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An asymptotic estimate for the hitting time of a half-line by two-dimensional Brownian motion*

By

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Abstract

The joint law of the first hitting time of a half-line, the first hitting place, and some local time for the two-dimensional Brownian motion is studied. The method is based on the continuous-time version of the fluctuation identity (the Wiener-Hopf decomposition) in two dimension.

1. Introduction and the result

Let $S(N) = (S_1(N), S_2(N))$ be a random walk on $\mathbb{Z}^2$ starting from the origin, V_+ be the nonnegative half of the first coordinate axis:

$$V_+ = \left\{(n, 0) \in \mathbb{Z}^2 | n \geq 0\right\}$$

and $\tau_{V+} = \inf\{N \geq 1 | S(N) \in V_+\}$. Lawler [7] obtained the following estimate when $S(N)$ is the simple random walk on $\mathbb{Z}^2$:

$$P[\tau_{V+} > k] \asymp k^{-1/4} \quad \text{as } k \to \infty$$

where $\asymp$ means the ratio of the both sides remains bounded, away from both 0 and ∞. This result is applied to estimates for the Hausdorff dimension of the clusters in the diffusion limited aggregation model. Then Fukai [3] showed that

$$P[\tau_{V+} > k] \sim \frac{C_0(\phi)}{k^{1/4}} \quad \text{as } k \to +\infty$$

for a wider class of random walks under a symmetry condition and an integrability condition where $\sim$ means the ratio of both sides converges to 1. The constant $C_0(\phi)$ is expressed in terms of the characteristic function $\phi(\theta_1, \theta_2) = E[\exp(i\theta_1 S_1(1) + i\theta_2 S_2(1))]$.

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Let $L(0) = 0$ and $L(N) = \#\{1 \le j \le N | S_2(j) = 0\}$. In [6], the author studied a trivariate asymptotic estimate involving τ_{V+} as well as the first hitting place $S_1(\tau_{V+})$ and the sojourn time $L(\tau_{V+})$ on the first axis up to τ_{V+}. Assume that $S(N)$ is genuinely two-dimensional, centered, and square-integrable and satisfies either $\phi(\theta_1, \theta_2) = \phi(-\theta_1, \theta_2)$ or $\phi(\theta_1, \theta_2) = \phi(-\theta_1, -\theta_2)$. Then for any $\mu_0 \ge 0$, $\mu_1 \ge 0$, $\mu_2 \ge 0$ such that $\mu_0 + \mu_1 + \mu_2 > 0$, there exists $C_1(\mu_0, \mu_1, \mu_2, \phi) > 0$ such that

$$1 - E[e^{-\mu_0 s^2 L(\tau_{V+}) - \mu_1 s^2 S_1(\tau_{V+}) - \mu_2 s^4 \tau_{V+}}] \sim C_1(\mu_0, \mu_1, \mu_2, \phi)s, \tag{1.1}$$

as $s \to +0$. Moreover, we have

$$C_1(\mu_0, \mu_1, \mu_2, \phi) = C_3^{-1/4} \Gamma\left(\frac{3}{4}\right) C_0(\phi) \exp\left(I\left(\mu_0, \sqrt{C_2}\mu_1, \frac{C_3 \mu_2}{2}\right)\right).$$

Here $I(\mu_0, \sqrt{C_2}\mu_1, \frac{C_3\mu_2}{2})$, C_2 and C_3 are as follows. Let $q_{ij} = E[S_i(1)S_j(1)]$ for $i, j \in \{1, 2\}$ and $2\pi/A_2$ be the smallest positive θ_2 such that $\phi(0, \theta_2) = 1$. Then we set $C_2 := (q_{11}q_{22} - q_{12}^2)/A_2^2 \equiv (\det \mathrm{Cov}(S, S))/A_2^2$, $C_3 := (2q_{22})/A_2^2$, and

$$I(\mu_0, \mu_1, \mu_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log\left(\mu_0 + \sqrt{\mu_1^2 t^2 + 2\mu_2}\right)}{t^2 + 1} dt. \tag{1.2}$$

Let us remark that making $s \to +0$ in (1.1) is equivalent to observing the long-time behavior of the random walk $S(N)$, i.e., the event that $S(N)$ travels away from V_+ for a long time and finally hits V_+ at $S_1(\tau_{V+})$ after $L(\tau_{V+})$ visits to the negative first axis.

In view of Donsker's invariance principle we can ignore some detail of the sample paths of $S(N)$ and this event should be associated with the event that a two-dimensional Brownian motion hits a half-line after a long excursion.

It then may be a natural question to ask if there is an estimate concerning the two-dimensional Brownian motion similar to (1.1). In the present paper we determine the joint law and obtain an estimate involving the first hitting time of the nonnegative half of the first axis in the plane by the Brownian motion as well as the first hitting place and the local time on the negative half of the first axis.

Let $(B_1(t), B_2(t))$ be a standard Brownian motion on $\mathbb{R}^2$ starting from (x, y). Its law and expectation are denoted by $P_{(x,y)}$ and $E_{(x,y)}$ respectively. Let $L_2(t)$ be the local time at 0 for $B_2(\cdot)$: $L_2(t) = \lim_{\varepsilon \to +0} \frac{1}{2\varepsilon} \int_0^t 1_{(-\varepsilon,\varepsilon)}(B_2(s))ds$. For $a \in \mathbb{R}$, we set

$$\tau(a) = \inf\{t \ge 0 | B_2(t) = 0, B_1(t) \ge a\}. \tag{1.3}$$

Then $\tau(0)$ is the first hitting time of the non-negative half of the first coordinate axis. Furthermore, $L_2(\tau(0))$ and $B_1(\tau(0))$ are respectively the local time spent before $\tau(0)$ on the negative half of the first coordinate axis and the first hitting place on the positive half.

Our main result is the following.

Let $\mathbb{C}_+ = \{z \in \mathbb{C} | \Im z > 0\}$, $\overline{\mathbb{C}_+} = \{z \in \mathbb{C} | \Im z \geq 0\}$ and set

$$\varphi(z;\mu_0,\mu_2) = \exp\left(\frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{z}{t^2 - z^2} \log(\mu_0 + \sqrt{t^2 + 2\mu_2})dt\right) \tag{1.4}$$

for $z \in \mathbb{C}_+$ and $\mu_i \geq 0$ $(i = 0, 2)$ such that $\mu_0 + \mu_2 > 0$. We extend $\varphi(z;\mu_0,\mu_2)$ for $z \in \mathbb{R}$ by continuity, as is shown in Lemma 2.2. We also set

$$\varphi(z;0,0) = 1/\sqrt{-iz}$$

for $z \in \overline{\mathbb{C}_+} \setminus \{0\}$ where we emply the branch such that $\sqrt{1} = 1$.

Theorem 1.1. *Let $a > 0$, $\mu_i \geq 0$ $(i = 0, 1, 2)$, and $\mu_0 + \mu_1 + \mu_2 > 0$.*
(i) *It holds*

$$\begin{aligned} &E_{(-a,0)}\left[e^{-\mu_0 L_2(\tau(0)) - \mu_1 B_1(\tau(0)) - \mu_2 \tau(0)}\right] \\ &\quad = e^{\mu_1 a} - \frac{e^{\mu_1 a}}{\varphi(i\mu_1;\mu_0,\mu_2)}\int_{-\infty}^{\infty} d\theta \frac{1 - e^{-(\mu_1 + i\theta)a}}{2\pi(\mu_1 + i\theta)}\varphi(\theta;\mu_0,\mu_2). \end{aligned}$$

(ii) *As $s \to +0$*

$$1 - E_{(-a,0)}\left[e^{-\mu_0 s^2 L_2(\tau(0)) - \mu_1 s^2 B_1(\tau(0)) - \mu_2 s^4 \tau(0)}\right] \sim \frac{2\sqrt{a}}{\sqrt{\pi}}\exp(I(\mu_0,\mu_1,\mu_2))s,$$

where $\sim$ means the ratio of the both sides converges to 1.

Corollary 1.1. *Setting $\mu_1 = 0$ and $\mu_1 = \mu_2 = 0$ respectively, we have by a Tauberian theorem*

$$1 - E_{(-a,0)}\left[e^{-\mu_0 s^2 L_2(\tau(0)) - \mu_2 s^4 \tau(0)}\right] \sim \frac{2\sqrt{a}\sqrt{\mu_0 + \sqrt{2\mu_2}}}{\sqrt{\pi}}s \qquad \text{as } s \to +0$$

for $\mu_0 + \mu_2 > 0$ and

$$P_{(-a,0)}\left[\tau(0) > A\right] \sim \frac{2^{5/4}\sqrt{a}}{\sqrt{\pi}\Gamma(3/4)}A^{-1/4} \qquad \text{as } A \to +\infty.$$

Let us make several remarks here.

The law of $B_1(\tau(0))$ is equivalent to what is known as the overshoot for the Cauchy process. See the remark after the proof of Lemma 2.3.

We could not find the definite integral $I(\mu_0,\mu_1,\mu_2)$ as in (1.2) in the huge volume [4].

The reason why $I(\mu_0,\mu_1,\mu_2)$ appears in (1.1) seems different from that in Theorem 1.1(ii). In [6], what is studied is the logarithm of some characteristic function and its integral with respect to the Poisson kernel. In a limiting procedure, $1/(t^2+1)$ in (1.2) appears from the Poisson kernel. In the present paper, we prove in Lemma 2.2(ii) that $I(\mu_0,\mu_1,\mu_2)$ is associated with the multiple Laplace transform of the inverse local time and the supremum process of

Brownian motion. We prove it using the fluctuation identities in two dimensions, (2.8) and (2.9), and the Frullani integral (2.12) that yields a logarithm.

Comparing Theorem 1.1 with (1.1), we could imagine that the two dimensional Brownian motion falls in the same 'class' as the random walks with $C_0(\phi) = 2^{5/4}\pi^{-1/2}\Gamma(\frac{3}{4})^{-1}$, $C_2 = 1$ and $C_3 = 2$ but we have not obtained precise formulation.

This paper is organized as follows. In Section 2, we prove a series of lemmas and the main theorem concering the Brownian motion starting from $(-a, 0)$. In the proof of Lemmas 2.1–2.4, we use some arguments that are similar to those in [5]. The most important one among them is the fluctuation identity in two-dimension: (2.8) and (2.9) that we quote from [5], Theorem 1.

In Section 3, we study the case of other starting points.

2. Proof

Let $\tau_2(t)$ be the right-continuous inverse of the local time $L_2(t)$ at 0 for $B_2(\cdot)$:

$$\tau_2(t) = \inf\{u \geq 0 | L_2(u) > t\}$$

and let $\xi(t) = B_1(\tau_2(t))$, $\eta(t) = \tau_2(t)$. Then $(\xi(t), \eta(t))$ is a two dimensional Lévy process with the following Fourier-Laplace exponent:

$$E_{(x,0)}[e^{i\theta(\xi(t)-x)-\mu_2\eta(t)}] = e^{-t\sqrt{2\mu_2+\theta^2}} \quad \text{for } \theta \in \mathbb{R} \text{ and } \mu_2 \geq 0. \tag{2.1}$$

Indeed, the following is a local martingale.

$$M(t) = \exp\Big((L_2(t) - |B_2(t)|)\sqrt{\theta^2 + 2\mu_2} + i\theta(B_1(t) - x) - \mu_2 t\Big).$$

We stop $M(t)$ at $\tau_2(t)$ to obtain a bounded martingale. Then we have

$$1 = E_{(x,0)}[e^{t\sqrt{\theta^2+2\mu_2}+i\theta(\xi(t)-x)-\mu_2\eta(t)}],$$

which is equivalent to (2.1).

The process $(\xi(t), \eta(t))$ inherits the scaling property from $(B_1(t), B_2(t))$; for any $c > 0$ it holds

$$t \mapsto (\xi(ct), \eta(ct)) \text{ under } P_{(cx,0)} \stackrel{\text{law}}{=} t \mapsto (c\xi(t), c^2\eta(t)) \text{ under } P_{(x,0)}. \tag{2.2}$$

It also possesses the following translation invariance:

$$t \mapsto (\xi(t), \eta(t)) \text{ under } P_{(x,0)} \stackrel{\text{law}}{=} t \mapsto (\xi(t) + x, \eta(t)) \text{ under } P_{(0,0)}. \tag{2.3}$$

We next set $\overline{\xi}(t) = \sup_{0 \leq u \leq t} \xi(u)$ and $\sigma(a) = \inf\{t \geq 0 | \xi(t) \geq a\}$ for $a \in \mathbb{R}$. Then it is immediately seen that $L_2(\tau(0)) = \sigma(0)$, $B_1(\tau(0)) = \overline{\xi}(\sigma(0))$, and $\tau(0) = \eta(\sigma(0))$, where $\tau(a)$ is defined in (1.3).

Lemma 2.1. *Let $a > 0$, $\mu_i \geq 0$ $(i = 0,1,2)$, and $\mu_0 + \mu_1 + \mu_2 > 0$. Then*

$$\left(1 - E_{(0,0)}\left[e^{-\mu_0\sigma(a)-\mu_1\overline{\xi}(\sigma(a))-\mu_2\eta(\sigma(a))}\right]\right) \cdot \left(\int_0^\infty dt\, E_{(0,0)}\left[e^{-\mu_0 t-\mu_1\overline{\xi}(t)-\mu_2\eta(t)}\right]\right) = \int_0^\infty dt\, E_{(0,0)}\left[e^{-\mu_0 t-\mu_1\overline{\xi}(t)-\mu_2\eta(t)}; \overline{\xi}(t) < a\right].$$

Proof. We have only to use the strong Markov property of $(\xi(t), \eta(t))$ at $\sigma(a)$. □

We now redefine the function $\varphi(z;\mu_0,\mu_2)$. The coincidence of two definitions will be shown in Lemma 2.2(iii). Let $\mathbb{C}_+ = \{z \in \mathbb{C} | \Im z > 0\}$, $\overline{\mathbb{C}_+} = \{z \in \mathbb{C} | \Im z \geq 0\}$ and set

$$\varphi(z;\mu_0,\mu_2) = \sqrt{\mu_0 + \sqrt{2\mu_2}}\int_0^\infty dt\, E_{(0,0)}\left[e^{-\mu_0 t+iz\overline{\xi}(t)-\mu_2\eta(t)}\right] \tag{2.4}$$

for $z \in \overline{\mathbb{C}_+}$ and $\mu_i \geq 0$ $(i = 0,2)$ such that $\mu_0 + \mu_2 > 0$. We set

$$\varphi(z;0,0) = 1/\sqrt{-iz} \tag{2.5}$$

for $z \in \overline{\mathbb{C}_+} \setminus \{0\}$ where we emply the branch such that $\sqrt{1} = 1$.

Lemma 2.2. *Let $\mu_i \geq 0$ $(i = 0,1,2)$ and $\mu_0 + \mu_2 > 0$.*

(i) *$z \mapsto \varphi(z;\mu_0,\mu_2)$ is holomorphic on $\mathbb{C}_+$, continuous and non-zero on $\overline{\mathbb{C}_+}$ and is bounded by $\varphi(0;\mu_0,\mu_2) = \frac{1}{\sqrt{\mu_0+\sqrt{2\mu_2}}}$ on $\overline{\mathbb{C}_+}$. Moreover it obeys the following parity law: $\Re\varphi(-x+iy;\mu_0,\mu_2) = \Re\varphi(x+iy;\mu_0,\mu_2)$, and $\Im\varphi(-x+iy;\mu_0,\mu_2) = -\Im\varphi(x+iy;\mu_0,\mu_2)$ for any $x \in \mathbb{R}$ and $y \geq 0$.*

(ii) *On the positive imaginary axis we have*

$$\varphi(iy;\mu_0,\mu_2) = \exp\left(\frac{-1}{2\pi}\int_{-\infty}^\infty \frac{1}{t^2+1}\log(\mu_0 + \sqrt{y^2t^2 + 2\mu_2})dt\right) \tag{2.6}$$

for $y \geq 0$ and, in an equivalent form,

$$\varphi(i\mu_1;\mu_0,\mu_2) = \exp(-I(\mu_0,\mu_1,\mu_2)). \tag{2.7}$$

(iii) *For any $z \in \mathbb{C}_+$,*

$$\varphi(z;\mu_0,\mu_2) = \exp\left(\frac{-1}{2\pi i}\int_{-\infty}^\infty \frac{z}{t^2-z^2}\log(\mu_0 + \sqrt{t^2 + 2\mu_2})dt\right).$$

(iv) *We have $|\varphi(\theta;\mu_0,\mu_2)| = \frac{1}{\sqrt{\mu_0+\sqrt{\theta^2+2\mu_2}}}$ if $\theta \in \mathbb{R}$ and as $|\theta| \to \infty$,*

$$\varphi(\theta;\mu_0,\mu_2) \sim \frac{e^{(\operatorname{sgn}\theta)\frac{\pi}{4}i}}{\sqrt{|\theta|}}.$$

Proof. The statements (iii) and (iv) will imply that $\varphi(z;\mu_0,\mu_2)$ never vanishes on $\mathbb{C}_+$ and on $\mathbb{R}$, respectively. (2.1) implies $\varphi(0;\mu_0,\mu_2) = \frac{1}{\sqrt{\mu_0+\sqrt{2\mu_2}}}$. The other statements in (i) can be verified by standard arguments.

To prove (ii), we first quote Theorem 1 in [5]:

$$\begin{aligned}&\sqrt{\mu_0+\sqrt{2\mu_2}}\varphi(z;\mu_0,\mu_2)\\ &\quad=\exp\left(\int_0^\infty \frac{e^{-\mu_0 t}dt}{t}E\left[\left(e^{iz\xi(t)}-1\right)e^{-\mu_2\eta(t)};\xi(t)>0\right]\right),\end{aligned} \tag{2.8}$$

$$|\varphi(\theta;\mu_0,\mu_2)|^2 = \varphi(\theta;\mu_0,\mu_2)\varphi(-\theta;\mu_0,\mu_2) = \frac{1}{\mu_0+\sqrt{2\mu_2+\theta^2}}, \tag{2.9}$$

for any $z \in \overline{\mathbb{C}_+}$ and any $\theta \in \mathbb{R}$.

When $y = 0$, the formula (2.6) is reduced to the following. On one hand, $\varphi(0;\mu_0,\mu_2) = \frac{1}{\sqrt{\mu_0+\sqrt{2\mu_2}}}$. On the other hand,

$$\frac{-1}{2\pi}\int_{-\infty}^{\infty}\frac{\log\left(\mu_0+\sqrt{2\mu_2}\right)}{t^2+1}dt = \frac{-1}{2}\log(\mu_0+\sqrt{2\mu_2})$$

and the equality in (2.6) holds.

We then assume $y > 0$. We need

$$e^{-yx}1_{\{x>0\}} = \int_{-\infty}^{\infty} d\theta \frac{1}{2\pi(y+i\theta)}e^{i\theta x} \quad \text{for } x \neq 0 \tag{2.10}$$

along with

$$\sup_{x\in\mathbb{R},A\geq 2y}\left|\int_{-A}^{A} d\theta\frac{1}{2\pi(y+i\theta)}e^{i\theta x}\right| < +\infty. \tag{2.11}$$

These can be proven by some elementary argument using the residue theorem. We also need the Frullani integral:

$$\int_0^\infty \frac{1}{t}\left(e^{-\alpha t}-e^{-\beta t}\right)dt = \log\beta - \log\alpha. \tag{2.12}$$

By (2.8),

$$\begin{aligned}&\log\left(\sqrt{\mu_0+\sqrt{2\mu_2}}\varphi(iy;\mu_0,\mu_2)\right)\\ &\quad=\int_0^\infty\frac{e^{-\mu_0 t}dt}{t}E_{(0,0)}\left[\left(e^{-y\xi(t)}-1\right)e^{-\mu_2\eta(t)};\xi(t)>0\right]\\ &\quad=\int_0^\infty\frac{e^{-\mu_0 t}dt}{t}\Big(E_{(0,0)}\left[e^{-y\xi(t)-\mu_2\eta(t)};\xi(t)>0\right]\\ &\qquad-E_{(0,0)}\left[e^{-\mu_2\eta(t)};\xi(t)>0\right]\Big).\end{aligned}$$

Then by virtue of (2.10), (2.11) and symmetry of $\xi(t)$, the above is equal to

$$\begin{aligned}&\int_0^\infty \frac{e^{-\mu_0 t}dt}{t}\left(E_{(0,0)}\left[\int_{-\infty}^\infty \frac{e^{i\theta\xi(t)-\mu_2\eta(t)}}{2\pi(y+i\theta)}d\theta\right]-\frac{1}{2}E_{(0,0)}\left[e^{-\mu_2\eta(t)}\right]\right)\\ &=\int_0^\infty \frac{e^{-\mu_0 t}dt}{t}\left(\int_{-\infty}^\infty \frac{e^{-t\sqrt{\theta^2+2\mu_2}}}{2\pi(y+i\theta)}d\theta-\frac{1}{2}e^{-t\sqrt{2\mu_2}}\right).\end{aligned}$$

Extracting the real part in the integrand, we obtain

$$\int_0^\infty \frac{e^{-\mu_0 t}dt}{t}\int_{-\infty}^\infty \frac{y}{2\pi(y^2+\theta^2)}\left(e^{-t\sqrt{\theta^2+2\mu_2}}-e^{-t\sqrt{2\mu_2}}\right)d\theta.$$

Here we have used $\int_{-\infty}^\infty \frac{y}{2\pi(y^2+\theta^2)}d\theta=1/2$. Then by interchange of integrations and by (2.12), the above is equal to

$$\begin{aligned}&\int_{-\infty}^\infty \frac{y\,d\theta}{2\pi(y^2+\theta^2)}\left(\log(\mu_0+\sqrt{2\mu_2})-\log(\mu_0+\sqrt{\theta^2+2\mu_2})\right)\\ &\quad=\frac{1}{2}\log(\mu_0+\sqrt{2\mu_2})-\int_{-\infty}^\infty \frac{y\,d\theta}{2\pi(y^2+\theta^2)}\log(\mu_0+\sqrt{\theta^2+2\mu_2})\\ &\quad=\frac{1}{2}\log(\mu_0+\sqrt{2\mu_2})-I(\mu_0,y,\mu_2).\end{aligned}$$

Rearrangement of terms gives us (2.6).

(iii) follows from the uniqueness theorem for holomorphic functions.

The first statement in (iv) follows from (2.9). To prove the second we fix $\mu_0\ge 0$ and $\mu_2\ge 0$ such that $\mu_0+\mu_2>0$ and set

$$f(z)=\exp\left(\frac{-2}{2\pi i}\int_0^\infty \frac{z}{t^2-z^2}\log(\mu_0+\sqrt{t^2+2\mu_2})dt\right)$$

for $z\in\mathbb{C}_+$ so that $f(z)=\varphi(z;\mu_0,\mu_2)$ on $\mathbb{C}_+$. We also set

$$g(z)=\exp\left(\frac{-2}{2\pi i}\int_0^\infty \frac{\exp(-\frac{\pi}{4}i)z}{-iu^2-z^2}\log(\mu_0+\sqrt{-iu^2+2\mu_2})du\right),$$

which is holomorphic on $\{z\in\mathbb{C}|-\pi/4<\arg z<3\pi/4\}$. If $0<\arg z<3\pi/4$ $g(z)$ is obtained from $f(z)$ by substituting $t=u\exp(-\frac{\pi}{4}i)$. Hence $g(z)=f(z)=\varphi(z;\mu_0,\mu_2)$ on $\{z\in\mathbb{C}|0<\arg z<3\pi/4\}$. Since $\varphi(z;\mu_0,\mu_2)$ is continuous on $\mathbb{R}$, $\varphi(\theta;\mu_0,\mu_2)=g(\theta)$ if $\theta\in\mathbb{R}$ and $\theta>0$.

By substituting $u=\theta x$, we have

$$\varphi(\theta;\mu_0,\mu_2)=\exp\left(\frac{-2}{2\pi i}\int_0^\infty \frac{\exp(-\frac{\pi}{4}i)}{-ix^2-1}\log(\mu_0+\sqrt{-i\theta^2x^2+2\mu_2})dx\right).$$

It can be shown by standard arguments that

$$\frac{\exp(\frac{\pi}{4}i)}{\sqrt{\theta}}=\exp\left(\frac{-2}{2\pi i}\int_0^\infty \frac{\exp(-\frac{\pi}{4}i)}{-ix^2-1}\log(\sqrt{-i\theta^2x^2})dx\right)$$

if $\theta\in\mathbb{R}$ and $\theta>0$. It is then elementary to deduce $\sqrt{\theta}\exp(-\frac{\pi}{4}i)\varphi(\theta;\mu_0,\mu_2)\to 1$ as $\theta\to+\infty$. If we make $\theta\to-\infty$ we take the complex conjugate. $\square$

Remark. As a check of (ii) and (iii) in Lemma 2.2, we present a convincing argument to deduce (1.4) from (2.4) assuming (i) and

$$|\log \varphi(z;\mu_0,\mu_2)| \le C_1 + C_2|z|^\nu \quad \text{for any } z \in \overline{\mathbb{C}_+} \tag{2.13}$$

with some $C_1 > 0$, $C_2 > 0$, and $0 < \nu < 1$. We employ the branch of $\log z$ such that $\log \varphi(0;\mu_0,\mu_2)$ is real. Note that $\varphi(z;\mu_0,\mu_2)$ in (1.4) falls in this class.

Let $z \in \mathbb{C}_+$, $R > 1 + |z|$ and C_R be the half-circle $t = Re^{is}$, $0 \le s \le \pi$ and set $r_z(t) = \frac{1}{t-z} - \frac{1}{t+z} = \frac{2z}{t^2-z^2}$. By the residue theorem we have

$$\frac{1}{2\pi i}\left(\int_{C_R} + \int_{-R}^{R}\right) r_z(t)\log\varphi(t;\mu_0,\mu_2)dt = \log\varphi(z;\mu_0,\mu_2),$$

where $\int_{-R}^{R}$ denotes the integral along the real axis. Since $r_z(t) = O(\frac{1}{t^2})$ as $t \to \pm\infty$, $\int_{C_R} r_z(t)\log\varphi(t;\mu_0,\mu_2)dt = O(R^{\nu-1})$ and hence

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} r_z(t)\log\varphi(t;\mu_0,\mu_2)dt = \log\varphi(z;\mu_0,\mu_2).$$

Note that the first statement in (iv) is a consequence of (2.9), independent of (ii) and (iii), and we may rely on $\Re\log\varphi(t;\mu_0,\mu_2) = -\frac{1}{2}\log(\mu_0 + \sqrt{t^2+2\mu_2})$.

Since $r_z(-t) = r_z(t)$ and $\log\varphi(-t;\mu_0,\mu_2) = \overline{\log\varphi(t;\mu_0,\mu_2)}$,

$$\begin{aligned}\log\varphi(z;\mu_0,\mu_2) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} r_z(-t)\log\varphi(-t;\mu_0,\mu_2)dt \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} r_z(t)\overline{\log\varphi(t;\mu_0,\mu_2)}dt.\end{aligned}$$

By taking the mean, we have

$$\begin{aligned}\log\varphi(z;\mu_0,\mu_2) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} r_z(t)\Re\log\varphi(t;\mu_0,\mu_2)dt \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{-z}{t^2-z^2}\log(\mu_0+\sqrt{t^2+2\mu_2})dt\end{aligned}$$

for any $z \in \mathbb{C}_+$, which coincides with (1.4).

We should be careful when we determine $\log\varphi(z;\mu_0,\mu_2)$ on $\mathbb{C}_+$ by using the real part of its boundary value on $\mathbb{R}$, i.e., neglecting the imaginary part. For any $c \in \mathbb{R}$, $z \mapsto icz + \log\varphi(z;\mu_0,\mu_2)$ has the same real part on $\mathbb{R}$ but is excluded by the condition (2.13). Since we have not deduced (2.13) from (2.4), we can not dispense the argument in the proof of (ii) in Lemma 2.2.

Note also that, for any $c > 0$, $e^{icz}\varphi(z;\mu_0,\mu_2)$ is bounded, non-zero on $\overline{\mathbb{C}_+}$, and $|e^{icz}\varphi(z;\mu_0,\mu_2)| = |\varphi(z;\mu_0,\mu_2)|$ on $\mathbb{R}$, which notifies that being bounded and non-zero on $\overline{\mathbb{C}_+}$ is insufficient for uniqueness of $\varphi(z;\mu_0,\mu_2)$ satisfying $|\varphi(\theta;\mu_0,\mu_2)| = 1/\sqrt{\mu_0+\sqrt{\theta^2+2\mu_2}}$ for $\theta \in \mathbb{R}$.

It would be natural to imagine that the second statement in Lemma 2.2(iv) i.e. $\lim_{\theta\to\pm\infty}\arg\varphi(\theta;\mu_0,\mu_2) = \pm\frac{\pi}{4}$, instead of (2.13), might be helpful in proving uniqueness. In fact, Lemma 2.2(iv) can be deduced from (2.4) and (2.8) by an argument similar to [5]. But the author has not succeeded in incorporating it.

At the end of this long remark, we note yet another possible approach based on uniqueness of the Wiener-Hopf factorization, see e.g. [1, p. 165]. It is elementary to show that $\varphi(z;\mu_0,\mu_2)$ defined in (1.4) satisfies $\lim_{\varepsilon\to+0}\Re\log\varphi(\theta+i\varepsilon;\mu_0,\mu_2) = \frac{-1}{2}\log\left(\mu_0+\sqrt{\theta^2+2\mu_2}\right)$. If one could prove that $\varphi(\theta;\mu_0,\mu_2)$ is the characteristic function of a non-negative infinitely divisible distribution without drift, then the factorization $\varphi(\theta;\mu_0,\mu_2)\varphi(-\theta;\mu_0,\mu_2) = \frac{1}{\mu_0+\sqrt{\theta^2+2\mu_2}}$ in (2.9) should be unique and one conclude that (1.4) and (2.4) coincide with each other.

Lemma 2.3. *For any $a>0$ and $\mu_i\geq 0$ $(i=0,1,2)$ such that $\mu_0+\mu_1+\mu_2>0$,*

$$\begin{aligned}(2.14)\qquad &1-E_{(0,0)}\left[e^{-\mu_0\sigma(a)-\mu_1\overline{\xi}(\sigma(a))-\mu_2\eta(\sigma(a))}\right]\\ &\quad=\frac{1}{\varphi(i\mu_1;\mu_0,\mu_2)}\int_{-\infty}^{\infty}d\theta\frac{1-e^{-(\mu_1+i\theta)a}}{2\pi(\mu_1+i\theta)}\varphi(\theta;\mu_0,\mu_2).\end{aligned}$$

Proof. In view of Lemmas 2.1 and 2.2, the statement is equivalent to the following if $\mu_0+\mu_2>0$.

$$\begin{aligned}(2.15)\qquad &\int_0^{\infty}dt\,E_{(0,0)}\left[e^{-\mu_0 t-\mu_1\overline{\xi}(t)-\mu_2\eta(t)};\overline{\xi}(t)<a\right]\\ &\quad=\int_{-\infty}^{\infty}d\theta\frac{1-e^{-(\mu_1+i\theta)a}}{2\pi(\mu_1+i\theta)}\left(\int_0^{\infty}dt\,E_{(0,0)}\left[e^{-\mu_0 t+i\theta\overline{\xi}(t)-\mu_2\eta(t)}\right]\right).\end{aligned}$$

To begin with, assume $\mu_0+\mu_2>0$ and observe that the right hand side of (2.15) is convergent by using Lemma 2.2(iv) and

$$(2.16)\qquad \sup_{\mu_1\geq 0}\left|\frac{1-e^{-(\mu_1+i\theta)a}}{2\pi(\mu_1+i\theta)}\right| = O\left(\frac{1}{1+|\theta|}\right).$$

In particular, the right hand side of (2.15) is equal to

$$\lim_{A\to+\infty}\int_{-A}^{A}d\theta\frac{1-e^{-(\mu_1+i\theta)a}}{2\pi(\mu_1+i\theta)}\left(\int_0^{\infty}dtE_{(0,0)}\left[e^{-\mu_0 t+i\theta\overline{\xi}(t)-\mu_2\eta(t)}\right]\right).$$

Since

$$\int_0^{\infty}dte^{-\mu_0 t-\mu_2\eta(t)}\cdot P_{(0,0)}$$

is a finite measure with the total mass $\frac{1}{\mu_0+\sqrt{2\mu_2}}$ and

$$\int_{-A}^{A}d\theta\left|\frac{1-e^{-(\mu_1+i\theta)a}}{2\pi(\mu_1+i\theta)}\right| = O(\log A)$$

by (2.16), we have by the Fubini theorem

$$\int_{-A}^{A} d\theta \frac{1-e^{-(\mu_1+i\theta)a}}{2\pi(\mu_1+i\theta)} \left(\int_0^\infty dt E_{(0,0)} \left[e^{-\mu_0 t + i\theta\overline{\xi}(t) - \mu_2\eta(t)} \right] \right)$$
$$= \int_0^\infty dt E_{(0,0)} \left[e^{-\mu_0 t - \mu_2\eta(t)} \int_{-A}^{A} d\theta \frac{1-e^{-(\mu_1+i\theta)a}}{2\pi(\mu_1+i\theta)} e^{i\theta\overline{\xi}(t)} \right].$$

We then need

$$e^{-\mu_1 x} 1_{\{0<x<a\}} = \lim_{A\to+\infty} \int_{-A}^{A} d\theta \frac{1-e^{-(\mu_1+i\theta)a}}{2\pi(\mu_1+i\theta)} e^{i\theta x} \quad \text{for } x \notin \{0,a\} \tag{2.17}$$

and

$$\sup_{x\in\mathbb{R}, A\geq 2\mu_1} \left| \int_{-A}^{A} d\theta \frac{1-e^{-(\mu_1+i\theta)a}}{2\pi(\mu_1+i\theta)} e^{i\theta x} \right| < +\infty. \tag{2.18}$$

Here (2.17) and (2.18) can be proven by some elementary argument using the residue theorem. Lemma 2.2(iv) also implies the law of $\overline{\xi}(t)$ has no point mass. Putting these together we can conclude

$$\lim_{A\to+\infty} \int_0^\infty dt E_{(0,0)} \left[e^{-\mu_0 t - \mu_2\eta(t)} \int_{-A}^{A} d\theta \frac{1-e^{-(\mu_1+i\theta)a}}{2\pi(\mu_1+i\theta)} e^{i\theta\overline{\xi}(t)} \right]$$
$$= \int_0^\infty dt E_{(0,0)} \left[e^{-\mu_0 t - \mu_1\overline{\xi}(t) - \mu_2\eta(t)}; \overline{\xi}(t) < a \right],$$

which is equal to the left hand side of (2.15) and the proof for the case $\mu_0+\mu_2 > 0$ is complete.

For the case $\mu_0 = \mu_2 = 0$, we replace μ_0 and μ_2 with $c\mu_0$ and $c^2\mu_2$, respectively, and then make $c \to +0$. So we start with $\mu_i > 0$ $(i = 0,1,2)$. By (2.2),

$$\varphi(z; c\mu_0, c^2\mu_2) = c^{-1/2}\varphi(c^{-1}z; \mu_0, \mu_2) \quad \text{for any } c > 0 \text{ and } z \in \overline{\mathbb{C}_+}.$$

Then

$$\varphi(\theta; c\mu_0, c^2\mu_2) \leq \frac{1}{\sqrt{c}\sqrt{\mu_0+\sqrt{2\mu_2}}} \quad \text{for any } \theta \in \mathbb{R},$$

$$\lim_{c\to+0} \varphi(\theta; c\mu_0, c^2\mu_2) = \frac{e^{(\operatorname{sgn}\theta)\frac{\pi}{4}i}}{\sqrt{|\theta|}} \quad \text{for any } \theta \in \mathbb{R}. \tag{2.19}$$

by Lemma 2.2(iv). Note that the ratio of the both sides of (2.19) converges uniformly to 1 on $\theta \in (-\infty,-\varepsilon)\cup(\varepsilon,\infty)$ for any $\varepsilon > 0$. It is then elementary to verify

$$\begin{aligned} &\lim_{c\to+0} \frac{1}{\varphi(i\mu_1; c\mu_0, c^2\mu_2)} \int_{-\infty}^{\infty} d\theta \frac{1-e^{-(\mu_1+i\theta)a}}{2\pi(\mu_1+i\theta)} \varphi(\theta; c\mu_0, c^2\mu_2) \\ &= \sqrt{\mu_1} \int_{-\infty}^{\infty} d\theta \frac{1-e^{-(\mu_1+i\theta)a}}{2\pi(\mu_1+i\theta)} \frac{e^{(\operatorname{sgn}\theta)\frac{\pi}{4}i}}{\sqrt{|\theta|}} \end{aligned} \tag{2.20}$$

since $\lim_{c\to+0} I(c\mu_0,\mu_1,c^2\mu_2) = \log\sqrt{\mu_1}$. On the other hand, by this Lemma for the case $\mu_0+\mu_2>0$, already verified, the above is equal to

$$\lim_{c\to+0}\left(1-E_{(0,0)}\left[e^{-c\mu_0\sigma(a)-\mu_1\overline{\xi}(\sigma(a))-c^2\mu_2\eta(\sigma(a))}\right]\right) = 1-E_{(0,0)}\left[e^{-\mu_1\overline{\xi}(\sigma(a))}\right].$$

Since $\mu_1>0$ and $\varphi(z;0,0)$ is defined in (2.5), the proof is complete. □

Remark. In the terminology of Chap. VI in [1], $\overline{\xi}(\sigma(a))-a$ is the overshoot for the Cauchy process $\xi(t)$. Adopting Exercise VI.1 and Lemma VIII.1 in [1], we have the following double Laplace transform:

$$\int_0^\infty da\, e^{-qa}\left(1-E_{(0,0)}\left[e^{-\mu_1\overline{\xi}(\sigma(a))}\right]\right) = \frac{\sqrt{\mu_1}}{q\sqrt{q+\mu_1}}.$$

On the other hand, the Laplace transform of (2.20) yields

$$\begin{aligned}\int_0^\infty da\, e^{-qa}\sqrt{\mu_1}\int_{-\infty}^\infty d\theta\frac{1-e^{-(\mu_1+i\theta)a}}{2\pi(\mu_1+i\theta)}\frac{e^{(\operatorname{sgn}\theta)\frac{\pi}{4}i}}{\sqrt{|\theta|}}\\ = \frac{\sqrt{\mu_1}}{q}\int_{-\infty}^\infty d\theta\frac{e^{(\operatorname{sgn}\theta)\frac{\pi}{4}i}}{2\pi(q+\mu_1+i\theta)\sqrt{|\theta|}}.\end{aligned}$$

The coincidence of these is verified by simple application of the residue theorem.

Lemma 2.4. *For any $\mu_i\ge 0$ $(i=0,1,2)$ such that $\mu_0+\mu_2>0$ we have*

$$1-E_{(0,0)}\left[e^{-\mu_0\sigma(a)-\mu_1\overline{\xi}(\sigma(a))-\mu_2\eta(\sigma(a))}\right]\sim\frac{2}{\sqrt{\pi}}\exp(I(\mu_0,\mu_1,\mu_2))\sqrt{a},$$

as $a\to+0$.

Proof. Combining Lemmas 2.2 and 2.3 we obtain, as $a\to+0$,

$$\begin{aligned}&1-E_{(0,0)}\left[e^{-\mu_0\sigma(a)-\mu_1\overline{\xi}(\sigma(a))-\mu_2\eta(\sigma(a))}\right]\\ &=\frac{1}{\varphi(i\mu_1;\mu_0,\mu_2)}\int_{-\infty}^\infty d\theta\frac{1-e^{-(\mu_1+i\theta)a}}{2\pi(\mu_1+i\theta)}\varphi(\theta;\mu_0,\mu_2)\\ &\sim\exp(I(\mu_0,\mu_1,\mu_2))\int_{-\infty}^\infty d\theta\frac{e^{i\frac{\pi}{4}(\operatorname{sgn}\theta)}\left(1-e^{-(\mu_1+i\theta)a}\right)}{2\pi(\mu_1+i\theta)\sqrt{|\theta|}}\\ &\sim\sqrt{a}\exp(I(\mu_0,\mu_1,\mu_2))\int_{-\infty}^\infty dx\frac{e^{i\frac{\pi}{4}(\operatorname{sgn}x)}\left(1-e^{-ix}\right)}{2\pi ix\sqrt{|x|}}\\ &=\frac{2}{\sqrt{\pi}}\exp(I(\mu_0,\mu_1,\mu_2))\sqrt{a},\end{aligned}$$

where we set $x=a\theta$ in the second approximation. □

Proof of Theorem 1.1. Recall that $L_2(\tau(0))=\sigma(0)$, $B_1(\tau(0))=\overline{\xi}(\sigma(0))$, $\tau(0)=\eta(\sigma(0))$. Using this relation and by the translation (2.3) of the whole process as well as the half-line to be hit, we have

$$\begin{aligned} &E_{(-a,0)}\left[e^{-\mu_0 L_2(\tau(0))-\mu_1 B_1(\tau(0))-\mu_2\tau(0)}\right] \\ &= E_{(-a,0)}\left[e^{-\mu_0\sigma(0)-\mu_1\overline{\xi}(\sigma(0))-\mu_2\eta(\sigma(0))}\right] \\ &= E_{(0,0)}\left[e^{-\mu_0\sigma(a)-\mu_1\left(\overline{\xi}(\sigma(a))-a\right)-\mu_2\eta(\sigma(a))}\right]. \end{aligned}$$

Then Lemma 2.3 completes the proof of (i).

By the scaling property (2.2) and by the translation (2.3) of the whole process as well as the half-line to be hit, we have

$$\begin{aligned} &1 - E_{(-a,0)}\left[e^{-\mu_0 s^2 L_2(\tau(0))-\mu_1 s^2 B_1(\tau(0))-\mu_2 s^4\tau(0)}\right] \\ &= 1 - E_{(-as^2,0)}\left[e^{-\mu_0 L_2(\tau(0))-\mu_1 B_1(\tau(0))-\mu_2\tau(0)}\right] \\ &= 1 - e^{\mu_1 as^2}E_{(0,0)}\left[e^{-\mu_0\sigma(as^2)-\mu_1\overline{\xi}(\sigma(as^2))-\mu_2\eta(\sigma(as^2))}\right] \\ &= (1-e^{\mu_1 as^2}) + e^{\mu_1 as^2}\left(1 - E_{(0,0)}\left[e^{-\mu_0\sigma(as^2)-\mu_1\overline{\xi}(\sigma(as^2))-\mu_2\eta(\sigma(as^2))}\right]\right). \end{aligned}$$

By Lemma 2.4 we obtain

$$\begin{aligned} &1 - E_{(-a,0)}\left[e^{-\mu_0 s^2 L_2(\tau(0))-\mu_1 s^2 B_1(\tau(0))-\mu_2 s^4\tau(0)}\right] \\ &= O(s^2) + e^{\mu_1 as^2}\frac{2\sqrt{a}s(1+o(1))}{\sqrt{\pi}}\exp(I(\mu_0,\mu_1,\mu_2)) \\ &= \frac{2\sqrt{a}}{\sqrt{\pi}}\exp(I(\mu_0,\mu_1,\mu_2))s(1+o(1)) \quad \text{as } s\to +0. \end{aligned}$$

□

3. Starting points outside the first axis

We close this note by extending Theorem 1.1 to the starting points outside the first coordinate axis. We set $f(x) = E_{(x,0)}\left[e^{-\mu_0 L_2(\tau(0))-\mu_1 B_1(\tau(0))-\mu_2\tau(0)}\right]$.

Let $T_{2,0}$ be the first hitting time of 0 by $B_2(\cdot)$. Then

$$P_{(x,y)}\left[T_{2,0}\in dt, B_1(T_{2,0})\in dX\right] = \frac{|y|e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t^3}}\cdot\frac{e^{-\frac{(X-x)^2}{2t}}}{\sqrt{2\pi t}}dt\,dX,$$

for $y\neq 0$ and $x\in\mathbb{R}$, (cf. e.g. [2], p. 163, formula 2.0.2) and note that integration in t yields a scaled and shifted Cauchy distribution:

$$P_{(x,y)}\left[B_1(T_{2,0})\in dX\right] = \frac{|y|}{\pi(y^2+(X-x)^2)}dX.$$

Note that $T_{2,0}=\tau(0)$ if $B_1(T_{2,0})\geq 0$ and $T_{2,0}<\tau(0)$ if $B_1(T_{2,0})<0$. By the strong Markov property, we have

$$E_{(x,y)}\left[e^{-\mu_0 L_2(\tau(0))-\mu_1 B_1(\tau(0))-\mu_2\tau(0)}\right] = E_{(x,y)}\left[e^{-\mu_2 T_{2,0}}f(B_1(T_{2,0}))\right]$$

for $x \in \mathbb{R}$ and $y \neq 0$. Hence the joint law is determined by

$$
\begin{aligned}
&E_{(x,y)}\left[e^{-\mu_0 L_2(\tau(0))-\mu_1 B_1(\tau(0))-\mu_2\tau(0)}\right] \\
&= \int_0^\infty dX \int_0^\infty dt\, e^{-\mu_2 t}\frac{|y|e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t^3}}\cdot\frac{e^{-\frac{(X-x)^2}{2t}}}{\sqrt{2\pi t}} \\
&\quad + \int_{-\infty}^0 dX \int_0^\infty dt\, e^{-\mu_2 t} f(X,0)\frac{|y|e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t^3}}\cdot\frac{e^{-\frac{(X-x)^2}{2t}}}{\sqrt{2\pi t}}
\end{aligned}
$$

since $f(x)$ is explicitly defined in (1.4). We then obtain an asymptotic estimate.

Theorem 3.1. *If $x \in \mathbb{R}$, $y \neq 0$, $\mu_i \geq 0$ $(i=0,1,2)$ and $\mu_0+\mu_1+\mu_2 > 0$ we have as $s \to +0$*

$$
\begin{aligned}
&1 - E_{(x,y)}\left[e^{-\mu_0 s^2 L_2(\tau(0))-\mu_1 s^2 B_1(\tau(0))-\mu_2 s^4\tau(0)}\right] \\
&\sim s\tfrac{2}{\sqrt{\pi}}e^{I(\mu_0,\mu_1,\mu_2)}\int_{-\infty}^0 \frac{|y|\sqrt{|X|}}{\pi(y^2+(X-x)^2)}dX.
\end{aligned}
$$

Proof. By Theorem 1.1 and the scaling property, there exists a constant $C_4(\mu_0,\mu_1,\mu_2) > 0$ such that

$$
0 < 1 - E_{(-a,0)}\left[e^{-\mu_0 L_2(\tau(0))-\mu_1 B_1(\tau(0))-\mu_2\tau(0)}\right] < C_4(\mu_0,\mu_1,\mu_2)\sqrt{a}
$$

for any $a \in (0,1)$.

We divide the expectation in the statement of this theorem into three parts:

$$
\begin{aligned}
&1 - E_{(x,y)}\left[e^{-\mu_0 s^2 L_2(\tau(0))-\mu_1 s^2 B_1(\tau(0))-\mu_2 s^4\tau(0)}\right] \\
&= E_{(x,y)}\left[1 - e^{-\mu_2 s^4 T_{2,0}}E_{(B_1(T_{2,0}),0)}\left[e^{-\mu_0 s^2 L_2(\tau(0))-\mu_1 s^2 B_1(\tau(0))-\mu_2 s^4\tau(0)}\right]\right] \\
&= E_{(x,y)}\left[1 - e^{-\mu_2 s^4 T_{2,0}}f(s^2 B_1(T_{2,0}))\right] \\
&= E_{(x,y)}\left[1 - e^{-\mu_2 s^4 T_{2,0}}; B_1(T_{2,0}) \geq 0\right] \\
&\quad + E_{(x,y)}\left[1 - e^{-\mu_2 s^4 T_{2,0}}f(s^2 B_1(T_{2,0})); -\tfrac{1}{s^2} < B_1(T_{2,0}) < 0\right] \\
&\quad + E_{(x,y)}\left[1 - e^{-\mu_2 s^4 T_{2,0}}f(s^2 B_1(T_{2,0})); B_1(T_{2,0}) \leq -\tfrac{1}{s^2}\right] \\
&=: A_1 + A_2 + A_3.
\end{aligned}
$$

Then $A_1 \leq E_{(x,y)}\left[1 - e^{-\mu_2 s^4 T_{2,0}}\right] = O(s^2)$ and

$$
A_3 \leq P_{(x,y)}\left[B_1(T_{2,0}) \leq -\frac{1}{s^2}\right] = \int_{-\infty}^{-\frac{1}{s^2}-x}\frac{|y|}{\pi(y^2+\xi^2)}d\xi = O(s^2).
$$

We devide A_2 as follows:

$$\begin{aligned}\frac{A_2}{s} &= \frac{1}{s}E_{(x,y)}\left[\left(1-e^{-\mu_2 s^4 T_{2,0}}\right)\right.\\ &\quad\left.+e^{-\mu_2 s^4 T_{2,0}}\left(1-f(s^2B_1(T_{2,0}))\right); -\frac{1}{s^2}<B_1(T_{2,0})<0\right]\\ &= \frac{1}{s}\cdot O(s^2)+E_{(x,y)}\left[e^{-\mu_2 s^4 T_{2,0}}\frac{1-f(s^2B_1(T_{2,0}))}{s}; -\frac{1}{s^2}<B_1(T_{2,0})<0\right].\end{aligned}$$

The integrand in the last line converges a.s to $\frac{2}{\sqrt{\pi}}e^{I(\mu_0,\mu_1,\mu_2)}\sqrt{|0\wedge B_1(T_{2,0})|}$ and is bounded by

$$\begin{aligned}&e^{-\mu_2 s^4 T_{2,0}}\frac{C_4(\mu_0,\mu_1,\mu_2)\sqrt{|0\wedge s^2B_1(T_{2,0}),0)|}}{s}\\ &\qquad\le C_4(\mu_0,\mu_1,\mu_2)\sqrt{|0\wedge B_1(T_{2,0}),0)|},\end{aligned}$$

which is integrable being the square root of the absolute value of a random variable having the Cauchy distribution. Hence $\frac{A_2}{s}$ converges to

$$E_{(x,y)}\left[\frac{2}{\sqrt{\pi}}e^{I(\mu_0,\mu_1,\mu_2)}\sqrt{|0\wedge B_1(T_{2,0})|}\right].$$

□

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