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Composition factors of polynomial representation of DAHA and q-decomposition numbers

By

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Abstract

We determine the composition factors of the polynomial representation of DAHA, conjectured by M. Kasatani in [Kasa, Conjecture 6.4.]. He constructed an increasing sequence of subrepresentations in the polynomial representation of DAHA using the "multi-wheel condition", and conjectured that it is a composition series. On the other hand, DAHA has two degenerate versions called the "degenerate DAHA" and the "rational DAHA". The category $\mathcal{O}$ of modules over these three algebras and the category of modules over the v-Schur algebra are closely related. By using this relationship, we reduce the determination of composition factors of polynomial representations of DAHA to the determination of the composition factors of the Weyl module $W_v^{(n)}$ for the v-Schur algebra. By using the LLT-Ariki type theorem of v-Schur algebra proved by Varagnolo-Vasserot, we determine the composition factors of $W_v^{(n)}$ by calculating the upper global basis and crystal basis of Fock space of $U_q(\widehat{\mathfrak{sl}}_\ell)$ when v is a primitive ℓ-th root of unity.

This result gives a different way from the determination of decomposition number of $W_v^{(n)}$ by H. Miyachi or B. Ackermann via the modular representation theory of the general linear groups.

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1. Introduction

Double Affine Hecke Algebra

The double affine Hecke algebra (DAHA) is a 2-parameter analogue of the Iwahori-Hecke algebra introduced by I. Cherednik in [Ch1]. This algebra is closely related to the symmetric or orthogonal polynomials. In [Ch2], Cherednik proved the Macdonald inner product conjecture by using DAHA.

The DAHA $\mathcal{H}_{n,\zeta,\tau}$ of type GL_n is generated by

$$T_i \ (1 \leq i \leq n-1), X_j^{\pm 1}, Y_j^{\pm 1} \ (1 \leq j \leq n),$$

and has two parameters ζ and τ. The generators T_i satisfy the Hecke relation $(T_i - \zeta^{1/2})(T_i + \zeta^{-1/2}) = 0$ and generate the Iwahori-Hecke algebra of type A. The two subalgebras $\langle T_i(1 \leq i \leq n-1), X_j(1 \leq j \leq n)\rangle$ and $\langle T_i(1 \leq i \leq n-1), Y_j(1 \leq j \leq n)$ are both isomorphic to the affine Hecke algebra of type GL_n. The parameter τ appears in some relations between X and Y.

The DAHA has a faithful representation on $\mathbb{C}(\zeta^{1/2}, \tau)[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$ defined by difference Dunkl operators. This representation is called the polynomial representation. If ζ and τ are generic, the non-symmetric Macdonald polynomials are simultaneous Y-eigenvectors.

The category consisting of Y-locally finite modules of DAHA is called the "category $\mathcal{O}$".

The Kasatani Conjecture

Recall that the DAHA has a faithful representation on the Laurent polynomial ring, called the polynomial representation. If ζ and τ are generic, this representation is irreducible and Y-semisimple. But if the two parameters of DAHA specialized at $\zeta^\ell \tau^r = 1$, then the polynomial representation is not any more irreducible and Y-semisimple in general.

M. Kasatani constructed in [Kasa] the increasing sequence of the subrepresentation of the polynomial representation by using the "multi-Wheel condition". He conjectured that this sequence is a composition series of the polynomial representation of DAHA.

Main results

In this paper we prove Kasatani's conjecture when

$$(\ell, r) = 1 \text{ and } \ell \neq 2.$$

In the rest of this introduction, we explain the strategy of our proof of this main results.

The DAHA has two degenerate versions, simply called the "degenerate DAHA" and the "rational DAHA".

Degenerate DAHA

Roughly speaking the degenerate DAHA $\mathbb{H}^{\deg}_{n,h}$ is obtained from DAHA by degenerating T and Y. This algebra is generated by the following three subalgebras:

$$\mathbb{C}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}],\ \mathbb{C}\mathfrak{S}_n,\ \mathbb{C}[y_1, \cdots, y_n],$$

and this algebra has one parameter h.

The degenerate DAHA has the category $\mathbb{O}^{\deg}_h$ consisting of y-locally finite modules. By T. Suzuki in [Su1], [Su2], this category and the irreducible modules are studied. Especially this category has the standard modules induced from the irreducible representations of the symmetric group, and irreducible modules as a unique simple quotient of the these standard modules.

Rational DAHA

Roughly speaking the rational DAHA $\mathbb{H}^{\mathrm{rat}}_{n,h}$ is obtained from DAHA by degenerating T, X and Y. This algebra is generated by the following three subalgebras:

$$\mathbb{C}[x_1, \cdots, x_n],\ \mathbb{C}\mathfrak{S}_n,\ \mathbb{C}[y_1, \cdots, y_n],$$

and this algebra has one parameter h.

The rational DAHA has also the category $\mathbb{O}^{\mathrm{rat}}$ consisting of y-locally nilpotent modules. The category $\mathbb{O}^{\mathrm{rat}}_h$ are studied in [GGOR]. Especially this category has the standard modules induced from the irreducible representations of the symmetric group, and irreducible modules as a unique simple quotient of these standard modules.

v-Schur algebra

The v-Schur algebra $\mathbb{S}_v(n)$ has two different definitions.

First is the quotient of the quantum universal enveloping algebra $U_v(\mathfrak{gl}_n)$ in the tensor representation of the vector representation, introduced by A. Beilinson, G. Lusztig, R. MacPherson in [BLM]. By using their result, J. Du [Du] established the surjective map $U_{\mathcal{A}}(\mathfrak{gl}_n) \to \mathbb{S}_v(n)$. Here $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and $U_{\mathcal{A}}$ is Lusztig's $\mathcal{A}$-form of $U_v(\mathfrak{gl}_n)$. For a partition λ of n, we define a $\mathbb{S}_v(n)$-module W_v^λ which has a highest weight $\lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_n\varepsilon_n$ as a representation of $U_{\mathcal{A}}(\mathfrak{gl}_n)$ via the above quotient map. The dimension of W_v^λ is equal to the number of semi-standard tableau of shape λ. We call W_v^λ the Weyl module associated to λ. Specializing v at a primitive ℓ-th root of unity, W_v^λ has a unique simple quotient L_v^λ. Then $\{L_v^\lambda \mid \lambda \vdash n\}$ gives a complete set of representatives of the irreducible representations of $\mathbb{S}_v(n)$.

Second definition is the endomorphism ring of the permutation module of the Iwahori-Hecke algebra, introduced by R. Dipper and G. James in [DJ]. Moreover $\mathbb{S}_v(n)$ is a quasi-hereditary cover of the Iwahori-Hecke algebra $H_{n,v}$, see [Rou2]. Then the Weyl modules W_v^λ gives standard objects of the category $\mathbb{S}_v(n)$-mod as a highest weight category in the sense of [CPS].

Relationships of these algebras

The category $\mathcal{O}$ of DAHA and its two degenerate versions and the category of the v-Schur algebra are related as seen in the following figure;

$$\begin{tikzcd}
\boxed{\text{category } \mathcal{O}_{(\ell,r)}} \arrow[d, dotted, dash, "(1)"] & \\
\boxed{\text{category } \mathbb{O}^{deg}_{r/\ell}} & \\
\boxed{\text{category } \mathbb{O}^{rat}_{r/\ell}} \arrow[u, hook, "(2)"'] \arrow[r, phantom, "\cong" above, "(3)" below] & \boxed{\text{category } \mathbb{S}(n)\text{-mod}}
\end{tikzcd}$$

(1) **Varagnolo-Vasserot's block equivalence [VV2]**
The category $\mathcal{O}$ of the DAHA and the category of $\mathbb{O}^{\mathrm{deg}}$ are not equivalent. But we can consider the specialized DAHA $\mathcal{H}_n^{(\ell,r)}$ where the two parameters specialized at $\zeta^\ell \tau^r = 1$ and $(\zeta, \tau) = 1$. Let $\mathcal{O}^{(\ell,r)}$ be the category $\mathcal{O}$ of $\mathcal{H}_n^{(\ell,r)}$. Further, we can consider the subcategory ${}^\chi\mathcal{O}^{(\ell,r)}$ of $\mathcal{O}^{(\ell,r)}$ consisting the modules such that all the Y-weights of them belong to the affine Weyl group orbit of χ. And we can consider the similar full subcategory ${}^\chi\mathbb{O}^{\mathrm{deg}}_{r/\ell}$ of $\mathbb{O}^{\mathrm{deg}}_{r/\ell}$. If χ satisfies some conditions, then the categories ${}^\chi\mathcal{O}^{(\ell,r)}$ and ${}^\chi\mathbb{O}_{r/\ell}$ are equivalent. This equivalence is a direct generalization of the equivalence between some category of representations of the affine Hecke algebra and the one of the degenerate affine Hecke algebra proved by G. Lusztig in [Lus].

(2) **T. Suzuki's embedding [Su1]**
T. Suzuki proved that the rational DAHA can be embedded in the degenerate DAHA. Moreover the functor $\mathbb{H}^{\mathrm{deg}}_{n,h} \otimes_{\mathbb{H}^{\mathrm{rat}}_{n,h}} -$ is fully faithful and exact. It is known that the standard module and its (unique) simple quotient are sent to the standard module and its (unique) simple quotient under the above functor.

(3) **R. Rouquier's equivalence [Rou2]**
R. Rouquier proved that the category $\mathbb{O}^{\mathrm{rat}}_{r/\ell}$ is equivalent to the category of $\mathbb{S}_v(n)$-mod at $v = \sqrt[\ell]{1}$ unless $\ell \neq 2$. And he proved that the standard modules are sent to the Weyl modules of the v-Schur algebra. This had been conjectured in [GGOR].

(4) **Lascoux-Leclerc-Thibon type conjecture**
Since S. Ariki proved the LLT conjecture on the decomposition numbers of the cyclotomic Hecke algebra in [Ari1], the modular representation theory of Hecke algebras are closely related to the representation theory of the quantum enveloping algebra and the theory of the crystal and global basis. Moreover M. Varagnolo and E. Vasserot proved the extended version of the LLT conjecture about the decomposition numbers of v-Schur algebra in [VV1]. By this result, the decomposition numbers $[W_v^\lambda : L_v^\mu]$ for v-Schur algebras are described by the transition matrix of the standard basis and the global basis of the Fock space of $U_q(\widehat{\mathfrak{sl}}_\ell)$.

Therefore we can reduce some problems of the representation theory of DAHA to some calculations of the global basis of the Fock space. The main result of this paper is based on this strategy.

By using the above strategy, we can reduce the determination of the composition factors of the polynomial representations to the one of the Weyl module $W^{(n)}$. It is again equivalent to the determination of the coefficients of the upper global basis $G^{\mathrm{up}}(\mu)$ in the expansion of specific standard base $|(n)\rangle$ in the Fock space. We calculate this coefficient and obtain the following

$$[W_v^{(n)} : L_v^\mu] = \begin{cases} 1 & \text{if } \mu = \mu_i^{(n)} \left(0 \le i \le N = \left\lfloor \dfrac{n}{\ell} \right\rfloor\right) \\ 0 & \text{otherwise} \end{cases} .$$

On the definition of the partition $\mu_i^{(n)}$, see Definition 6.1. Note that this result is true for $\ell = 2$. However since Rouquier's equivalence (3) is not proved at $\ell = 2$, we cannot prove the Kasatani conjecture at $\ell = 2$ by our method.

This paper is organized as follows.

In Section 2, we review the double affine Hecke algebras and Kasatani's conjectures for the composition multiplicities of the polynomial representations of DAHA. In Section 3, we recall two degeneration of DAHA, the trigonometric DAHA and the rational DAHA, and their category O. In Section 4, we describe connections between the category O and the module category of the v-Schur algebras. In Section 5, we recall the notion of the crystal basis and the lower and upper global basis on the Fock space representation of $U_q(\widehat{\mathfrak{sl}}_\ell)$. In Section 6, we state and prove the main theorem of this paper.

Remark 1. After writing this paper, the author found some prior works for the determination of decomposition numbers via the modular representation theory of the general liner groups.

The results for $d_{(n),\mu}(1) = [W_v^{(n)} : L_v^\mu]$ were proved in [Mi, Lemma 12.2.4 and Corollary 12.2.6]. He described the composition factors of the Steinberg modules for the general linear groups. The q-decomposition numbers $d_{(n),\mu}(q)$ were also conjectured in [Mi, Conjecture 12.2.19]. In not only [Mi], the determination of $[W_v^{(n)} : L_v^\mu]$ was also obtained by B. Ackermann [Ack]. He determined the whole Loewy (radical) series of the projective cover of the Steinberg modules.

Remark 2. After submitting this paper, P. Etingof and E. Stoica [ES] established Kasatani's conjecture in full generality, especially containing the case of $\ell = 2$. Their approach are different from our strategy and Miyachi or Ackermann's studies. They used some parabolic induction and restriction functors for rational Cherednik algebras by R. Bezrukavnikov and P. Etingof [BE].

2. DAHA and the Kasatani Conjecture

2.1. Notations for affine root systems and affine Weyl groups

We will use the following notations for the affine root system and the affine Weyl group of type A.

Let $\mathfrak{h}$ be an $(n+2)$-dimensional vector space over $\mathbb{C}$ with basis

$$\mathfrak{h} = \bigoplus_{i=1}^{n} \mathbb{C}\varepsilon_i^\vee \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$, where ε_i, Λ and δ are the dual basis of $\varepsilon_i^\vee, c$ and d;

$$\mathfrak{h}^* = \bigoplus_{i=1}^{n} \mathbb{C}\varepsilon_i \oplus \mathbb{C}\Lambda \oplus \mathbb{C}\delta.$$

There exists a non-degenerate symmetric bilinear form $(\ |\ )$ on $\mathfrak{h}$ defined by

$$(\varepsilon_i^\vee|\varepsilon_j^\vee) = \delta_{ij}, \quad (\varepsilon_i^\vee|c) = (\varepsilon_i^\vee|d) = 0, \quad (c|d) = 1, \quad (c|c) = (d|d) = 0.$$

The natural pairing is denoted by $\langle\ |\ \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$. There exists an isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ such that

$$\varepsilon_i \mapsto \varepsilon_i^\vee, \quad \delta \mapsto c, \quad \Lambda \mapsto d.$$

We denote by $h^\vee \in \mathfrak{h}$ the image of $h \in \mathfrak{h}^*$ under this isomorphism. We can introduce the bilinear form $(\ |\ )$ on $\mathfrak{h}^*$ through this isomorphism, and then

$$(h|k) = \langle h|k^\vee\rangle = (h^\vee|k^\vee) \text{ for } h, k \in \mathfrak{h}^*.$$

Put

$$\alpha_{ij} = \varepsilon_i - \varepsilon_j,\ (1 \le i \ne j \le n) \quad \alpha_i = \alpha_{i,i+1}\ (1 \le i \le n-1).$$

Then

$$\begin{aligned} R &= \{\alpha_{ij} | 1 \le i \ne j \le n\} \subset \mathfrak{h}^*, \\ R^+ &= \{\alpha_{ij} \in R | i < j\}, \\ \Pi &= \{\alpha_1, \cdots, \alpha_{n-1}\} \end{aligned}$$

give the set of roots, positive roots and simple roots of type A_{n-1}, respectively.

Put

$$\alpha_0 = -\alpha_{1n} + \delta.$$

Then

$$\begin{aligned}
\widehat{R} &= \{\alpha + k\delta | \alpha \in R,\ k \in \mathbb{Z}\} \subset \mathfrak{h}^*, \\
\widehat{R}^+ &= \{\alpha + k\delta | \alpha \in R^+, k \in \mathbb{Z}_{\geq 0}\} \sqcup \{-\alpha + k\delta | \alpha \in R^+,\ k \in \mathbb{Z}_{\geq 1}\}, \\
\widehat{\Pi} &= \{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}\}
\end{aligned}$$

give the set of real roots, positive real roots and simple roots of type $A_{n-1}^{(1)}$, respectively.

Let P and $P^\vee$ be the weight lattice and co-weight lattice defined by

$$P = \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i \subset \mathfrak{h}^*, \quad P^\vee = \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i^\vee \subset \mathfrak{h}.$$

We introduce the affine Weyl group of type $A_{n-1}^{(1)}$.

Definition 2.1. The extended affine Weyl group W_n of type $A_{n-1}^{(1)}$ is the group defined by the following generators and relations;

$$\begin{aligned}
\text{generators} : \ & s_0, s_1, \cdots, s_{n-1}, \pi^{\pm 1}, \\
\text{relations} : \ & s_i^2 = 1 \quad (0 \leq i \leq n-1), \\
& s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (i \in \mathbb{Z}/n\mathbb{Z}, n > 2), \\
& s_i s_j = s_j s_i \quad (j \not\equiv i, i \pm 1), \\
& \pi s_i = s_{i+1} \pi \quad (i \in \mathbb{Z}/n\mathbb{Z}), \\
& \pi^{-1}\pi = \pi\pi^{-1} = 1.
\end{aligned}$$

The subgroup $\langle s_1, \cdots, s_{n-1}\rangle$ is isomorphic to the symmetric groups $\mathfrak{S}_n$. The subgroup $\langle s_0, s_1, \cdots, s_{n-1}\rangle$ is called the (non-extended) affine Weyl group of type $A_{n-1}^{(1)}$.

We can describe the extended affine Weyl group as a semi-direct product group. Put

$$X_{\varepsilon_1} = \pi s_{n-1} s_{n-2} \cdots s_1, \quad X_{\varepsilon_i} = \pi^{i-1} X_{\varepsilon_1} \pi^{-i+1}\ (2 \leq i \leq n).$$

Then there exists an embedding $P \hookrightarrow W_n$ defined by $\varepsilon_i \mapsto X_{\varepsilon_i}$. We denote by X_η the image of $\eta \in P$ under this embedding. Then there exists an isomorphism

$$W_n \cong P \rtimes \mathfrak{S}_n = \langle X_{\varepsilon_i}\ (1 \leq i \leq n), s_1, \cdots, s_{n-1}\rangle$$

such that

$$w X_\eta w^{-1} = X_{w(\eta)}\ (w \in \mathfrak{S}_n,\ \eta \in P).$$

Here the symmetric group $\mathfrak{S}_n$ acts on P by $s_i : \varepsilon_i \leftrightarrow \varepsilon_{i+1}$.

The extended affine Weyl group W_n acts on $\mathfrak{h}^*$ by

$$\begin{aligned}
s_i(h) &= h - (\alpha_i|h)\alpha_i \quad (h \in \mathfrak{h}^*),\\
X_\eta(h) &= h + (\delta|h)\eta - \left\{(\eta|h) + \frac{1}{2}(\eta|\eta)(\delta|h)\right\}\delta,\\
\pi(\varepsilon_i) &= \varepsilon_{i+1} \quad (1 \le i \le n-1),\\
\pi(\varepsilon_n) &= \varepsilon_n - \delta,\\
\pi(\Lambda) &= \Lambda,\\
\pi(\delta) &= \delta.
\end{aligned}$$

The dual action on $\mathfrak{h}$ is the following;

$$\begin{aligned}
s_i(h^\vee) &= h^\vee - \langle\alpha_i|h^\vee\rangle\alpha_i^\vee \quad (h^\vee \in \mathfrak{h}),\\
X_\eta(h^\vee) &= h^\vee + \langle\delta|h^\vee\rangle\eta^\vee - \left\{\langle\eta|h^\vee\rangle + \frac{1}{2}(\eta|\eta)\langle\delta|h^\vee\rangle\right\}c,\\
\pi(\varepsilon_i^\vee) &= \varepsilon_{i+1}^\vee \quad (1 \le i \le n-1),\\
\pi(\varepsilon_n^\vee) &= \varepsilon_n^\vee - c,\\
\pi(c) &= c,\\
\pi(d) &= d.
\end{aligned}$$

2.2. Double affine Hecke algebra and its Polynomial representation

2.2.1. Double affine Hecke algebra of type GL_n

Let $\mathbb{K}$ be a field $\mathbb{C}(\zeta^{1/2}, \tau)$. The double affine Hecke algebra of type GL_n is defined as follows.

Definition 2.2. The double affine Hecke algebra $\mathcal{H}_{n,\zeta,\tau}$ of type GL_n is an associative algebra over $\mathbb{K}$ generated by

$$T_i \ (0 \le i \le n-1), \quad Y_\eta \ (\eta \in P \oplus \mathbb{Z}\delta), \quad \pi^{\pm 1}$$

satisfying the following relations

$$\begin{array}{ll}
Y_\delta = \tau, & \\
(T_i - \zeta^{1/2})(T_i + \zeta^{-1/2}) = 0 & (0 \le i \le n-1),\\
T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} & (i \in \mathbb{Z}/n\mathbb{Z}),\\
T_iT_j = T_jT_i & (\text{otherwise}),\\
T_iY_\eta - Y_{s_i(\eta)}T_i = (\zeta^{1/2} - \zeta^{-1/2})\dfrac{Y_{s_i\eta} - Y_\eta}{Y_{\alpha_i} - 1} & (0 \le i \le n-1),\\
\pi T_i = T_{i+1}\pi & (i \in \mathbb{Z}/n\mathbb{Z}),\\
\pi Y_\eta = Y_{\pi(\eta)}\pi, & \\
Y_\eta Y_\xi = Y_{\eta+\xi}. &
\end{array}$$

The two subalgebras

$$\begin{aligned}
H_{n,\zeta} &= \langle T_1, \cdots, T_{n-1}\rangle,\\
H_{n,\zeta}^{\mathrm{aff}} &= \langle T_1, \cdots, T_{n-1}, Y_\eta \ (\eta \in P)\rangle
\end{aligned}$$

are isomorphic to the Iwahori-Hecke algebra of type A_{n-1} and the affine Hecke algebra of type GL_n, respectively.

Remark 3. Put $Y_i = Y_{\varepsilon_i}$ and

$$X_1 = T_1 \cdots T_{n-1}\pi^{-1}, \quad X_i = \pi^{i-1}X_1\pi^{-i+1}.$$

There is another description of generators and relations as the following;

$$\begin{array}{lll}
\text{generators :} & T_i \ (1 \le i \le n-1), \quad Y_j^{\pm 1}, X_j^{\pm 1} \ (1 \le j \le n), & \\
\text{relations :} & (T_i - \zeta^{1/2})(T_i + \zeta^{-1/2}) = 0 & (1 \le i \le n-1), \\
& T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & (1 \le i \le n-1), \\
& T_i T_j = T_j T_i & (|i-j| \ge 2), \\
& T_i X_{i+1} T_i = X_i & (1 \le i \le n-1), \\
& T_i X_j = X_j T_i & (j \ne i, i+1), \\
& T_i Y_i T_i = Y_{i+1} & (1 \le i \le n-1), \\
& T_i Y_j = Y_j T_i & (j \ne i, i+1), \\
& X_2^{-1} Y_1 X_2 Y_1^{-1} = T_1^2, & \\
& X_j \left(\prod_{k=1}^{n} Y_k \right) = \tau \left(\prod_{k=1}^{n} Y_k \right) X_j & (1 \le j \le n), \\
& Y_j \left(\prod_{k=1}^{n} X_k \right) = \tau \left(\prod_{k=1}^{n} X_k \right) Y_j & (1 \le j \le n), \\
& X_i X_j = X_j X_i, X_i X_i^{-1} = 1 & (1 \le i, j \le n), \\
& Y_i Y_j = Y_j Y_i, Y_i Y_i^{-1} = 1 & (1 \le i, j \le n).
\end{array}$$

2.2.2. Polynomial representation

The DAHA $\mathcal{H}_n$ has a representation on $\mathbb{K}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$ defined by the difference Dunkl operators. This representation is called the polynomial representation.

Proposition 2.1.

(1) *The DAHA $\mathcal{H}_{n,\zeta,\tau}$ has a representation on $\mathbb{K}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$ defined by the following,*

$$\begin{aligned}
X_j &\mapsto x_j \ (\textit{multiplication}), \\
T_i &\mapsto \zeta^{1/2} s_i + \frac{\zeta^{1/2} - \zeta^{-1/2}}{x_{i+1}x_i^{-1} - 1}(s_i - 1), \\
Y_j &\mapsto T_j^{-1} \cdots T_{n-1}^{-1} \omega T_1 \cdots T_{j-1},
\end{aligned}$$

where s_i is the permutation of x_i and x_{i+1}, and

$$(\omega f)(x_1, \cdots, x_n) = f(\tau^{-1}x_n, x_1, \cdots, x_{n-1}).$$

(Note that $\pi = X_1^{-1} T_1 \cdots T_{n-1}$.)

(2) *This representation is the induced representation from the one-dimensional representation of* $H^{\mathrm{aff}}_{n,\zeta}$ *defined by the following,*

$$T_i \mapsto \zeta^{1/2}, \quad Y_j \mapsto \zeta^{\rho_j}.$$

Here

$$\rho = (\rho_1, \cdots, \rho_n) = \left(-\frac{n-1}{2}, -\frac{n-3}{2}, \cdots, \frac{n-1}{2}\right).$$

Remark 4. If ζ, τ are generic, this polynomial representation is irreducible and Y-semisimple, namely the action of the commutative operators Y_i $(1 \le i \le n)$ are simultaneously diagonalizable. The simultaneous eigenvectors for Y_i, are the non-symmetric Macdonald polynomials. For more details, see [Kasa].

2.2.3. Category $\mathcal{O}$ of DAHA

Let $\mathcal{H}_{n,\zeta,\tau}$-mod be the category of finitely generated $\mathcal{H}_{n,\zeta,\tau}$-modules. Put $\mathbb{C}[Y]$ to be the subalgebra of $\mathcal{H}_{n,\zeta,\tau}$ generated by Y_η $(\eta \in P)$.

Definition 2.3. The category $\mathcal{O}$ is the full subcategory of $\mathcal{H}_{n,\zeta,\tau}$-mod consisting of modules which are locally finite with respect to $\mathbb{C}[Y]$. Here we say that a module $M \in \mathcal{H}_{n,\zeta,\tau}$-mod is locally finite with respect to $\mathbb{C}[Y]$ if $\mathbb{C}[Y]v$ is finite-dimensional for any $v \in M$.

Note that the polynomial representation of $\mathcal{H}_{n,\zeta,\tau}$ belongs to $\mathcal{O}$.

Suppose $M \in \mathcal{O}$. Then M has a generalized weight decomposition

$$M = \bigoplus_{\chi \in \mathfrak{h}^*} M_\chi,$$
$$\text{where } M_\chi = \bigcup_{k \ge 1} \left\{ v \in M | (Y_\eta - \zeta^{(\eta|\chi)})^k v = 0 \text{ for any } \eta \in P \right\}.$$

Let us set $\mathrm{Supp}(M) = \{\chi \in \mathfrak{h}^* | M_\chi \neq 0\}$.

Definition 2.4. The category ${}^\chi\mathcal{O}$ is the full subcategory consisting of the modules M such that $\mathrm{Supp}(M) \subset W_n \cdot \chi$.

Note that the polynomial representation of $\mathcal{H}_{n,\zeta,\tau}$ belongs to ${}^\rho\mathcal{O}$.

2.2.4. Specialized parameters

In this paper, we will specialize the two parameters ζ and τ at

$$\zeta^\ell \tau^r = 1 \quad (2 \le \ell \le n,\ 1 \le r,\ (\ell, r) = 1).$$

We assume that ζ, τ are not roots of unity. Let $\mathcal{H}^{(\ell,r)}_n$ be the algebra $\mathcal{H}_{n,\zeta,\tau}$ with the parameters specialized as above, and $V^{(\ell,r)}_n$ the polynomial representation of $\mathcal{H}_n^{(\ell,r)}$. Generally, the representation $V_n^{(\ell,r)}$ is not irreducible and not Y-semisimple in general.

Let us define the full subcategories $\mathcal{O}^{(\ell,r)}$ and ${}^{\chi}\mathcal{O}^{(\ell,r)}$ of $\mathcal{H}_n^{(\ell,r)}$-mod similarly to the generic case.

2.3. Kasatani's Conjecture on the polynomial representation of DAHA

M. Kasatani constructed in [Kasa] the subrepresentations of $V_n^{(\ell,r)}$ using "wheels" of variables with length ℓ. This is called the "multi-wheel condition". In this section, we will recall his construction based on [Kasa].

Definition 2.5. Let $Z_m^{(\ell,r)}$ be the subset of $\mathbb{K}^n$ contained in $(z_1, \cdots, z_n) \in \mathbb{K}^n$ satisfying the following conditions;

there exist distinct indices

$$i_{j,1}, \cdots, i_{j,\ell} \in \{1, \cdots, n\} \quad (1 \leq j \leq m)$$

and positive integers

$$s_{j,1}, \cdots, s_{j,\ell} \in \mathbb{Z}_{\geq 0} \quad (1 \leq j \leq m)$$

such that

$$\begin{aligned}
&z_{i_{j,a}} = \zeta \tau^{s_{j,a}} z_{i_{j,a+1}} \quad (1 \leq j \leq m,\ 1 \leq i \leq \ell),\\
&\sum_{a=1}^{\ell} s_{j,a} = r \quad (1 \leq j \leq m),\\
&i_{j,a+1} > i_{j,a} \text{ if } s_{j,a} = 0.
\end{aligned}$$

Let us define the ideals

$$I_m^{(\ell,r)} = \{f \in \mathbb{K}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]; f(z) = 0 \text{ for all } z \in Z_m^{(\ell,r)}\}.$$

We call the defining relation of $I_m^{(\ell,r)}$ the multi-wheel condition.

We introduce Kasatani's result and conjecture.

Theorem 2.1 ([Kasa, Theorem 6.3]). *Let $N = \lfloor \frac{n}{\ell} \rfloor$. The sequence*

$$0 = I_0^{(\ell,r)} \subsetneq I_1^{(\ell,r)} \subsetneq I_2^{(\ell,r)} \subsetneq \cdots I_N^{(\ell,r)} \subsetneq I_{N+1}^{(\ell,r)} = V_n^{(\ell,r)}.$$

is an increasing sequence of subrepresentations of $V_m^{(\ell,r)}$.

We call the following conjecture the Kasatani Conjecture on the polynomial representation of DAHA.

Conjecture 2.1 ([Kasa, Conjecture 6.4]). *The above increasing sequence of subrepresentations of $V_n^{(\ell,r)}$ is a composition series, namely the quotient representations*

$$I_{a+1}^{(\ell,r)}/I_a^{(\ell,r)} \quad (0 \le a \le N)$$

are irreducible. Especially note that the number of composition factors of $V_n^{(\ell,r)}$ is equal to $N+1 = \lfloor \frac{n}{\ell} \rfloor + 1$.

3. Two degenerate versions of DAHA and their category $\mathbb{O}$

3.1. Trigonometric degeneration of DAHA

In this section, we will recall two degenerate versions of DAHA and their category $\mathbb{O}$. They are called the "trigonometric degeneration of DAHA" and "rational degeneration of DAHA".

3.1.1. Trigonometric degeneration of DAHA

Let us define the trigonometric degeneration of DAHA, simply called degenerate DAHA.

Definition 3.1. The degenerate DAHA $\mathbb{H}_{n,h}^{\deg}$ of type GL_n is an associative algebra over $\mathbb{C}$ generated by

$$\pi^{\pm 1}, s_0, s_1, \cdots, s_{n-1}, \quad y_\eta^{\deg} \ (\eta \in P \oplus \mathbb{Z}\delta)$$

satisfying the following relations

$$\begin{aligned}
&y_\delta^{\deg} = 1, \\
&y_\eta^{\deg} + y_\xi^{\deg} = y_{\eta+\xi}^{\deg} && (\eta, \xi \in P), \\
&\langle \pi^{\pm 1}, s_0, s_1, \cdots, s_{n-1} \rangle \cong \mathbb{C}W_n, \\
&s_i y_\eta^{\deg} - y_{s_i\eta}^{\deg} s_i = h \frac{y_{s_i\eta}^{\deg} - y_\eta^{\deg}}{y_{\alpha_i}} && (0 \le i \le n-1,\ \eta \in P), \\
&\pi y_\eta^{\deg} = y_{\pi\eta}^{\deg} \pi,
\end{aligned}$$

for $h \in \mathbb{C} \backslash \{0\}$.

Recall another generators $X_1, \cdots, X_n, s_1, \cdots, s_n$ of the extended affine Weyl group W_n, namely

$$X_1 = \pi s_{n-1} s_{n-2} \cdots s_1, \quad X_i = \pi^{i-1} X_1 \pi^{-i+1}.$$

3.1.2. Standard modules and Category $\mathbb{O}^{\deg}$ of degenerate DAHA

Let $\mathbb{H}_{n,h}^{\deg}$-mod be the category of finitely generated $\mathbb{H}_{n,h}^{\deg}$-modules. Let $\mathbb{C}[y^{\deg}]$ be the subalgebra of $\mathbb{H}_{n,h}^{\deg}$ generated by $y_\eta^{\deg}$ $(\eta \in P)$.

Definition 3.2. The category $\mathbb{O}_h^{\deg}$ is the full subcategory of $\mathbb{H}_{n,h}^{\deg}$-mod consisting of modules which are locally finite with respect to $\mathbb{C}[y^{\deg}]$.

Suppose $M \in \mathbb{O}_h^{\mathrm{deg}}$. Then M has the generalized weight decomposition

$$M = \bigoplus_{\chi \in \mathfrak{h}^*} M_\chi,$$
$$\text{where } M_\chi = \bigcup_{k \geq 1} \left\{ v \in M | (y_\eta^{\mathrm{deg}} - (\eta|\chi))^k v = 0 \text{ for any } \eta \in P \right\}.$$

Let us set $\mathrm{Supp}(M) = \{\chi \in \mathfrak{h}^* | M_\chi \neq 0\}$.

Definition 3.3. The category ${}^\chi\mathbb{O}_h^{\mathrm{deg}}$ is the full subcategory consisting of the modules M such that $\mathrm{Supp}(M) \subset W_n \cdot \chi$.

The degenerate DAHA $\mathbb{H}_{n,h}^{\mathrm{deg}}$ has the polynomial representation on $\mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$. More generally, we can introduce the standard modules.

Definition 3.4. Let S be a irreducible $\mathfrak{S}_n$-module. Then S becomes a $\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}[y^{\mathrm{deg}}]$-module by the following action of y^{deg};

$$y_i^{\mathrm{deg}} \mapsto \sum_{j<i} s_{ji} - \frac{n-1}{2}.$$

The standard module $\Delta_h^{\mathrm{deg}}(S)$ associated to S is the induced module of the $\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}[y^{\mathrm{deg}}]$-module S to $\mathbb{H}_{n,h}^{deg}$;

$$\Delta_h^{\mathrm{deg}}(S) = \mathrm{Ind}_{\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}[y^{\mathrm{deg}}]}^{\mathbb{H}_{n,h}^{\mathrm{deg}}} S.$$

Especially, the standard module $\Delta_h^{\mathrm{deg}}(\mathrm{triv})$ associated with the trivial representation of $\mathfrak{S}_n$ is isomorphic to the polynomial representation of $\mathbb{H}_{n,h}^{\mathrm{deg}}$. Namely, the polynomial representation of $\mathbb{H}_{n,h}^{\mathrm{deg}}$ is induced module from the one-dimensional $\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}[y^{\mathrm{deg}}]$-module:

$$s_i \mapsto 1 \ (1 \leq i \leq n-1), \quad y_j^{\mathrm{deg}} \mapsto \rho_j \ (1 \leq j \leq n).$$

Recall that $\rho = (\rho_1, \cdots, \rho_n) = \left(-\frac{n-1}{2}, -\frac{n-3}{2}, \cdots, \frac{n-1}{2}\right)$.

Note that the polynomial representation $\Delta_h^{\mathrm{deg}}(\mathrm{triv})$ belongs to the category ${}^\rho\mathbb{O}_h^{\mathrm{deg}}$.

Proposition 3.1 ([Su1]). *For each irreducible representation S of $\mathfrak{S}_n$, the standard module $\Delta_h^{\mathrm{deg}}(S)$ has a unique simple quotient denoted by $L_h^{\mathrm{deg}}(S)$.*

3.2. Rational degeneration of DAHA

3.2.1. Rational degeneration of DAHA

Let us define the rational degeneration of DAHA, simply called rational DAHA.

Definition 3.5. The degenerate DAHA $\mathbb{H}^{\mathrm{rat}}_{n,h}$ of type GL_n is the associative algebra over $\mathbb{C}$ generated by

$$x_{\eta^\vee}\ (\eta^\vee \in P^\vee),\quad s_1,\cdots,s_{n-1},\quad y^{\mathrm{rat}}_\eta\ (\eta\in P)$$

satisfying the following relations

$$\begin{array}{ll} y^{\mathrm{rat}}_\eta + y^{\mathrm{rat}}_\xi = y^{\mathrm{rat}}_{\eta+\xi} & (\eta,\xi\in P) \\ x_{\eta^\vee}+x_{\xi^\vee} = x_{\eta^\vee+\xi^\vee} & (\eta^\vee,\xi^\vee\in P^\vee) \\ \langle s_1,\cdots,s_{n-1}\rangle \cong \mathbb{C}\mathfrak{S}_n, & \\ wx_{\eta^\vee} = x_{w\eta^\vee}w & (w\in\mathfrak{S}_n), \\ wy^{\mathrm{rat}}_\eta = y^{\mathrm{rat}}_{w\eta}w, & \\ [x_i, y^{\mathrm{rat}}_j] = \begin{cases} hs_{ij} & (\text{if } i\neq j) \\ 1-h\sum\limits_{k\neq i} s_{ik} & (\text{if } i=j) \end{cases}, & \end{array}$$

for $h\in\mathbb{C}\backslash\{0\}$, where $x_i = x_{\varepsilon_i^\vee}, y^{\mathrm{rat}}_i = y^{\mathrm{rat}}_{\varepsilon_i}$.

3.2.2. Standard modules and category $\mathbb{O}^{\mathrm{rat}}$ of rational DAHA

Let $\mathbb{H}^{\mathrm{rat}}_{n,h}$-mod be the category of finitely generated $\mathbb{H}^{\mathrm{rat}}_{n,h}$-modules. Let $\mathbb{C}[y^{\mathrm{rat}}]$ be the subalgebra of $\mathbb{H}^{\mathrm{rat}}_{n,h}$ generated by y^{rat}_η $(\eta\in P)$.

Definition 3.6. The category $\mathbb{O}^{\mathrm{rat}}_h$ is the full subcategory of $\mathbb{H}^{\mathrm{rat}}_{n,h}$-mod consisting of modules which are locally nilpotent with respect to y^{rat}. Here a module $M\in\mathbb{H}^{\mathrm{rat}}_{n,h}$-mod is locally nilpotent with respect to y^{rat} if for any $v\in M$ there exists $N>0$ such that $(y^{\mathrm{rat}}_i)^N v = 0$ $(1\le i\le n)$.

The rational DAHA $\mathbb{H}^{\mathrm{rat}}_n$ has the polynomial representation on $\mathbb{C}[x_1,\cdots,x_n]$. More generally, we can introduce the standard modules.

Definition 3.7. Let S be a irreducible $\mathfrak{S}_n$-module. Then S becomes a $\mathbb{C}\mathfrak{S}_n\otimes\mathbb{C}[y^{\mathrm{rat}}]$-module by the following action of y^{rat}_i;

$$y^{\mathrm{rat}}_i S = 0.$$

The standard module $\Delta^{\mathrm{rat}}_h(S)$ associated to S is the $\mathbb{H}^{\mathrm{rat}}_{n,h}$-module induced by the $\mathbb{C}\mathfrak{S}_n\otimes\mathbb{C}[y^{\mathrm{rat}}]$-module;

$$\Delta^{\mathrm{rat}}_h(S) = \mathrm{Ind}^{\mathbb{H}^{\mathrm{rat}}_{n,h}}_{\mathbb{C}\mathfrak{S}_n\otimes\mathbb{C}[y^{\mathrm{rat}}]} S.$$

The standard module $\Delta^{rat}_h(\mathrm{triv})$ associated to the trivial representation of $\mathfrak{S}_n$ is isomorphic to the polynomial representation of $\mathbb{H}^{\mathrm{rat}}_{n,h}$.

Note that the polynomial representation $\Delta^{\mathrm{rat}}_h(\mathrm{triv})$ belongs to the category $\mathbb{O}^{\mathrm{rat}}_h$.

Proposition 3.2.

(1) *The standard modules* $\Delta_h^{\mathrm{rat}}(S)$ *have a unique simple quotient denoted by* $L_h^{\mathrm{rat}}(S)$.

(2) *The set* $\{L_h^{\mathrm{rat}}(S) \mid S \in \operatorname{Irr}\mathfrak{S}_n\}$ *is a complete set of representative of irreducible representations of* $\mathbb{H}_{n,h}^{\mathrm{rat}}$.

3.2.3. Embedding to degenerate DAHA

The rational DAHA can be embedded to the degenerate DAHA proved by T. Suzuki.

Proposition 3.3 ([Su1]). *The following homomorphism from* $\mathbb{H}_{n,h}^{\mathrm{rat}}$ *to* $\mathbb{H}_{n,h}^{\mathrm{deg}}$ *is an embedding*;

$$\begin{aligned} s_i &\mapsto s_i, \\ x_j^{\vee} &\mapsto X_j = \pi^{j-1} X_1 \pi^{-j+1} \text{ where } X_1 = \pi s_{n-1} \cdots s_1, \\ y_j^{\mathrm{rat}} &\mapsto X_j^{-1} \left(y_j^{\mathrm{deg}} - \sum_{1 \le k < j} s_{kj} + \frac{n-1}{2} \right). \end{aligned}$$

4. Relationships between category $\mathcal{O}$ and v-Schur algebras

In this section, we will explain the relationships of some categories.

4.1. Relationship of $\mathcal{O}$ and $\mathbb{O}^{\mathrm{deg}}$

The categories $\mathcal{O}$ and $\mathbb{O}^{\mathrm{deg}}$ are not equivalent. But there exists an equivalence of categories between their full subcategories when the parameters are special. The following theorem proved by Varagnolo-Vasserot [VV2] using an analogous way by Lusztig [Lus].

Theorem 4.1 ([VV2], [Lus]). *If* $\chi \in \mathfrak{h}^*$ *satisfies that*

$$(\eta|\chi) \in \mathbb{Z} \text{ and } (\delta|\chi) \in \mathbb{Z} \text{ for any } \eta \in P,$$

then the categories ${}^{\chi}\mathcal{O}^{(\ell,r)}$ *of the specialized DAHA* $\mathcal{H}_n^{(\ell,r)}$ *and* ${}^{\chi}\mathbb{O}_{r/\ell}^{\mathrm{deg}}$ *of the degenerate DAHA* $\mathbb{H}_{n,r/\ell}^{\mathrm{deg}}$ *are equivalent.*

4.2. Embedding of $\mathbb{O}^{\mathrm{rat}}$ to $\mathbb{O}^{\mathrm{deg}}$

By the embedding $\mathbb{H}_{n,h}^{\mathrm{rat}} \hookrightarrow \mathbb{H}_{n,h}^{\mathrm{deg}}$, we can define the induction functor $\mathbb{H}_{n,h}^{\mathrm{rat}}$-mod to $\mathbb{H}_{n,h}^{\mathrm{deg}}$-mod.

Theorem 4.2 ([Su1]).

(1) *The functor*

$$\mathbb{O}_h^{\mathrm{rat}} \to \mathbb{O}_h^{\mathrm{deg}}; M \mapsto \mathbb{H}_{n,h}^{\mathrm{deg}} \otimes_{\mathbb{H}_{n,h}^{\mathrm{rat}}} M$$

is fully faithful and exact.

(2) *The above functor sends the standard modules to standard modules, namely*

$$\Delta_h^{\deg}(S) = \mathbb{H}_{n,h}^{\deg} \otimes_{\mathbb{H}_{n,h}^{\mathrm{rat}}} \Delta_h^{\mathrm{rat}}(S).$$

Especially, $\Delta_h^{\deg}(\mathrm{triv}) = \mathbb{H}_{n,h}^{\deg} \otimes_{\mathbb{H}_{n,h}^{\mathrm{rat}}} \Delta_h^{rat}(\mathrm{triv})$.

(3) *The above functor sends the simple module* $L_H^{\mathrm{rat}}(S)$ *to* $L_h^{\deg}(S)$*, namely*

$$L_h^{\deg}(S) = \mathbb{H}_{n,h}^{\deg} \otimes_{\mathbb{H}_{n,h}^{\mathrm{rat}}} L_h^{\mathrm{rat}}(S).$$

Thus we obtain $[\Delta_h^{\deg}(S) : L_h^{\deg}(S')] = [\Delta_h^{\mathrm{rat}}(S) : L_h^{\mathrm{rat}}(S')]$ *for each irreducible representations* S *and* S'.

4.3. v-Schur algebra $\mathbb{S}_v(n)$

The v-Schur algebra was introduced by Dipper and James [DJ]. They used the Iwahori-Hecke algebra $H_{n,v}$ of type A. For a composition $\mu = (\mu_1, \cdots, \mu_n)$ of n, namely $\mu \in \mathbb{Z}_{\geq 0}^n$ and $\mu_1 + \mu_2 + \cdots + \mu_n = n$, let us denote by $\mathfrak{S}_\mu$ the Young subgroup $\mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_n}$. Set $m_\mu = \sum_{w \in \mathfrak{S}_\mu} T_w \in H_{n,v}$. We define the left $H_{n,v}$-module $M^\mu = H_{n,v} m_\mu$. We call this module the permutation module for μ. We consider the direct sum M of permutation modules;

$$M = \bigoplus_\mu M^\mu.$$

The v-Schur algebra $\mathbb{S}_v(n)$ is the endomorphism ring of M;

$$\mathbb{S}_v(n) = \mathrm{End}_{H_{n,v}}(M)^{\mathrm{op}}.$$

On the other hand, Beilinson-Lusztig-MacPherson ([BLM]) constructed the v-Schur algebra by using a geometry of partial flag varieties. Let us consider the quantum enveloping algebra $U_v(\mathfrak{gl}_n)$ and its vector representation $\mathbb{C}^n$. Let $(\mathbb{C}^n)^{\otimes n}$ be the quantum tensor product representation;

$$U_v(\mathfrak{gl}_n) \to \mathrm{End}((\mathbb{C}^n)^{\otimes n}).$$

Then the v-Schur algebra $\mathbb{S}_v(n)$ is isomorphic to the image of this homomorphism, namely we have a surjective algebra homomorphism $U_v(\mathfrak{gl}_n) \twoheadrightarrow \mathbb{S}_v(n)$. Set $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. By using the above result, J. Du [Du] established the surjective map $U_{\mathcal{A}}(\mathfrak{gl}_n) \to \mathbb{S}_v(n)$. Here $U_{\mathcal{A}}(\mathfrak{gl}_n)$ is Lusztig's $\mathcal{A}$-form of $U_v(\mathfrak{gl}_n)$.

For a partition λ of n, the Weyl module W_v^λ is a $\mathbb{S}_v(n)$-module which has a highest weight $\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$ as a representation of $U_{\mathcal{A}}(\mathfrak{gl}_n)$ via the above quotient map. Its dimension is equal to the number of semi-standard tableau of shape λ. If we specialize v at a primitive ℓ-th root of unity, W_v^λ has a unique simple quotient L_v^λ. Then $\{L_v^\lambda \mid \lambda \vdash n\}$ gives a complete set of representatives of the irreducible representations of $\mathbb{S}_v(n)$.

The v-Schur algebra $\mathbb{S}_v(n)$ has a cellular algebra structure in the sense that Graham-Lehrer ([GL], e.g. [Mat]). The Weyl modules W_v^λ gives the cell modules of $\mathbb{S}_v(n)$.

Since the permutation module M naturally becomes an $\mathbb{S}_v(n)$-module, we can define the following functor;

$$\mathcal{S} : \mathbb{S}_v(n)\text{-mod} \to H_{n,v}\text{-mod}; \text{ given by } N \mapsto M \otimes_{\mathbb{S}_v(n)} N.$$

We call this functor the Schur functor.

When we specialize v at a primitive ℓ-th root of unity, $\mathcal{S}(L_v^\lambda) \neq 0$ if and only if λ is ℓ-restricted, i.e. $\lambda_{i-1} - \lambda_i < \ell$ for any i. Set $D_v^\lambda = \mathcal{S}(L_v^\lambda)$. Then we obtain the set of $H_{n,v}$-modules $\{D_v^\lambda \mid \lambda : \ell\text{-restricted}\}$. This gives a complete representatives of the irreducible representations of $H_{n,v}$ at $v = \sqrt[\ell]{1}$.

On the other hand, we can obtain $\mathfrak{S}_n$-modules S^λ for $S_v^\lambda = \mathcal{S}(W_v^\lambda)$ by specializing v at 1. If $\lambda = (n)$, S^λ gives the trivial representation of $\mathfrak{S}_n$.

4.4. Equivalence of $\mathbb{S}_v(n)$-mod and $\mathbb{O}^{\text{rat}}$

R. Rouquier proved in [Rou2] that there exists an equivalence of categories between $\mathbb{O}^{\text{rat}}$ and $\mathbb{S}_v(n)$-mod. Set $\Delta_{r/\ell}^{\text{rat}}(\lambda) = \Delta_{r/\ell}^{\text{rat}}(S^\lambda)$.

Theorem 4.3. *Let us consider the categories $\mathbb{O}_h^{\text{rat}}$ and $\mathbb{S}_v(n)$-mod at $v = \sqrt[\ell]{1}$. When $h = \dfrac{r}{\ell} \notin \dfrac{1}{2} + \mathbb{Z}$, there exists an equivalence of categories*

$$\Psi^{\text{Rouq}} \ : \ \mathbb{S}_v(n)\text{-mod} \ \xrightarrow{\ \sim\ } \ \mathbb{O}_{r/\ell}^{\text{rat}}$$

such that if $h \leq 0$ the Weyl module W_v^λ are sent to the standard modules $\Delta_{r/\ell}^{\text{rat}}(\lambda)$.

4.5. Summary on the case of polynomial representation

We defined three functors in preceding sections:

$$\begin{gathered} {}^\rho\Psi^{\text{deg}} : {}^\rho\mathbb{O}_{r/\ell}^{\text{deg}} \xrightarrow{\ \sim\ } {}^\rho\mathcal{O}_{r/\ell}, \\ \Psi^{\text{rat}} : \mathbb{O}_{r/\ell}^{\text{rat}} \longrightarrow \mathbb{O}_{r/\ell}^{\text{deg}}, \\ \Psi^{\text{Rouq}} : \mathbb{S}(n)\text{-mod} \xrightarrow{\ \sim\ } \mathbb{O}_{r/\ell}^{\text{rat}}. \end{gathered}$$

The two functors ${}^\rho\Psi^{\text{deg}}$ and Ψ^{Rouq} are equivalences of categories, and the functor Ψ^{rat} is fully faithful and exact. The representation

$${}^\rho\Psi^{\text{deg}}(\Psi^{\text{rat}} \circ \Psi^{\text{Rouq}}(W^{(n)}))$$

is isomorphic to the polynomial representation of $\mathcal{H}_n^{(\ell,r)}$.

We can reduce the determination of the composition factors of the polynomial representation of $\mathcal{H}_n^{(\ell,r)}$ to the one of the composition factors of $\mathbb{S}(n)$-module $W^{(n)}$ by using the above correspondence.

5. Global and crystal basis of fock space

5.1. Quantum enveloping algebra $U_q(\widehat{\mathfrak{sl}}_\ell)$

Let us recall the quantum enveloping algebra $U_q(\widehat{\mathfrak{sl}}_\ell)$. Set $[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}}$. Let the matrix $A = (a_{ij})_{0 \le i \le \ell-1}$ be the Cartan matrix of type $A^{(1)}_{\ell-1}$. Namely, if $\ell \ge 3$,

$$a_{ij} = \begin{cases} 2 & i = j \\ -1 & i \equiv j \pm 1 (\operatorname{mod} \ell) \\ 0 & \text{otherwise} \end{cases}$$

and if $\ell = 2$

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Definition 5.1. The quantum enveloping algebra $U_q(\widehat{\mathfrak{sl}}_\ell)$ is the associate algebra over $\mathbb{C}(q)$ generated by

$$E_i, F_i, K_i \quad (0 \le i \le \ell - 1),$$

satisfying the following relations

$$\begin{aligned}
K_iK_j &= K_jK_i, \\
K_iE_j &= q^{a_{ij}} E_jK_i, \\
K_iF_j &= q^{-a_{ij}} F_jK_i, \\
E_iF_j - F_jE_i &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
E_iE_j &= E_jE_i \quad (\text{if } i \ne j \pm 1), \\
F_iF_j &= F_jF_i \quad (\text{if } i \ne j \pm 1),
\end{aligned}$$

and the q-Serre relations,

if $\ell \ge 3$,

$$E_i^2E_{i\pm1} - (q + q^{-1})E_iE_{i\pm1}E_i + E_{i\pm1}E_i^2 = 0,$$
$$F_i^2F_{i\pm1} - (q + q^{-1})F_iF_{i\pm1}F_i + F_{i\pm1}F_i^2 = 0,$$

if $\ell = 2$,

$$E_i^3E_{i\pm1} - [3]E_i^2E_{i\pm1}E_i + [3]E_iE_{i\pm1}E_i^2 - E_{i\pm1}E_i^2 = 0,$$
$$F_i^3F_{i\pm1} - [3]F_i^2F_{i\pm1}F_i + [3]F_iF_{i\pm1}F_i^2 - F_{i\pm1}E_i^2 = 0,$$

The indices in the above relations are to be read modulo ℓ.

$U_q(\widehat{\mathfrak{sl}}_\ell)$ is a Hopf algebra with a coproduct given by

$$\begin{aligned}
\Delta^-(E_i) &= 1 \otimes E_i + E_i \otimes K_i^{-1}, \\
\Delta^-(F_i) &= F_i \otimes 1 + K_i \otimes F_i, \\
\Delta^-(K_i) &= K_i \otimes K_i.
\end{aligned}$$

There exists another coproduct Δ^+ on $U_q(\widehat{\mathfrak{sl}}_n)$ given by

$$\begin{aligned}\Delta^+(E_i) &= E_i \otimes 1 + K_i \otimes E_i,\\ \Delta^+(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i,\\ \Delta^+(K_i) &= K_i \otimes K_i.\end{aligned}$$

When we consider the lower and upper global basis, we will use these two coproducts.

5.2. Two realizations of Fock space

In this subsection, we will describe two realizations of the Fock space of $U_q(\widehat{\mathfrak{sl}}_\ell)$, the "Hayashi realization" and semi-infinite wedge space.

5.2.1. Hayashi realization

Let us recall some notations and definitions.

A partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a non-increasing sequence of natural numbers. The corresponding Young diagram is a collection of rows of square boxes which are left justified and λ_i boxes in the i-th row. Let $\mathcal{P}$ be the set of all partitions.

Definition 5.2.

(1) The content of box $x \in \lambda$ is defined by

$$c(x) = \mathrm{col}(x) - \mathrm{row}(x).$$

Assume that we are given a positive number ℓ. The ℓ-residue of a box x is defined by $\mathrm{res}_\ell(x) = c(x)$ modulo ℓ.

(2) If a partition μ is obtained by removing a box x from a partition λ, the box x is called a removable box in λ. Conversely, if a partition λ is obtained by adding a box x to a partition μ, the box x is call an addable box in μ. If a removable [resp. addable] box x has residue i, we call the box x an i-removable [resp. i-addable] box.

We will use the following notations for the description of the Hayashi realization.

Definition 5.3. Let ℓ be a fixed positive number. Assume that a partition λ is obtained by adding a box x to a partition μ.

$$\begin{aligned}N_i^b(\lambda,\mu) &= \sharp\{y \in \lambda | y \text{ is an } i\text{-addable box below } x\}\\ &\quad -\sharp\{y \in \lambda | y \text{is an } i\text{-removable box below } x\},\\ N_i^a(\lambda,\mu) &= \sharp\{y \in \lambda | y \text{ is an } i\text{-addable box above } x\}\\ &\quad -\sharp\{y \in \lambda | y \text{ is an } i\text{-removable box above } x\},\\ N_i(\lambda) &= \sharp\{y \in \lambda | y \text{ is an } i\text{-addable box}\} - \sharp\{y \in \lambda | y \text{ is a } i\text{-removable box}\},\end{aligned}$$

Let $|\lambda\rangle$ be a symbol associated to the partition $\lambda \in \mathcal{P}$. The Fock space of $U_q(\widehat{\mathfrak{sl}}_\ell)$ is defined

$$\mathcal{F} = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{C}(q)|\lambda\rangle.$$

In [Ha], T. Hayashi defined the $U_q(\widehat{\mathfrak{sl}}_\ell)$-action on the Fock space $\mathcal{F}$.

Theorem 5.1 (the Hayashi realization of the Fock space). *The Fock space $\mathcal{F}$ becomes a $U_q(\widehat{\mathfrak{sl}}_n)$-module via the following action;*

$$\begin{aligned}
E_i|\lambda\rangle &= \sum_{\mathrm{res}_\ell(\lambda/\nu)=i} q^{-N_i^a(\nu,\lambda)}|\nu\rangle,\\
F_i|\lambda\rangle &= \sum_{\mathrm{res}_\ell(\mu/\lambda)=i} q^{N_i^b(\lambda,\mu)}|\mu\rangle,\\
K_i|\lambda\rangle &= q^{N_i(\lambda)}|\lambda\rangle \quad (0 \le i \le \ell-1).
\end{aligned}$$

Note that the operator E_i removes one box from λ, and F_i adds one box to λ.

A proof of this theorem can be found in [Ari2].

5.2.2. Wedge space

We will recall the realization of the Fock space as a semi-infinite wedge space. This section is based on [KMS].

Let $V = \mathbb{C}^\ell$ with basis $v_1, \cdots, v_\ell$, and $V(z) = V \otimes \mathbb{C}(q)[z, z^{-1}]$ with basis $u_{j-a\ell} = z^a v_j$. Then $U_q(\widehat{\mathfrak{sl}}_\ell)$ acts on $V(z)$ by the following way;

$$E_i u_m = \delta(m-1 \equiv i \mod \ell) u_{m-1}, \tag{5.1}$$

$$F_i u_m = \delta(m \equiv i \mod \ell) u_{m+1}, \tag{5.2}$$

$$K_i u_m = q^{\delta(m\equiv i \mod \ell)-\delta(m\equiv i+1 \mod \ell)} u_m. \tag{5.3}$$

This module $V(z)$ is called the evaluation module of $U_q(\widehat{\mathfrak{sl}}_\ell)$.

Let $I = (\cdots, i_2, i_1, i_0)$ be a semi-infinite sequence of integers such that $i_0 > i_1 > i_2 > \cdots$ and $i_k = -k$ if $k \gg 0$. Let u_I be a semi-infinite wedge product,

$$u_I = \cdots \wedge u_{i_2} \wedge u_{i_1} \wedge u_{i_0}.$$

This wedge product satisfies the following relations; if $k > m$,

$$\begin{aligned}
&u_k \wedge u_m = -u_m \wedge u_k \quad (k \equiv m \mod \ell),\\
&u_k \wedge u_m = -q u_m \wedge u_k\\
&\qquad + (q^2-1)\left\{u_{m-i} \wedge u_{k+i} - q u_{m-\ell} \wedge u_{k+\ell} + q^2 u_{m-\ell+i} \wedge u_{k+\ell+i} - \cdots\right\}\\
&\qquad\qquad (m-k \equiv i \mod \ell, 0 < i < \ell).
\end{aligned}$$

Let vac_{-k} be the k-th vacuum vector defined by

$$\mathrm{vac}_{-k} = \cdots \wedge u_{-(k+2)} \wedge u_{-(k+1)} \wedge u_{-k}.$$

Definition 5.4. The Fock space of $U_q(\widehat{\mathfrak{sl}}_\ell)$ is defined by

$$\mathcal{F} = \bigoplus_I \mathbb{C}(q) u_I,$$

where I runs over the set of semi-infinite increasing sequences of integers such that $i_k = -k+1$ $(k \gg 0)$.

The Fock space $\mathcal{F}$ becomes a $U_q(\widehat{\mathfrak{sl}}_\ell)$-module by the coproduct Δ^-. We have

$$E_i \operatorname{vac}_{-k} = 0, \tag{5.4}$$

$$F_i \operatorname{vac}_{-k} = \begin{cases} \operatorname{vac}_{-k-1} \wedge u_{-k+1} & (i \equiv -k \mod \ell) \\ 0 & \text{otherwise} \end{cases}, \tag{5.5}$$

$$K_i \operatorname{vac}_{-k} = \begin{cases} q \operatorname{vac}_{-k} & (i \equiv -k \mod \ell) \\ \operatorname{vac}_{-k} & \text{otherwise} \end{cases}, \tag{5.6}$$

and the coproduct Δ^-.

Proposition 5.1. *These two realizations are isomorphic as $U_q(\widehat{\mathfrak{sl}}_n)$-modules by the one-to-one correspondence*

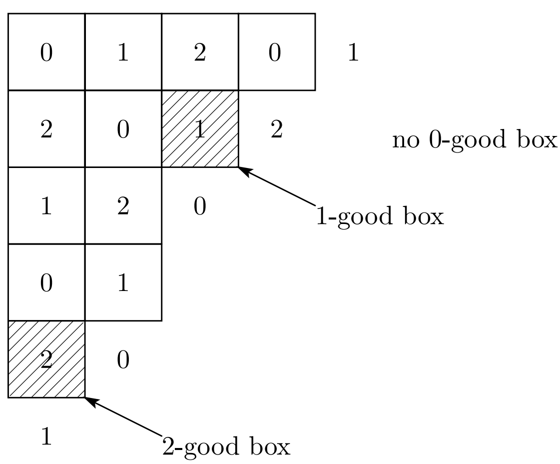
$$|\lambda = (\lambda_0 \geq \lambda_1 \geq \cdots)\rangle \leftrightarrow \cdots u_{\lambda_2 - 2} \wedge u_{\lambda_1 - 1} \wedge u_{\lambda_0}.$$

5.3. Crystal basis : Misra-Miwa's Theorem

We will use the notion of "i-good box" for the description of the crystal structure of the Fock space $\mathcal{F}$.

Definition 5.5. Let ℓ be a fixed positive number, and λ be a partition. Reading the i-addable boxes and the i-removable boxes in λ from bottom up, we can obtain the sequence of A and R. Next, delete all occurrences of AR from this sequence and keep doing this until no such subsequences remain. Then the i-good box in λ is the corresponding i-removable box to the right most R in this sequence.

$\lambda = (4, 3, 2, 2, 1),\ \ell = 3$



Let R be the subring of rational functions in $\mathbb{C}(q)$ which do not have a pole at 0. Let

$$L = \bigoplus_{\lambda \in \mathcal{P}} R|\lambda\rangle, \quad B = \{|\lambda\rangle \pmod{qL}\}.$$

Theorem 5.2 (Misra-Miwa [MM]). *The (L, B) is a crystal basis of $\mathcal{F}$ by the following action of Kashiwara operators $\widetilde{e_i}, \widetilde{f_i}$ $(1 \leq i \leq \ell - 1)$;*

(1) *If a partition λ has no i-good box, then $\widetilde{e_i}|\lambda\rangle = 0 \pmod{qL}$.*

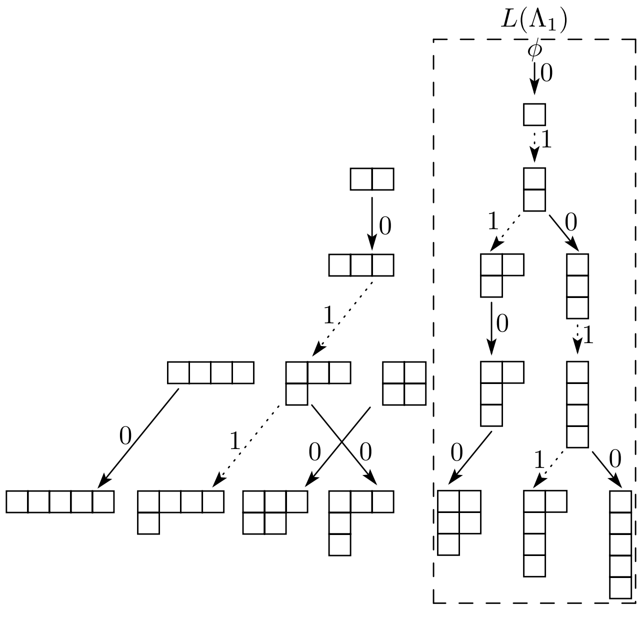
(2) *If x is a i-good box of λ and $\mu = \lambda \backslash \{x\}$,*

$$\widetilde{e_i}|\lambda\rangle = |\mu\rangle \pmod{qL}, \quad \widetilde{f_i}|\mu\rangle = |\lambda\rangle. \pmod{qL}.$$

(3) *If a partition μ has no i-addable box x which is i-good box in $\mu \cup \{x\}$, then $\widetilde{f_i}|\mu\rangle = 0 \pmod{qL}$.*

A proof of this theorem can be found in [Ari2].

Remark 5. The connected component of ϕ in B consists of $|\lambda\rangle \pmod{qL}$ such that λ is ℓ-restricted. This is a crystal basis of $L(\Lambda_1) := U_q^-(\widehat{\mathfrak{sl}}_\ell)\phi$.



5.4. Global basis

In this section, we will introduce the lower and upper global basis of $\mathcal{F}$. We consider the Fock space $\mathcal{F}$ as a wedge space.

In [KMS], the operator B_k $(k \in \mathbb{Z}, k \neq 0)$ on $\mathcal{F}$ is defined by the following;

$$\begin{aligned} B_k u_I = (\cdots \wedge u_{i_2} \wedge u_{i_1} \wedge u_{i_0 - \ell k}) + (\cdots \wedge u_{i_2} \wedge u_{i_1 - \ell k} \wedge u_{i_0}) \\ + (\cdots \wedge u_{i_2 - \ell k} \wedge u_{i_1} \wedge u_{i_0}) + \cdots . \end{aligned}$$

Note that $\cdots u_{i_\nu - \ell k} \wedge \cdots \wedge u_{i_0} = 0$ for $\nu \gg 0$.

5.4.1. Lower global basis

First, we will introduce the bar involution on $\mathcal{F}$.

Proposition 5.2 (e.g. [Kas1], [LT]). *There exists a unique bar involution* $^- : \mathcal{F} \to \mathcal{F}$ *satisfying the following four properties*;
(1) $\overline{F_i v} = F_i \overline{v} \quad (v \in \mathcal{F},\ 0 \le i \le \ell - 1)$,
(2) $\overline{B_k v} = B_k \overline{v} \quad (k > 0)$,
(3) $\overline{\mathrm{vac}_0} = \mathrm{vac}_0$.
(4) $\overline{qv} = q^{-1}\overline{v}$.

Theorem 5.3 (e.g. [Kas1], [LT]). *There exists a unique basis*

$$\{G^{\mathrm{low}}(\mu) \in \mathcal{F} | \mu \in \mathcal{P}\}$$

on $\mathcal{F}$ *satisfying the following two properties*;

(1) (*"bar invariance"*)

$$\overline{G^{\mathrm{low}}(\mu)} = G^{\mathrm{low}}(\mu).$$

(2) *If* μ *is a partition of* n, *there exists some polynomials* $d_{\lambda\mu}(q) \in q\mathbb{Z}[q]$, *then*

$$G^{\mathrm{low}}(\mu) = |\mu\rangle + \sum_{\mu \lhd \lambda \in \mathcal{P}_n} d_{\lambda\mu}(q)|\lambda\rangle,$$

where the ordering $\rhd$ *is the dominance ordering of partitions.*

This $\{G^{\mathrm{low}}(\mu)\}$ *is called the lower global basis of* $\mathcal{F}$.

Example 5.1. We will calculate some lower global bases for $\ell = 2$.
(0) $\mathrm{vac}_0 = |\phi\rangle$ is bar-invariant. Thus

$$G^{\mathrm{low}}(\phi) = |\phi\rangle.$$

(1) Since F_0 is bar-invariant,

$$F_0 \,\mathrm{vac}_0 = \mathrm{vac}_{-1} \wedge u_1 = |(1)\rangle$$

is bar-invariant. Thus

$$G^{\mathrm{low}}((1)) = |(1)\rangle.$$

(2) Since F_1 is bar-invariant,

$$\begin{aligned} F_1(\mathrm{vac}_{-1} \wedge u_1) &= (F_1 \,\mathrm{vac}_{-1}) \wedge u_1 + (K_1 \,\mathrm{vac}_{-1}) \wedge (F_1 u_1) \\ &= \mathrm{vac}_{-2} \wedge u_0 \wedge u_1 + q \,\mathrm{vac}_{-1} \wedge u_2 \\ &= |(1^2)\rangle + q|(2)\rangle \end{aligned}$$

is bar-invariant. Thus

$$G^{\mathrm{low}}((1^2)) = |(1^2)\rangle + q|(2)\rangle.$$

It is clear that $G^{\mathrm{low}}((2)) = |(2)\rangle$. Therefore

$$D = (d_{\lambda\mu}) = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}.$$

(3) Since F_0 is bar-invariant,

$$\begin{aligned} &F_0(\mathrm{vac}_{-2} \wedge u_0 \wedge u_1 + q\,\mathrm{vac}_{-1} \wedge u_2) \\ &\quad = (F_0\,\mathrm{vac}_{-2}) \wedge u_0 \wedge u_1 + (K_0\,\mathrm{vac}_{-2}) \wedge F_0(u_1 \wedge u_0) + q(F_0\,\mathrm{vac}_{-1}) \wedge u_2 \\ &\qquad + q(K_0\,\mathrm{vac}_{-1}) \wedge F_0 u_2 \\ &\quad = \mathrm{vac}_{-3} \wedge u_{-1} \wedge u_0 \wedge u_1 + q\,\mathrm{vac}_{-2} \wedge F_0(u_0 \wedge u_1) + q\,\mathrm{vac}_{-1} \wedge u_3 \\ &\quad = \mathrm{vac}_{-3} \wedge u_{-1} \wedge u_0 \wedge u_1 + q\,\mathrm{vac}_{-1} \wedge u_3 \\ &\quad = |(1^3)\rangle + q|(3)\rangle \end{aligned}$$

is bar-invariant. Thus

$$G^{\mathrm{low}}((1^3)) = |(1^3)\rangle + q|(3)\rangle.$$

Since F_1 is bar-invariant,

$$F_1(\mathrm{vac}_{-2} \wedge u_0 \wedge u_1 + q\,\mathrm{vac}_{-1} \wedge u_2) = (q + q^{-1})(\mathrm{vac}_{-2} \wedge u_0 \wedge u_2)$$

are bar-invariant. Moreover, $\mathrm{vac}_{-2} \wedge u_0 \wedge u_2 = |(2,1)\rangle$ is bar-invariant because $(q + q^{-1})$ is bar-invariant. Thus

$$G^{\mathrm{low}}((2,1)) = |(2,1)\rangle.$$

It is clear that $G^{\mathrm{low}}((3)) = |(3)\rangle$. Therefore

$$D = (d_{\lambda\mu}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q & 0 & 1 \end{pmatrix}.$$

5.4.2. Upper global basis

Let $\{\langle\lambda|\}_{\lambda\in\mathcal{P}}$ be the dual basis of $\{|\lambda\rangle\}_{\lambda\in\mathcal{P}}$ with respect to the scalar product $\langle\lambda|\mu\rangle = \delta_{\lambda\mu}$. The $U_q(\widehat{\mathfrak{sl}}_\ell)$-module

$$\mathcal{F}^\vee = \bigoplus_{\lambda\in\mathcal{P}} \mathbb{C}(q)\langle\lambda|$$

is isomorphic to the wedge space

$$\bigoplus_I \mathbb{C}(q) u_I$$

with the action of $U_q(\widehat{\mathfrak{sl}}_n)$ defined by (5.4), (5.5), (5.6) and coproduct Δ^+.

Proposition 5.3 (e.g. [Kas1], [LT]). *There exists a unique bar involution* $^{-} : \mathcal{F}^\vee \to \mathcal{F}^\vee$ *satisfying the following four properties*;
(1) $\overline{F_i v} = F_i \overline{v} \quad (v \in \mathcal{F}^\vee,\ 0 \le i \le \ell - 1)$,
(2) $\overline{B_k v} = B_k \overline{v} \quad (k < 0)$,
(3) $\overline{\mathrm{vac}_0} = \mathrm{vac}_0$.
(4) $\overline{qv} = q^{-1}\overline{v}$

Theorem 5.4 (e.g. [Kas1], [LT]). *There exists a unique basis*

$$\{G^{\mathrm{up}}(\mu) \in \mathcal{F}^\vee | \mu \in \mathcal{P}\}$$

on $\mathcal{F}^\vee$ *satisfying the following two properties*;

(1) (*"bar invariance"*)

$$\overline{G^{\mathrm{up}}(\mu)} = G^{\mathrm{up}}(\mu).$$

(2) *If* λ *is a partition of* n, *there exists some polynomials* $d'_{\lambda\mu}(q) \in q\mathbb{Z}[q]$ *such that*

$$\langle\lambda| = G^{\mathrm{up}}(\lambda) + \sum_{\lambda \rhd \mu \in \mathcal{P}_n} d'_{\lambda\mu}(q) G^{\mathrm{up}}(\mu).$$

Moreover $d'_{\lambda\mu}(q)$ *is equal to* $d_{\lambda\mu}(q)$ *in Theorem* 5.3. *Especially the basis* $\{G^{\mathrm{up}}(\mu)\}$ *is the dual basis of* $\{G^{\mathrm{low}}(\mu)\}$.

This $\{G^{\mathrm{up}}(\mu)\}$ *is called the upper global basis of* $\mathcal{F}^\vee$.

Example 5.2. We will calculate some upper global basis for $\ell = 2$.
(0) Since vac_0 is bar-invariant, $G^{\mathrm{up}}(\phi) = \langle\phi|$.
(1) Since $F_0(\mathrm{vac}_0) = \mathrm{vac}_{-1} \wedge u_1$ is bar-invariant, $G^{\mathrm{up}}((1)) = \langle(1)|$.
(2) Note that the action of F_1 is defined by the coproduct Δ^+. We obtain

$$\begin{aligned} F_1(\mathrm{vac}_{-1} \wedge u_1) &= F_1 \mathrm{vac}_{-1} \wedge K_1^{-1} u_1 + \mathrm{vac}_{-1} \wedge F_1 u_1 \\ &= q^{-1} \mathrm{vac}_{-2} \wedge u_0 \wedge u_1 + \mathrm{vac}_{-1} \wedge u_2. \end{aligned} \tag{5.7}$$

This is bar-invariant, but not $G^{\mathrm{up}}(\mathrm{vac}_{-1} \wedge u_2) = G^{\mathrm{up}}((2))$ because the right hand side does not satisfy the condition (2) of the upper global basis, i.e. $q^{-1} \notin q\mathbb{Z}[q]$.

B_{-1} is bar-invariant. Thus

$$\begin{aligned} B_{-1} \mathrm{vac}_0 &= \mathrm{vac}_{-2} \wedge u_1 \wedge u_0 + \mathrm{vac}_{-2} \wedge u_{-1} \wedge u_2 \\ &= -q \,\mathrm{vac}_{-2} \wedge u_0 \wedge u_1 + \mathrm{vac}_{-1} \wedge u_2 \end{aligned} \tag{5.8}$$

is bar-invariant.

$$(5.7) - (5.8) = (q + q^{-1}) \,\mathrm{vac}_{-2} \wedge u_0 \wedge u_1 \tag{5.9}$$

is bar-invariant.

$$(5.7) - (5.9) = \mathrm{vac}_{-1} \wedge u_2 - q \,\mathrm{vac}_{-2} \wedge u_0 \wedge u_1$$

is bar-invariant. Thus

$$G^{\mathrm{up}}((1^2)) = \langle (1^2)|, \quad G^{\mathrm{up}}((2)) = \langle (2)| - q\langle (1^2)|,$$

and

$$\langle (2)| = G^{\mathrm{up}}((2)) + qG^{\mathrm{up}}((1^2)).$$

Therefore the matrix $D = (d_{\lambda\mu})$ coincides with Example 5.1(2).
(3) By using the bar-invariance of F_0, F_1, we can obtain the bar-invariance of

$$F_0(\mathrm{vac}_{-2} \wedge u_0 \wedge u_1),\ F_1(\mathrm{vac}_{-2} \wedge u_0 \wedge u_1),\ F_0(\mathrm{vac}_{-1} \wedge u_2 - q\,\mathrm{vac}_{-2} \wedge u_0 \wedge u_1).$$

It is easy to compute G^{up} and see that the matrix D is coincide the Example 5.1(3).

5.5. LLT-Ariki type theorem on the v-Schur algebras

We consider the v-Schur algebra $\mathbb{S}_v(n)$. If v is a primitive ℓ-th root of unity, the Weyl modules W_v^λ has a unique simple quotient L_v^λ. The set $\{L_v^\lambda \mid \lambda \vdash n\}$ gives a complete representatives of simple modules. The composition multiplicities

$$d_{\lambda\mu} = [W_v^\lambda : L_v^\mu]$$

are called the decomposition numbers.

Varagnolo-Vasserot proved in [VV1] that the decomposition numbers coincide $d_{\lambda\mu}(1)$. This is an extended result of the LLT-Ariki type theorem of the Hecke algebra for the v-Schur algebras.

Theorem 5.5 (Varagnolo-Vasserot[VV1])**.** *Consider the Fock space $\mathcal{F}$ of $U_q(\widehat{\mathfrak{sl}}_\ell)$ and its lower global basis $\{G^{low}(\mu)|\mu \in \mathcal{P}\}$ and crystal basis $\{|\lambda\rangle|\lambda \in \mathcal{P}\}$. Let us consider the coefficients of*

$$G^{\mathrm{low}}(\mu) = |\mu\rangle + \sum_{\mu \lhd \lambda \in \mathcal{P}_n} d_{\lambda\mu}(q)|\lambda\rangle.$$

Then we have

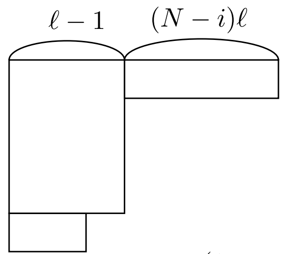
$$d_{\lambda\mu} = [W_v^\lambda : L_v^\mu] = d_{\lambda\mu}(1).$$

6. Main Theorem

6.1. Main Theorem

We will state the Main theorem on the coefficient $d_{(n),\mu}(q)$.
First, let us define the following partitions of n.

Definition 6.1. Put $N = \left[\frac{n}{\ell}\right]$. The partitions $\mu_i^{(n)}$ is defined by the following,

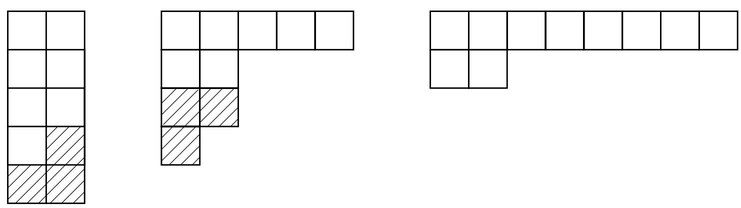
$\mu_i^{(n)} =$ 

$(1 \leq i \leq N)$

and $\mu_0^{(n)} = (n)$.

Remark 6. These partitions of n are constructed by removing a rim ℓ-hook from the bottom to the first row. For example, let us consider the case $n = 10, \ell = 3$. Then $N = 3$. In this case, $\mu_i^{(10)}$ $(i = 3, 2, 1, 0)$ are the following,

$$\mu_3^{(10)} = (2^5),\ \mu_2^{(10)} = (5, 2^2, 1),\ \mu_1^{(10)} = (8, 2),\ \mu_0^{(10)} = (10).$$



$\mu_3^{(10)}$ $\mu_2^{(10)}$ $\mu_1^{(10)}$

Theorem 6.1. *The crystal basis element $\langle (n)|$ is expanded by G^{up} as the following*;

$$\langle (n)| = \sum_{i=0}^{N} q^i G^{\mathrm{up}}(\mu_i^{(n)}),$$

i.e.

$$d_{(n),\mu}(q) = \begin{cases} q^i & \textit{if } \mu = \mu_i^{(n)} \\ 0 & \textit{otherwise} \end{cases}.$$

Corollary 6.1. *If v is a primitive ℓ-th root of unity, the decomposition numbers*

$$d_{(n),\mu} = [W_v^{(n)} : L_v^{\mu}] = \begin{cases} 1 & \textit{if } \mu = \mu_i^{(n)} \\ 0 & \textit{otherwise} \end{cases}.$$

6.2. Proof of Main Theorem

6.2.1. Key Lemmas

First, we can describe the action of E_i on the upper global base $\{G^{\mathrm{up}}\}$. Let us set

$$\varepsilon_i(\mu) = \max\{k \geq 0 | \tilde{e}_i^{\,k} \mu \neq 0\}.$$

Lemma 6.1 (Kashiwara [Kas2]). *The element E_i in $U_q(\widehat{sl}_\ell)$ acts on G^{up} as the following,*

$$E_i G^{\mathrm{up}}(\mu) = [\varepsilon_i(\mu)] G^{\mathrm{up}}(\widetilde{e}_i \mu) + \sum_{\varepsilon_i(\nu) < \varepsilon_i(\mu) - 1} b^i_{\mu\nu} G^{\mathrm{up}}(\nu).$$

Especially, if $\varepsilon_i(\mu) = 1$, then

$$E_i G^{\mathrm{up}}(\mu) = G^{\mathrm{up}}(\widetilde{e}_i \mu).$$

Secondly, we have the following lemma on a property of $\bigcap_j \mathrm{Ker}(E_j)$. This follows from the irreducible decomposition of $\mathcal{F}$ e.g. [LLT], [KMS].

Lemma 6.2. *For any $x \in \bigcap_j \mathrm{Ker}(E_j) \subset \mathcal{F}^\vee$, we have the following expansion;*

$$x = \sum_{\lambda \in \mathcal{P}} b_{x,\lambda} G^{\mathrm{up}}(\ell\lambda).$$

Thirdly, we consider the coefficients of $G^{\mathrm{up}}(\ell\lambda)$ in the expansion of $\langle (n)|$.

Lemma 6.3.
(1) (*Kashiwara* [Kas1]) *The following expansions hold,*

$$\begin{aligned} &G^{\mathrm{low}}(\mathrm{vac}_{-m-1} \wedge u_{\ell\lambda_m - m} \wedge \cdots \wedge u_{\ell\lambda_1 - 1} \wedge u_{\ell\lambda_0}) \\ &\quad = \sum a_{j_m, j_{m-1}, \cdots, j_0}(q)\, \mathrm{vac}_{-m-1} \wedge u_{j_m + \ell\lambda_m - \ell m} \\ &\qquad\qquad \wedge u_{j_{m-1} + \ell\lambda_{m-1} - \ell(m-1)} \wedge \cdots \wedge u_{j_0 + \ell\lambda_0} \end{aligned}$$

where the sum runs on the index $(j_m, j_{m-1}, \cdots, j_0)$ satisfied

$$\begin{aligned} (0, \ell-1, 2(\ell-1), \cdots, m(\ell-1)) &\le (j_m, j_{m-1}, \cdots, j_0) \\ &\le (m(\ell-1), (m-1)(\ell-1), \cdots, 0) \end{aligned}$$

and

$$(-m, -m+1, \cdots, 0) \le (j_m + \ell\lambda_m - \ell m, j_{m-1} + \ell\lambda_{m-1} - \ell(m-1), \cdots, j_0 + \ell\lambda_0).$$

(2) *For $\lambda \in \mathcal{P}$ such that $\ell\lambda \neq (n)$, we have $d_{\ell\lambda,(n)}(q) = 0$.*

Fourthly, we can describe the action of the Kashiwara operators $\widetilde{e}_j$ on $\mu_i^{(n)}$ by using the Misra-Miwa's Theorem 5.2.

Lemma 6.4. *The action of $\widetilde{e}_j$ on $\mu_i^{(n)}$ is obtained by the following;*
(1) *If $n \not\equiv 0 \pmod{\ell}$, then*

$$\widetilde{e}_j(\mu_i^{(n)}) = \begin{cases} 0 & \text{if } j \not\equiv n-1 \\ \mu_i^{(n-1)} & \text{if } j \equiv n-1 \end{cases}$$

for $0 \leq i \leq N$.

(2) *If* $n \equiv 0 \pmod{\ell}$, *then*

$$\widetilde{e_j}(\mu_i^{(n)}) = \begin{cases} 0 & \text{if } j \not\equiv n-1 \\ \mu_{i-1}^{(n-1)} & \text{if } j \equiv n-1 \end{cases}$$

for $1 \leq i \leq N$. *And* $\widetilde{e_j}\mu_0^{(n)} = 0$ *for any* $0 \leq j \leq \ell - 1$.

(3) *Especially,*

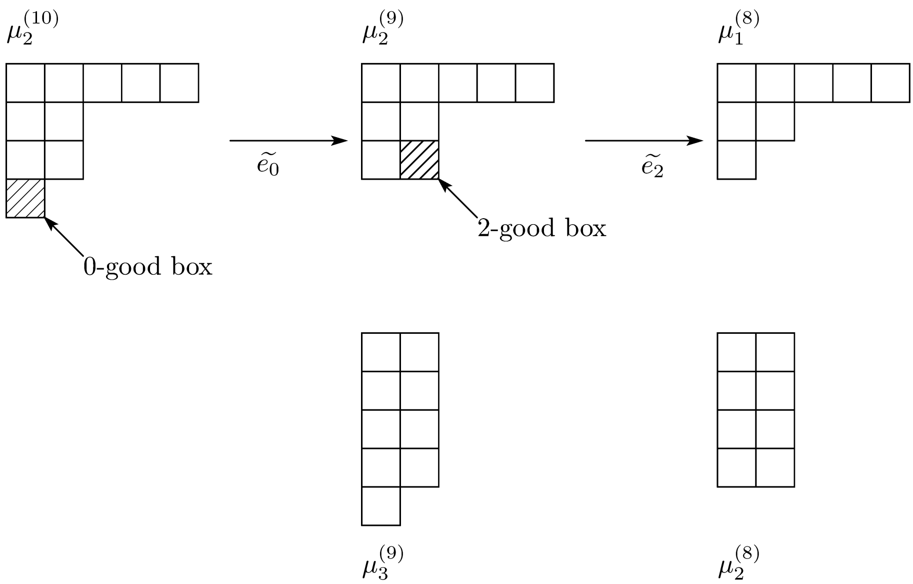
$$\varepsilon_j(\mu_i^{(n)}) = \begin{cases} 1 & \text{if } j \equiv n-1 \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq N$. *And*

$$\varepsilon_j(\mu_0^{(n)}) = \begin{cases} 1 & \text{if } j \equiv n-1 \text{ and } n \not\equiv 0 \\ 0 & \text{otherwise} \end{cases}.$$

Example 6.1. We consider the case $n = 10, \ell = 3$. Then

$$\widetilde{e_0}\mu_2^{(10)} = (5,2,2) = \mu_2^{(9)},$$
$$\widetilde{e_0}\mu_2^{(9)} = (5,2,1) = \mu_1^{(8)}.$$



6.2.2. Proof of Main Theorem

We set

$$D_n := \mathrm{vac}_{-1} \wedge u_n - \sum_{i=0}^{N} q^i G^{\mathrm{up}}(\mu_i^{(n)})$$

We show that $D_n = 0$ by the induction on n.

We assume that $D_i = 0$ for $i < n$.

Then we first claim that

$$D_n \in \bigcap_{j=0}^{\ell-1} \mathrm{Ker}(E_j). \tag{6.1}$$

If $n \equiv 0 \pmod{\ell}$, we have

$$E_j(\mathrm{vac}_{-1} \wedge u_n) = 0 \ (0 \le j \le \ell - 2)$$

by Δ^+ and the action of E_j on u_m's at (5.1). We also obtain

$$E_j G^{\mathrm{up}}(\mu_i^{(n)}) = 0$$

for $0 \le j \le \ell - 2$ and $0 \le i \le N$ by Lemma 6.1 and Lemma 6.4. For $j = \ell - 1$, by using Δ^+, (5.1), (5.3), Lemma 6.1 and Lemma 6.4 again, we have

$$E_{\ell-1}(D_n) = q\,\mathrm{vac}_{-1} \wedge u_{n-1} - \sum_{i=1}^{N} q^i G^{\mathrm{up}}(\mu_{i-1}^{(n-1)}) = qD_{n-1}.$$

Then $E_{\ell-1}(D_n) = 0$ by the induction hypothesis. Thus $E_j(D_n) = 0$ for $0 \le j \le \ell - 1$ and $n \equiv 0 \pmod{\ell}$.

If $n \not\equiv 0 \pmod{\ell}$, similarly as above, we have

$$E_j(D_n) = 0$$

for $j \not\equiv n - 1 \pmod{\ell}$, and

$$E_j(D_n) = \mathrm{vac}_{-1} \wedge u_{n-1} - \sum_{i=0}^{N} q^i G^{\mathrm{up}}(\mu_i^{(n-1)}) = D_{n-1}$$

for j such that $j \equiv n-1 \pmod{\ell}$. Then $E_j(D_n) = 0$ by the induction hypothesis.

Hence, in any cases, we deduce the claim (6.1).

By Lemma 6.2, D_n is expanded by $\{G^{\mathrm{up}}(\ell\lambda) | \lambda \in \mathcal{P}\}$;

$$D_n = \sum_{\lambda \in \mathcal{P}} b_{\lambda,n} G^{\mathrm{up}}(\ell\lambda).$$

Note that there is no λ such that $\mu_i^{(n)} = \ell\lambda$ for $1 \le i \le N$. Then, by Lemma 6.3, we have $b_{\lambda,n} = 0$ for $\lambda \in \mathcal{P}$ such that $\ell\lambda \ne (n)$. On the other hand, note that $\mu_0^{(n)} = (n)$. Then, for λ such that $\ell\lambda = (n)$, we have $d_{\ell\lambda,(n)} = 1 + b_{\lambda,n}$. By the upper unitriangularity at Theorem 5.4, we have $b_{\lambda,n} = 0$. We conclude $D_n = 0$. Thus the proof of theorem is complete.

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