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Prescribing Gevrey singularities for solutions of pseudodifferential operators

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

By

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Abstract

We show a criterion to establish the existence of solutions with prescribed Gevrey wave front sets for pseudodifferential operators of finite order.

1. Introduction

The problem of the wave front sets of solutions of differential and pseudodifferential operators has been widely investigated. In particular, several authors have been interested to the question whether one can find a distribution–solution u of a given differential or pseudodifferential operator P so that Pu is smooth or analytic or Gevrey of some order $s > 1$ and u has prescribed singularities or prescribed wave front sets, see [2, 3, 7, 10, 12, 14, 15, 19, 21, 24] and the references quoted therein. This question is closely related to the problems of hypoellipticity and solvability, local or global, smooth or analytic or Gevrey of some order $s > 1$. In fact, in order to prove that the differential or pseudodifferential operator under study is not hypoelliptic in the category of some function space, the corresponding method consists in showing the existence of singular solutions (see, e.g., [22, 13] and the references quoted therein).

The purpose of this paper is to give a criterion in terms of a priori estimates to establish the existence of distribution or smooth-solutions with prescribed Gevrey wave front sets for pseudodifferential operators of finite order (see Theorems 2.1, 2.3). The results are inspired by the work of Ivrii [15] and of Duistermaat and Hörmander [7] in the C^∞ category and their proofs are along the lines of [15]. We point out that the intricate topology of the Gevrey spaces $G^s(\Omega)$ does not permit to simply rephrase Ivrii's proof which relies on applying Baire's theorem to a suitable intersection of two Fréchet spaces. We recall that intersections of complete metrizable locally convex spaces are again complete

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metrizable locally convex spaces. While, in the framework of not metrizable locally convex spaces like the Gevrey spaces several topological properties of the whole space are not stable with respect to intersections (see [1, 4]).

In the sequel, we fix some notations and give some definitions and results.

The Gevrey classes $G^s(\Omega)$ and $g^s(\Omega)$ are defined as follows. Let K be a regular compact subset of $\mathbf{R}^n$ (i.e., K is the closure of the set of its interior points) and $\eta > 0$, $s \geq 1$; we denote by

$$G^s(K,\eta) = \left\{ \phi \in C^\infty(K);\, |\phi; K, s, \eta| := \sup_{\alpha \in \mathbf{N}_0^n} \sup_{x \in K} |\partial^\alpha \phi(x)| \frac{\eta^{|\alpha|}}{(\alpha!)^s} < \infty \right\}.$$

$G^s(K,\eta)$ is a Banach space with respect to the norm $|\ ; K, s, \eta|$, or equivalently with respect to the norm

$$\|\phi; K, s, \eta\| := \sum_{\alpha \in \mathbf{N}_0^n} \frac{\eta^{|\alpha|}}{(\alpha!)^s} |\partial^\alpha \phi|_{L^2(K)}.$$

Then we define

$$G^s(K) = \operatorname*{ind\,lim}_{\eta \to 0} G^s(K,\eta), \qquad g^s(K) = \operatorname*{proj\,lim}_{\eta \to +\infty} G^s(K,\eta),$$
$$G^s(\Omega) = \operatorname*{proj\,lim}_{K \subset \Omega} G^s(K), \qquad g^s(\Omega) = \operatorname*{proj\,lim}_{K \subset \Omega} g^s(K).$$

In particular $G^1(\Omega) = A(\Omega)$ is the space of all real analytic functions in Ω.

If $s > 1$ $G_0^s(K,\eta) = G^s(K,\eta) \cap C_0^\infty(K) \neq \emptyset$ is a Banach space with respect to the norm $|\ ; K, s, \eta|$, and then the Gevrey classes $G_0^s(\Omega)$ and $g_0^s(\Omega)$ can be also defined by setting

$$G_0^s(K) = \operatorname*{ind\,lim}_{\eta \to 0} G_0^s(K,\eta), \qquad g_0^s(K) = \operatorname*{proj\,lim}_{\eta \to \infty} G_0^s(K,\eta),$$
$$G_0^s(\Omega) = \operatorname*{ind\,lim}_{K \subset \Omega} G_0^s(K), \qquad g_0^s(\Omega) = \operatorname*{ind\,lim}_{K \subset \Omega} g_0^s(K).$$

We emphasize that there are in fact many different but equivalent ways to define these spaces; see [15], [18], [22]. We remark that the spaces $G^s(K)$ and $G_0^s(K)$ are inductive limits of Banach spaces with compact linking maps and then they are dual Fréchet–Schwartz spaces, i.e. (DFS)–spaces (a (DFS)–space is the strong dual of a reflexive Fréchet space). While, the spaces $g^s(K)$ and $g_0^s(K)$ are nuclear Fréchet spaces; hence $g^s(\Omega)$ is a nuclear Fréchet space. We also recall that $g_0^s(K) \hookrightarrow G_0^s(K)$ and $g^s(K) \hookrightarrow G^s(K)$ continuously, the first one with dense range; hence $g_0^s(\Omega) \hookrightarrow G_0^s(\Omega)$ and $g^s(\Omega) \hookrightarrow G^s(\Omega)$ continuously too. Moreover, if $1 < s < s'$, $G_0^s(\Omega) \hookrightarrow g_0^{s'}(\Omega)$ and $G^s(\Omega) \hookrightarrow g^{s'}(\Omega)$ continuously.

The elements of the topological dual of these spaces are called ultradistributions for $s > 1$ and real analytic functionals for $s = 1$ ([18], [23]). In particular, we denote by $\mathcal{D}'_{\{s\}}(\Omega) = (G_0^s(\Omega))'$, $\mathcal{D}'_{\{s\}}(K) = (G_0^s(K))'$, $\mathcal{E}'_{\{s\}}(\Omega) = (G^s(\Omega))'$ and $\mathcal{E}'_{\{s\}}(K) = (G^s(K))'$ endowed with the corresponding strong topology; on the other hand, denote by $\mathcal{D}'_{(s)}(\Omega) = (g_0^s(\Omega))'$, $\mathcal{D}'_{(s)}(K) = (g_0^s(K))'$,

$\mathcal{E}'_{(s)}(\Omega) = (g^s(\Omega))'$ and $\mathcal{E}'_{(s)}(K) = (g^s(K))'$ endowed with the corresponding strong topology.

Next, denote by $T^n = \mathbf{R}^n/2\pi\mathbf{Z}^n$ the n–dimensional torus. For each $s \geq 1$ let $G^s(T^n)$ be the space of all G^s–functions on T^n, which are identified with the G^s–functions on $\mathbf{R}^n$ that are 2π–periodic in each variable. Moreover, by setting $K_\pi = [-\pi, \pi]^n$ and

$$G^s(T^n, \eta) = \left\{ \phi \in G^s(T^n);\ |\phi; K_\pi, s, \eta| = \sum_{\alpha \in \mathbf{N}_0^n} \sup_{x \in K_\pi} |\partial^\alpha \phi(x)| \frac{\eta^{|\alpha|}}{(\alpha!)^s} < \infty \right\}$$

for each $\eta > 0$, we have

$$G^s(T^n) = \operatorname*{ind\,lim}_{\eta \to 0} G^s(T^n, \eta)$$

when $G^s(T^n)$ is endowed with the topology induced on it by $G^s(\mathbf{R}^n)$. Therefore $G^s(T^n)$ is also a (DFS)–space. We denote by $\mathcal{E}'_s(T^n) = (G^s(T^n))'$ equipped with the corresponding strong topology. We point out that the spaces $G^s(K)$ and $G^s(T^n)$ are endowed with the same topological structure.

Finally, we recall a Gevrey version of the Schwartz space $\mathcal{S}$. We denote by $\mathcal{S}^{(s)}$ the space of all $f \in \mathcal{S}$ such that for any $\eta > 0$ and any integer $N \geq 1$

$$\|f\|_{\eta,N} := \sup_{\alpha \in \mathbf{N}_0^n} \sup_{x \in \mathbf{R}^n} \eta^{|\alpha|} (\alpha!)^{-s} (1 + |x|^N) |\partial^\alpha f(x)| < \infty.$$

We have obviuosly $g_0^s(\mathbf{R}^n) \subset \mathcal{S}^{(s)} \subset g^s(\mathbf{R}^n)$. While, we denote by $\mathcal{S}_{(s)}$ the space of all $f \in \mathcal{S}$ such that for any $\epsilon > 0$ and any integer $m \geq 1$

$$\|f\|^*_{\epsilon,m} := \sup_{|\alpha| \leq m} \sup_{x \in \mathbf{R}^n} \exp(\epsilon |x|^{1/s}) |\partial^\alpha f(x)| < \infty.$$

The Fourier transform $f \to \hat{f}$ is a topological isomorphism of $\mathcal{S}^{(s)}$ onto $\mathcal{S}_{(s)}$ and, in view of the inversion formula, of $\mathcal{S}_{(s)}$ onto $\mathcal{S}^{(s)}$. For more details on such spaces we refer to [9, 20].

For $u \in \mathcal{D}'_{\{s\}}(\Omega)$ ($u \in \mathcal{D}'_{(s)}(\Omega)$) we denote by $\{s\} - \operatorname{singsupp} u$ ($(s) - \operatorname{singsupp} u$) the smallest closed subset of Ω in the complement of which u is a G^s function (u is a g^s function). The set $\{s\} - \operatorname{singsupp} u$ (the set $(s) - \operatorname{singsupp} u$, resp.) is called the G^s*–singular support* of u (the g^s*–singular support* of u, resp.).

Moreover, for $u \in \mathcal{D}'_{\{s\}}(\Omega)$ we denote by $WF_{\{s\}}(u)$ the $\{s\}$*–wave front set* of u. For $u \in \mathcal{D}'_{(s)}(\Omega)$ we denote by $WF_{(s)}(u)$ the (s)*–wave front set* of u.

We recall that $WF_{\{s\}}(u)$ is the complement in $\Omega \times \mathbf{R}^n \setminus \{0\}$ ($=T^*(\Omega) \setminus 0$) of the set of all $\rho = (x_0, \xi_0)$ such that there exist a neighborhood U of x_0, a conic neighborhood Γ of ξ_0 in $\mathbf{R}^n \setminus \{0\}$ and a function $\phi \in G_0^s(\Omega)$ ($\phi \in C_0^\infty(\Omega)$ if $s = 1$) equal to one in U such that for some $C > 0, \varepsilon > 0$

$$|\widehat{(\phi u)}(\xi)| \leq C \exp(-\varepsilon |\xi|^{1/s}), \qquad \xi \in \Gamma. \tag{1.1}$$

For $1 < r < s$, $WF_{\{s\}}(u) \subset WF_{\{r\}}(u)$ for all $u \in \mathcal{D}'_{\{s\}}(\Omega)$.

While, $WF_{(s)}(u)$ is the complement in $\Omega \times \mathbf{R}^n \setminus \{0\}$ $(=T^*(\Omega) \setminus 0)$ of the set of all $\rho = (x_0, \xi_0)$ such that there exist a neighborhood U of x_0, a conic neighborhood Γ of ξ_0 in $\mathbf{R}^n \setminus \{0\}$ and a function $\phi \in g_0^s(\Omega)$ equal to one in U such that for each $n \in \mathbf{N}$ there is $C_n > 0$ for which

$$|(\hat{\phi u})(\xi)| \leq C_n \exp(-n|\xi|^{1/s}), \qquad \xi \in \Gamma. \tag{1.2}$$

Clearly, $WF_{\{s\}}(u) \subset WF_{(s)}(u)$ for all $u \in \mathcal{D}'_{\{s\}}(\Omega)$, and for $1 < r < s$, $WF_{(s)}(u) \subset WF_{(r)}(u)$ for all $u \in \mathcal{D}'_{(s)}(\Omega)$.

Let $s \geq 1$ and $m \in \mathbf{R}$. We denote by $S^{m,s}(\Omega)$ the space of all functions $p(x, \xi) \in C^\infty(\Omega \times \mathbf{R}^n)$ satisfying the following condition: for every compact subset $K \subset \Omega$ there exist two constants B, $C > 0$ such that

$$|D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C^{|\alpha|+|\beta|+1} (\alpha!)^s (\beta!) (1 + |\xi|)^{m-|\beta|} \tag{1.3}$$

for every α, $\beta \in \mathbf{Z}_+^n$ and $x \in K$, $\xi \in \mathbf{R}^n$ with $1+|\xi| \geq B|\beta|^s$ (see [22, Definition 3.3.1]).

The conic support of p, denoted by $\operatorname{conesupp} p$, is defined by the closure in $\Omega \times \mathbf{R}^n$ of the set $\{(x, t\xi) \colon (x, \xi) \in \operatorname{supp} p,\ t \geq 0\}$.

If $p \in S^{m,s}(\Omega)$, the *G^s–pseudodifferential operator of order m with symbol* p is defined for $\phi \in G_0^s(\Omega)$ by

$$p(x, D)\phi(x) = (2\pi)^{-n} \int e^{ix.\xi} p(x, \xi) \hat{\phi}(\xi) d\xi. \tag{1.4}$$

Then $P = p(x, D)$ defined by (1.4) is a continuous linear operator from $G_0^s(\Omega)$ to $G^s(\Omega)$, which can be extended as a continuous linear operator from $\mathcal{E}'_{\{s\}}(\Omega)$ to $\mathcal{D}'_{\{s\}}(\Omega)$, and P is $\{s\}$–pseudo–local, i.e., $\{s\}$–singsupp $Pu \subset \{s\}$–singsupp u for all $u \in \mathcal{E}'_{\{s\}}(\Omega)$. We denote by $OPS^{m,s}(\Omega)$ the space of all operators of this form. The class $OPS^{m,s}(\Omega)$ is contained in the class $L^m(\Omega)$ of Hörmander [13]. Actually, for every $r \geq s$, $P = p(x, D)$ is also a continuous linear operator from $g_0^r(\Omega)$ to $g^r(\Omega)$, which can be extended as a continuous linear operator from $\mathcal{E}'_{(r)}(\Omega)$ to $\mathcal{D}'_{(r)}(\Omega)$, and P is (r)–pseudo–local, i.e., (r)–singsupp $Pu \subset (r)$–singsupp u for all $u \in \mathcal{E}'_{(r)}(\Omega)$.

The operator $P = p(x, D)$ is called *properly supported* if $P \colon G_0^s(\Omega) \to G_0^s(\Omega)$, $P \colon G^s(\Omega) \to G^s(\Omega)$, $P \colon \mathcal{E}'_s(\Omega) \to \mathcal{E}'_s(\Omega)$ and $P \colon \mathcal{D}'_s(\Omega) \to \mathcal{D}'_s(\Omega)$ continuously.

If $P = p(x, D)$ is a properly supported G^s–pseudodifferential operator of order m with symbol p, then for each compact subset K of Ω there exists a compact subset K' of Ω such that $K \subset K'$ and $P \colon G_0^s(K) \to G_0^s(K')$ continuously.

The other notations are standard. We refer the reader for functional analysis to [17], and for the theory of linear partial differential operators to [13, 22].

2. Statement of the main results

In the following we will give criterions in terms of a priori estimates to establish the existence of distribution or smooth–solutions with prescribed Gevrey wave front sets for pseudodifferential operators of finite order. For this we need further definitions.

Let Ω be an open subset of $\mathbf{R}^n$ and $s > 1$. For each $t \in \mathbf{R}$ we denote by $H^t(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$|u|_t = \left(\int_{\mathbf{R}^n} (1 + |\xi|^2)^t |\hat{u}(\xi)|^2 d\xi \right)^{1/2}.$$

For $u \in \mathcal{D}'_s(\Omega)$ and $t \in \mathbf{R}$, we denote by $WF_t(u)$ the H^t*–Sobolev wavefront set of* u. $WF_t(u)$ can be defined as the set of all $\rho = (x, \xi) \in T^*(\Omega) \setminus 0$ for which the assumptions $a \in L^0(\Omega)$ and $au \in H^t(\Omega)$ imply that $a_0(\rho) = 0$, where a_0 is the principal symbol of a (see [13]). Clearly, $WF_t(u) \subseteq WF_{\{s\}}(u)$.

Let $P \in OPS^{m,s}(\Omega)$ be a classical properly supported pseudodifferential operator of order $m \in \mathbf{R}$ and of class $s > 1$, briefly $P \in OPS^{m,s}_{cl}(\Omega)$. Then its symbol $p \in S^{m,s}(\Omega)$ satisfies

$$p(x, \xi) \simeq \sum_{j=0}^{\infty} p_{m-j}(x, \xi),$$

where $p_{m-j}(x, t\xi) = t^{m-j} p_{m-j}(x, \xi)$ for $t > 0$ and $(x, \xi) \in T^*(\Omega) \setminus 0$, p_m is its principal symbol and the characteristic manifold of $P = p(x, D)$ is the subset $Char\, P$ of $T^*(\Omega) \setminus 0$ defined by

$$Char\, P = \{(x, \xi) \in T^*(\Omega) \setminus 0 \colon p_m(x, \xi) = 0\}.$$

In particular, $WF_{\{s\}}(u) \subset WF_{\{s\}}(Pu) \cup Char\, P$ (see [22, Corollary 3.4.14]).

Theorem 2.1. *Let $t, t', s \in \mathbf{R}_+$, $t' < t$, $s > 1$ and $P \in OPS^{m,s}_{cl}(\Omega)$ be properly supported. Let V be a conical open subset of $T^*(\Omega) \setminus 0$, $\emptyset \neq N$ be a conical closed subset of $T^*(\Omega) \setminus 0$ such that $N \subset V$.*

Assume that it does not exists a function $u \in H^{t'}(\Omega)$ such that

$$WF_{\{s\}}(Pu) \cap V = \emptyset, \; WF_t(u) \cap V = WF_{\{s\}}(u) \cap V = N.$$

Then for each compact subset K of Ω and $\eta > 0$ there exist $\rho_0 \in N$ and $\psi, \varphi, \varphi' \in S^{0,s}(\Omega)$, with conesupp $\varphi \subset V \setminus N$, conesupp $\varphi' \subset V$, $\psi \equiv 1$ in a small conical neighborhood of ρ_0, and $C > 0$, $\eta', \eta'' > \eta$ and a compact set $K' \supset K$, such that

$$(2.1) \qquad |\psi(x, D)u|_t \le C \left(|u|_{t'} + |\varphi(x, D)u; K, s, \eta'| + |\varphi'(x, D)Pu; K', s, \eta''| \right)$$

for all $u \in G^s_0(K, \eta)$.

Recalling that for $s \geq 1$ the spaces $G^s(K)$ and $G^s(T^n)$ are endowed with the same topological structure and every $P \in OPS^{m,s}_{cl}(T^n)$ continuously acts from $G^s(T^n)$ into itself, Theorem 2.1 easily extends to the case $s \geq 1$ on the n–dimensional torus T^n. Indeed, we have

Theorem 2.2. *Let $t, t', s \in \mathbf{R}_+$, $t' < t$, $s \geq 1$ and $P \in OPS^{m,s}_{cl}(T^n)$. Let V be a conical open subset of $T^*(T^n) \setminus 0$, $\emptyset \neq N$ be a conical closed subset of $T^*(T^n) \setminus 0$ such that $N \subset V$.*

Assume that it does not exists a function $u \in H^{t'}(T^n)$ such that

$$WF_{\{s\}}(Pu) \cap V = \emptyset,\ WF_t(u) \cap V = WF_{\{s\}}(u) \cap V = N.$$

Then for each $\eta > 0$ there exist $\rho_0 \in N$ and $\psi, \varphi, \varphi' \in S^{0,s}(T^n)$, with conesupp $\varphi \subset V \setminus N$, conesupp $\varphi' \subset V$, $\psi \equiv 1$ in a small conical neighborhood of ρ_0, and $C > 0$, $\eta', \eta'' > \eta$ such that

(2.2)
$$|\psi(x,D)u|_{t,K_\pi} \leq C\left(|u|_{t',K_\pi} + |\varphi(x,D)u; K_\pi, s, \eta'| + |\varphi'(x,D)Pu; K_\pi, s, \eta''|\right)$$

for all $u \in G^s(T^n, \eta)$.

We also have the analogous result of Theorem 2.1 for the sets $WF_{(t)}$ instead of the sets WF_t.

In order to state and show such a result, we recall that a fundamental sequence of seminorms for the space $g^t(\Omega)$ can be defined as follows.

Let $K_1 \subset \overset{\circ}{K}_2 \subset K_2 \subset \ldots \subset \Omega$ be an exhaustion of Ω by compact sets ($\overset{\circ}{K}_2$ denotes the interior of K_2) and choose $\chi_j \in g^t_0(K_j)$ with $0 \leq \chi_j \leq 1$ and $\chi_j \equiv 1$ on K_{j-1}. Then the increasing sequence of seminorms defined by

$$|u|_{t,j} := \left(\int_{\mathbf{R}^n} |\widehat{(\chi_j u)}(\xi)|^2 e^{j|\xi|^{1/t}} d\xi\right)^{1/2}, \tag{2.3}$$

is a fundamental system of continuous seminorms on $g^t(\Omega)$ (see, e.g., [5, Lemmas 3.3 and 3.5, Proposition 4.5]).

Observe that for every $j \in \mathbf{N}$ the completion $H^j_{(t)}(\Omega)$ of $g^t_0(\Omega)$ with respect to the norm

$$|u|_{t,j} := \left(\int_{\mathbf{R}^n} |\hat{u}(\xi)|^2 e^{j|\xi|^{1/t}} d\xi\right)^{1/2},$$

is a Hilbert space which is continuously included in $L^2(\mathbf{R}^n)$ and hence in $\mathcal{D}'_{\{s\}}(\Omega)$ (also in $\mathcal{D}'_{(s)}(\Omega)$).

In the sequel, for any $t > 1$ we denote simply by $(|\ |_{t,i})_{i \in \mathbf{N}}$ a fundamental increasing sequence of continuous seminorms on the Fréchet space $g^t(\Omega)$ defined as in (2.3).

Theorem 2.3. *Let $1 < s < t < t'$ and $P \in OPS^{m,s}_{cl}(\Omega)$ be properly supported. Let V be a conical open subset of $T^*(\Omega) \setminus 0$, $\emptyset \neq N$ be a conical closed subset of $T^*(\Omega) \setminus 0$ such that $N \subset V$.*

Assume that it does not exists a function $u \in g^{t'}(\Omega)$ *such that* $WF_{\{s\}}(Pu) \cap V = \emptyset$, $WF_{(t)}(u) \cap V = WF_{\{s\}}(u) \cap V = N$.

Then for each compact subset K *of* Ω, $\eta > 0$ *and* $i \in \mathbf{N}$ *there exist* $\rho_0 \in N$ *and* $\psi, \varphi, \varphi' \in S^{0,s}(\Omega)$, *with conesupp* $\varphi \subset V \setminus N$, *conesupp* $\varphi' \subset V$, $\psi \equiv 1$ *in a small conical neighborhood of* ρ_0, *and* $C > 0$, $j \in \mathbf{N}$, $\eta', \eta'' > \eta$ *and a compact subset* $K' \supset K$, *such that*

$$(2.4)\quad |\psi(x,D)u|_{t,i} \leq C\left(|u|_{t',j} + |\varphi(x,D)u; K, s, \eta'| + |\varphi'(x,D)Pu; K', s, \eta''|\right)$$

for all $u \in G_0^s(K, \eta)$.

We point out that the interesting case in Theorems 2.1, 2.2 and 2.3 appears when $N \subset Char\, P$. In fact, if there exists a point $\rho_0 \in N \setminus Char\, P$, then the operator P is microlocally hypoelliptic (elliptic) near ρ_0 and if Pu is regular at ρ_0 the same is true with respect to u.

We also observe that Theorems 2.1 and 2.3 are local, while Theorem 2.2 is of global type. In particular, Theorems 2.1 and 2.3 will be illustrated at the end of this paper by a simple example. More complicated examples, including illustration of the non–local Theorem 2.2, will be given in a forthcoming paper.

3. Proof of the main results

Proof of Theorem 2.1. Let $K \subset \Omega$ be a compact set. Since P is properly supported, there exists a compact subset $K' \subset \Omega$ such that $K \subset K'$ and P continuously maps $G_0^s(K)$ into $G_0^s(K')$.

Let $(\eta_\mu)_{\mu \in \mathbf{N}}$ be a decreasing sequence of positive numbers such that $\eta_\mu \to 0$. Then

$$G_0^s(K) = \operatorname{ind}\lim_{\mu} G_0^s(K, \eta_\mu) \quad \text{and} \quad G^s(K) = \operatorname{ind}\lim_{\mu} G^s(K, \eta_\mu),$$
$$G_0^s(K') = \operatorname{ind}\lim_{\mu} G_0^s(K', \eta_\mu) \quad \text{and} \quad G^s(K') = \operatorname{ind}\lim_{\mu} G^s(K', \eta_\mu).$$

Let M be a conical closed set in $T^*(\Omega) \setminus 0$ such that $N \subset M \subset V$. Consider an increasing sequence $(\Gamma_\nu)_{\nu \in \mathbf{N}}$ of conical closed sets in $T^*(\Omega) \setminus 0$ such that $V \setminus M = \cup_{\nu \in \mathbf{N}} \Gamma_\nu$, and choose a sequence $(\varphi_\nu)_{\nu \in \mathbf{N}}$ in $S^{0,s}(\Omega)$ such that $\Gamma_\nu \subset \{\rho \in T^*(\Omega) \setminus 0 \colon \varphi_\nu(\rho) = 1\} \subset \operatorname{conesupp} \varphi_\nu \subset \Gamma_{\nu+1}$.

Since each $\varphi_\nu(x,D)$ continuously maps $G_0^s(K)$ into $G^s(K)$, there exists $(\eta_\mu^\nu)_{\mu \in \mathbf{N}} \subset (\eta_\mu^{\nu-1})_{\mu \in \mathbf{N}} \subset (\eta_\mu)_{\mu \in \mathbf{N}}$ such that for every $\mu \in \mathbf{N}$

$$(3.1)\qquad \varphi_\nu(x,D) \colon G_0^s(K, \eta_\mu) \to G^s(K, \eta_\mu^\nu) \quad \text{continuously.}$$

For each $\mu \in \mathbf{N}$ we introduce the space

$$F_\mu := \{u \in \mathcal{E}'_{\{s\}}(\Omega) \colon \operatorname{supp} u \subset K,\ u \in H^{t'}(\Omega),\ WF_{\{s\}}(u) \cap (V \setminus M) = \emptyset\,, \text{and}$$
$$\varphi_\nu(x,D)u \in G^s(K, \eta_\mu^\nu) \ \text{ for all } \nu \in \mathbf{N}\}$$

and observe that $G_0^s(K, \eta_\mu) \subset F_\mu$ (we point out that $WF_{\{s\}}(u) \cap \Gamma_{\nu+1} = \emptyset$ and $\operatorname{conesupp} \varphi_\nu \subset \Gamma_{\nu+1}$ imply that $\varphi_\nu(x,D)u \in G^s(\Omega)$ and hence $\varphi_\nu(x,D)u \in$

$G^s(K)$). Indeed, if $u \in G_0^s(K, \eta_\mu)$ then $\operatorname{supp} u \subset K$, $u \in H^{t'}(\Omega)$, $WF_{\{s\}}(u) \cap (V \setminus M) = \emptyset$, and by (3.1) $\varphi_\mu(x, D)u \in G^s(K, \eta_\mu^\nu)$ for all $\nu \in \mathbf{N}$.

Moreover, we endow the space F_μ with the locally convex topology generated by the following sequence of norms

$$p_{t',\mu,\nu}(u) = |u|_{t'} + |\varphi_\nu(x, D)u; K, s, \eta_\mu^\nu|, \quad \nu \in \mathbf{N}.$$

Then $(F_\mu, (p_{t',\mu,\nu})_{\nu\in N})$ is a Fréchet space. Indeed, let $(u_j)_{j\in\mathbf{N}}$ be a Cauchy sequence in F_μ. Then $(u_j)_{j\in\mathbf{N}}$ is a Cauchy sequence in $H^{t'}(\Omega)$ with $\operatorname{supp} u_j \subset K$ for all $j \in \mathbf{N}$, and $(\varphi_\nu(x, D)u_j)_{j\in\mathbf{N}}$ is a Cauchy sequence in $G^s(K, \eta_\mu^\nu)$ for all $\nu \in \mathbf{N}$: therefore $u_j \to u \in H^{t'}(\Omega)$ in $H^{t'}(\Omega)$ with $\operatorname{supp} u \subset K$, and $\varphi_\nu(x, D)u_j \to u_\nu \in G^s(K, \eta_\mu^\nu)$ in $G^s(K, \eta_\mu^\nu)$ for all $\nu \in \mathbf{N}$. Since the space $H_c^{t'}(\Omega)$ (c means with compact support) is continuously embedded in $\mathcal{E}'_{\{s\}}(\Omega)$, we obtain that $u_j \to u$ in $\mathcal{E}'_{\{s\}}(\Omega)$ too. This implies that $\varphi_\nu(x, D)u_j \to \varphi_\nu(x, D)u$ in $\mathcal{D}'_{\{s\}}(\Omega)$ because $\varphi_\nu(x, D)\colon \mathcal{E}'_{\{s\}}(\Omega) \to \mathcal{D}'_{\{s\}}(\Omega)$ continuously. On the other hand, since $G^s(K, \eta_\mu^\nu)$ is continuously embedded in $\mathcal{D}'_{\{s\}}(\Omega)$, $\varphi_\nu(x, D)u_j \to u_\nu$ in $\mathcal{D}'_{\{s\}}(\Omega)$. Thus, $\varphi_\nu(x, D)u = u_\nu \in G^s(K, \eta_\mu^\nu) \subset G^s(K)$ for all $\nu \in \mathbf{N}$. Consequently, for every $\nu \in \mathbf{N}$

$$WF_{\{s\}}(\varphi_\nu(x, D)u_{|\mathring{K}}) = WF_{\{s\}}(\varphi_\nu(x, D)u)_{|\mathring{K}} = \emptyset,$$

where $\mathring{K}$ denotes the interior of K. This implies that

$$\begin{aligned} WF_{\{s\}}(u_{|\mathring{K}}) &= WF_{\{s\}}(u)_{|\mathring{K}} \\ &\subset WF_{\{s\}}(\varphi_\nu(x, D)u)_{|\mathring{K}} \cup (Char\,\varphi_\nu)_{|\mathring{K}} \\ &= (Char\,\varphi_\nu)_{|\mathring{K}}. \end{aligned} \tag{3.2}$$

Since $\operatorname{supp} u \subset K$, we have $WF_{\{s\}}(u_{|\mathring{K}}) = WF_{\{s\}}(u)$ and so (3.2) yields that

$$WF_{\{s\}}(u) \cap \Gamma_\nu \subset (Char\,\varphi_\nu)_{|\mathring{K}} \cap \Gamma_\nu = \emptyset$$

for all $\nu \in \mathbf{N}$; hence $WF_{\{s\}}(u) \cap (V \setminus M) = \cup_\nu (WF_{\{s\}}(u) \cap \Gamma_\nu) = \emptyset$. Thus, $u \in F_\mu$ and $u_j \to u$ in F_μ.

Next, we consider an increasing sequence $(\Gamma'_\nu)_{\nu\in\mathbf{N}}$ of conical closed sets in $T^*(\Omega) \setminus 0$ such that $V = \cup_{\nu\in\mathbf{N}} \Gamma'_\nu$, and choose a sequence $(\varphi'_\nu)_{\nu\in\mathbf{N}}$ in $S^{0,s}(\Omega)$ such that $\Gamma'_\nu \subset \{\rho \in T^*(\Omega) \setminus 0 \colon \varphi'_\nu(\rho) = 1\} \subset \operatorname{conesupp} \varphi'_\nu \subset \Gamma'_{\nu+1}$.

Since each $\varphi'_\nu(x, D)$ continuously maps $G_0^s(K')$ into $G^s(K')$, there exists $(\tau_\mu^\nu)_{\mu\in\mathbf{N}} \subset (\tau_\mu^{\nu-1})_{\mu\in\mathbf{N}} \subset (\tau_\mu)_{\mu\in\mathbf{N}}$ such that for each $\mu \in \mathbf{N}$

$$\varphi'_\nu(x, D)\colon G_0^s(K', \tau_\mu) \to G^s(K', \tau_\mu^\nu) \quad \text{continuously} \tag{3.3}$$

(here we assume that $P\colon G_0^s(K, \eta_\mu) \to G^s(K', \tau_\mu)$ continuously, where $(\tau_\mu)_{\mu\in\mathbf{N}}$ is a suitable subsequence of $(\eta_\mu)_{\mu\in\mathbf{N}}$).

We denote by

$$G_\mu := \{v \in \mathcal{E}'_{\{s\}}(\Omega)\colon \operatorname{supp} v \subset K',\ v \in H^{t'-m}(\Omega),\ WF_{\{s\}}(v) \cap V = \emptyset\,,\ \text{and}\\ \varphi'_\nu(x,D)v \in G^s(K',\tau_\mu^\nu)\ \text{for all}\ \nu \in \mathbf{N}\}$$

and observe that $G_0^s(K',\tau_\mu) \subset G_\mu$ (this follows as in the case of the space F_μ).

We endow the space G_μ with the locally convex topology generated by the following sequence of norms

$$q_{t'-m,\mu,\nu}(v) = |v|_{t'-m} + |\varphi'_\nu(x,D)v; K', s, \tau_\mu^\nu|, \quad \nu \in \mathbf{N}.$$

Then $(G_\mu, (q_{t'-m,\mu,\nu})_{\nu\in N})$ is a Fréchet space (this follows by arguing as in the case of the space F_μ).

Thus, the space

$$X_\mu := \{u \in F_\mu\colon Pu \in G_\mu\}$$

is a Fréchet space with respect to the intersection topology defined by the following sequence of norms

$$r_{\mu,\nu}(u) = p_{t',\mu,\nu}(u) + q_{t'-m,\mu,\nu}(Pu), \quad \nu \in \mathbf{N}.$$

Indeed, if $(u_j)_{j\in\mathbf{N}}$ is a Cauchy sequence in X_μ, then $u_j \to u \in F_\mu$ in F_μ and $Pu_j \to v \in G_\mu$ in G_μ. It follows that $u_j \to u$ and $Pu_j \to v$ in $\mathcal{E}'_{\{s\}}(\Omega)$; hence $Pu_j \to v$, and $Pu_j \to Pu$ in $\mathcal{D}'_{\{s\}}(\Omega)$ as $P\colon \mathcal{E}'_{\{s\}}(\Omega) \to \mathcal{D}'_{\{s\}}(\Omega)$ continuously. Consequently, $Pu = v$ so that $u \in X_\mu$ and $u_j \to u$ in X_μ.

Suppose that N is a compact subset of $\Omega \times S^*(\Omega)$, where $S^*(\Omega) = \{\xi \in \mathbf{R}^n\colon |\xi| = 1\}$. Then for each $j \in \mathbf{N}$ there exists a finite set $\{\rho_k^j = (x_k^j, \xi_k^j) \in N\colon |\xi_k^j| = 1\}_{k=1}^{K_j}$ such that the sets

$$\Gamma_k^j := \{\rho = (x,\xi) \in T^*(\Omega) \setminus 0\colon |x - x_k^j| + |\xi|\xi|^{-1} - \xi_k^j| < 1/j\}\,,\ k = 1, \dots, K_j,$$

form a covering of N.

Now, for each $j \in \mathbf{N}$ and $k = 1, \dots, K_j$ let $\psi_k^j \in S^{0,s}(\Omega)$ with $\Gamma_k^{2j} \subset \{\rho \in T^*(\Omega) \setminus 0\colon \psi_k^j(\rho) = 1\} \subset \operatorname{conesupp} \psi_k^j \subset \Gamma_k^j$. Then we introduce the following subsets of X_μ

$$G_{jk\widetilde{M}} := \{u \in X_\mu\colon \psi_k^j(x,D)u \in H^t(\Omega),\ |\psi_k^j(x,D)u|_t \le \widetilde{M}\},$$

for $j, \widetilde{M} \in \mathbf{N}$ and $k = 1, \dots, K_j$. Each set $G_{jk\widetilde{M}}$ is absolutely convex (clearly) and closed. Indeed, if $(u_r)_{r\in\mathbf{N}} \subset G_{jk\widetilde{M}}$ converges to some u in X_μ, then $u_r \to u \in F_\mu$ in F_μ (hence $u_r \to u$ in $\mathcal{E}'_{\{s\}}(\Omega)$) and $Pu_r \to Pu \in G_\mu$ in G_μ (hence $Pu_r \to Pu$ in $\mathcal{E}'_{\{s\}}(\Omega)$ and in $\mathcal{D}'_{\{s\}}(\Omega)$).

Moreover, $\psi_k^j(x,D)u_r \in H^t(\Omega)$ and $|\psi_k^j(x,D)u_r|_t \le \widetilde{M}$ for all $r \in \mathbf{N}$, thereby implying that $\psi_k^j(x,D)u_r \to u_k^j \in H^t(\Omega)$ weakly (eventually by passing to a subsequence) as the bounded sets in the Hilbert space $H^t(\Omega)$ are weakly compact, where $|u_k^j|_t \le \underline{\lim}_r |\psi_k^j(x,D)u_r|_t \le \widetilde{M}$; hence $\psi_k^j(x,D)u_r \to u_k^j$ in $\mathcal{D}'_{\{s\}}(\Omega)$. On the other hand, the fact that $u_r \to u$ in $\mathcal{E}'_{\{s\}}(\Omega)$ implies that $\psi_k^j(x,D)u_r \to \psi_k^j(x,D)u$ as $\psi_k^j(x,D)\colon \mathcal{E}'_{\{s\}}(\Omega) \to \mathcal{D}'_{\{s\}}(\Omega)$ continuously. Thus, $\psi_k^j(x,D)u = u_k^j \in H^t(\Omega)$ and $|\psi_k^j(x,D)u|_t \le \widetilde{M}$. This means that $G_{jk\widetilde{M}}$ is a closed subset of X_μ.

By assumption we have

$$X_\mu = \cup_{j,\widetilde{M}\in\mathbf{N}} \cup_{k=1}^{K_j} G_{jk\widetilde{M}}\,.$$

Indeed, if $u \in X_\mu$, then $u \in H^{t'}(\Omega)$, $WF_{\{s\}}(u) \cap (V \setminus M) = \emptyset$ and $WF_{\{s\}}(Pu) \cap V = \emptyset$. Thus, $WF_{\{s\}}(u) \cap V \subset M$. On the other hand, by assumption $N \not\subset WF_t(u) \cap V \subset WF_{\{s\}}(u) \cap V \subset M$. Thus, there exists $\rho_0 \in N$ such that $\rho_0 \notin WF_t(u) \cap V$, thereby implying that $\rho_0 \notin WF_t(u)$ as $\rho_0 \in N \subset V$. Since $\rho_0 \in N$ and $\rho_0 \notin WF_t(u)$, we have $\rho_0 \in \Gamma_k^{2j}$ for some $j \in \mathbf{N}$ and $k \in \{1,\dots,K_j\}$, and hence $\psi_k^j(x,D)u \in H^t(\Omega)$; so $|\psi_k^j(x,D)u|_t \le \widetilde{M}$ for some $\widetilde{M} \in \mathbf{N}$.

As X_μ is a Fréchet space, there exist $j \in \mathbf{N}$, $k \in \{1,\dots,K_j\}$ and $\widetilde{M} \in \mathbf{N}$ such that $G_{jk\widetilde{M}}$ is a neighborhood of 0 (recall that it is absolutely convex), i.e., there exist $\varepsilon > 0$ and $\nu \in \mathbf{N}$ such that

$$U = \{u \in X_\mu \colon r_{\mu,\nu}(u) < \varepsilon\} \subset G_{jk\widetilde{M}}.$$

This implies that there exists $C_\mu > 0$ so that

$$|\psi_k^j(x,D)u|_t \le C\left(|u|_{t'} + |\varphi_\nu(x,D)u; K, s, \eta_\mu^\nu| + |\varphi'_\nu(x,D)Pu; K', s, \tau_\mu^\nu|\right)$$

for all $u \in X_\mu$, and hence for all $u \in G_0^s(K,\eta_\mu)$ (as $P\colon H^{t'}(\Omega) \to H^{t'-m}(\Omega)$ continuously).

Since μ is arbitrary and $\eta_\mu \to 0$, the result follows for every $\eta > 0$.

In the case N is not a compact set of $\Omega \times S^*(\Omega)$, we can write

$$N = \cup_{h=1}^{\infty} N_h\,,$$

where each N_h is a compact set of $\Omega \times S^*(\Omega)$. Then we can construct a finite covering $(\Gamma_{hk}^j)_{j,k}$ of N_h for every $h \in \mathbf{N}$, thereby obtaining a countable covering $(\Gamma_{hk}^j)_{j,h,k}$ of N. Thus, to complete the proof we can proceed as in the case in which N is a compact set with minor changes. $\square$

Proof of Theorem 2.3. We would like argue as in the proof of Theorem 2.1, but we have to do some changes in the proof due to the topology of the spaces $g^t(\Omega)$ that is generated by a sequence of seminorms.

We follow the same notations of the proof of Theorem 2.1. In particular, we assume that (3.1) and (3.3) hold for the sequences $(\eta_\mu^\nu)_{\mu\in\mathbf{N}} \subset (\eta_\mu^{\nu-1})_{\mu\in\mathbf{N}} \subset (\eta_\mu)_{\mu\in\mathbf{N}}$ and $(\tau_\mu^\nu)_{\mu\in\mathbf{N}} \subset (\tau_\mu^{\nu-1})_{\mu\in\mathbf{N}} \subset (\tau_\mu)_{\mu\in\mathbf{N}}$, where $(\tau_\mu)_{\mu\in\mathbf{N}}$ is a suitable subsequence of $(\eta_\mu)_{\mu\in\mathbf{N}}$, and that $P\colon G_0^s(K,\eta_\mu) \to G^s(K',\tau_\mu)$ continuously.

For each μ we define the space

$$F_\mu = \{u \in \mathcal{E}'_{\{s\}}(\Omega)\colon \operatorname{supp} u \subset K,\ u \in g^{t'}(\Omega),\ WF_{\{s\}}(u) \cap (V \setminus M) = \emptyset\,,\ \text{and}\\ \varphi_\nu(x,D)u \in G^s(K,\eta_\mu^\nu)\ \text{ for all } \nu \in \mathbf{N}\}$$

and observe that $G_0^s(K,\eta_\mu) \subset F_\mu$. Indeed, if $u \in G_0^s(K,\eta_\mu)$ then $\operatorname{supp} u \subset K$, $u \in g^{t'}(\Omega)$ as $s < t'$, $WF_{\{s\}}(u) \cap (V \setminus M) = \emptyset$, and by (3.1) $\varphi_\mu(x,D)u \in G^s(K,\eta_\mu^\nu)$ for all $\nu \in \mathbf{N}$.

Moreover, we endow the space F_μ with the locally convex topology generated by the following sequence of seminorms

$$p_{t',i,\mu,\nu}(u) = |u|_{t',i} + |\varphi_\nu(x,D)u; K, s, \eta_\mu^\nu|\,, \quad i,\nu \in \mathbf{N}.$$

Then $(F_\mu, (p_{t',i,\mu,\nu})_{i,\nu\in N})$ is a Fréchet space. Indeed, let $(u_j)_{j\in\mathbf{N}}$ be a Cauchy sequence in F_μ. Then $(u_j)_{j\in\mathbf{N}}$ is a Cauchy sequence in $g^{t'}(\Omega)$ with $\operatorname{supp} u_j \subset K$ for all $j \in \mathbf{N}$, and $(\varphi_\nu(x,D)u_j)_{j\in\mathbf{N}}$ is a Cauchy sequence in $G^s(K,\eta_\mu^\nu)$ for all $\nu \in \mathbf{N}$. Therefore $u_j \to u \in g^{t'}(\Omega)$ in $g^{t'}(\Omega)$, where $\operatorname{supp} u \subset K$ again (hence $u_j \to u$ in $g_0^{t'}(K)$ and in $g_0^{t'}(\Omega)$), and $\varphi_\nu(x,D)u_j \to u_\nu \in G^s(K,\eta_\mu^\nu)$ in $G^s(K,\eta_\mu^\nu)$ for all $\nu \in \mathbf{N}$. Since the space $g_0^{t'}(\Omega)$ is continuously embedded in $\mathcal{E}'_{\{s\}}(\Omega)$, we have that $u_j \to u$ in $\mathcal{E}'_{\{s\}}(\Omega)$ too. This implies that $\varphi_\nu(x,D)u_j \to \varphi_\nu(x,D)u$ in $\mathcal{D}'_{\{s\}}(\Omega)$ because $\varphi_\nu(x,D)$ continuously maps $\mathcal{E}'_{\{s\}}(\Omega)$ into $\mathcal{D}'_{\{s\}}(\Omega)$. Since $G^s(K,\eta_\mu^\nu)$ is continuously embedded in $\mathcal{D}'_{\{s\}}(\Omega)$, we also have $\varphi_\nu(x,D)u_j \to u_\nu$ in $\mathcal{D}'_{\{s\}}(\Omega)$. Thus, $\varphi_\nu(x,D)u = u_\nu \in G^s(K,\eta_\mu^\nu) \subset G^s(K)$ for all $\nu \in \mathbf{N}$. Consequently, for each $\nu \in \mathbf{N}$

$$WF_{\{s\}}(\varphi_\nu(x,D)u_{|\mathring{K}}) = WF_{\{s\}}(\varphi_\nu(x,D)u)_{|\mathring{K}} = \emptyset\,,$$

where $\mathring{K}$ denotes the interior of K. This implies that

$$\begin{aligned} WF_{\{s\}}(u_{|\mathring{K}}) &= WF_{\{s\}}(u)_{|\mathring{K}} \\ &\subset WF_{\{s\}}(\varphi_\nu(x,D)u)_{|\mathring{K}} \cup (Char\,\varphi_\nu)_{|\mathring{K}} \\ &= (Char\,\varphi_\nu)_{|\mathring{K}}. \end{aligned} \tag{3.4}$$

Since $\operatorname{supp} u \subset K$, $WF_{\{s\}}(u_{|\mathring{K}}) = WF_{\{s\}}(u)$, and so by (3.4)

$$WF_{\{s\}}(u) \cap \Gamma_\nu \subset (Char\,\varphi_\nu)_{|\mathring{K}} \cap \Gamma_\nu = \emptyset$$

for all $\nu \in \mathbf{N}$; hence $WF_{\{s\}}(u) \cap (V \setminus M) = \cup_\nu (WF_{\{s\}}(u) \cap \Gamma_\nu) = \emptyset$. Thus, $u \in F_\mu$, and $u_j \to u$ in F_μ.

Next, we denote by

$$G_\mu = \{v \in \mathcal{E}'_{\{s\}}(\Omega)\colon \operatorname{supp} v \subset K',\ v \in g^{t'}(\Omega),\ WF_{\{s\}}(v) \cap V = \emptyset\,,\ \text{and}\\ \varphi'_\nu(x,D)v \in G^s(K',\tau_\mu^\nu)\ \text{ for all } \nu \in \mathbf{N}\}$$

and observe that $G_0^s(K', \tau_\mu) \subset G_\mu$ (this follows as in the case of the space F_μ).

We endow the space G_μ with the locally convex topology generated by the following sequence of seminorms

$$q_{t',i,\mu,\nu}(v) = |v|_{t',i} + |\varphi'_\nu(x,D)v; K', s, \tau_\mu^\nu|, \quad i, \nu \in \mathbf{N}.$$

Then $(G_\mu, (q_{t',i,\mu,\nu})_{i,\nu \in N})$ is a Fréchet space and the proof is again similar to the case of the space F_μ.

Thus, the space

$$X_\mu = \{u \in F_\mu \colon Pu \in G_\mu\}$$

is a Fréchet space with respect to the intersection topology defined by the following sequence of seminorms

$$r_{i,\mu,\nu}(u) = p_{t',i,\mu,\nu}(u) + q_{t',i,\mu,\nu}(Pu), \quad i, \nu \in \mathbf{N}.$$

Indeed, if $(u_j)_{j \in \mathbf{N}}$ is a Cauchy sequence in X_μ, then $u_j \to u \in F_\mu$ in F_μ, and $Pu_j \to v \in G_\mu$ in G_μ. It follows that $u_j \to u$ and $Pu_j \to v$ in $\mathcal{E}'_{\{s\}}(\Omega)$; hence $Pu_j \to v$ and $Pu_j \to Pu$ in $\mathcal{D}'_{\{s\}}(\Omega)$, where the last statement follows from the fact that $P \colon \mathcal{E}'_{\{s\}}(\Omega) \to \mathcal{D}'_{\{s\}}(\Omega)$ continuously. Consequently, $Pu = v$ so that $u \in X_\mu$ and $u_j \to u$ in X_μ. We point out that $X_\mu \neq \{0\}$ because $P = p(x,D)$ acts also continuously from $g^{t'}(\Omega)$ into $g^{t'}(\Omega)$ (and from $g_0^{t'}(\Omega)$ into $g_0^{t'}(\Omega)$) as $s < t'$.

Following the proof of Theorem 2.1, for a fixed $i \in \mathbf{N}$ we introduce the following subsets of X_μ

$$G_{jk\widetilde{M}} = \{u \in X_\mu \colon \ \psi_k^j(x,D)u \in H^i_{(t)}(\Omega), \ |\psi_k^j(x,D)u|_{t,i} \leq \widetilde{M}\}$$

for every $j, \widetilde{M} \in \mathbf{N}$ and $k = 1, \ldots, K_j$, where the functions $(\psi_k^j)_{j,k \in \mathbf{N}} \in S^{0,s}(\Omega)$ are chosen as in the proof of Theorem 2.1.

Each set $G_{jk\widetilde{M}}$ is absolutely convex (clearly) and closed. Indeed, if the sequence $(u_r)_{r \in \mathbf{N}} \subset G_{jk\widetilde{M}}$ converges to some u in X_μ, then $u_r \to u \in F_\mu$ in F_μ (hence $u_r \to u$ in $\mathcal{E}'_{\{s\}}(\Omega)$) and $Pu_r \to Pu \in G_\mu$ in G_μ (hence $Pu_r \to Pu$ in $\mathcal{E}'_{\{s\}}(\Omega)$ and in $\mathcal{D}'_{\{s\}}(\Omega)$).

Moreover, since $\psi_k^j(x,D)u_r \in H^i_{(t)}(\Omega)$, and $|\psi_k^j(x,D)u_r|_{t,i} \leq \widetilde{M}$ for all $r \in \mathbf{N}$, there exists $u_k^j \in H^i_{(t)}(\Omega)$ such that $\chi_i \psi_k^j(x,D)u_r \to u_k^j$ weakly (eventually by passing to a subsequence) as the bounded sets in the Hilbert space $H^i_{(t)}(\Omega)$ are weakly compact, and hence $|u_k^j|_{t,i} \leq \underline{\lim}_r |\psi_k^j(x,D)u_r|_{t,i} \leq \widetilde{M}$, where the functions χ_i have been defined before (2.3). This implies that $\chi_i \psi_k^j(x,D)u_r \to u_k^j$ in $\mathcal{D}'_{\{s\}}(\Omega)$ too. On the other hand, the fact that $u_r \to u$ in $\mathcal{E}'_{\{s\}}(\Omega)$ implies that $\psi_k^j(x,D)u_r \to \psi_k^j(x,D)u$ because $\psi_k^j(x,D)$ continuously maps $\mathcal{E}'_{\{s\}}(\Omega)$ into $\mathcal{D}'_{\{s\}}(\Omega)$. Thus, $\chi_i \psi_k^j(x,D)u = u_k^j$ so that $|\psi_k^j(x,D)u|_{t,i} \leq \widetilde{M}$. This means that $G_{jk\widetilde{M}}$ is a closed subset of X_μ.

By assumption we have

$$X_\mu = \cup_{j,\widetilde{M}\in \mathbf{N}} \cup_{k=1}^{K_j} G_{jk\widetilde{M}}.$$

Indeed, if $u \in X_\mu$, then $u \in g^{t'}(\Omega)$, $WF_{\{s\}}(u) \cap (V \setminus M) = \emptyset$ and $WF_{\{s\}}(Pu) \cap V = \emptyset$. Thus, $WF_{\{s\}}(u) \cap V \subset (WF_{\{s\}}(Pu) \cap V) \cup (\operatorname{Char} P \cap V) = (\operatorname{Char} P \cap V)$ and $WF_{\{s\}}(u) \cap V \subset M$. On the other hand, by assumption $N \not\subset WF_{(t)}(u) \cap V \subset WF_{\{s\}}(u) \cap V \subset M$. Then there exists $\rho_0 \in N$ such that $\rho_0 \notin WF_{(t)}(u) \cap V$, thereby implying that $\rho_0 \notin WF_{(t)}(u)$ as $\rho_0 \in N \subset V$. Since $\rho_0 \in N$ and $\rho_0 \notin WF_{(t)}(u)$, $\rho_0 \in \Gamma_k^{2j}$ for some $j \in \mathbf{N}$ and $k \in \{1, \ldots, K_j\}$, and $\psi_k^j(x, D)u \in g^t(\Omega)$ so that $|\psi_k^j(x, D)u|_{t,i} \le \widetilde{M}$ for some $\widetilde{M} \in \mathbf{N}$.

Since X_μ is a Fréchet space, there exist $j \in \mathbf{N}$, $k \in \{1, \ldots, K_j\}$ and $\widetilde{M} \in \mathbf{N}$ such that $G_{jk\widetilde{M}}$ is a neighborhood of 0 (recall that it is absolutely convex), i.e., there exist $\varepsilon > 0$ and $i', \nu \in \mathbf{N}$ such that

$$U = \{u \in X_\mu \colon r_{i',\mu,\nu}(u) < \varepsilon\} \subset G_{jk\widetilde{M}}.$$

This yields that there exists $C_\mu > 0$ so that

$$|\psi_k^j(x, D)u|_{t,i} \le C_\mu \left(|u|_{t',i'} + |\varphi_\nu(x, D)u; K, s, \eta_\mu^\nu| + |\varphi'_\nu(x, D)Pu; K', s, \tau_\mu^\nu|\right)$$

for all $u \in X_\mu$ and hence for all $u \in G_0^s(K, \eta_\mu)$ (as $P\colon g^{t'}(\Omega) \to g^{t'}(\Omega)$ continuously).

Since μ is arbitrary and $\eta_\mu \to 0$, the results follows for every $\eta > 0$. $\square$

Example 3.1. Consider the Mizohata's operator in $\mathbf{R}^2$

$$P = \frac{\partial}{\partial x_1} + ib(x_1)\frac{\partial}{\partial x_2}\,, \tag{3.5}$$

where $b(t) = -2(n+1)t^{2n+1}$, $t \in \mathbf{R}$, $n \in \mathbf{N}_0$. Then $\operatorname{Char} P = \{(0, x_2; 0, \xi_2) : x_2 \in \mathbf{R},\ \xi_2 \ne 0\}$.

Let $s > 1$ and $\rho > 0$. Let us consider the compact sets $K = \{x \in \mathbf{R}^2 : |x_1| \le \rho,\ |x_2| \le \rho\}$, $K_1 = \{x \in \mathbf{R}^2 :\ |x_1| \le \rho/2,\ |x_2| \le \rho/2\}$ of $\mathbf{R}^2$, and set $H_0 = \overline{K \setminus K_1}$. Moreover, consider the conical sets $N = \{(0, \tilde{a}; 0, \xi_2) :\ \xi_2 > 0\}$ where $\tilde{a} \in \mathbf{R}$, and $V = T^*(\mathbf{R}^2) \setminus 0$.

For each $\tau > 0$ and $x \in \mathbf{R}^2$ set

$$f_\tau(x) = h_\tau(x) g(x)\,, \tag{3.6}$$

where $h_\tau(x) = e^{\tau q(x)}$ with $q(x) = -x_1^{2(n+1)} + ix_2 + \varepsilon(-x_1^{2(n+1)} + ix_2)^2$ for $x = (x_1, x_2) \in \mathbf{R}^2$.

Choose $\varepsilon > 0$ small enough such that $1 - \varepsilon x_1^{2(n+1)} > 0$ for $|x_1| \le \rho$. Moreover, suppose that

$$g(x) = g_0(x_1) g_0(x_2)\,, \tag{3.7}$$

where $g_0 \in G_0^s(\mathbf{R})$, $g_0(t) = 1$ for $t \in [-\rho/2, \rho/2]$ and $g_0(t) = 0$ for $|t| \geq \rho$, $0 \leq g_0 \leq 1$, $\int_{\mathbf{R}} g_0(t)dt = 1$. Then $\max_{x\in K} |f_\tau(x)| = 1$, therefore implying that $(f_\tau)_\tau$ does not converge to 0 in $G^s(\mathbf{R}^2)$ (hence in $G_0^s(\mathbf{R}^2)$) if $\tau \to \infty$. On the other hand, $f_\tau \to 0$ in $G_0^s(\mathbf{R}^2 \setminus \{(0,0)\})$ as $\tau \to \infty$. In order to show this, we recall from [6, Proposition 1.3] that for every compact set H of $\mathbf{R}^2$ and for every $\eta > \epsilon > 0$ there exists $c > 0$ such that

$$|fh; s, H, \eta - \epsilon| \leq c|f; s, H, \eta||h; s, H, \eta| \tag{3.8}$$

for all $f, h \in G^s(H, \eta)$ or in $G_0^s(H, \eta)$.

Since $\operatorname{supp} f_\tau \subset K$, it suffices to show that $f_\tau \to 0$ in $G_0^s(H)$ for every compact subset of $K \setminus \{(0,0)\}$.

Let H be a compact set of $K \setminus \{(0,0)\}$. Then there exists $0 < r < \rho$ such that H is a subset of the compact set $\bar{H} = K \setminus \{x \in K : \ |x_1| < r, \ |x_2| < r\}$. So, for every $f \in G^s(\mathbf{R}^2)$

$$|f; s, H, \eta| \leq |f; s, \bar{H}, \eta| \tag{3.9}$$

if $|f; s, \bar{H}, \eta| < \infty$.

Since q, $g \in G^s(\mathbf{R}^2)$, we have q, $g \in G^s(\bar{H}, \eta')$ for some $\eta' > 0$; this implies that $h_\tau, \ f_\tau \in G^s(\bar{H}, \eta)$ for every $\tau > 0$ and $0 < \eta < \eta'$.

Let $\epsilon > 0$ such that $\eta > \epsilon$. Then by (3.8) we obtain that for every $\tau > 0$

$$|f_\tau; s, \bar{H}, \eta - \epsilon| \leq c|h_\tau; s, \bar{H}, \eta||g; s, \bar{H}, \eta|\,, \tag{3.10}$$

where by [6, Proposition 3.2]

$$|h_\tau; s, \bar{H}, \eta| \leq Ce^{a\tau + d(\tau\eta)^{\frac{1}{s}} + d\eta^{\frac{1}{s-1}}} \tag{3.11}$$

with C and d positive constants depending only on q, and

$$a = \sup_{(x_1,x_2)\in\bar{H}} (-x_1^{2(n+1)}(1 - \varepsilon x_1^{2(n+1)}) - \varepsilon x_2^2) < 0.$$

Thus, (3.11) yields that $h_\tau \to 0$ in $G^s(\bar{H}, \eta)$ as $\tau \to \infty$, and hence by (3.9) and (3.10) we have $f_\tau \to 0$ in $G_0^s(H, \eta - \epsilon)$ too. This means that $f_\tau \to 0$ in $G_0^s(H)$ letting $\tau \to \infty$.

Now, observe that for every $\tau > 0$

$$Ph_\tau = \tau e^{\tau q(x)} Pq = 0$$

and hence

$$Pf_\tau = (Ph_\tau)g + h_\tau(Pg) = h_\tau(Pg)\,,$$

where $Pg \equiv 0$ in K_1. This together (3.8) imply that for every $\tau > 0$

$$|Pf_\tau; s, K, \eta - \epsilon| = |h_\tau(Pg); s, H_0, \eta - \epsilon| \leq c|h_\tau; s, H_0, \eta||Pg; s, H_0, \eta|\,, \tag{3.12}$$

where $h_\tau \to 0$ in $G^s(H_0, \eta)$ by (3.11) (observe that H_0 is a compact set of the same type of $\bar{H}$ above defined). Therefore, by (3.12) $Pf_\tau \to 0$ in $G_0^s(K, \eta -$

ϵ) too. This means that $Pf_\tau \to 0$ in $G_0^s(K)$ and hence in $G_0^s(\mathbf{R}^2)$ because $\operatorname{supp} Pf_\tau \subset K$. Consequently,

$$\varphi'(x,D)Pf_\tau \to 0 \tag{3.13}$$

in $G^s(\mathbf{R}^2)$ for every G^s–pseudodifferential operator, in particular with symbol $\varphi' \in S^{0,s}(\mathbf{R}^2)$.

Next, we observe that $f_\tau(x) = e^{i\tau x_2} G_\tau(x_1 \tau^{1/\tau^{2(n+1)}}, x_2\tau^{1/2})$, where

$$\begin{aligned}
G_\tau(x) &= g\left(\frac{x_1}{\tau^{1/2(n+1)}}, \frac{x_2}{\tau^{1/2}}\right) e^{\tau[\varepsilon \frac{x_1^{4(n+1)}}{\tau^2} - \frac{x_1^{2(n+1)}}{\tau} - \varepsilon \frac{x_2^2}{\tau} - 2i\varepsilon \frac{x_1^{2(n+1)} x_2}{\tau\sqrt{\tau}}]} \\
&= g\left(\frac{x_1}{\tau^{1/2(n+1)}}, \frac{x_2}{\tau^{1/2}}\right) e^{-(x_1^{2(n+1)} + \varepsilon x_2^2)} e^{\varepsilon \frac{x_1^{4(n+1)}}{\tau} - 2i\varepsilon \frac{x_1^{2(n+1)} x_2}{\sqrt{\tau}}}
\end{aligned}$$

converges to $G(x) = e^{-(x_1^{2(n+1)} + \varepsilon x_2^2)}$ in $\mathcal{S}_{(s)}$ (hence in $\mathcal{S}$) letting $\tau \to \infty$ if $\varepsilon < (1/2\rho)^{2(n+1)}$. In fact, $G_\tau(x) \neq 0$ implies $\frac{|x_1|}{\tau^{1/2(n+1)}} \le \rho$ so that $x_1^{4(n+1)} \le x_1^{2(n+1)} \rho^{2(n+1)} \tau$ and therefore $e^{\frac{\varepsilon}{\tau} x_1^{4(n+1)}} \le e^{\varepsilon x_1^{2(n+1)} \rho^{2(n+1)}}$.

Thus, we have

$$\hat{f}_\tau(\xi) = \tau^{-\frac{n+2}{2(n+1)}} \hat{G}_\tau\left(\frac{\xi_1}{\tau^{\frac{1}{2(n+1)}}}, \frac{\xi_2 - \tau}{\sqrt{\tau}}\right) \tag{3.14}$$

for every $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$.

Since $G_\tau \to G$ in $\mathcal{S}_{(s)}$ (so that $\hat{G}_\tau \to \hat{G}$ in $\mathcal{S}^{(s)}$ and hence in $\mathcal{S}$), for any $\eta > 0$ and any integer $N \ge 0$ we have $\sup_{\alpha \in \mathbf{N}_0^2} (1+|\xi|^N) \eta^{|\alpha|} (\alpha!)^{-s} |\partial^\alpha \hat{f}_\tau(\xi)| \to 0$ uniformly on any compact subset of $\mathbf{R}^2$ and outside any conic neighborhood of $e_2 := (0,1)$ by (3.14). So,

$$\varphi(x,D) f_\tau \to 0 \tag{3.15}$$

in $G^s(K)$ for any properly supported G^s–pseudodifferential operator $\varphi(x,D)$, in particular with symbol $\varphi \in S^{0,s}(\mathbf{R}^2)$, such that $(0,e_2) \notin WF_{\{s\}}(\varphi)$, hence if $\operatorname{conesupp} \varphi \subset V \setminus N$ (in such a case we have $(0,e_2) \notin WF_{\{s\}}(\varphi)$). We also have that for every $t \in \mathbf{R}$

$$\begin{aligned}
|f_\tau|_t^2 &= \int |\hat{f}_\tau(\xi)|^2 (1+|\xi|^2)^t d\xi \\
&= \tau^{-\frac{n+2}{n+1}} \int \left|\hat{G}_\tau\left(\xi_1 \tau^{-\frac{1}{2(n+1)}}, \frac{\xi_2 - \tau}{\sqrt{\tau}}\right)\right|^2 (1+|\xi|^2)^t d\xi \\
&= \tau^{-\frac{n+2}{2(n+1)}} \int |\hat{G}_\tau(\eta)|^2 (1 + [\eta_1^2 \tau^{\frac{1}{n+1}} + (\sqrt{\tau}\eta_2 + \tau)^2])^t d\eta \\
&= \tau^{-\frac{n+2}{2(n+1)}} \int |\hat{G}_\tau(\eta)|^2 (1 + \tau|(\eta_1 \tau^{-\frac{n}{2(n+1)}}, \eta_2) + \sqrt{\tau} e_2|^2)^t d\eta,
\end{aligned}$$

so

$$\begin{aligned}
&|f_\tau|_t^2 \tau^{-2t + \frac{n+2}{2(n+1)}} = \\
&= \tau^{-2t} \int |\hat{G}_\tau(\eta)|^2 (1 + \tau|(\eta_1 \tau^{-\frac{n}{2(n+1)}}, \eta_2) + \sqrt{\tau} e_2|^2)^t d\eta \to |G|_{L^2(\mathbf{R}^2)}^2.
\end{aligned} \tag{3.16}$$

We claim that (3.13), (3.15) and (3.16) imply that (2.1) does not hold. Indeed, if we suppose that (2.1) is satisfied, we have that for every $\tau > 0$

$$|\psi(x,D)f_\tau|_t \le C\left(|f_\tau|_{t'} + |\varphi(x,D)f_\tau;K,s,\eta'| + |\varphi'(x,D)Pf_\tau;K,s,\eta''|\right). \tag{3.17}$$

Therefore, for every $\tau > 0$

$$\begin{aligned}|\psi(x,D)f_\tau|_t^2\tau^{-2t+\frac{n+2}{2(n+1)}} &\le C^2(\tau^{-t+t'}|f_\tau|_{t'}\tau^{-t'+\frac{n+2}{4(n+1)}}+\\ &\quad+\tau^{-t+\frac{n+2}{4(n+1)}}(|\varphi(x,D)f_\tau;K,s,\eta'|+|\varphi'(x,D)Pf_\tau;K',s,\eta''|))^2;\end{aligned}$$

letting $\tau \to \infty$, we obtain that

$$\limsup_{\tau\to\infty}|\psi(x,D)f_\tau|_t^2\tau^{-2t+\frac{n+2}{2(n+1)}} = 0$$

if $t' \ge \frac{n+2}{4(n+1)}$ as $t > t'$.

On the other hand, as $f_\tau \in \mathcal{S}$ and $\psi_x(\eta) := \psi(x,\eta) \in \mathcal{S}'$ for every $x \in \mathbf{R}^2$, we have

$$\psi(x,D)f_\tau(x) = \int e^{ix.\eta}\psi(x,\eta)\hat{f}_\tau(\eta)d\eta = \int f_\tau(y)\hat{\psi}_x(y-x)dy$$

for every $x \in \mathbf{R}^2$, where the integrals denote action of distributions; hence, for every $\xi \in \mathbf{R}^2$

$$\begin{aligned}\widehat{\psi(x,D)}f_\tau(\xi) &= \int\int e^{-ix.\xi}f_\tau(y)\hat{\psi}_x(y-x)dxdy\\ &= \int\int e^{-i\xi.(y-z)}f_\tau(y)\hat{\psi}_{y-z}(z)dydz\\ &= \int e^{-iy.\xi}f_\tau(y)dy\int e^{iz.\xi}\hat{\psi}_{y-z}(z)dz\\ &= (2\pi)^{-2}\int e^{-i\xi.y}f_\tau(y)\psi(y-\xi,\xi)dy\\ &= (2\pi)^{-2}\hat{f}_\tau(\xi) + (2\pi)^{-2}\int e^{-i\xi.y}f_\tau(y)[\psi(y-\xi,\xi)-1]dy.\end{aligned} \tag{3.18}$$

Thus, $|\psi(x,D)f_\tau|_t^2\tau^{-2t+\frac{n+2}{2(n+1)}} \to (2\pi)^{-4}|G|^2_{L^2(\mathbf{R}^n)} \ne 0$ because $\psi \equiv 1$ in a conical neighborhood of N. This is a contradiction.

Thus, we can conclude that there exists a function $u \in H^{t'}(\mathbf{R}^2)$ such that $WF_{\{s\}}(Pu) = \emptyset$ and hence $Pu \in G^s(\mathbf{R}^2)$, and $WF_t(u) = WF_{\{s\}}(u) = N$ and so $\{s\} - \operatorname{singsupp} u = t - \operatorname{singsupp} u = \{(0,\tilde{a})\}$.

We also have, for every $t \in \mathbf{R}$,

$$\begin{aligned}|f_\tau|^2_{t,j} &= \int|\hat{f}_\tau(\xi)|^2e^{j|\xi|^{1/t}}d\xi = \tau^{-\frac{n+2}{n+1}}\int\left|\hat{G}_\tau\left(\xi_1\tau^{-\frac{1}{2(n+1)}},\frac{\xi_2-\tau}{\sqrt{\tau}}\right)\right|^2e^{j|\xi|^{1/t}}d\xi\\ &= \tau^{-\frac{n+2}{2(n+1)}}\int|\hat{G}_\tau(\eta)|^2e^{j(\eta_1^2\tau^{\frac{1}{n+1}}+(\sqrt{\tau}\eta_2+\tau)^2)^{1/2t}}d\eta,\end{aligned}$$

so

$$
\begin{aligned}
&|f_\tau|^2_{t,j} e^{-j\tau^{1/t}} \tau^{\frac{n+2}{2(n+1)}} = \\
(3.19) \qquad &= e^{-j\tau^{1/t}} \int |\hat{G}_\tau(\eta)|^2 e^{j(\eta_1^2 \tau^{\frac{1}{n+1}} + (\sqrt{\tau}\eta_2+\tau)^2)^{1/2t}} d\eta \to |G|^2_{L^2(\mathbf{R}^2)}
\end{aligned}
$$

if $t > 2$. Thus, Example 3.1 illustrates Theorem 2.3 for $2 < s < t < t'$. For $t = 2$, we have

$$\lim_{\tau\to\infty} |f_\tau|^2_{t,j} e^{-j\tau^{1/t}} \tau^{\frac{n+2}{2(n+1)}} = \int |\hat{G}(\eta)|^2 e^{2j\eta_2} d\eta > 0.$$

We claim that (3.13), (3.15) and (3.19) imply that (2.4) does not hold. Indeed, if we assume that (2.4) is satisfied, fixed any $i \in \mathbf{N}$, we have that for every $\tau > 0$

(3.20)
$$|\psi(x,D)f_\tau|_{t,i} \le C\left(|f_\tau|_{t',j} + |\varphi(x,D)f_\tau; K, s, \eta'| + |\varphi'(x,D)Pf_\tau; K', s, \eta''|\right).$$

Therefore, for every $\tau > 0$

$$
\begin{aligned}
|\psi(x,D)f_\tau|^2_{t,i} e^{-i\tau^{1/t}} \tau^{\frac{n+2}{2(n+1)}} \le C^2 (&e^{\frac{j}{2}\tau^{1/t'} - \frac{i}{2}\tau^{1/t}} |f_\tau|_{t',j} e^{-\frac{j}{2}\tau^{1/t'}} \tau^{\frac{n+2}{4(n+1)}} + \\
&+ e^{-\frac{i}{2}\tau^{1/t}} \tau^{\frac{n+2}{4(n+1)}} (|\varphi(x,D)f_\tau; K, s, \eta'| + |\varphi'(x,D)Pf_\tau; K', s, \eta''|))^2;
\end{aligned}
$$

letting $\tau \to \infty$, we obtain that

$$\limsup_{\tau\to\infty} |\psi(x,D)f_\tau|^2_{t,i} e^{-i\tau^{1/t}} \tau^{\frac{n+2}{2(n+1)}} = 0\,,$$

as $1 < t < t'$, i.e. $\tau^{\frac{1}{t} - \frac{1}{t'}} \to 0$ for $\tau \to \infty$.

On the other hand, $|\psi(x,D)f_\tau|^2_{t,i} e^{-i\tau^{1/t}} \tau^{\frac{n+2}{2(n+1)}} \to (2\pi)^{-4}|G|^2_{L^2(\mathbf{R}^n)} \neq 0$ (see (3.18)): this is a contradiction.

Thus, we can conclude that there exists a function $u \in g^{t'}(\mathbf{R}^2)$ (hence $u \in G^{t'}(\mathbf{R}^2)$) such that $WF_{\{s\}}(Pu) = \emptyset$ and hence $Pu \in G^s(\mathbf{R}^2)$, and $WF_{(t)}(u) = WF_{\{s\}}(u) = N$ and so $\{s\} - \operatorname{singsupp} u = (t) - \operatorname{singsupp} u = \{(0, \tilde{a})\}$.

Appendix.

Let $1 < t < 2$, i.e., $1 > \frac{1}{t} > \frac{1}{2}$ in (3.19). Observe that

$$
\begin{aligned}
|f_\tau|^2_{t,j} e^{-j\tau^{1/t}} \tau^{\frac{n+2}{2(n+1)}} &= e^{-j\tau^{1/t}} \int |\hat{G}_\tau(\eta)|^2 e^{j(\eta_1^2 \tau^{\frac{1}{n+1}} + (\sqrt{\tau}\eta_2+\tau)^2)^{1/2t}} d\eta \\
&\ge e^{-j\tau^{1/t}} \int_{\eta_2 \ge 1} |\hat{G}_\tau(\eta)|^2 e^{j(\eta_1^2 \tau^{\frac{1}{n+1}} + (\sqrt{\tau}\eta_2+\tau)^2)^{1/2t}} d\eta \\
&\ge e^{-j\tau^{1/t}} \int_{\eta_2 \ge 1} |\hat{G}_\tau(\eta)|^2 e^{j(\tau+\sqrt{\tau})^{1/t}} d\eta.
\end{aligned}
$$

On the other hand, $j(\tau+\sqrt{\tau})^{1/t} = j\tau^{1/t}(1+\frac{\tau^{-1/2}}{t}+O(\tau^{-1}))$ as $\tau \to \infty$. Thus,

$$|f_\tau|^2_{t,j} e^{-j\tau^{1/t}} \tau^{\frac{n+2}{2(n+1)}} \geq e^{\frac{j}{t}\tau^{1/t-1/2}+O(\tau^{1/t-1})} \cdot \int_{\eta_2 \geq 1} |\hat{G}_\tau(\eta)|^2 d\eta,$$

where $e^{O(\tau^{1/t-1})} \to 1$ and $\int_{\eta_2\geq 1} |\hat{G}_\tau(\eta)|^2 d\eta \to \int_{\eta_2 \geq 1} |\hat{G}(\eta)|^2 d\eta$ as $\tau \to \infty$. Consequently, we have

$$\lim_{\tau\to\infty} |f_\tau|^2_{t,j} e^{-j\tau^{1/t}} \tau^{\frac{n+2}{2(n+1)}} = \infty.$$

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