = \bigcup_{j=1}^{r} W_j$. Since each $(W_j - U_j)$ is a ($\overline{\mathbb{Q}}$-)Zariski closed set in $V_{X(\mathfrak{a})}$, we obtain $W(X(\mathfrak{a}), Y(\mathfrak{a},\mathfrak{b}')) \subset \left(\bigcup_{j=1}^{s}(W_j - U_j)\right) \cup \left(\bigcup_{s+1 \le j \le r} W_j\right)$. Hence we have $\dim W(X(\mathfrak{a}), Y(\mathfrak{a},\mathfrak{b}')) < \dim W(X(\mathfrak{a}), Y(\mathfrak{a},\mathfrak{b}))$.

By an induction, we can choose $\hat{Y} = Y(\mathfrak{a}, \hat{\mathfrak{b}})$ with some integral ideal $\hat{\mathfrak{b}}$ $(\subset \mathfrak{a})$ in such a way that $E_{\hat{Y}X(\mathfrak{a})}$ is set theorically an injective map from $V_{X(\mathfrak{a})}$ to $V^{(m)}_{\hat{Y}}$.

Since $E_{\hat{Y}X(\mathfrak{a})}$ is injective, we can define the inverse rational map $E^{-1}_{\hat{Y}X(\mathfrak{a})}$ on $\overline{E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})}$, which is regular on some non-empty Zariski open subset of $\overline{E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})}$. Hence $E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})$ contains some non-empty Zariski open subset of $\overline{E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})}$. This implies that $E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})$ is dense in $\overline{E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})}$ with respect to the topology of $V^{(m)}_{\hat{Y}}$ as a complex manifold, since $\overline{E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})}$ is a variety.

Obviously we can get

$$(\varphi^{(m)}_{\hat{Y}})^{-1}\left(E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})\right) = \bigcup_{\gamma \in \Gamma^{(m)}_{\hat{Y}}} \gamma \circ \varepsilon_\delta(\mathfrak{D}).$$

Each $\gamma \circ \varepsilon_\delta(\mathfrak{D})$ is an analytic set in $\mathfrak{H}_m^{\mathbf{a}}$ whose dimension is equal to that of $\mathfrak{D}$. For $\gamma_1, \gamma_2 \in \Gamma_{\hat{Y}}^{(m)}$, if $\gamma_1 I_\delta(\Gamma_{X(\mathfrak{a})}) \cap \gamma_2 I_\delta(\Gamma_{X(\mathfrak{a})}) = \phi$, then $\gamma_1 \circ \varepsilon_\delta(\mathfrak{D}) \cap \gamma_2 \circ \varepsilon_\delta(\mathfrak{D}) = \phi$ holds (since the map $E_{\hat{Y}X(\mathfrak{a})}$ is injective and any non-scalar element of $\Gamma_{\hat{Y}}^{(m)}$ has no fixed points in $\mathfrak{H}_m^{\mathbf{a}}$.)

Now we have

$$(\varphi_{\hat{Y}}^{(m)})^{-1}\left(\overline{E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})}\right) = \overline{\bigcup_{\gamma \in \Gamma_{\hat{Y}}^{(m)}} \gamma \circ \varepsilon_\delta(\mathfrak{D})},$$

since $\varphi_{\hat{Y}}^{(m)}$ is locally biholomorphic. (The bar in the right hand side means the closure with respect to the topology of $\mathfrak{H}_m^{\mathbf{a}}$ as a complex manifold and that in the left hand side denotes the Zariski closure in $V_{\hat{Y}}^{(m)}$.) Clearly each $\gamma \circ \varepsilon_\delta(\mathfrak{D})$ is an analytic set in $\mathfrak{H}_m^{\mathbf{a}}$. Moreover, the set $\overline{\bigcup_{\gamma \in \Gamma_{\hat{Y}}^{(m)}} \gamma \circ \varepsilon_\delta(\mathfrak{D})}$ is also an analytic set in $\mathfrak{H}_m^{\mathbf{a}}$ of same dimension. Now we can prove that there exists no limit point of infinitely different $\gamma \circ \varepsilon_\delta(\mathfrak{D})$. If such a point exists, it must be contained in infinitely different $\gamma \circ \varepsilon_\delta(\mathfrak{D})$ by using an analytic continuation, and this contradicts the disjointness of different $\gamma \circ \varepsilon_\delta(\mathfrak{D})$ (stated above). Since each $\gamma \circ \varepsilon_\delta(\mathfrak{D})$ is closed in $\mathfrak{H}_m^{\mathbf{a}}$ and there is no limit point of infinitely different $\gamma \circ \varepsilon_\delta(\mathfrak{D})$, we have

$$\bigcup_{\gamma \in \Gamma_{\hat{Y}}^{(m)}} \gamma \circ \varepsilon_\delta(\mathfrak{D}) = \overline{\bigcup_{\gamma \in \Gamma_{\hat{Y}}^{(m)}} \gamma \circ \varepsilon_\delta(\mathfrak{D})}.$$

This implies $E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})}) = \overline{E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})}$. Moreover, at each $z \in \gamma_0 \circ \varepsilon_\delta(\mathfrak{D})$ (with $\gamma_0 \in \Gamma_{\hat{Y}}^{(m)}$), we can take an open neighborhood U_z of z in $\mathfrak{H}_m^{\mathbf{a}}$ which satisfies $U_z \cap \gamma \circ \varepsilon_\delta(\mathfrak{D}) = \phi$ for any $\gamma \in \Gamma_{\hat{Y}}^{(m)}$ so that $\gamma I_\delta(\Gamma_{X(\mathfrak{a})}) \neq \gamma_0 I_\delta(\Gamma_{X(\mathfrak{a})})$. This implies $E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})$ is a non-singular subvariety of $V_{\hat{Y}}^{(m)}$, since $\varphi_{\hat{Y}}^{(m)}$ is locally biholomorphic and the Jacobian of ε_δ is non-zero at each $\mathfrak{z} \in \mathfrak{D}$.

From the definition, it is clear that $E_{\hat{Y}X(\mathfrak{a})} = \varphi_{\hat{Y}}^{(m)} \circ \varepsilon_\delta \circ \varphi_{X(\mathfrak{a})}^{-1}$, and the Jacobian of the right hand side is non-zero at each $Q \in V_{X(\mathfrak{a})}$. Hence $E_{\hat{Y}X(\mathfrak{a})}$ is a biholomorphic map from $V_{X(\mathfrak{a})}$ to $E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})$. This implies $E_{\hat{Y}X(\mathfrak{a})}^{-1}$ is regular on whole $E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})$, since the variety $E_{\hat{Y}X(\mathfrak{a})}(V_{X(\mathfrak{a})})$ is non-singular. □

5. The construction of a Galois action

In this section we will construct a certain Galois action on holomorphic modular forms with respect to arbitrary unitary groups. This is a generalization of the main theorem of [15].

Let K, F be as above and $T \in K_m^m$ be a "normal" skew-hermitian matrix with respect to a CM-type $\Psi = (\Psi_v)_{v \in \mathbf{a}}$ as in (1.1). We define CM-types

$\Psi(T,1), \Psi(T,2), \ldots, \Psi(T, m-2q)$ as in Section 1. Set

$$C_{(T,\Psi)}(\mathbb{C}) = \left\{ (\sigma; T, \Psi; \underline{a}) \;\middle|\; \begin{array}{l} \sigma \in \mathrm{Aut}(\mathbb{C}), \\ \underline{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-2q} \end{pmatrix} \in (K_{\mathbf{h}}^{\times})^{m-2q+1}, \\ \text{where } a_0 \in g_{\Psi}(\sigma), \\ \text{and } a_j \in g_{\Psi(T,j)}(\sigma) \text{ for } 1 \leq j \leq m-2q \end{array} \right\},$$

where $K_{\mathbf{h}}^{\times}$ denotes the non-archimedean component of the idele group $K_A^{\times}$. Note that, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, there exists some $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$. If $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$, we can easily see that $(\rho\sigma\rho; T, \Psi; \underline{a}^{\rho}) \in C_{(T,\Psi)}(\mathbb{C})$, where $\underline{a}^{\rho} = \begin{pmatrix} a_0^{\rho} \\ a_1^{\rho} \\ \vdots \\ a_{m-2q}^{\rho} \end{pmatrix}$.

For any $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$, take $B(\sigma; T, \Psi; \underline{a}) \in \mathrm{GL}(m, K_{\mathbf{h}})$ as

$$B(\sigma; T, \Psi; \underline{a}) = \begin{pmatrix} (\frac{1}{2} + \frac{a_0 a_0^{\rho}}{2})1_q & & & & (\frac{1}{2} - \frac{a_0 a_0^{\rho}}{2})1_q \\ & a_1^{\rho} & & & \\ & & \ddots & & \\ & & & a_{m-2q}^{\rho} & \\ (\frac{1}{2} - \frac{a_0 a_0^{\rho}}{2})1_q & & & & (\frac{1}{2} + \frac{a_0 a_0^{\rho}}{2})1_q \end{pmatrix}.$$

The following theorem is the main theorem of this paper.

Theorem 5.1. *Let T be a "normal" skew-hermitian matrix with respect to a CM-type Ψ, which is expressed as in* (1.1). *For any $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$, take $\tilde{T} \in K_m^m$ as*

$$\tilde{T} = \begin{pmatrix} \iota(\sigma, a_0)\tau \cdot 1_q & & & & \\ & \iota(\sigma, a_1)t_1 & & & \\ & & \ddots & & \\ & & & \iota(\sigma, a_{m-2q})t_{m-2q} & \\ & & & & \iota(\sigma, a_0)\tau^{\rho} \cdot 1_q \end{pmatrix}.$$

Then $\tilde{T}$ is a "normal" skew-hermitian matrix with respect to the CM-type $\Psi\sigma$. Given any $f \in \mathcal{M}_k(T, \Psi)$, take an open compact subgroup $D_{\mathbf{h}}^1$ of $\mathrm{U}_1(T,\Psi)_{\mathbf{h}}$ so that $f \in \mathcal{M}_k(T,\Psi)\left((\mathrm{U}_1(T,\Psi)_{\mathbf{a}} \times D_{\mathbf{h}}^1) \cap \mathrm{U}_1(T,\Psi)\right)$. Then there exists $f^{(\sigma;T,\Psi;\underline{a})} \in \mathcal{M}_{k^{\sigma}}(\tilde{T}, \Psi\sigma)$ which satisfies the following property.

(i) *In case $q > 0$, for any $\tilde{\alpha} \in \mathrm{U}(\tilde{T}, \Psi\sigma)$, we have*

$$(f^{(\sigma;T,\Psi;\underline{a})}|_{k^{\sigma}}\tilde{\alpha})|\varepsilon_0(\tilde{T}, \Psi\sigma) = ((f|_k\alpha)|\varepsilon_0(T,\Psi))^{\sigma}. \tag{5.1}$$

Here $\alpha \in \mathrm{U}(T, \Psi)$ *is an element such that*

$$\alpha_{\mathbf{h}} \in D^1_{\mathbf{h}} \cdot B(\sigma; T, \Psi; \underline{a}) \tilde{\alpha}_{\mathbf{h}} B(\sigma; T, \Psi; \underline{a})^{-1}. \tag{5.2}$$

The action of σ *in the right hand side of* (5.1) *is as defined in* (2.2).
(ii) *In case* $q = 0$, *for any* $\tilde{\alpha} \in \mathrm{U}(\tilde{T}, \Psi\sigma)$, *we have*

$$(f^{(\sigma; T, \Psi; \underline{a})}|_{k^\sigma} \tilde{\alpha})(\mathbf{0}) = \{(f|_k \alpha)(\mathbf{0})\}^\sigma ,$$

where α *is as in* (5.2).

Remark 1. Using Lemma 3.1, we can easily prove that $\tilde{T}$ is "normal" with respect to $\Psi\sigma$. Moreover, the dimension of a maximal isotropic subspace with respect to $\tilde{T}$ is also q, the signature of $-\sqrt{-1} \cdot \tilde{T}^{\Psi_v \sigma}$ is (r_v, s_v) for each $v \in \mathbf{a}$, and we obtain $\Psi(\tilde{T}, j) = \Psi(T, j)\sigma$ for $1 \leq j \leq m - 2q$.

Remark 2. The map $\tilde{x}_{\mathbf{h}} \to B(\sigma; T, \Psi; \underline{a}) \tilde{x}_{\mathbf{h}} B(\sigma; T, \Psi; \underline{a})^{-1}$ gives an isomorphism of $\mathrm{U}(\tilde{T}, \Psi\sigma)_{\mathbf{h}}$ onto $\mathrm{U}(T, \Psi)_{\mathbf{h}}$. It is because we have

$$B(\sigma; T, \Psi; \underline{a}) \tilde{T}_{\mathbf{h}} {}^t B(\sigma; T, \Psi; \underline{a})^\rho = \chi(\sigma) T_{\mathbf{h}},$$

where $\tilde{T}_{\mathbf{h}}$ and $T_{\mathbf{h}}$ denote the non-archimedean components of $\tilde{T}$ and T.

Remark 3. Clearly the modular form $f^{(\sigma; T, \Psi; \underline{a})}$ is uniquely determined, since the set $\bigcup_{\tilde{\alpha} \in \mathrm{U}(\tilde{T}, \Psi\sigma)} \tilde{\alpha} \circ \varepsilon_0(\mathfrak{H}^{\mathbf{a}}_q)$ (or $\left\{ \tilde{\alpha}(\mathbf{0}) \,\middle|\, \tilde{\alpha} \in \mathrm{U}(\tilde{T}, \Psi\sigma) \right\}$ if $q = 0$) is dense in $\mathfrak{D}(\tilde{T}, \Psi\sigma)$.

Remark 4. For any $\tilde{\alpha} \in \mathrm{U}(\tilde{T}, \Psi\sigma)$, there exists $\alpha \in \mathrm{U}(T, \Psi)$ which satisfies (5.2). Indeed, this follows from the strong approximation property of $\mathrm{U}_1(T, \Psi)$, since we have $\begin{pmatrix} \det(\tilde{\alpha}) & \\ & 1_{m-1} \end{pmatrix} \in \mathrm{U}(T, \Psi)$ and

$$B(\sigma; T, \Psi; \underline{a}) \tilde{\alpha}_{\mathbf{h}} B(\sigma; T, \Psi; \underline{a})^{-1} \begin{pmatrix} \det(\tilde{\alpha}) & \\ & 1_{m-1} \end{pmatrix}^{-1} \in \mathrm{U}_1(T, \Psi)_A.$$

Before proving this theorem, we need some preparation. First, for $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$, take $A(\sigma; T, \Psi; \underline{a}) \in \mathrm{GSp}(m, F)_{A+}$ and $C(\sigma; T, \Psi; \underline{a}) \in \mathrm{GSp}(m - 2q, F)_{A+}$ as

$$A(\sigma; T, \Psi; \underline{a}) = \begin{pmatrix} 1_q & & & & & \\ & C_1 & & & C_2 & \\ & & 1_q & & & \\ & & & \chi(\sigma) \cdot 1_q & & \\ & C_3 & & & C_4 & \\ & & & & & \chi(\sigma) \cdot 1_q \end{pmatrix},$$

$$C(\sigma; T, \Psi; \underline{a}) = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix},$$

where

$$C_1 = \begin{pmatrix} \frac{a_1+a_1^\rho}{2} & & & \\ & \frac{a_2+a_2^\rho}{2} & & \\ & & \ddots & \\ & & & \frac{a_{m-2q}+a_{m-2q}^\rho}{2} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} \frac{t_1\chi(\sigma)(a_1-a_1^\rho)}{2a_1a_1^\rho} & & & \\ & \frac{t_2\chi(\sigma)(a_2-a_2^\rho)}{2a_2a_2^\rho} & & \\ & & \ddots & \\ & & & \frac{t_{m-2q}\chi(\sigma)(a_{m-2q}-a_{m-2q}^\rho)}{2a_{m-2q}a_{m-2q}^\rho} \end{pmatrix},$$

$$C_3 = \begin{pmatrix} \frac{t_1^{-1}(a_1-a_1^\rho)}{2} & & & \\ & \frac{t_2^{-1}(a_2-a_2^\rho)}{2} & & \\ & & \ddots & \\ & & & \frac{t_{m-2q}^{-1}(a_{m-2q}-a_{m-2q}^\rho)}{2} \end{pmatrix},$$

$$C_4 = \begin{pmatrix} \frac{\chi(\sigma)(a_1+a_1^\rho)}{2a_1a_1^\rho} & & & \\ & \frac{\chi(\sigma)(a_2+a_2^\rho)}{2a_2a_2^\rho} & & \\ & & \ddots & \\ & & & \frac{\chi(\sigma)(a_{m-2q}+a_{m-2q}^\rho)}{2a_{m-2q}a_{m-2q}^\rho} \end{pmatrix}.$$

Then we have $\nu(A(\sigma;T,\Psi;\underline{a})) = \nu(C(\sigma;T,\Psi;\underline{a})) = \chi(\sigma)$ and hence $A(\sigma;T,\Psi;\underline{a}) \in \mathcal{G}_+^{(m)}$, $C(\sigma;T,\Psi;\underline{a}) \in \mathcal{G}_+^{(m-2q)}$. By a computation, we obtain

$$\begin{aligned} I_\delta(T,\Psi)\left(B(\sigma;T,\Psi;\underline{a})\tilde{x}_{\mathbf{h}}B(\sigma;T,\Psi;\underline{a})^{-1}\right) \\ = A(\sigma;T,\Psi;\underline{a})I_\delta(\tilde{T},\Psi\sigma)(\tilde{x}_{\mathbf{h}})A(\sigma;T,\Psi;\underline{a})^{-1}, \end{aligned} \tag{5.3}$$

for any $\tilde{x}_{\mathbf{h}} \in \mathrm{U}(\tilde{T},\Psi\sigma)_{\mathbf{h}}$. It is obvious that

(5.4)

$$A(\sigma;T,\Psi;\underline{a}) = \begin{pmatrix} 1_m & 0 \\ 0 & \chi(\sigma)1_m \end{pmatrix} I_{q,m-2q,q}\left(1_{2q}, \begin{pmatrix} 1_{m-2q} & 0 \\ 0 & \chi(\sigma)1_{m-2q} \end{pmatrix}^{-1} C(\sigma;T,\Psi;\underline{a}), 1_{2q}\right).$$

We have the following lemma.

Lemma 5.2. (i) *In case $q > 0$, take $\tilde{Y}, Y \in \mathcal{Z}^{(m)}$, $\tilde{U}, U \in \mathcal{Z}^{(q)}$ and a congruence subgroup $\tilde{\Gamma}^{(q)}$ of $\mathrm{Sp}(q,F)$ so that*

$$\begin{aligned} Y &= A(\sigma;T,\Psi;\underline{a})\tilde{Y}A(\sigma;T,\Psi;\underline{a})^{-1}, \\ U &= \begin{pmatrix} 1_q & 0 \\ 0 & \chi(\sigma)1_q \end{pmatrix} \tilde{U} \begin{pmatrix} 1_q & 0 \\ 0 & \chi(\sigma)1_q \end{pmatrix}^{-1}, \end{aligned} \tag{5.5}$$

and

$$\mathcal{O}_F^\times \tilde{\Gamma}^{(q)} \supset \Gamma_{\tilde{U}}^{(q)} = \tilde{U} \cap \mathrm{GSp}(q,F),$$
$$I_{q,m-2q,q}\left(\tilde{\Gamma}^{(q)}, 1_{2m-4q}, \begin{pmatrix} 1_q & 0 \\ 0 & -\delta^2 1_q \end{pmatrix}^{-1} \tilde{\Gamma}^{(q)} \begin{pmatrix} 1_q & 0 \\ 0 & -\delta^2 1_q \end{pmatrix}\right)$$
$$\subset \Gamma_{\tilde{Y}}^{(m)} = \tilde{Y} \cap \mathrm{GSp}(m,F).$$

For $z, \tilde{z} \in \mathfrak{H}_q^{\mathbf{a}}$ *which satisfy*

$$\varphi_{\tilde{U}}^{(q)}(\tilde{z}) = \left[J_{\tilde{U}U}^{(q)}\left(\begin{pmatrix} 1_q & 0 \\ 0 & \chi(\sigma) 1_q \end{pmatrix}^{-1}\right)\left(\varphi_U^{(q)}(z)\right)\right]^\sigma,$$

we have

$$J_{Y\tilde{Y}}^{(m)}(A(\sigma;T,\Psi;\underline{a}))\left(\varphi_{\tilde{Y}}^{(m)}\left(\varepsilon_\delta(\tilde{T},\Psi\sigma)\circ\varepsilon_0(\tilde{T},\Psi\sigma)(\tilde{z})\right)\right)$$
$$= \varphi_Y^{(m)}\left(\varepsilon_\delta(T,\Psi)\circ\varepsilon_0(T,\Psi)(z)\right)^\sigma.$$

(ii) *In case* $q=0$, *we have*

$$J_{Y\tilde{Y}}^{(m)}(A(\sigma;T,\Psi;\underline{a}))\left(\varphi_{\tilde{Y}}^{(m)}\left(\varepsilon_\delta(\tilde{T},\Psi\sigma)(\mathbf{0})\right)\right) = \varphi_Y^{(m)}\left(\varepsilon_\delta(T,\Psi)(\mathbf{0})\right)^\sigma,$$

where $Y, \tilde{Y} \in \mathcal{Z}^{(m)}$ *are as in* (5.5).

We can easily prove this lemma combining (5.4), Lemma 3.4 and Lemma 5.3 below.

Lemma 5.3. *Put*

$$\hat{z_0} = \begin{pmatrix} t_1^{\Psi(T,1)} & & & \\ & t_2^{\Psi(T,2)} & & \\ & & \ddots & \\ & & & t_{m-2q}^{\Psi(T,m-2q)} \end{pmatrix} \in \mathfrak{H}_{m-2q}^{\mathbf{a}},$$

and

$$\hat{\tilde{z}}_0 = \begin{pmatrix} (\iota(\sigma,a_1)t_1)^{\Psi(\tilde{T},1)} & & \\ & \ddots & \\ & & (\iota(\sigma,a_{m-2q})t_{m-2q})^{\Psi(\tilde{T},m-2q)} \end{pmatrix} \in \mathfrak{H}_{m-2q}^{\mathbf{a}}.$$

Then for any $\tilde{Y} \in \mathcal{Z}^{(m-2q)}$, *we have*

$$J_{Y\tilde{Y}}^{(m-2q)}(C(\sigma;T,\Psi;\underline{a}))\left(\varphi_{\tilde{Y}}^{(m-2q)}(\hat{\tilde{z}}_0)\right) = \varphi_Y^{(m-2q)}(\hat{z_0})^\sigma,$$

where $Y = C(\sigma;T,\Psi;\underline{a})\tilde{Y}C(\sigma;T,\Psi;\underline{a})^{-1} \in \mathcal{Z}^{(m-2q)}$.

This lemma is easily verified by using Proposition 3.2 and Lemma 3.4.

Now, using Lemma 5.2, we obtain the following proposition.

Proposition 5.4. *For any $\tilde{X} \in \mathcal{Z}(\tilde{T}, \Psi\sigma)$ and $\tilde{Y} \in \mathcal{Z}^{(m)}$ so that $I_\delta(\tilde{T}, \Psi\sigma)(\tilde{X}) \subset \tilde{Y}$, we have*

$$J^{(m)}_{Y\tilde{Y}}(A(\sigma; T, \Psi; \underline{a})) \left(\overline{E_{\tilde{Y}\tilde{X}}(V_{\tilde{X}})}\right) = \left(\overline{E_{YX}(V_X)}\right)^\sigma.$$

Here the bars mean the Zariski closures in $V^{(m)}_{\tilde{Y}}$ and $V^{(m)}_Y$, and

$$\begin{aligned} X_{\mathbf{h}} &= B(\sigma; T, \Psi; \underline{a}) \tilde{X}_{\mathbf{h}} B(\sigma; T, \Psi; \underline{a})^{-1}, \\ Y &= A(\sigma; T, \Psi; \underline{a}) \tilde{Y} A(\sigma; T, \Psi; \underline{a})^{-1} \qquad \in \mathcal{Z}^{(m)}, \end{aligned}$$

where $X_{\mathbf{h}}$ and $\tilde{X}_{\mathbf{h}}$ denote the non-archimedean components of $X \in \mathcal{Z}(T, \Psi)$ and $\tilde{X} \in \mathcal{Z}(\tilde{T}, \Psi\sigma)$ respectively.

Proof. Note that $\overline{E_{\tilde{Y}\tilde{X}}(V_{\tilde{X}})}$, $\overline{E_{YX}(V_X)}$ are subvarieties of $V^{(m)}_{\tilde{Y}}$, $V^{(m)}_Y$ and $\dim\left(\overline{E_{\tilde{Y}\tilde{X}}(V_{\tilde{X}})}\right) = \dim\left(\overline{E_{YX}(V_X)}\right)$. In case $q > 0$, consider the set

$$\tilde{P} = \left\{ \tilde{\alpha}(\varepsilon_0(\tilde{T}, \Psi\sigma)(\tilde{z})) \,\middle|\, \tilde{z} \in \mathfrak{H}^{\mathbf{a}}_q,\ \tilde{\alpha} \in \mathrm{U}(\tilde{T}, \Psi\sigma) \right\}.$$

Then $\tilde{P}$ is dense in $\mathfrak{D}(\tilde{T}, \Psi\sigma)$. In the same way, take a dense subset P of $\mathfrak{D}(T, \Psi)$ as

$$P = \left\{ \alpha(\varepsilon_0(T, \Psi)(z)) \,\middle|\, z \in \mathfrak{H}^{\mathbf{a}}_q,\ \alpha \in \mathrm{U}(T, \Psi) \right\}.$$

Using (5.3) and Lemma 5.2, we easily obtain

$$J^{(m)}_{Y\tilde{Y}}(A(\sigma; T, \Psi; \underline{a})) \left(\varphi^{(m)}_{\tilde{Y}}\left(\varepsilon_\delta(\tilde{T}, \Psi\sigma)(\tilde{P})\right)\right) = \left[\varphi^{(m)}_Y\left(\varepsilon_\delta(T, \Psi)(P)\right)\right]^\sigma.$$

Hence we obtain our assertion since $\tilde{P}, P$ are dense in $\mathfrak{D}(\tilde{T}, \Psi\sigma)$, $\mathfrak{D}(T, \Psi)$.

In case $q = 0$, we can prove this proposition in the same way, since the set $\left\{ \tilde{\alpha}(\mathbf{0}) \,\middle|\, \tilde{\alpha} \in \mathrm{U}(\tilde{T}, \Psi\sigma) \right\}$ is dense in $\mathfrak{D}(\tilde{T}, \Psi\sigma)$. □

Proof of Theorem 5.1.

From now on we will prove Theorem 5.1 when $q > 0$. The proof in the case where $q = 0$ is easier.

First take any $\theta \in \mathcal{M}^{(m)}_{\mathbf{1}}(\overline{\mathbb{Q}})$ such that $\theta|\varepsilon_\delta(T, \Psi) \not\equiv 0$ and consider $\theta^{(A(\sigma; T, \Psi; \underline{a}), \sigma)} \in \mathcal{M}^{(m)}_{\mathbf{1}}(\overline{\mathbb{Q}})$ (where the action of $(A(\sigma; T, \Psi; \underline{a}), \sigma)$ is as defined in Proposition 3.5.) It is possible since $\nu(A(\sigma; T, \Psi; \underline{a})) = \chi(\sigma)$. For $\hat{z_0}, \tilde{\hat{z_0}} \in \mathfrak{H}^{\mathbf{a}}_{m-2q}$ as in Lemma 5.3, take $h \in \mathcal{M}^{(m-2q)}_{\mathbf{1}}(\overline{\mathbb{Q}})$ so that $h(\hat{z_0}) \neq 0$. Then this implies $h^{(C(\sigma; T, \Psi; \underline{a}), \sigma)}(\tilde{\hat{z_0}}) \neq 0$.

We first prove this theorem in case $f \in \mathcal{M}_{\kappa\cdot\mathbf{1}}(T,\Psi)$ where κ is a positive even integer. For such an f, define $\tilde{f} \in \mathcal{A}_{\kappa\cdot\mathbf{1}}(\tilde{T},\Psi\sigma)$ by

$$\begin{aligned} \tilde{f}(\tilde{\mathfrak{z}}) &= \Phi_X\left(h(\hat{z_0})^{\kappa/2} f(\theta|\varepsilon_\delta(T,\Psi))^{-\kappa/2}\right)^\sigma \circ (E_{YX}^{-1})^\sigma \\ &\quad \circ J_{Y\tilde{Y}}^{(m)}(A(\sigma;T,\Psi;\underline{a})) \circ \left(\varphi_{\tilde{Y}}^{(m)}(\varepsilon_\delta(\tilde{T},\Psi\sigma)(\tilde{\mathfrak{z}}))\right) \\ &\quad \times h^{(C(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z}}_0)^{-\kappa/2} \times \left(\left(\theta^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}|\varepsilon_\delta(\tilde{T},\Psi\sigma)\right)(\tilde{\mathfrak{z}})\right)^{\kappa/2}. \end{aligned} \tag{5.6}$$

Here $X \in \mathcal{Z}(T,\Psi)$, $\tilde{X} \in \mathcal{Z}(\tilde{T},\Psi\sigma)$ and $Y,\tilde{Y} \in \mathcal{Z}^{(m)}$ which satisfy the following conditions (1)–(6).
(1) $\tilde{X}_{\mathbf{h}} = B(\sigma;T,\Psi;\underline{a})^{-1}X_{\mathbf{h}}B(\sigma;T,\Psi;\underline{a})$, where $X_{\mathbf{h}}$ and $\tilde{X}_{\mathbf{h}}$ denote the non-archimedean components of X and $\tilde{X}$.
(2) $\tilde{Y} = A(\sigma;T,\Psi;\underline{a})^{-1}YA(\sigma;T,\Psi;\underline{a})$.
(3) $I_\delta(T,\Psi)(X) \subset Y$, $I_\delta(\tilde{T},\Psi\sigma)(\tilde{X}) \subset \tilde{Y}$.
(4) $K^\times\Gamma_X/K^\times$ and $F^\times\Gamma_Y^{(m)}/F^\times$ are torsion free.
(5) $E_{YX}(V_X) = \overline{E_{YX}(V_X)}$ and E_{YX}^{-1} is regular on whole $E_{YX}(V_X)$.
(6) $h(\hat{z_0})^{\kappa/2}f(\theta|\varepsilon_\delta(T,\Psi))^{-\kappa/2}$ $(\in \mathcal{A}_0(T,\Psi))$ is written as
$\Phi_X\left(h(\hat{z_0})^{\kappa/2}f(\theta|\varepsilon_\delta(T,\Psi))^{-\kappa/2}\right) \circ \varphi_X$ with a rational function
$\Phi_X\left(h(\hat{z_0})^{\kappa/2}f(\theta|\varepsilon_\delta(T,\Psi))^{-\kappa/2}\right)$ on V_X.

We can take such $X,\tilde{X},Y,\tilde{Y}$ by virtue of Theorem 4.1. Now we have constructed a regular morphism

$$(E_{YX}^{-1})^\sigma \circ J_{Y\tilde{Y}}^{(m)}(A(\sigma;T,\Psi;\underline{a})) \circ E_{\tilde{Y}\tilde{X}} : V_{\tilde{X}} \to V_X^\sigma \tag{5.7}$$

(for any $\sigma \in \mathrm{Aut}(\mathbb{C})$), which is essentially same as the one constructed in [6].

Clearly $\tilde{f}$ does not depend on the choice of $X,\tilde{X},Y,\tilde{Y}$. Moreover, using Lemma 5.3, we easily see that $\tilde{f}$ is independent of h. Let us prove that $\tilde{f}$ is also independent of θ.

It suffices to prove

$$\begin{aligned} &\left[\Phi_X\left((\theta_1|\varepsilon_\delta(T,\Psi))/(\theta_2|\varepsilon_\delta(T,\Psi))\right) \circ E_{YX}^{-1}\right]^\sigma \\ &\quad \circ J_{Y\tilde{Y}}^{(m)}(A(\sigma;T,\Psi;\underline{a})) \circ \varphi_{\tilde{Y}}^{(m)} \circ \varepsilon_\delta(\tilde{T},\Psi\sigma) \\ &\quad = (\theta_1^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}/\theta_2^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}) \circ \varepsilon_\delta(\tilde{T},\Psi\sigma) \end{aligned} \tag{5.8}$$

on $\mathfrak{D}(\tilde{T},\Psi\sigma)$ for $\theta_1,\theta_2 \in \mathcal{M}_{\mathbf{1}}^{(m)}(\overline{\mathbb{Q}})$ so that $\theta_1|\varepsilon_\delta(T,\Psi) \not\equiv 0$, $\theta_2|\varepsilon_\delta(T,\Psi) \not\equiv 0$. Note that $\theta_2^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}|\varepsilon_\delta(\tilde{T},\Psi\sigma) \not\equiv 0$ then. The content in the square blacket of (5.8) is equal to the restriction of $\Phi_Y^{(m)}(\theta_1/\theta_2)$ on $E_{YX}(V_X)$. We can easily verify (5.8) since we obtain

$$\Phi_Y^{(m)}(\theta_1/\theta_2)^\sigma \circ J_{Y\tilde{Y}}^{(m)}(A(\sigma;T,\Psi;\underline{a})) \circ \varphi_{\tilde{Y}}^{(m)} = \theta_1^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}/\theta_2^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}$$

from (6) of Proposition 3.5.

At each $\tilde{\mathfrak{z}} \in \mathfrak{D}(\tilde{T},\Psi\sigma)$, take $\theta \in \mathcal{M}_{\mathbf{1}}^{(m)}(\overline{\mathbb{Q}})$ so that $\theta^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}(\varepsilon_\delta(\tilde{T},\Psi\sigma)(\tilde{\mathfrak{z}})) \neq 0$. Then $\theta|\varepsilon_\delta(T,\Psi)$ is non-zero on $\varphi_X^{-1}\Big(E_{YX}^{-1} \circ \Big[J_{Y\tilde{Y}}^{(m)}(A(\sigma;T,\Psi;\underline{a}))$

$\left(\varphi_{\tilde{Y}}^{(m)}(\varepsilon_\delta(\tilde{T},\Psi\sigma)(\tilde{\mathfrak{z}}))\right)\Big]^{\sigma^{-1}}\Big)$ for sufficiently small $X, \tilde{X}, Y, \tilde{Y}$ satisfying the conditions in (5.6). This implies that $\tilde{f}$ is holomorphic at $\tilde{\mathfrak{z}}$. Hence we have $\tilde{f} \in \mathcal{M}_{\kappa\cdot\mathbf{1}}(\tilde{T},\Psi\sigma)$.

Let us prove that the modular form $\tilde{f}$ satisfies the property required for $f^{(\sigma;T,\Psi;\underline{a})}$. Put

$$(\theta \circ \varepsilon_{q,m-2q,q})(z_1,z_2,z_3) = \sum_{i=1}^{s} \theta_{1,i}(z_1)\theta_{2,i}(z_2)\theta_{3,i}(z_3),$$

with $\theta_{1,i}, \theta_{3,i} \in \mathcal{M}_{\mathbf{1}}^{(q)}(\overline{\mathbb{Q}})$ and $\theta_{2,i} \in \mathcal{M}_{\mathbf{1}}^{(m-2q)}(\overline{\mathbb{Q}})$. Then from (5.4) and Lemma 3.3, we have

$$(\theta^{(A(\sigma;T,\Psi;\underline{a}),\sigma)} \circ \varepsilon_{q,m-2q,q})(z_1,z_2,z_3) = \sum_{i=1}^{s} \theta_{1,i}^{\sigma}(z_1)\theta_{2,i}^{(C(\sigma;T,\Psi;\underline{a}),\sigma)}(z_2)\theta_{3,i}^{\sigma}(z_3).$$

From (2.4) and (2.5), we obtain

$$\left((\theta|\varepsilon_\delta(T,\Psi))|\varepsilon_0(T,\Psi)\right)(z) = \sum_{i=1}^{s} \theta_{2,i}(\hat{z_0})\theta_{1,i}(z)\theta_{3,i}((-\delta^2)z),$$

where $(-\delta^2) = ((-\delta^2)_v)_{v\in\mathbf{a}} \in \mathbb{R}^{\mathbf{a}}$ and the both sides are viewed as modular forms with respect to $z \in \mathfrak{H}_q^{\mathbf{a}}$. Using Lemma 3.3 and Lemma 5.3, we can get

$$\begin{aligned} &h^{(C(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z_0}})^{-1}\left((\theta^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}|\varepsilon_\delta(\tilde{T},\Psi\sigma))|\varepsilon_0(\tilde{T},\Psi\sigma)\right) \\ &\quad = \left\{h(\hat{z_0})^{-1}(\theta|\varepsilon_\delta(T,\Psi))|\varepsilon_0(T,\Psi)\right\}^{\sigma}, \end{aligned} \tag{5.9}$$

where the action of σ in the right hand side is as defined in (2.2). Substituting $\varepsilon_0(\tilde{T},\Psi\sigma)(z)$ for $\tilde{\mathfrak{z}}$, we view (5.6) as a function in $z \in \mathfrak{H}_q^{\mathbf{a}}$. Take $\theta \in \mathcal{M}_{\mathbf{1}}^{(m)}(\overline{\mathbb{Q}})$ so that $(\theta|\varepsilon_\delta(T,\Psi))|\varepsilon_0(T,\Psi) \not\equiv 0$ on $\mathfrak{H}_q^{\mathbf{a}}$. Then from (3.1) and Lemma 5.2, we have

$$\begin{aligned} &(\tilde{f}|\varepsilon_0(\tilde{T},\Psi\sigma))/\left(h^{(C(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z_0}})^{-1}\left((\theta^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}|\varepsilon_\delta(\tilde{T},\Psi\sigma))|\varepsilon_0(\tilde{T},\Psi\sigma)\right)\right)^{\kappa/2} \\ &\quad = (f|\varepsilon_0(T,\Psi))^{\sigma}/\left[\left\{h(\hat{z_0})^{-1}(\theta|\varepsilon_\delta(T,\Psi))|\varepsilon_0(T,\Psi)\right\}^{\kappa/2}\right]^{\sigma}. \end{aligned}$$

Note that $(\theta^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}|\varepsilon_\delta(\tilde{T},\Psi\sigma))|\varepsilon_0(\tilde{T},\Psi\sigma) \not\equiv 0$ because of (5.9). Combining this with (5.9), we obtain

$$\left(\tilde{f}|\varepsilon_0(\tilde{T},\Psi\sigma)\right) = (f|\varepsilon_0(T,\Psi))^{\sigma}. \tag{5.10}$$

Fix $\theta \in \mathcal{M}_{\mathbf{1}}^{(m)}(\overline{\mathbb{Q}})$ and take an open compact subgroup $D_{\mathbf{h}}^1$ (resp. $W_{\mathbf{h}}$) of $\mathrm{U}_1(T,\Psi)_{\mathbf{h}}$ (resp. $\mathrm{Sp}(m,F)_{\mathbf{h}}$) so that $f \in \mathcal{M}_{\kappa\cdot\mathbf{1}}(T,\Psi)((\mathrm{U}_1(T,\Psi)_{\mathbf{a}} \times D_{\mathbf{h}}^1) \cap \mathrm{U}_1(T,\Psi))$, $\theta \in \mathcal{M}_{\mathbf{1}}^{(m)}((\mathrm{Sp}(m,F)_{\mathbf{a}} \times W_{\mathbf{h}}) \cap \mathrm{Sp}(m,F))$ and $I_\delta(T,\Psi)(D_{\mathbf{h}}^1) \subset W_{\mathbf{h}}$.

For such $D^1_{\mathbf{h}}$, $W_{\mathbf{h}}$ and any $\tilde{\alpha} \in \mathrm{U}(\tilde{T}, \Psi\sigma)$, take $\alpha \in \mathrm{U}(T, \Psi)$ as in (5.2). Then $f|_{\kappa\cdot\mathbf{1}}\alpha$ and $\theta|_{\mathbf{1}}I_\delta(T, \Psi)(\alpha)$ are independent of the choice of α. Moreover, we obtain

$$(\theta|_{\mathbf{1}}I_\delta(T, \Psi)(\alpha))^{(A(\sigma;T,\Psi;\underline{a}),\sigma)} = \theta^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}|_{\mathbf{1}}I_\delta(\tilde{T}, \Psi\sigma)(\tilde{\alpha}),$$

from Proposition 3.5 and (5.3). Substituting $f|_{\kappa\cdot\mathbf{1}}\alpha$, $\theta|_{\mathbf{1}}I_\delta(T, \Psi)(\alpha)$ for f, θ in (5.6) (and taking X, Y in (5.6) sufficiently small), we obtain $\tilde{f}|_{\kappa\cdot\mathbf{1}}\tilde{\alpha}$ instead of $\tilde{f}$. This means

$$(\tilde{f}|_{\kappa\cdot\mathbf{1}}\tilde{\alpha})|\varepsilon_0(\tilde{T}, \Psi\sigma) = \{(f|_{\kappa\cdot\mathbf{1}}\alpha)|\varepsilon_0(T, \Psi)\}^\sigma .$$

Hence we have proved that $\tilde{f}$ satisfies the property of $f^{(\sigma;T,\Psi;\underline{a})}$.

Given $f \in \mathcal{M}_k(T, \Psi)$ of an arbitrary weight $k \in \mathbb{Z}^{\mathbf{a}}$, let us prove the existence of $f^{(\sigma;T,\Psi;\underline{a})}$. To prove this, we need to construct an example $\xi_v \in \mathcal{M}_{v+2s_v\mathbf{1}}(T, \Psi)$ such that there exists $\xi_v^{(\sigma;T,\Psi;\underline{a})} \in \mathcal{M}_{v\sigma+2s_v\mathbf{1}}(\tilde{T}, \Psi\sigma)$ for each $v \in \mathbf{a}$.

Put $|\mathbf{a}| = [F : \mathbb{Q}]$ and express $\mathbf{a}$ as $\mathbf{a} = \{v_1, v_2, \ldots, v_{|\mathbf{a}|}\}$. In Section 3 of [15] (or essentially in [9]), we constructed a $\mathbb{C}^{m|\mathbf{a}|}_{m|\mathbf{a}|}$-valued holomorphic function Δ on $\mathfrak{H}^{\mathbf{a}}_m$ which satisfies

$$\Delta(\gamma(z)) = \left(\prod_{v\in\mathbf{a}} \det(\mu_v^{(m)}(\gamma, z))\right) \begin{pmatrix} \mu_{v_1}^{(m)}(\gamma, z) & & \\ & \ddots & \\ & & \mu_{v_{|\mathbf{a}|}}^{(m)}(\gamma, z) \end{pmatrix} \Delta(z),$$

for any $\gamma \in \Gamma^{(m)}$, where $\Gamma^{(m)}$ is a certain congruence subgroup of $\mathrm{Sp}(m, F)$. The Fourier coefficients of Δ are all $\overline{\mathbb{Q}}$-rational. Express Δ in the form

$$\Delta = \begin{pmatrix} \Delta_{v_1} \\ \Delta_{v_2} \\ \vdots \\ \Delta_{v_{|\mathbf{a}|}} \end{pmatrix},$$

with $\mathbb{C}^m_{m|\mathbf{a}|}$-valued functions Δ_v $(v \in \mathbf{a})$ on $\mathfrak{H}^{\mathbf{a}}_m$. Then, for each $v \in \mathbf{a}$ and any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have

$$\Delta_v^\sigma = \Delta_{v\sigma},$$

where the action of σ in the left hand side is that on each Fourier coefficient (in the sense of (2.2)). At each $z_0 \in \mathfrak{H}^{\mathbf{a}}_m$, we can choose such Δ so that $\det(\Delta(z_0)) \neq 0$ (which is shown in section 3 of [15]).

The function Δ_v satisfies

$$\Delta_v(\gamma(z)) = \left(\prod_{v'\in\mathbf{a}} \det(\mu_{v'}^{(m)}(\gamma, z))\right) \mu_v^{(m)}(\gamma, z)\Delta_v(z)$$

for any $\gamma \in \Gamma^{(m)}$. For each $v \in \mathbf{a}$, we define the $\mathbb{C}^m_{m|\mathbf{a}|}$-valued holomorphic function $\hat{\Delta}_v$ on $\mathfrak{D}(T,\Psi)$ by

$$\hat{\Delta}_v(\mathfrak{z}) = \left[\prod_{v' \in \mathbf{a}} \left\{ \left(-\frac{\delta^{\Psi_{v'}}}{2}\right)^{-q} \det\left(\omega_{v'}(\mathfrak{z})^{-1}\mu_{v'}^{(m)}(C(T,\delta)^{-1}, \varepsilon_\delta(\mathfrak{z}))\right)\right\}\right]$$
$$\times\, \omega_v(\mathfrak{z})^{-1}(T^{\Psi_v})\mu_v^{(m)}(C(T,\delta)^{-1}, \varepsilon_\delta(\mathfrak{z}))\Delta_v(\varepsilon_\delta(\mathfrak{z})).$$

Using (2.3), we obtain

$$\hat{\Delta}_v(\alpha(\mathfrak{z})) = \left\{\prod_{v' \in \mathbf{a}} \det(\mu_{v'}(\alpha,\mathfrak{z}))^2\right\} \begin{pmatrix} \lambda_v(\alpha,\mathfrak{z}) & 0 \\ 0 & \mu_v(\alpha,\mathfrak{z}) \end{pmatrix} \hat{\Delta}_v(\mathfrak{z})$$

for any $\alpha \in \Gamma$, where Γ is a certain congruence subgroup of $\mathrm{U}_1(T,\Psi)$.

For each $v \in \mathbf{a}$, take $\hat{z}_{(1,v)} = (\hat{z}_{(1,v),v'})_{v' \in \mathbf{a}}$, $\tilde{\hat{z}}_{(1,v)} = (\tilde{\hat{z}}_{(1,v),v'})_{v' \in \mathbf{a}} \in \mathfrak{H}^{\mathbf{a}}_{r_v - q}$ and $\hat{z}_{(2,v)} = (\hat{z}_{(2,v),v'})_{v' \in \mathbf{a}}$, $\tilde{\hat{z}}_{(2,v)} = (\tilde{\hat{z}}_{(2,v),v'})_{v' \in \mathbf{a}} \in \mathfrak{H}^{\mathbf{a}}_{s_v - q}$ as

$$\hat{z_0} = \begin{pmatrix} \hat{z}_{(1,v)} & 0 \\ 0 & \hat{z}_{(2,v)} \end{pmatrix}, \qquad \tilde{\hat{z_0}} = \begin{pmatrix} \tilde{\hat{z}}_{(1,v)} & 0 \\ 0 & \tilde{\hat{z}}_{(2,v)} \end{pmatrix},$$

where $\hat{z_0}$ and $\tilde{\hat{z_0}}$ are as defined in Lemma 5.3. Moreover, we define $C_{(1,v)}(\sigma;T,\Psi;\underline{a}) \in \mathcal{G}_+^{(r_v-q)}$, $C_{(2,v)}(\sigma;T,\Psi;\underline{a}) \in \mathcal{G}_+^{(s_v-q)}$ by

$$C_{(1,v)}(\sigma;T,\Psi;\underline{a}) = \begin{pmatrix} C_{(1,v),1} & C_{(1,v),2} \\ C_{(1,v),3} & C_{(1,v),4} \end{pmatrix},$$
$$C_{(2,v)}(\sigma;T,\Psi;\underline{a}) = \begin{pmatrix} C_{(2,v),1} & C_{(2,v),2} \\ C_{(2,v),3} & C_{(2,v),4} \end{pmatrix},$$

where $C_{(1,v),1}$, $C_{(1,v),2}$, $C_{(1,v),3}$, $C_{(1,v),4} \in (F_A)^{r_v-q}_{r_v-q}$ and $C_{(2,v),1}$, $C_{(2,v),2}$, $C_{(2,v),3}$, $C_{(2,v),4} \in (F_A)^{s_v-q}_{s_v-q}$ so that

$$C(\sigma;T,\Psi;\underline{a}) = \left(\begin{array}{cc|cc} C_{(1,v),1} & 0 & C_{(1,v),2} & 0 \\ 0 & C_{(2,v),1} & 0 & C_{(2,v),2} \\ \hline C_{(1,v),3} & 0 & C_{(1,v),4} & 0 \\ 0 & C_{(2,v),3} & 0 & C_{(2,v),4} \end{array}\right).$$

Clearly we have

$$\nu(C_{(1,v)}(\sigma;T,\Psi;\underline{a})) = \nu(C_{(2,v)}(\sigma;T,\Psi;\underline{a})) = \chi(\sigma).$$

Take $h_{(1,v)} \in \mathcal{M}^{(r_v-q)}_{s_v \cdot \mathbf{1}}(\overline{\mathbf{Q}})$ and $h_{(2,v)} \in \mathcal{M}^{(s_v-q)}_{v+s_v \cdot \mathbf{1}}(\overline{\mathbf{Q}})$ so that $h_{(1,v)}(\hat{z}_{(1,v)}) \neq 0$ and $h_{(2,v)}(\hat{z}_{(2,v)}) \neq 0$. We construct a holomorphic function ξ_v on $\mathfrak{D}(T,\Psi)$ by

$$\xi_v(\mathfrak{z}) = \left(-\frac{1}{2}\tau^{\Psi_v}\right)^{-q} \det(\hat{z}_{(2,v),v})^{-1} h_{(1,v)}(\hat{z}_{(1,v)})^{-1} h_{(2,v)}(\hat{z}_{(2,v)})^{-1}$$
$$\times \det\left((0^{s_v}_{r_v}\ 1_{s_v})\hat{\Delta}_v(\mathfrak{z})Q\right),$$

with any $Q \in \mathbb{Q}_{s_v}^{m|\mathbf{a}|}$, where $0_{r_v}^{s_v}$ denotes the zero matrix of size $s_v \times r_v$. Then we can easily verify $\xi_v \in \mathcal{M}_{v+2s_v\cdot\mathbf{1}}(T,\Psi)$. At each $\mathfrak{z}_0 \in \mathfrak{D}(T,\Psi)$, if $\det(\Delta(\varepsilon_\delta(\mathfrak{z}_0))) \neq 0$, we can take $Q \in \mathbb{Q}_{s_v}^{m|\mathbf{a}|}$ so that $\xi_v(\mathfrak{z}_0) \neq 0$ (since the rank of $\hat{\Delta}_v(\mathfrak{z}_0)$ is exactly m then).

In the same way as the definition of $\hat{\Delta}_v$, we define the $\mathbb{C}_{m|\mathbf{a}|}^m$-valued function $\tilde{\hat{\Delta}}_v$ on $\mathfrak{D}(\tilde{T},\Psi\sigma)$ by

$$\tilde{\hat{\Delta}}_v(\tilde{\mathfrak{z}}) = \left[\prod_{v'\in\mathbf{a}}\left\{\left(-\frac{1}{2}\delta^{\Psi_{v'}\sigma}\right)^{-q}\det\left(\omega_{v'}(\tilde{\mathfrak{z}})^{-1}\mu_{v'}^{(m)}(C(\tilde{T},\delta)^{-1},\varepsilon_\delta(\tilde{\mathfrak{z}}))\right)\right\}\right] \\ \times\,\omega_{v\sigma}(\tilde{\mathfrak{z}})^{-1}(\tilde{T}^{\Psi_v\sigma})\mu_{v\sigma}^{(m)}(C(\tilde{T},\delta)^{-1},\varepsilon_\delta(\tilde{\mathfrak{z}}))\Delta_v^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}(\varepsilon_\delta(\tilde{\mathfrak{z}})),$$

where $\tilde{\mathfrak{z}} \in \mathfrak{D}(\tilde{T},\Psi\sigma)$, ε_δ means $\varepsilon_\delta(\tilde{T},\Psi\sigma)$, and the action of $(A(\sigma;T,\Psi;\underline{a}),\sigma)$ is as in Proposition 3.5 (on vector-valued modular forms, stated in 10.2 of [14]). Take $\mathbb{C}$-valued function $\tilde{\xi}_v$ on $\mathfrak{D}(\tilde{T},\Psi\sigma)$ as

$$\tilde{\xi}_v(\tilde{\mathfrak{z}}) = \left(-\frac{(\iota(\sigma,a_0)\tau)^{\Psi_v\sigma}}{2}\right)^{-q}\det(\tilde{\hat{z}}_{(2,v),v\sigma})^{-1} \\ \times\, h_{(1,v)}^{(C_{(1,v)}(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z}}_{(1,v)})^{-1}h_{(2,v)}^{(C_{(2,v)}(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z}}_{(2,v)})^{-1} \\ \times\det\left((0_{r_v}^{s_v}\ 1_{s_v})\tilde{\hat{\Delta}}_v(\tilde{\mathfrak{z}})Q\right),$$

where Q is same as in the definition of ξ_v. Note that $h_{(1,v)}^{(C_{(1,v)}(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z}}_{(1,v)}) \neq 0$ and $h_{(2,v)}^{(C_{(2,v)}(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z}}_{(2,v)}) \neq 0$ since $h_{(1,v)}(\hat{z}_{(1,v)}) \neq 0$ and $h_{(2,v)}(\hat{z}_{(2,v)}) \neq 0$. It is clear that $\tilde{\xi}_v \in \mathcal{M}_{v\sigma+2s_v\cdot\mathbf{1}}(\tilde{T},\Psi\sigma)$.

Now let us prove that $\tilde{\xi}_v$ satisfies the property of $\xi_v^{(\sigma;T,\Psi;\underline{a})}$. First let us consider $\xi_v|\varepsilon_0(T,\Psi)$ and $\tilde{\xi}_v|\varepsilon_0(\tilde{T},\Psi\sigma)$, which are holomorphic modular forms on $\mathfrak{H}_q^{\mathbf{a}}$. By a computation, we have

$$(\xi_v|\varepsilon_0)(z) = h_{(1,v)}(\hat{z}_{(1,v)})^{-1}h_{(2,v)}(\hat{z}_{(2,v)})^{-1} \\ \times\det\left(\begin{pmatrix} 0_q^{s_v-q} & 0_{r_v-q}^{s_v-q} & 1_{s_v-q} & 0_q^{s_v-q} \\ 1_q & 0_{r_v-q}^q & 0_{s_v-q}^q & \delta^{\Psi_v}\cdot 1_q \end{pmatrix}\Delta_v(\varepsilon_\delta\circ\varepsilon_0(z))Q\right).$$

In the same way as in Lemma 3.3, we can put

$$\det\left(\begin{pmatrix} 0_q^{s_v-q} & 0_{r_v-q}^{s_v-q} & 1_{s_v-q} & 0_q^{s_v-q} \\ 1_q & 0_{r_v-q}^q & 0_{s_v-q}^q & \delta^{\Psi_v}\cdot 1_q \end{pmatrix}\Delta_v\left(\begin{pmatrix} z & & & \\ & z_1 & & \\ & & z_2 & \\ & & & (-\delta^2)z \end{pmatrix}\right)Q\right) \\ = \sum_{i=1}^r h_{1,i}(z_1)h_{2,i}(z_2)g_i(z),$$

where $h_{1,i} \in \mathcal{M}_{s_v\cdot\mathbf{1}}^{(r_v-q)}(\overline{\mathbb{Q}})$, $h_{2,i} \in \mathcal{M}_{v+s_v\cdot\mathbf{1}}^{(s_v-q)}(\overline{\mathbb{Q}})$, $g_i \in \mathcal{M}_{v+2s_v\cdot\mathbf{1}}^{(q)}(\overline{\mathbb{Q}})$ and $z \in \mathfrak{H}_q^{\mathbf{a}}$,

$z_1 \in \mathfrak{H}^{\mathbf{a}}_{(r_v-q)}$, $z_2 \in \mathfrak{H}^{\mathbf{a}}_{(s_v-q)}$. Then we have

$$\begin{aligned}
\det\Bigg(&\begin{pmatrix} 0_q^{s_v-q} & 0_{r_v-q}^{s_v-q} & 1_{s_v-q} & 0_q^{s_v-q} \\ 1_q & 0_{r_v-q}^{q} & 0_{s_v-q}^{q} & \delta^{\Psi_v\sigma}\cdot 1_q \end{pmatrix} \\
&\times \Delta_v^{\sigma}\left(\left(\begin{pmatrix} z & & & \\ & z_1 & & \\ & & z_2 & \\ & & & (-\delta^2)z \end{pmatrix}\right)Q\right) \\
&= \sum_{i=1}^{r} h_{1,i}^{\sigma}(z_1)h_{2,i}^{\sigma}(z_2)g_i^{\sigma}(z).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\det\Bigg[&\begin{pmatrix} 0_q^{s_v-q} & 0_{r_v-q}^{s_v-q} & 1_{s_v-q} & 0_q^{s_v-q} \\ 1_q & 0_{r_v-q}^{q} & 0_{s_v-q}^{q} & \delta^{\Psi_v\sigma}\cdot 1_q \end{pmatrix} \\
&\times \Delta_v^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}\left(\left(\begin{pmatrix} z & & & \\ & z_1 & & \\ & & z_2 & \\ & & & (-\delta^2)z \end{pmatrix}\right)Q\right]\Bigg] \\
&= \sum_{i=1}^{r} h_{1,i}^{(C_{(1,v)}(\sigma;T,\Psi;\underline{a}),\sigma)}(z_1)h_{2,i}^{(C_{(2,v)}(\sigma;T,\Psi;\underline{a}),\sigma)}(z_2)g_i^{\sigma}(z).
\end{aligned}$$

Combining these, we obtain

$$\begin{aligned}
(\xi_v|\varepsilon_0(T,\Psi))(z) &= h_{(1,v)}(\hat{z}_{(1,v)})^{-1}h_{(2,v)}(\hat{z}_{(2,v)})^{-1} \\
&\quad\times \sum_{i=1}^{r} h_{1,i}(\hat{z}_{(1,v)})h_{2,i}(\hat{z}_{(2,v)})g_i(z),
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{\xi}_v|\varepsilon_0(\tilde{T},\Psi\sigma))(z) &= h_{(1,v)}^{(C_{(1,v)}(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z}}_{(1,v)})^{-1}h_{(2,v)}^{(C_{(2,v)}(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z}}_{(2,v)})^{-1} \\
&\quad\times \sum_{i=1}^{r} h_{1,i}^{(C_{(1,v)}(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z}}_{(1,v)})h_{2,i}^{(C_{(2,v)}(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z}}_{(2,v)})g_i^{\sigma}(z),
\end{aligned}$$

for $z \in \mathfrak{H}^{\mathbf{a}}_q$. From Lemma 3.4 and Lemma 5.3, we have

$$\begin{aligned}
&h_{(1,v)}^{(C_{(1,v)}(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z}}_{(1,v)})^{-1}h_{1,i}^{(C_{(1,v)}(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z}}_{(1,v)}) \\
&\quad= \left\{h_{(1,v)}(\hat{z}_{(1,v)})^{-1}h_{1,i}(\hat{z}_{(1,v)})\right\}^{\sigma}, \\
&h_{(2,v)}^{(C_{(2,v)}(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z}}_{(2,v)})^{-1}h_{2,i}^{(C_{(2,v)}(\sigma;T,\Psi;\underline{a}),\sigma)}(\tilde{\hat{z}}_{(2,v)}) \\
&\quad= \left\{h_{(2,v)}(\hat{z}_{(2,v)})^{-1}h_{2,i}(\hat{z}_{(2,v)})\right\}^{\sigma}.
\end{aligned}$$

Hence we obtain

$$\tilde{\xi}_v|\varepsilon_0(\tilde{T},\Psi\sigma) = (\xi_v|\varepsilon_0(T,\Psi))^\sigma\,. \tag{5.11}$$

For any $\alpha \in \mathrm{U}(T,\Psi)$ and $\tilde{\alpha} \in \mathrm{U}(\tilde{T},\Psi\sigma)$, let us consider $\xi_v|_{v+2s_v\cdot\mathbf{1}}\alpha$ and $\tilde{\xi}_v|_{v\sigma+2s_v\cdot\mathbf{1}}\tilde{\alpha}$. Substituting $\Delta_v^{(I_\delta(\alpha),1)}$ (resp. $\Delta_v^{(A(\sigma;T,\Psi;\underline{a})I_\delta(\tilde{\alpha}),\sigma)}$) for Δ_v (resp. $\Delta_v^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}$), we get $\xi_v|_{v+2s_v\cdot\mathbf{1}}\alpha$ (resp. $\tilde{\xi}_v|_{v\sigma+2s_v\cdot\mathbf{1}}\tilde{\alpha}$) instead of ξ_v (resp. $\tilde{\xi}_v$). Take open compact subgroups $D^1_{\mathbf{h}}$, $W_{\mathbf{h}}$ of $\mathrm{U}_1(T,\Psi)_{\mathbf{h}}$, $\mathrm{Sp}(m,F)_{\mathbf{h}}$ respectively, satisfying the following conditions (1)–(3).
(1) $\xi_v \in \mathcal{M}_{v+2s_v\cdot\mathbf{1}}(T,\Psi)((\mathrm{U}_1(T,\Psi)_{\mathbf{a}} \times D^1_{\mathbf{h}}) \cap \mathrm{U}_1(T,\Psi))$.
(2) $I_\delta(T,\Psi)(D^1_{\mathbf{h}}) \subset W_{\mathbf{h}}$.
(3) $\Delta_v^{(\gamma,1)} = \Delta_v$ for any $\gamma \in (\mathrm{Sp}(m,F)_{\mathbf{a}} \times W_{\mathbf{h}}) \cap \mathrm{Sp}(m,F)$.
For such $D^1_{\mathbf{h}}$, $W_{\mathbf{h}}$, take $\alpha \in \mathrm{U}(T,\Psi)$ and $\tilde{\alpha} \in \mathrm{U}(\tilde{T},\Psi\sigma)$ as in (5.2). Then we obtain

$$(\tilde{\xi}_v|_{v\sigma+2s_v\cdot\mathbf{1}}\tilde{\alpha})|\varepsilon_0(\tilde{T},\Psi\sigma) = \{(\xi_v|_{v+2s_v\cdot\mathbf{1}}\alpha)|\varepsilon_0(T,\Psi)\}^\sigma\,,$$

instead of (5.11), since we have

$$\Delta_v^{(A(\sigma;T,\Psi;\underline{a})I_\delta(\tilde{\alpha}),\sigma)} = \Delta_v^{(I_\delta(\alpha)A(\sigma;T,\Psi;\underline{a}),\sigma)}.$$

This means that $\tilde{\xi}_v$ satisfies the property of $\xi_v^{(\sigma;T,\Psi;\underline{a})}$.

For any $k \in \mathbb{Z}^{\mathbf{a}}$ and any $f \in \mathcal{M}_k(T,\Psi)$, take $l = (l_v)_{v\in\mathbf{a}} \in (\mathbb{N}\cup\{0\})^{\mathbf{a}}$ so that $k + \sum_{v\in\mathbf{a}} l_v(v + 2s_v\cdot\mathbf{1}) = \kappa\cdot\mathbf{1}$ for some positive even integer κ. Taking (non-zero) $(\xi_v)_{v\in\mathbf{a}}$ as above, we define

$$\tilde{f} = \left(f\prod_{v\in\mathbf{a}}\xi_v^{l_v}\right)^{(\sigma;T,\Psi;\underline{a})} \times \prod_{v\in\mathbf{a}}\left(\xi_v^{(\sigma;T,\Psi;\underline{a})}\right)^{-l_v}.$$

Note that $\xi_v^{(\sigma;T,\Psi;\underline{a})}$ ($v \in \mathbf{a}$) are non-zero since so are ξ_v. Hence we have $\tilde{f} \in \mathcal{A}_{k^\sigma}(\tilde{T},\Psi\sigma)$. Let us prove that $\tilde{f}$ does not depend on the choice of $(\xi_v)_{v\in\mathbf{a}}$. Take different couples $(\xi_{1,v})_{v\in\mathbf{a}}$ and $(\xi_{2,v})_{v\in\mathbf{a}}$ for $(\xi_v)_{v\in\mathbf{a}}$. Put

$$g = f\prod_{v\in\mathbf{a}}\left(\xi_{1,v}^{l_v}\xi_{2,v}^{l_v}\right) \quad \in \mathcal{M}_{2\kappa\cdot\mathbf{1}-k}(T,\Psi).$$

Then both of

$$\left(f\prod_{v\in\mathbf{a}}\xi_{1,v}^{l_v}\right)^{(\sigma;T,\Psi;\underline{a})} \cdot \prod_{v\in\mathbf{a}}\left(\xi_{2,v}^{(\sigma;T,\Psi;\underline{a})}\right)^{l_v} \tag{5.12}$$

and

$$\left(f\prod_{v\in\mathbf{a}}\xi_{2,v}^{l_v}\right)^{(\sigma;T,\Psi;\underline{a})} \cdot \prod_{v\in\mathbf{a}}\left(\xi_{1,v}^{(\sigma;T,\Psi;\underline{a})}\right)^{l_v} \tag{5.13}$$

satisfy the property of $g^{(\sigma;T,\Psi;\underline{a})}$. Thus the two functions above coincide, which shows our claim.

Next let us prove that $\tilde{f}$ is holomorphic. At each $\tilde{\mathfrak{z}} \in \mathfrak{D}(\tilde{T}, \Psi\sigma)$, take $(\xi_v)_{v\in\mathbf{a}}$ so that $\prod_{v\in\mathbf{a}} \xi_v^{(\sigma;T,\Psi;\underline{a})}(\tilde{\mathfrak{z}}) \neq 0$. It is possible if we take Δ so that $\det\left(\Delta^{(A(\sigma;T,\Psi;\underline{a}),\sigma)}(\varepsilon_\delta(\tilde{\mathfrak{z}}))\right) \neq 0$, and choose suitable $Q \in \mathbb{Q}_{s_v}^{m|\mathbf{a}|}$ for each $v \in \mathbf{a}$. Then $\tilde{f}$ is holomorphic at $\tilde{\mathfrak{z}}$. Hence we have $\tilde{f} \in \mathcal{M}_{k^\sigma}(\tilde{T}, \Psi\sigma)$. We can easily prove that $\tilde{f}$ satisfies the property of $f^{(\sigma;T,\Psi;\underline{a})}$, since we can take $(\xi_v)_{v\in\mathbf{a}}$ so that $\prod_{v\in\mathbf{a}} \xi_v^{(\sigma;T,\Psi;\underline{a})}$ is non-zero at a given point in $\mathfrak{D}(\tilde{T}, \Psi\sigma)$. □

The following lemma is an immediate consequence of the theorem above.

Lemma 5.5. *For any $f \in \mathcal{M}_k(T, \Psi)$, take an open compact subgroup $D^1_{\mathbf{h}}$ of $\mathrm{U}_1(T, \Psi)_{\mathbf{h}}$ so that $f \in \mathcal{M}_k(T, \Psi)((\mathrm{U}_1(T, \Psi)_{\mathbf{a}} \times D^1_{\mathbf{h}}) \cap \mathrm{U}_1(T, \Psi))$. Then we have*

$$(f|_k\beta)^{(\sigma;T,\Psi;\underline{a})} = f^{(\sigma;T,\Psi;\underline{a})}|_{k^\sigma}\tilde{\beta},$$

for any $\beta \in \mathrm{U}(T, \Psi)$ and $\tilde{\beta} \in \mathrm{U}(\tilde{T}, \Psi\sigma)$ so that

$$\beta_{\mathbf{h}} \in D^1_{\mathbf{h}} \cdot B(\sigma; T, \Psi; \underline{a})\tilde{\beta}_{\mathbf{h}} B(\sigma; T, \Psi; \underline{a})^{-1},$$

where $\beta_{\mathbf{h}}$ and $\tilde{\beta}_{\mathbf{h}}$ denote the non-archimedean parts of β and $\tilde{\beta}$.

6. Relation with Hecke operators

In this section we will define Hecke operators and prove that the Galois action constructed in the previous section is compatible with them.

First of all, we need to define cusp forms. For $k \in \mathbb{Z}^{\mathbf{a}}$, we denote by $\mathcal{S}_k(T, \Psi)$ the set of all $f \in \mathcal{M}_k(T, \Psi)$ such that $(f|_k\alpha)|\varepsilon_0(T, \Psi)$ are cusp forms in $\mathcal{M}_k^{(q)}$ for any $\alpha \in \mathrm{U}(T, \Psi)$. In case $q = 0$, we set $\mathcal{S}_k(T, \Psi) = \mathcal{M}_k(T, \Psi)$. Then this definition of cusp form is equivalent to that of Section 10 in [12]. Obviously we have $\mathcal{S}_k(T, \Psi)^{(\sigma;T,\Psi;\underline{a})} = \mathcal{S}_{k^\sigma}(\tilde{T}, \Psi\sigma)$.

We also need to define inner products of modular forms. There exists a unique measure $\mathbf{d}\mathfrak{z}_v$ on $\mathfrak{D}(T, \Psi)_v$ which is invariant under the action of $\mathrm{U}(T, \Psi)_v$, up to positive constant multiples. Take a measure $\mathbf{d}\mathfrak{z} = \prod_{v\in\mathbf{a}} \mathbf{d}\mathfrak{z}_v$ on $\mathfrak{D}(T, \Psi)$. Then $\mathbf{d}\mathfrak{z}$ is clearly invariant under the action of $\mathrm{U}(T, \Psi)_{\mathbf{a}}$.

Let $f, g \in \mathcal{M}_k(T, \Psi)$ and assume that either f or g belongs to $\mathcal{S}_k(T, \Psi)$. Take a congruence subgroup Γ of $\mathrm{U}(T, \Psi)$ so that $f, g \in \mathcal{M}_k(T, \Psi)(\Gamma)$ and put

$$\langle f, g\rangle = \mathrm{meas}(\Gamma \setminus \mathfrak{D}(T, \Psi))^{-1} \int_{\Gamma\setminus\mathfrak{D}(T,\Psi)} \overline{f(\mathfrak{z})} g(\mathfrak{z}) \left\{\prod_{v\in\mathbf{a}} \det(\eta_v(\mathfrak{z}))^{k_v}\right\} \mathbf{d}\mathfrak{z},$$

where η_v is as in section 1 and

$$\mathrm{meas}(\Gamma \setminus \mathfrak{D}(T, \Psi)) = \int_{\Gamma\setminus\mathfrak{D}(T,\Psi)} \mathbf{d}\mathfrak{z}.$$

Note that $\langle f, g\rangle$ is independent of the choice of Γ. It is clear that

$$\langle f|_k\alpha, g|_k\alpha\rangle = \langle f, g\rangle \tag{6.1}$$

for any $\alpha \in \mathrm{U}(T,\Psi)$.

To define Hecke operators, we have to consider adelized modular forms. Let D be a subgroup of $\mathrm{U}(T,\Psi)_A$ which is written as $D = \mathrm{U}(T,\Psi)_{\mathbf{a}} \times D_{\mathbf{h}}$ with some open compact subgroup $D_{\mathbf{h}}$ of $\mathrm{U}(T,\Psi)_{\mathbf{h}}$. For any $k \in \mathbb{Z}^{\mathbf{a}}$, we denote by $\mathcal{M}_k(T,\Psi)(D)$ the set of all functions $\mathbf{f} : \mathrm{U}(T,\Psi)_A \to \mathbb{C}$ satisfying the following conditions (1)–(3).

(1) $\mathbf{f}(xd_{\mathbf{h}}) = \mathbf{f}(x)$ for any $d_{\mathbf{h}} \in D_{\mathbf{h}}$.

(2) $\mathbf{f}(\beta x) = \mathbf{f}(x)$ for any $\beta \in \mathrm{U}(T,\Psi)$.

(3) For each $p \in \mathrm{U}(T,\Psi)_{\mathbf{h}}$, there exists an element $f_p \in \mathcal{M}_k(T,\Psi)$ such that $\mathbf{f}(py) = (f_p|_k y)(\mathbf{0})$ for any $y \in \mathrm{U}(T,\Psi)_{\mathbf{a}}$.

Then we easily have $f_p \in \mathcal{M}_k(T,\Psi)(pDp^{-1} \cap \mathrm{U}(T,\Psi))$. Using the strong approximation property of $\mathrm{U}_1(T,\Psi)$, we can take a finite subset $\mathcal{B}$ of $\mathrm{U}(T,\Psi)_{\mathbf{h}}$ so that

$$\mathrm{U}(T,\Psi)_A = \bigsqcup_{b\in\mathcal{B}} \mathrm{U}(T,\Psi)bD \qquad \text{(disjoint union)}. \tag{6.2}$$

Then the map $\mathbf{f} \to (f_b)_{b\in\mathcal{B}}$ gives a bijection from $\mathcal{M}_k(T,\Psi)(D)$ onto $\prod_{b\in\mathcal{B}} \mathcal{M}_k(T,\Psi)(bDb^{-1} \cap \mathrm{U}(T,\Psi))$.

We write simply $\mathbf{f} \leftrightarrow (f_p)_p$ or $\mathbf{f} \leftrightarrow (f_b)_{b\in\mathcal{B}}$ to indicate that f_p (resp. f_b) is determined by $\mathbf{f}$ for each $p \in \mathrm{U}(T,\Psi)_{\mathbf{h}}$ (resp. $b \in \mathcal{B}$) by (3) above. We denote by $\mathcal{S}_k(T,\Psi)(D)$ the set of all $\mathbf{f} \leftrightarrow (f_p)_p \in \mathcal{M}_k(T,\Psi)(D)$ so that $f_p \in \mathcal{S}_k(T,\Psi)$ for each $p \in \mathrm{U}(T,\Psi)_{\mathbf{h}}$.

Let $\mathbf{f} \leftrightarrow (f_b)_{b\in\mathcal{B}}, \mathbf{g} \leftrightarrow (g_b)_{b\in\mathcal{B}} \in \mathcal{M}_k(T,\Psi)(D)$ and assume that either $\mathbf{f}$ or $\mathbf{g}$ belongs to $\mathcal{S}_k(T,\Psi)(D)$. Then we define the inner product of $\mathbf{f}$ and $\mathbf{g}$ by

$$\langle \mathbf{f}, \mathbf{g} \rangle = |\mathcal{B}|^{-1} \sum_{b\in\mathcal{B}} \langle f_b, g_b \rangle,$$

where $|\mathcal{B}|$ denotes the number of elements in $\mathcal{B}$. We can easily verify that $\langle \mathbf{f}, \mathbf{g} \rangle$ is independent of the choice of $\mathcal{B}$.

The Galois action can also be constructed on the space of adelized modular forms.

Theorem 6.1. *For any $\mathbf{f} \leftrightarrow (f_p)_p \in \mathcal{M}_k(T,\Psi)(D)$ and any $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$, there exists $\mathbf{f}^{(\sigma;T,\Psi;\underline{a})} \leftrightarrow (\tilde{f}_{\tilde{p}})_{\tilde{p}} \in \mathcal{M}_{k^\sigma}(\tilde{T}, \Psi\sigma)(\tilde{D})$ such that $\tilde{f}_{\tilde{p}} = f_p^{(\sigma;T,\Psi;\underline{a})}$ if $\tilde{p} = B(\sigma;T,\Psi;\underline{a})^{-1} p B(\sigma;T,\Psi;\underline{a})$. Here $\tilde{T}$ is as in Theorem 5.1 and $\tilde{D}$ is a subgroup of $\mathrm{U}(\tilde{T}, \Psi\sigma)_A$ defined by $\tilde{D} = \mathrm{U}(\tilde{T}, \Psi\sigma)_{\mathbf{a}} \times \tilde{D}_{\mathbf{h}}$, where*

$$\tilde{D}_{\mathbf{h}} = B(\sigma;T,\Psi;\underline{a})^{-1} D_{\mathbf{h}} B(\sigma;T,\Psi;\underline{a}) \qquad (\subset \mathrm{U}(\tilde{T}, \Psi\sigma)_{\mathbf{h}}).$$

Proof. Take $\mathcal{B}$ as in (6.2) and put $\mathbf{f} \leftrightarrow (f_b)_{b\in\mathcal{B}}$. Set

$$\tilde{\mathcal{B}} = B(\sigma;T,\Psi;\underline{a})^{-1} \mathcal{B} B(\sigma;T,\Psi;\underline{a}) \subset \mathrm{U}(\tilde{T}, \Psi\sigma)_{\mathbf{h}}.$$

Then clearly

$$\mathrm{U}(\tilde{T}, \Psi\sigma)_A = \bigsqcup_{\tilde{b}\in\tilde{\mathcal{B}}} \mathrm{U}(\tilde{T}, \Psi\sigma)\tilde{b}\tilde{D} \qquad \text{(disjoint union)}.$$

We take $\mathbf{f}^{(\sigma;T,\Psi;\underline{a})} \leftrightarrow (\tilde{f}_{\tilde{b}})_{\tilde{b}\in\tilde{\mathcal{B}}}$ so that $\tilde{f}_{\tilde{b}} = f_b^{(\sigma;T,\Psi;\underline{a})}$ if $\tilde{b} = B(\sigma;T,\Psi;\underline{a})^{-1}b\, B(\sigma;T,\Psi;\underline{a})$.

Next let us consider f_p for any $p \in \mathrm{U}(T,\Psi)_{\mathbf{h}}$. Because of (6.2), p can be written as $p = \beta_{\mathbf{h}} b d_{\mathbf{h}}$ with some $b \in \mathcal{B}$, $d_{\mathbf{h}} \in D_{\mathbf{h}}$, and non-archimedean component $\beta_{\mathbf{h}}$ of some $\beta \in \mathrm{U}(T,\Psi)$. Then we have $f_p = f_b|_k\beta^{-1}$. Put $\tilde{p} = B(\sigma;T,\Psi;\underline{a})^{-1}pB(\sigma;T,\Psi;\underline{a}) \in \mathrm{U}(\tilde{T},\Psi\sigma)_{\mathbf{h}}$. From the strong approximation property of $\mathrm{U}_1(\tilde{T},\Psi\sigma)$, we can write $\tilde{p}$ as $\tilde{p} = \tilde{\beta}_{\mathbf{h}}\tilde{b}\tilde{d}_{\mathbf{h}}$ with $\tilde{b} = B(\sigma;T,\Psi;\underline{a})^{-1}b\, B(\sigma;T,\Psi;\underline{a})$, some $\tilde{d}_{\mathbf{h}} \in \tilde{D}_{\mathbf{h}}$, and non-archimedean component $\tilde{\beta}_{\mathbf{h}}$ of some $\tilde{\beta} \in \mathrm{U}(\tilde{T},\Psi\sigma)$ so that $\det(\tilde{\beta}) = \det(\beta)$. Then we have

$$\tilde{f}_{\tilde{p}} = \tilde{f}_{\tilde{b}}|_{k^\sigma}\tilde{\beta}^{-1},$$

where $\mathbf{f}^{(\sigma;T,\Psi;\underline{a})} \leftrightarrow (\tilde{f}_{\tilde{p}})_{\tilde{p}}$. Then clearly

$$\beta_{\mathbf{h}}^{-1} \in (bD_{\mathbf{h}}b^{-1} \cap \mathrm{U}_1(T,\Psi)_{\mathbf{h}})B(\sigma;T,\Psi;\underline{a})\tilde{\beta}_{\mathbf{h}}^{-1}B(\sigma;T,\Psi;\underline{a})^{-1}.$$

Using Lemma 5.5, we obtain

$$\tilde{f}_{\tilde{p}} = \tilde{f}_{\tilde{b}}|_{k^\sigma}\tilde{\beta}^{-1} = (f_b|_k\beta^{-1})^{(\sigma;T,\Psi;\underline{a})} = f_p^{(\sigma;T,\Psi;\underline{a})}.$$

□

Obviously, $\mathbf{f}^{(\sigma;T,\Psi;\underline{a})} \in \mathcal{S}_{k^\sigma}(\tilde{T},\Psi\sigma)(\tilde{D})$ if and only if $\mathbf{f} \in \mathcal{S}_k(T,\Psi)(D)$.

For D as above and any $y \in \mathrm{U}(T,\Psi)_{\mathbf{h}}$, put

$$DyD = \bigsqcup_{\eta\in Y} D\eta \qquad \text{(disjoint union)},$$

where Y is a finite set contained in $\mathrm{U}(T,\Psi)_{\mathbf{h}}$. For $\mathbf{f} \in \mathcal{M}_k(T,\Psi)(D)$, we define $\mathbf{f}|DyD$ by

$$(\mathbf{f}|DyD)(x) = \sum_{\eta\in Y}\mathbf{f}(x\eta^{-1}) \tag{6.3}$$

for any $x \in \mathrm{U}(T,\Psi)_A$. Note that $\mathbf{f}|DyD$ does not depend on the choice of Y, and $\mathbf{f}|DyD \in \mathcal{M}_k(T,\Psi)(D)$. By a computation, we obtain

$$(\mathbf{f}|DyD)^{(\sigma;T,\Psi;\underline{a})} = \mathbf{f}^{(\sigma;T,\Psi;\underline{a})}|\tilde{D}\tilde{y}\tilde{D} \tag{6.4}$$

for any $(\sigma;T,\Psi;\underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$, where $\tilde{y} = B(\sigma;T,\Psi;\underline{a})^{-1}yB(\sigma;T,\Psi;\underline{a}) \in \mathrm{U}(\tilde{T},\Psi\sigma)_{\mathbf{h}}$, and $\tilde{D}$ is as defined in Theorem 6.1. The following lemma is proved in 11.7 of [12].

Lemma 6.2. *Let $\mathbf{f},\mathbf{g} \in \mathcal{M}_k(T,\Psi)(D)$ and assume that either $\mathbf{f}$ or $\mathbf{g}$ belongs to $\mathcal{S}_k(T,\Psi)(D)$. Then we have*

$$\langle \mathbf{f}|DyD, \mathbf{g}\rangle = \langle \mathbf{f}, \mathbf{g}|Dy^{-1}D\rangle,$$

for any $y \in \mathrm{U}(T,\Psi)_{\mathbf{h}}$.

Hecke operators with respect to unitary groups are defined in Section 11 of [12]. Let V be a vector space over K of finite dimension and ϕ be a non-degenerate hermitian form on V over K. For each non-archimedean prime $\mathfrak{p}$ of F, put

$$V_{\mathfrak{p}} := V \otimes_{\mathcal{O}_F} \mathcal{O}_{\mathfrak{p}}.$$

Take an $\mathcal{O}_K$-lattice L in V. We denote by $\mu_\phi(L)$ the fractional ideal of $\mathcal{O}_F$ generated by $\{\phi(l,l) \,|\, l \in L\}$. Put $\mathcal{O}_{K_{\mathfrak{p}}} = \mathcal{O}_K \otimes_{\mathcal{O}_F} \mathcal{O}_{\mathfrak{p}}$ and take an $\mathcal{O}_{K_{\mathfrak{p}}}$-lattice $\hat{L}$ in $V_{\mathfrak{p}}$. We denote by $\mu_\phi(\hat{L})$ the fractional ideal of $\mathcal{O}_{\mathfrak{p}}$ generated by $\left\{\phi(l,l) \,\middle|\, l \in \hat{L}\right\}$. An $\mathcal{O}_K$-lattice L in V is called maximal if there exists no $\mathcal{O}_K$-lattice L' in V such that $L \subsetneqq L'$ and $\mu_\phi(L) = \mu_\phi(L')$. Similarly, an $\mathcal{O}_{K_{\mathfrak{p}}}$-lattice $\hat{L}$ in $V_{\mathfrak{p}}$ is called maximal if there exists no $\mathcal{O}_{K_{\mathfrak{p}}}$-lattice $\hat{L}'$ in $V_{\mathfrak{p}}$ such that $\hat{L} \subsetneqq \hat{L}'$ and $\mu_\phi(\hat{L}) = \mu_\phi(\hat{L}')$.

For an $\mathcal{O}_K$-lattice L in V, define the $\mathcal{O}_{K_{\mathfrak{p}}}$-lattice $L_{\mathfrak{p}}$ in $V_{\mathfrak{p}}$ by $L_{\mathfrak{p}} = L \otimes_{\mathcal{O}_F} \mathcal{O}_{\mathfrak{p}}$. Then the following lemma is proved in 8.10 of [12].

Lemma 6.3. *Take an $\mathcal{O}_K$-lattice L in V. For each finite prime $\mathfrak{p}$ of F, we have $\mu_\phi(L_{\mathfrak{p}}) = \mu_\phi(L)_{\mathfrak{p}}$, where $\mu_\phi(L)_{\mathfrak{p}}$ denotes the fractional ideal of $\mathcal{O}_{\mathfrak{p}}$ generated by $\mu_\phi(L)$. Moreover, L is maximal if and only if $L_{\mathfrak{p}}$ is maximal for each finite prime $\mathfrak{p}$ of F.*

For an $\mathcal{O}_K$-lattice L in V and $x \in \mathrm{GL}_K(V)_A$, there exists a unique $\mathcal{O}_K$-lattice Lx in V which satisfies $(Lx)_{\mathfrak{p}} = L_{\mathfrak{p}} x_{\mathfrak{p}}$ for each finite prime $\mathfrak{p}$ of F, where $x_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-component of x. This is proved in 8.3 of [12]. Clearly, we have $(Lx_1)x_2 = L(x_1x_2)$ for any $x_1, x_2 \in \mathrm{GL}_K(V)_A$.

For a "normal" skew-hermitian matrix T with respect to a CM-type Ψ and $s \in K^\times$ so that $s^\rho = -s$, we define a hermitian form $\phi_{T,s}$ on K^1_m as

$$\phi_{T,s}(y_1, y_2) = y_1 (s^{-1}T)^t y_2^\rho \qquad (y_1, y_2 \in K^1_m).$$

Let L be a maximal $\mathcal{O}_K$-lattice in K^1_m with respect to $\phi_{T,s}$. For any $x \in \mathrm{U}(T,\Psi)_A$, Lemma 6.3 shows that Lx is also a maximal $\mathcal{O}_K$-lattice with respect to $\phi_{T,s}$. Moreover, for any $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$, we can easily verify that the lattice $LB(\sigma; T, \Psi; \underline{a})$ is a maximal $\mathcal{O}_K$-lattice with respect to $\phi_{\tilde{T},s}$, where $\tilde{T}$ is as defined in Theorem 5.1.

As stated in 11.10 of [12], we can take a maximal $\mathcal{O}_K$-lattice M in K^1_m (with respect to $\phi_{T,s}$) such that $\mu_{\phi_{T,s}}(M) = \mathcal{O}_F$. Take the dual lattice $\hat{M}$ of M with respect to $\phi_{T,s}$, that is,

$$\hat{M} = \left\{x \in K^1_m \,\middle|\, \phi_{T,s}(x, M) \subset \mathfrak{d}^{-1}\right\},$$

where $\mathfrak{d}$ denotes the different of K relative to F. For any integral ideal $\mathfrak{c}$ of $\mathcal{O}_F$, take a subgroup $D = D(\mathfrak{c}, M)$ of $\mathrm{U}(T,\Psi)_A$ as

$$D = D(\mathfrak{c}, M) = \left\{ x \in \mathrm{U}(T,\Psi)_A \;\middle|\; \begin{array}{l} Mx = M \text{ and } \hat{M}_{\mathfrak{p}}(x_{\mathfrak{p}} - 1_m) \subset \mathfrak{c}_{\mathfrak{p}} M_{\mathfrak{p}} \\ \text{for any finite prime } \mathfrak{p} \text{ of } F \\ \text{such that } \mathfrak{p}|\mathfrak{c} \end{array} \right\}.$$

For such D and any finite prime $\mathfrak{p}$ of F, put

$$D_{\mathfrak{p}} = D \cap \mathrm{U}(T, \Psi)_{\mathfrak{p}},$$

where $\mathrm{U}(T, \Psi)_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-component of $\mathrm{U}(T, \Psi)_A$. Then we have

$$D = \mathrm{U}(T, \Psi)_{\mathbf{a}} \times \prod_{\mathfrak{p} \in \mathbf{h}_F} D_{\mathfrak{p}},$$

where $\mathbf{h}_F$ denotes the set of all finite primes of F. We define a subgroup $\mathfrak{X}$ of $\mathrm{U}(T, \Psi)_A$ by

$$\mathfrak{X}_{\mathbf{h}} = \{ y \in \mathrm{U}(T, \Psi)_{\mathbf{h}} \,|\, y_{\mathfrak{p}} \in D_{\mathfrak{p}} \text{ for every } \mathfrak{p} | \mathfrak{c} \},$$
$$\mathfrak{X} = \mathrm{U}(T, \Psi)_{\mathbf{a}} \times \mathfrak{X}_{\mathbf{h}}.$$

Let $\mathfrak{R}(D, \mathfrak{X})$ be the free $\mathbb{Z}$-module generated by the set $\{DxD \,|\, x \in \mathfrak{X}\}$. This is the so-called Hecke ring. (For details, see 11.10 of [12].) An element of $\mathfrak{R}(D, \mathfrak{X})$ acts on $\mathcal{M}_k(T, \Psi)(D)$ by (6.3).

Next let us consider the action of $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$. Put $\tilde{M} = M \cdot B(\sigma; T, \Psi; \underline{a})$. Then $\tilde{M}$ is a maximal $\mathcal{O}_K$-lattice in K_m^1 with respect to $\phi_{\tilde{T},s}$. In the same way, we take

$$\begin{aligned}
\hat{\tilde{M}} &= \left\{ x \in K_m^1 \,\middle|\, \phi_{\tilde{T},s}(x, \tilde{M}) \subset \mathfrak{d}^{-1} \right\} \\
&= \hat{M} \cdot B(\sigma; T, \Psi; \underline{a}), \\
\tilde{D} &= \tilde{D}(\mathfrak{c}, \tilde{M}) \\
&= \left\{ \tilde{x} \in \mathrm{U}(\tilde{T}, \Psi\sigma)_A \;\middle|\; \begin{array}{l} \tilde{M}\tilde{x} = \tilde{M} \text{ and } \hat{\tilde{M}}_{\mathfrak{p}}(\tilde{x}_{\mathfrak{p}} - 1_m) \subset \mathfrak{c}_{\mathfrak{p}} \tilde{M}_{\mathfrak{p}} \\ \text{for any finite prime } \mathfrak{p} \text{ of } F \\ \text{such that } \mathfrak{p}|\mathfrak{c} \end{array} \right\}.
\end{aligned}$$

We can easily verify $\tilde{D} = \mathrm{U}(\tilde{T}, \Psi\sigma)_{\mathbf{a}} \times \prod_{\mathfrak{p} \in \mathbf{h}_F} \tilde{D}_{\mathfrak{p}}$, where $\tilde{D}_{\mathfrak{p}} = \tilde{D} \cap \mathrm{U}(\tilde{T}, \Psi\sigma)_{\mathfrak{p}}$. Moreover, we have

$$\prod_{\mathfrak{p} \in \mathbf{h}_F} \tilde{D}_{\mathfrak{p}} = B(\sigma; T, \Psi; \underline{a})^{-1} \left(\prod_{\mathfrak{p} \in \mathbf{h}_F} D_{\mathfrak{p}} \right) B(\sigma; T, \Psi; \underline{a}),$$

and hence $\tilde{D}$ coincides with that defined in Theorem 6.1. Put

$$\begin{aligned}
\tilde{\mathfrak{X}}_{\mathbf{h}} &= \left\{ \tilde{y} \in \mathrm{U}(\tilde{T}, \Psi\sigma)_{\mathbf{h}} \,\middle|\, \tilde{y}_{\mathfrak{p}} \in \tilde{D}_{\mathfrak{p}} \text{ for every } \mathfrak{p}|\mathfrak{c} \right\} \\
&= B(\sigma; T, \Psi; \underline{a})^{-1} \mathfrak{X}_{\mathbf{h}} B(\sigma; T, \Psi; \underline{a}), \\
\tilde{\mathfrak{X}} &= \mathrm{U}(\tilde{T}, \Psi\sigma)_{\mathbf{a}} \times \tilde{\mathfrak{X}}_{\mathbf{h}}.
\end{aligned}$$

Hence the map $Dx_{\mathbf{h}}D \mapsto \tilde{D}B(\sigma; T, \Psi; \underline{a})^{-1} x_{\mathbf{h}} B(\sigma; T, \Psi; \underline{a})\tilde{D}$ (where $x_{\mathbf{h}} \in \mathfrak{X}_{\mathbf{h}}$) gives a ring isomorphism from $\mathfrak{R}(D, \mathfrak{X})$ onto $\mathfrak{R}(\tilde{D}, \tilde{\mathfrak{X}})$.

For any $\mathcal{O}_K$-ideal $\mathfrak{a}$, we denote by $\mathfrak{T}(\mathfrak{a}; D, \mathfrak{X}) \in \mathfrak{R}(D, \mathfrak{X})$ the operator $T(\mathfrak{a})$ defined in 26.10 of [14], corresponding to D and $\mathfrak{X}$. In the same way, we denote by $\mathfrak{T}(\mathfrak{a}; \tilde{D}, \tilde{\mathfrak{X}}) \in \mathfrak{R}(\tilde{D}, \tilde{\mathfrak{X}})$ the operator $T(\mathfrak{a})$ corresponding to $\tilde{D}$ and $\tilde{\mathfrak{X}}$. Now, if

$$\mathfrak{T}(\mathfrak{a}; D, \mathfrak{X}) = \sum_{x_{\mathbf{h}}} D x_{\mathbf{h}} D,$$

then we easily see

$$\mathfrak{T}(\mathfrak{a}^{\rho}; D, \mathfrak{X}) = \sum_{x_{\mathbf{h}}} D x_{\mathbf{h}}^{-1} D, \tag{6.5}$$

and

$$\mathfrak{T}(\mathfrak{a}; \tilde{D}, \tilde{\mathfrak{X}}) = \sum_{x_{\mathbf{h}}} \tilde{D} B(\sigma; T, \Psi; \underline{a})^{-1} x_{\mathbf{h}} B(\sigma; T, \Psi; \underline{a}) \tilde{D}. \tag{6.6}$$

Combining (6.5) and Lemma 6.2, we obtain

$$\langle \mathbf{f} | \mathfrak{T}(\mathfrak{a}; D, \mathfrak{X}), \mathbf{g} \rangle = \langle \mathbf{f}, \mathbf{g} | \mathfrak{T}(\mathfrak{a}^{\rho}; D, \mathfrak{X}) \rangle, \tag{6.7}$$

if $\mathbf{f}, \mathbf{g} \in \mathcal{M}_k(T, \Psi)(D)$ and either $\mathbf{f}$ or $\mathbf{g}$ belongs to $\mathcal{S}_k(T, \Psi)(D)$. We have the following proposition.

Proposition 6.4. *Take any* $\mathbf{f} \in \mathcal{M}_k(T, \Psi)(D)$ *and* $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$*, and let* $\tilde{T}$ *and* $\tilde{D}$ *be defined as above. Then the following assertions hold.*
(1) *For any* $\mathcal{O}_K$*-ideal* $\mathfrak{a}$*,*

$$(\mathbf{f} | \mathfrak{T}(\mathfrak{a}; D, \mathfrak{X}))^{(\sigma; T, \Psi; \underline{a})} = \mathbf{f}^{(\sigma; T, \Psi; \underline{a})} | \mathfrak{T}(\mathfrak{a}; \tilde{D}, \tilde{\mathfrak{X}}).$$

(2) *Assume* $\mathbf{f} | \mathfrak{T}(\mathfrak{a}; D, \mathfrak{X}) = \lambda(\mathfrak{a}) \mathbf{f}$ *with some* $\lambda(\mathfrak{a}) \in \mathbb{C}$*. Then we have* $\mathbf{f}^{(\sigma; T, \Psi; \underline{a})} | \mathfrak{T}(\mathfrak{a}; \tilde{D}, \tilde{\mathfrak{X}}) = \lambda(\mathfrak{a})^{\sigma} \mathbf{f}^{(\sigma; T, \Psi; \underline{a})}$.

The asssertion (1) follows immediately from (6.4), and (2) is an immediate consequence of (1). From this proposition, we easily obtain the following lemma.

Lemma 6.5. *Take an* $\mathcal{O}_K$*-ideal* $\mathfrak{a}$*. Assume that* $\mathbf{f} | \mathfrak{T}(\mathfrak{a}; D, \mathfrak{X}) = \lambda(\mathfrak{a}) \mathbf{f}$ *with some non-zero* $\mathbf{f} \in \mathcal{M}_k(T, \Psi)(D)$ *and* $\lambda(\mathfrak{a}) \in \mathbb{C}$*. Then* $\lambda(\mathfrak{a})$ *is an algebraic number.*

Proof. For any $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$, we can take $(\sigma; T, \Psi; 1^{m-2q+1}) \in C_{(T,\Psi)}(\mathbb{C})$, where $1^{m-2q+1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ $(\in (K_A^{\times})^{m-2q+1})$. Then we have $B(\sigma; T, \Psi; 1^{m-2q+1}) = 1_m$, $\mathbf{f}^{(\sigma; T, \Psi; 1^{m-2q+1})} \in \mathcal{M}_k(T, \Psi)(D)$, and

$$\mathbf{f}^{(\sigma; T, \Psi; 1^{m-2q+1})} | \mathfrak{T}(\mathfrak{a}; D, \mathfrak{X}) = \lambda(\mathfrak{a})^{\sigma} \mathbf{f}^{(\sigma; T, \Psi; 1^{m-2q+1})}.$$

Since $\mathcal{M}_k(T,\Psi)(D)$ is a finite dimensional vector space (over $\mathbb{C}$), $\mathfrak{T}(\mathfrak{a};D,\mathfrak{X})$ has only finitely many eigenvalues. This means that $\{\lambda(\mathfrak{a})^\sigma \,|\sigma\in\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})\}$ is a finite set. If $\lambda(\mathfrak{a})$ is not algebraic, it is an infinite set. Hence we have $\lambda(\mathfrak{a})\in\overline{\mathbb{Q}}$. $\square$

We have more precise results about Hecke eigenvalues as follows.

Theorem 6.6. *Take $0\neq\mathbf{f}\in\mathcal{S}_k(T,\Psi)(D)$ and assume that $\mathbf{f}|\mathfrak{T}(\mathfrak{a};D,\mathfrak{X}) = \lambda(\mathfrak{a})\mathbf{f}$ with $\lambda(\mathfrak{a})\in\mathbb{C}$ for any $\mathcal{O}_K$-ideal $\mathfrak{a}$. Then the field $\mathbb{Q}(\{\lambda(\mathfrak{a})\}_\mathfrak{a})$ is a CM-field or a totally real algebraic number field of finite degree over $\mathbb{Q}$.*

Proof. First let us prove that each $\lambda(\mathfrak{a})$ is contained in a CM-field. From Lemma 6.2 and (6.5), we have

$$\langle\mathbf{f}|\mathfrak{T}(\mathfrak{a};D,\mathfrak{X}),\mathbf{f}\rangle=\langle\mathbf{f},\mathbf{f}|\mathfrak{T}(\mathfrak{a}^\rho;D,\mathfrak{X})\rangle.$$

Since $\mathbf{f}$ is a common eigenform of all $\mathfrak{T}(\mathfrak{a};D,\mathfrak{X})$, we obtain $\overline{\lambda(\mathfrak{a})}=\lambda(\mathfrak{a}^\rho)$. For any $(\sigma;T,\Psi;\underline{a})\in C_{(T,\Psi)}(\mathbb{C})$, take $\tilde{D}$ and $\tilde{\mathfrak{X}}$ as above. Then we have

$$\begin{aligned}\mathbf{f}^{(\sigma;T,\Psi;\underline{a})}|\mathfrak{T}(\mathfrak{a};\tilde{D},\tilde{\mathfrak{X}})&=\lambda(\mathfrak{a})^\sigma\mathbf{f}^{(\sigma;T,\Psi;\underline{a})},\\ \mathbf{f}^{(\sigma;T,\Psi;\underline{a})}|\mathfrak{T}(\mathfrak{a}^\rho;\tilde{D},\tilde{\mathfrak{X}})&=(\overline{\lambda(\mathfrak{a})})^\sigma\mathbf{f}^{(\sigma;T,\Psi;\underline{a})}.\end{aligned}$$

On the other hand, we have

$$\langle\mathbf{f}^{(\sigma;T,\Psi;\underline{a})}|\mathfrak{T}(\mathfrak{a};\tilde{D},\tilde{\mathfrak{X}}),\mathbf{f}^{(\sigma;T,\Psi;\underline{a})}\rangle=\langle\mathbf{f}^{(\sigma;T,\Psi;\underline{a})},\mathbf{f}^{(\sigma;T,\Psi;\underline{a})}|\mathfrak{T}(\mathfrak{a}^\rho;\tilde{D},\tilde{\mathfrak{X}})\rangle.$$

Hence we have $\overline{\lambda(\mathfrak{a})^\sigma}=(\overline{\lambda(\mathfrak{a})})^\sigma$ for any $\sigma\in\mathrm{Aut}(\mathbb{C})$. This means that each field $\mathbb{Q}(\lambda(\mathfrak{a}))$ is a CM-field or a totally real algebraic number field since $\lambda(\mathfrak{a})$ is algebraic.

Next we will prove that the field $\mathbb{Q}(\{\lambda(\mathfrak{a})\}_\mathfrak{a})$ is finite degree over $\mathbb{Q}$. Let K' be the Galois closure of K over $\mathbb{Q}$. Then K' is also a CM-field. Let $\mathbf{h}'$ be the set of all non-archimedean primes of K' and we denote by $K'_{\mathbf{h}'}$ (resp. ${K'}^\times_{\mathbf{h}'}$) the non-archimedean component of the adele ring K'_A (resp. idele group ${K'}^\times_A$). The archimedean component of K'_A (resp. ${K'}^\times_A$) is denoted by K'_∞ (resp. ${K'}^\times_\infty$).

For $\sigma\in\mathrm{Aut}(\mathbb{C}/K')$ and $b\in{K'}^\times_{\mathbf{h}'}$ ($\subset{K'}^\times_A$) so that $[b^{-1},K']=\sigma|_{K'_{ab}}$, we define

$$N_{(T,\Psi)}(b)=\begin{pmatrix}N'_\Psi\circ N_{K'/K^*_\Psi}(b)\\ N'_{\Psi(T,1)}\circ N_{K'/K^*_{\Psi(T,1)}}(b)\\ \vdots\\ N'_{\Psi(T,m-2q)}\circ N_{K'/K^*_{\Psi(T,m-2q)}}(b)\end{pmatrix}\in(K^\times_A)^{m-2q+1}.$$

Then we have $(\sigma;T,\Psi;N_{(T,\Psi)}(b))\in C_{(T,\Psi)}(\mathbb{C})$. Moreover, we obtain $\Psi\sigma=\Psi$, $B(\sigma;T,\Psi;N_{(T,\Psi)}(b))\in\mathrm{GU}(T,\Psi)_\mathbf{h}$ and $f^{(\sigma;T,\Psi;N_{(T,\Psi)}(b))}\in\mathcal{M}_k(T,\Psi)$ for any $f\in\mathcal{M}_k(T,\Psi)$.

Take an open subgroup P_D of $\prod_{\mathfrak{p}'\in\mathbf{h}'}\mathcal{O}^\times_{\mathfrak{p}'}$ ($\subset{K'}^\times_{\mathbf{h}'}$) so that $B(\sigma;T,\Psi;N_{(T,\Psi)}(b))\in K^\times_A D$ for any $b\in P_D$ and $\sigma\in\mathrm{Aut}(\mathbb{C}/K')$ corresponding to b.

Then $P_D\overline{K'^\times K'^\times_\infty}/\overline{K'^\times K'^\times_\infty}$ is an open subgroup of $K'^\times_A/\overline{K'^\times K'^\times_\infty}$, where the bars mean the closures in $K'^\times_A$. Let K'_D be the finite abelian extension of K' corresponding to $P_D\overline{K'^\times K'^\times_\infty}/\overline{K'^\times K'^\times_\infty}$. For any $\sigma \in \mathrm{Aut}(\mathbb{C}/K'_D)$, take $b(\sigma) \in P_D$ so that $[b(\sigma)^{-1}, K'] = \sigma|_{K'_{ab}}$ and consider the action of $(\sigma; T, \Psi; N_{(T,\Psi)}(b(\sigma)))$ on $\mathcal{M}_k(T,\Psi)(D)$. Then its image is also contained in $\mathcal{M}_k(T,\Psi)(D)$, since we have $\tilde{T} = T$, $\Psi\sigma = \Psi$, $k^\sigma = k$ and $\tilde{D} = D$ from $B(\sigma; T, \Psi; N_{(T,\Psi)}(b(\sigma))) \in K^\times_A D$. Moreover, we also have $\tilde{\mathfrak{X}} = \mathfrak{X}$ in this case. Hence, if $\mathbf{f} \in \mathcal{S}_k(T,\Psi)(D)$ satisfies $\mathbf{f}|\mathfrak{T}(\mathfrak{a}; D, \mathfrak{X}) = \lambda(\mathfrak{a})\mathbf{f}$ with $\lambda(\mathfrak{a}) \in \mathbb{C}$ for each $\mathfrak{a}$, then

$$\mathbf{f}^{(\sigma;T,\Psi;N_{(T,\Psi)}(b(\sigma)))}|\mathfrak{T}(\mathfrak{a}; D, \mathfrak{X}) = \lambda(\mathfrak{a})^\sigma \mathbf{f}^{(\sigma;T,\Psi;N_{(T,\Psi)}(b(\sigma)))} \tag{6.8}$$

for any $\mathfrak{a}$. The space $\mathcal{S}_k(T,\Psi)(D)$ is of finite dimension over $\mathbb{C}$ and common eigenforms of $\{\mathfrak{T}(\mathfrak{a}; D, \mathfrak{X})\}_\mathfrak{a}$ are mutually orthogonal if they have different eigenvalues for some $\mathfrak{a}$. Hence there are only finitely many common eigenforms in $\mathcal{S}_k(T,\Psi)(D)$ whose eigenvalues are different. Now, the equation (6.8) implies that $\mathbb{Q}(\{\lambda(\mathfrak{a})\}_\mathfrak{a})$ is finite degree over K'_D, and hence over $\mathbb{Q}$. □

In a certain case (if $\mathfrak{c}$ is sufficiently small), the Hecke ring $\mathfrak{R}(D, \mathfrak{X})$ is commutative and hence the space $\mathcal{S}_k(T,\Psi)(D)$ is spanned by common eigenforms of $\{\mathfrak{T}(\mathfrak{a}; D, \mathfrak{X})\}_\mathfrak{a}$. (For details, see §20 of [12].) In this case, for any $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$, and corresponding $\tilde{T}$, $\tilde{D}$, $\tilde{\mathfrak{X}}$ as above, we easily see that $\mathcal{S}_{k^\sigma}(\tilde{T}, \Psi\sigma)$ is also spanned by common eigenforms of $\left\{\mathfrak{T}(\mathfrak{a}; \tilde{D}, \tilde{\mathfrak{X}})\right\}_\mathfrak{a}$, since $\mathfrak{R}(\tilde{D}, \tilde{\mathfrak{X}})$ is isomorphic to $\mathfrak{R}(D, \mathfrak{X})$.

We can easily see that $B(\sigma; T, \Psi; \underline{a})^{-1} B(\rho\sigma\rho; T, \Psi; \underline{a}^\rho) \in \mathrm{U}(\tilde{T}, \Psi\sigma)_\mathbf{h}$. For $\mathbf{f} \in \mathcal{M}_k(T,\Psi)(D)$, we define $\mathbf{f}^{(\sigma;T,\Psi;\underline{a}),-} \in \mathcal{M}_{k^\sigma}(\tilde{T}, \Psi\sigma)(\tilde{D})$ by

$$\mathbf{f}^{(\sigma;T,\Psi;\underline{a}),-}(\tilde{x}) = \mathbf{f}^{(\rho\sigma\rho;T,\Psi;\underline{a}^\rho)}\left(\tilde{x}(B(\sigma; T, \Psi; \underline{a})^{-1} B(\rho\sigma\rho; T, \Psi; \underline{a}^\rho))\right)$$

for each $\tilde{x} \in \mathrm{U}(\tilde{T}, \Psi\sigma)_A$. Then we have the following conjecture.

Conjecture. *Let $0 \neq \mathbf{f}, \mathbf{g}_1$ and $\mathbf{g}_2 \in \mathcal{S}_k(T,\Psi)(D)$ be common eigenforms of $\{\mathfrak{T}(\mathfrak{a}; D, \mathfrak{X})\}_\mathfrak{a}$ having the same eigenvalue for each $\mathfrak{a}$. For any $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$, we have $\langle \mathbf{f}^{(\sigma;T,\Psi;\underline{a}),-}, \mathbf{f}^{(\sigma;T,\Psi;\underline{a})}\rangle \neq 0$ and*

$$\frac{\langle \mathbf{g}_1^{(\sigma;T,\Psi;\underline{a}),-}, \mathbf{g}_2^{(\sigma;T,\Psi;\underline{a})}\rangle}{\langle \mathbf{f}^{(\sigma;T,\Psi;\underline{a}),-}, \mathbf{f}^{(\sigma;T,\Psi;\underline{a})}\rangle} = \left\{\frac{\langle \mathbf{g}_1, \mathbf{g}_2\rangle}{\langle \mathbf{f}, \mathbf{f}\rangle}\right\}^\sigma.$$

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