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Flips and variation of moduli scheme of sheaves on a surface

By

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Abstract

Let H be an ample line bundle on a non-singular projective surface X, and $M(H)$ the coarse moduli scheme of rank-two H-semistable sheaves with fixed Chern classes on X. We show that if H changes and passes through walls to get closer to K_X, then $M(H)$ undergoes natural flips with respect to canonical divisors. When X is minimal and $\kappa(X) \geq 1$, this sequence of flips terminates in $M(H_X)$; H_X is an ample line bundle lying so closely to K_X that the canonical divisor of $M(H_X)$ is nef. Remark that so-called Thaddeus-type flips somewhat differ from flips with respect to canonical divisors.

1. Introduction

Let X be a non-singular projective surface over $\mathbf{C}$, and H an ample line bundle on X. Denote by $M(H)$ (resp. $\bar{M}(H)$) the coarse moduli scheme of rank-two H-stable (resp. H-semistable) sheaves on X with Chern class $(c_1, c_2) \in \mathrm{Pic}(X) \times \mathbf{Z}$. We shall consider birational aspects of the problem how $\bar{M}(H)$ changes as H varies.

Let H_- and H_+ be generic ample line bundles separated by just one (c_1, c_2)-wall W. For $a \in (0, 1)$ one can define a-semistability of sheaves and the coarse moduli scheme $M(a)$ (resp. $\bar{M}(a)$) of rank-two a-stable (resp. a-semistable) sheaves with Chern classes (c_1, c_2) in such a way that $\bar{M}(\epsilon) = \bar{M}(H_+)$ and $\bar{M}(1-\epsilon) = \bar{M}(H_-)$ if $\epsilon > 0$ is sufficiently small. Let $a_- < a_+$ be generic parameters separated by only one miniwall a_0. Denote $\bar{M}_\pm = \bar{M}(a_\pm)$ and $\bar{M}_0 = \bar{M}(a_0)$. There are natural morphisms $f_- : \bar{M}_- \to \bar{M}_0$ and $f_+ : \bar{M}_+ \to \bar{M}_0$. After [4], let $f : X \to Y$ be a birational proper morphism such that K_X is $\mathbf{Q}$-Cartier and $-K_X$ is f-ample, and that the codimension of the exceptional set $\mathrm{Ex}(f)$ of f is more than 1. We say a birational proper morphism $f_+ : X_+ \to Y$ is a *flip* of f if (1) K_{X_+} is $\mathbf{Q}$-Cartier, (2) K_{X_+} is f_+-ample and (3) the codimension of the exceptional set $\mathrm{Ex}(f_+)$ is more than 1. The main result is the following.

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Theorem 1.1. *Assume c_2 is so large that $\bar{M}_-$ and $\bar{M}_+$ are normal and that the codimensions of*

$$\bar{M}_\pm \supset P_\pm = \big\{[E] \bigm| E \text{ is not } a_\mp\text{-semistable}\big\} \tag{1.1}$$

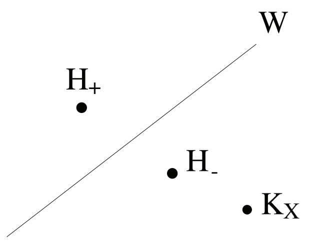
and

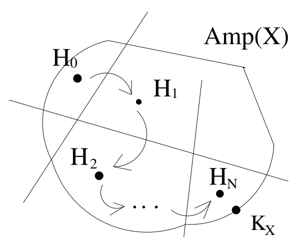
$$\bar{M}_\pm \supset \mathrm{Sing}(\bar{M}_\pm) = \{[E] \mid \dim \mathrm{Ext}^2(E,E)^0 \neq 0\}$$

are more than 1. *Suppose K_X does not lies in the wall W separating H_- and H_+, and that K_X and H_- lie in the same connected components of* $\mathrm{NS}(X)_{\mathbb{R}} \backslash W$. (*See the left figure below.*) *Then the birational map*

$$\begin{tikzcd}[column sep=small] M_+ \arrow[rr, dashed] \arrow[dr, "f_-"'] & & M_- \arrow[dl, "f_+"] \\ & f_-(M_-) = f_+(M_+) \subset \bar{M}_0, & \end{tikzcd} \tag{1.2}$$

where $f_-(M_-)$ is open in $\bar{M}_0$, is a flip.





Let us observe this theorem in case where X is minimal and $\kappa(X) \geq 1$. Then K_X is nef and so there is an ample line bundle H_X such that no wall of type (c_1, c_2) divides K_X and H_X. Suppose $M(H) = \bar{M}(H)$ for any generic polarization H, that is valid if $c_1 = 0$ and c_2 is odd for example. Fix a polarization H_0. If c_2 is sufficiently large, then $M(H)$ are isomorphic in codimension one for $H \in \mathcal{L} = \{(1-t)H_0 + tH_X | t \in [0,1]\}$. Hence when $H \in \mathcal{L}$ starts from a polarization H_0 and gets closer to K_X, one gets flips

$$M(H) = M(a_0 = \epsilon) \cdots > M(a_1) \quad \cdots > M(a_{N'} = 1 - \epsilon) = M(H')$$

when H passes through a wall to get closer to K_X by Theorem 1.1, where a_i are parameters lying in adjacent miniwalls. Hence the birational map $M(H) \cdots > M(H')$ is decomposed into finite sequence of natural flips. After we repeat it finitely many times, H reaches H_X and hence the sequence of birational morphisms

$$M(H = H_0) \cdots > M(H_1) \quad \cdots > M(H_N = H_X)$$

terminates in $M(H_X)$. (See the right figure above.) It is known that the canonical divisor of $M(H_X)$ is nef. Thus one can regard this "natural" process described in a moduli-theoretic way as an analogy of minimal model program of $M(H)$, although it is unknown whether $M(H_X)$ admits only terminal singularities. Note that $M(H)$ is of general type if X is of general type, $H^0(K_X)$ contains a reduced curve, and $\chi(\mathcal{O}_X)+c_1^2(E) \equiv 0(2)$ by [5]. Corollary 2.1 states that any sequence of flips occurring from the variation of moduli schemes of sheaves on a surface always stops, in relation to termination of flips.

We mention some characteristics of this paper compared with Thaddeus' work [8], which considered the variation of GIT quotients and linearizations. By [6] the rational map $M_- \cdots > M_+$ is a Thaddeus-flip, that is, a rational map which is an isomorphism in codimension 1 and comes from the variation of GIT quotient and linearizations. Relations about a flip with respect to the canonical divisor are not mentioned there. So-called Thaddeus-flip is weaker than a flip defined above. Moreover the birational map (1.2) is described in a moduli-theoretic way. Moduli schemes M_- and M_+ are connected by a natural blow-up and a blow-down described in moduli theory; see [9, Prop. 4.9].

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2. Proof of Theorem

There is a union of hyperplanes $W \subset \operatorname{Amp}(X)$ called (c_1, c_2)-walls in the ample cone $\operatorname{Amp}(X)$ such that $\bar{M}(H)$ changes only when H passes through walls ([7]). Let H_- and H_+ be ample line bundles separated by just one wall W, and $H_0 = \lambda H_- + (1-\lambda)H_+$ an ample line bundle contained in W. If c_2 is sufficiently large with respect to a compact subset $S \subset \operatorname{Amp}(X)$ containing $H_\pm$, then $M_\pm$ are normal and the codimension of $P_\pm \subset M_\pm$, which is defined at (1.1), are greater than 2 from [5] and [3, Thm. 4.C.7]. By [1, Sect. 3], for a number $a \in [0,1]$ one can define the a-stability of a torsion-free sheaf E using

$$P_a(E(n)) = \{(1-a)\chi(E(H_-)(nH_0)) + a\chi(E(H_+(nH_0)))\}/\operatorname{rk}(E).$$

There is the coarse moduli scheme $\bar{M}(a)$ of rank-two a-semistable sheaves on X with Chern classes (c_1, c_2). Denote by $M(a)$ its open subscheme of a-stable sheaves. There is a finite numbers a_i called miniwall such that, as a varies, $M(a)$ changes only when a passes through miniwalls. Let $a_- < a_+$ be parameters separated by only one miniwall a_0, and denote $\bar{M}_\pm = \bar{M}(a_\pm)$ and $\bar{M}_0 = \bar{M}(a_0)$. Since a rank-two a_--semistable sheaf of type (c_1, c_2) is a_0-semistable, there are natural morphisms $f_- : \bar{M}_- \to \bar{M}_0$ and $f_+ : \bar{M}_+ \to \bar{M}_0$ when $M = H_+ - H_-$ is effective and C equals $n_0 M$ with sufficiently large n_0, by [1, Prop. 3.14].

Remark that the canonical divisors of $M_\pm$ are $\mathbf{Q}$-Cartier. Indeed, M_- equal $R//\operatorname{SL}(N)$, where R is a subscheme of the Quot-scheme parameterizing quotient sheaves $\mathcal{O}_X(-M) \to E^-$ on X. Let E^-_R be the universal quotient sheaf on X_R. From descent lemma [3, Theorem 4.2.15], $\det RHom_{X_R/R}(E^-_R, E^-_R)$ descends to a line bundle on M_-, which we denote by $\det RHom_{X_{M_-}/M_-}(E^-, E^-)$. It is known that $K_{M_-}|_{M_- \setminus \operatorname{Sing}(M_-)}$ equals $\det RHom_{X_{M_-}/M_-}(E^-, E^-)$ from deformation theory. Since M_- is normal, we have

$$K_{M_-} = \det RHom_{X_{M_-}/M_-}(E^-, E^-), \tag{2.1}$$

so it is $\mathbf{Q}$-Cartier.

Let η be an element of

$$A^+(W) = \left\{\eta \in \mathrm{NS}(X) \;\middle|\; \eta \text{ defines } W,\, 4c_2 - c_1^2 + \eta^2 \geq 0 \text{ and } \eta \cdot H_+ > 0)\right\}.$$

After [1, Definition 4.2] we define

$$T_\eta = M(1, (c_1 + \eta)/2, n) \times M(1, (c_1 - \eta)/2, m),$$

where n and m are numbers defined by

$$n + m = c_2 - (c_1^2 - \eta^2)/4 \quad \text{and} \quad n - m = \eta \cdot (c_1 - K_X)/2 + (2a_0 - 1)\eta \cdot C,$$

and $M(1, (c_1 + \eta)/2)$ is the moduli scheme of rank-one torsion-free sheaves on X with Chern classes $((c_1 + \eta)/2, n)$. We also denote $T = \coprod T_\eta$, where η runs over $A^+(W)$. If F_{T_η} (resp. G_{T_η}) is the pull-back of a universal sheaf of $M(1, (c_1 + \eta)/2, n)$ (resp. $M(1, (c_1 - \eta)/2, m)$) to X_{T_η}, then we have the following.

Proposition 2.1 ([9], Section 5)**.** *We have isomorphisms*

$$P_- \simeq \coprod_{\eta \in A^+(W)} \mathbf{P}_{T_\eta}\left(Ext^1_{X_{T_\eta}/T_\eta}(F_{T_\eta}, G_{T_\eta}(K_X))\right) \quad \textit{and} \tag{2.2}$$

$$P_+ \simeq \coprod_{\eta \in A^+(W)} \mathbf{P}_{T_\eta}\left(Ext^1_{X_{T_\eta}/T_\eta}(G_{T_\eta}, F_{T_\eta}(K_X))\right). \tag{2.3}$$

There are line bundles L_i (resp. L'_i) on P_- (resp. P_+) with exact sequences

$$0 \longrightarrow F_T \otimes L_1 \longrightarrow E^-_{M_-}|_{P_-} \longrightarrow G_T \otimes L_2 \longrightarrow 0 \quad \textit{and}$$
$$0 \longrightarrow G_T \otimes L'_1 \longrightarrow E^+_{M_+}|_{P_+} \longrightarrow F_T \otimes L'_2 \longrightarrow 0$$

such that $L_1 \otimes L_2^{-1} = \mathcal{O}_{P_-}(1)$, which means the tautological line bundle of the right side of (2.2), and $L'_1 \otimes L_2^{'-1} = \mathcal{O}_{P_+}(1)$. Here $E^-_{M_-}$ is a universal family of M_-, which exists etale-locally.

Claim 1. It holds that

$$K_{M_-}|_{P_- \underset{T}{\times} T_\eta} = -(\eta \cdot K_X)\, \mathcal{O}_{P_-}(1) + (\text{some line bundle on } T), \quad \text{and}$$
$$K_{M_+}|_{P_+ \underset{T}{\times} T_\eta} = (\eta \cdot K_X)\, \mathcal{O}_{P_+}(1) + (\text{some line bundle on } T).$$

Proof. Suppose that $A^+(W) = \{\eta\}$ for simplicity. From the definition of walls, one can check $c_1(F_t) - c_1(G_t) = \eta$. By Proposition 2.1 and (2.1),

$$\begin{aligned} K_{M_-}|_{P_-} &= \det RHom_{X_{P_-}/P_-}(E^-_{M_-}|_{P_-}, E^-_{M_-}|_{P_-}) \\ &= \det RHom_{X_T/T}(F_T, F_T) + \det(RHom_{X_T/T}(F_T, G_T) \otimes L_2 \otimes L_1^{-1}) \\ &\quad + \det(RHom_{X_T/T}(G_T, F_T) \otimes L_1 \otimes L_2^{-1}) + \det RHom_{X_T/T}(G_T, G_T) \\ &= -\chi(F_t, G_t) \cdot \mathcal{O}_{P_-}(1) + \chi(G_t, F_t) \cdot \mathcal{O}_{P_-}(1) + (\text{line bundle on } T) \\ &= -(\eta \cdot K_X)\, \mathcal{O}_{P_-}(1) + (\text{line bundle on } T). \end{aligned}$$

One can calculate $K_{M_+}|_{P_+}$ similarly. □

Remark that, since $\eta \cdot H_+ > 0$, one can verify that $\eta \cdot K_X < 0$ if and only if K_X does not lie in $W^\eta = W$, and H_- and K_X lie in the same connected components of $\mathrm{NS}(X)_{\mathbb{R}} \setminus W$. The next lemma ends the proof of 1.1.

Lemma 2.1. *The map $f_+ : M_+ \to f_+(M_+)$ is proper.*

Proof. It suffices to show that $f_+^{-1} f_+(M_+) = M_+$. Suppose not. Then some $[E] \in M_+$ and $[E'] \in \bar{M}_+ \setminus M_+$ satisfies $f_+([E]) = f_+([E'])$. Since a_+ is generic, E' is denoted with an exact sequence

$$0 \longrightarrow F' \longrightarrow E' \longrightarrow G' \longrightarrow 0 \tag{2.4}$$

with rank-one torsion-free sheaves F' and G' such that $P_a(F'(n)) = P_a(G'(n))$ for any a. $[E]$ and $[E']$ are S-equivalent with respect to a_0-stability, so (2.4) implies that E cannot be a_+-stable, which is a contradiction. □

Corollary 2.1. *Any sequence of flips occurring from the variation of polarizations and moduli schemes of sheaves on a surface X always stops after finitely many modifications.*

Proof. In fixing a polarization H, we claim that only finitely many walls pass across the segment L connecting H and K_X. Indeed, $L \cap \partial \mathrm{Amp}(X)$ is empty or equals $\{H_0\}$ with a $\mathbf{Q}$-divisor H_0 from the cone theorem. Thus the claim follow from the fact [6, Lem. 1.5'] that only finitely many walls intersect with a fixed polyhedral cone in $\mathrm{Amp}(X)$ spanned by $\mathbf{Q}$-divisors. One can readily check the corollary from this claim. □

We end with proving some facts in Introduction.

Lemma 2.2. *Assume K_X is nef, and fix c_1 and a polarization H_0. If c_2 is sufficiently large with respect to X and H_0, then for $H \in \mathcal{L} = \{(1-t)H_0 + tK_X | t \in [0,1)\}$, $M(H)$ are mutually isomorphic in codimension one.*

Proof. Let H_- and H_+ be any ample line bundles lying in adjacent chambers and separated by a wall W^η passing through a point L on $\mathcal{L}$. Since K_X is nef, one can find an effective divisor $H \in H^0(\mathcal{O}_X(NK_X))$ with some $N > 0$ such that H is the disjoint union of some finite smooth curves. Then similarly to [7, Lem. 2.2] we can show that, for a divisor F with $2F - c_1 \sim \eta$,

$$h^0(\mathcal{O}_X(K_X - (2F - c_1))) \leq d_1(X) + N|K_X \cdot \eta|, \tag{2.5}$$

where d_1 is a constant depending only on X. By the proof of [7, Lem. 2.1], it holds that

$$|K_X \cdot \eta| \leq \left\{ (K_X^2)^{1/2} + \left(|K_X \cdot L| / (L^2)^{1/2} \right) \right\} (4c_2 - c_1^2)^{1/2}. \tag{2.6}$$

When $K_X^2 > 0$, $|K_X \cdot L|/(L^2)^{1/2}$ is bounded for any $L \in \mathcal{L}$. When $K_X^2 = 0$, one can check that

$$|(K_X \cdot L)^2 / L^2| \leq (K_X \cdot H_0)^2 / H_0^2.$$

Thus some d_2 depending only on X and H_0 satisfies

$$|K_X \cdot \eta| \leq d_2(X, H_0) \cdot (4c_2 - c_1^2)^{1/2}. \tag{2.7}$$

In the same way as [7, Thm. 2.3], we can show that

$$\dim(P_- \underset{T}{\times} T_\eta) \leq (3/4)(4c_2 - c_1^2) + d_3(X, H_0) + d_4(X, H_0) \cdot (4c_2 - c_1^2)^{1/2}$$

with some constant d_3 and d_4 depending only on X and H_0 by using (2.5), (2.6) and (2.7), and this implies the lemma. □

Claim 2. Suppose K_X is nef, and let H_X be an ample line bundle H_X such that no wall of type (c_1, c_2) divides K_X and H_X. Then the canonical divisor of $M(H_X)$ is nef.

Proof. From [3, Prop. 8.3.1] $2K_{M(H_X)} = p_*(\Delta(E_{M(H_X)}) \cdot K_X)$, and $p_*(\Delta(E_{M(H_X)}) \cdot H_X)$ is nef. When H_X is sufficiently close to K_X, the assertion holds. □

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References

[1] G. Ellingsrud and L. Göttsche, *Variation of moduli spaces and Donaldson invariants under change of polarization*, J. Reine Angew. Math. **467** (1995), 1–49.

[2] R. Friedman, Algebraic Surfaces and Holomorphic Vector Bundles, Springer-Verlag, New York, 1998.

[3] D. Huybrechts and M. Lehn, The Geometry of Moduli Spaces of Sheaves, Friedr. Vieweg & Sohn, 1997.

[4] J. Kollár and S. Mori, Birational Geometry of Algebraic Varieties, vol. 134, Cambridge University Press, 1998.

[5] J. Li, *Kodaira dimension of moduli space of vector bundles on surfaces*, Invent. Math. **115**-1 (1994), 1–40.

[6] K. Matsuki and R. Wentworth, *Mumford-Thaddeus principle on the moduli space of vector bundles on an algebraic surface*, Internat. J. Math. **8**-1 (1997), 97–148.

[7] Z. Qin, *Birational properties of moduli spaces of stable locally free rank-*2 *sheaves on algebraic surfaces*, Manuscripta Math. **72**-2 (1991), 163–180.

[8] M. Thaddeus, *Geometric invariant theory and flips*, J. Amer. Math. Soc. **9**-3 (1996), 691–723.

[9] K. Yamada, *A sequence of blowing-ups connecting moduli of sheaves and the Donaldson polynomial under change of polarization*, J. Math. Kyoto Univ. **43**-4 (2003), 829–878, math.AG/0704.2866.