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Convergence of dependent walks in a random scenery to fBm-local time fractional stable motions

By

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Abstract

It is classical to approximate the distribution of fractional Brownian motion by a renormalized sum S_n of dependent Gaussian random variables. In this paper we consider such a walk Z_n that collects random rewards ξ_j for $j \in \mathbb{Z}$, when the ceiling of the walk S_n is located at j. The random reward (or scenery) ξ_j is independent of the walk and with heavy tail. We show the convergence of the sum of independent copies of Z_n suitably renormalized to a stable motion with integral representation, whose kernel is the local time of a fractional Brownian motion (fBm). This work extends a previous work where the random walk S_n had independent increments limits.

1. Introduction

1.1. Motivations

Many stochastic processes have been proposed to model communication networks. We can refer to [13, 12, 9] for instance, where the limiting processes are either fractional Brownian motion or Lévy β-stable process. More recently, in [3] a process named H-fBm local time fractional stable motion was constructed. When $H = \frac{1}{2}$, the so called random-reward scheme, was also proposed, it is a discrete scheme, which could be thought of as a toy model for internet traffic, and which is converging to this process. The aim of this paper is to extend these results to the case $H \neq \frac{1}{2}$. In the proof of the convergence in [3] a strong approximation of the local time of standard Brownian motion was used. As far as we know no strong approximation of the local time of fractional Brownian motion is available and it was one the problems to overcome. In [4] discrete approximations of local time fractional stable motion have been obtained by Dombry and Guillotin-Plantard [4] where the fBm local time is replaced by the local time of an α-stable Lévy motion. But they did not use

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strong approximation of the local time. In this paper we use convergence of the local time of a classical walk with dependent increments to the local time of fractional Brownian motion and the technique in [4] to get our result. Please note that other approximations of fBm local time fractional stable motion have been considered in [8], but they are not related to walks in random sceneries.

1.2. Model and results

We recall first the definition of the so-called H-fBm local time fractional stable motion introduced in [3] and denoted by Γ in the sequel. This process is defined as the integral of a random kernel (the local time of a fBm) with respect to a symmetric β-stable random measure. Formally, let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a probability space supporting a fractional Brownian motion $(B_H(t))_{t\geq 0}$ with Hurst index $H \in (0,1)$ and let $L_t(x)$ be its jointly continuous local time process. Let M be a symmetric β-stable random measure on the space $\Omega' \times \mathbb{R}$ with control measure $\mathbb{P}' \times \lambda$ (where λ is the Lebesgue measure on $\mathbb{R}$). The random measure itself lives on some other probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the definition and properties of stable random measures and integrals with respect with these measures, we refer to the monography of Samorodnitsky and Taqqu [10]. We consider the process Γ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\Gamma(t) = \int_{\Omega' \times \mathbb{R}} \sigma L_t(x)(\omega') M(\mathrm{d}\omega', \mathrm{d}x), \quad t \geq 0. \tag{1.1}$$

This process has a continuous version (Hölder properties are discussed in [3]) and is a β-stable stationary increments δ-self-similar process with $\delta = 1 - H + H\beta^{-1}$. The aim of this work is to prove the convergence of the so called random reward scheme to the process Γ.

This discrete scheme is based on random walks in random sceneries that we introduce now. Let $\xi = (\xi_x)_{x\in\mathbb{Z}}$ denote a sequence of independent, identically distributed, symmetric real-valued random variables. The sequence ξ is called a *random scenery.* Suppose that it belongs to the normal domain of attraction of a stable symmetric distribution Z_β of index $\beta \in (0,2]$. This means that the following weak convergence holds:

$$n^{-\frac{1}{\beta}} \sum_{x=0}^{n} \xi_x \underset{n\to\infty}{\overset{\mathcal{L}}{\Longrightarrow}} Z_\beta, \tag{1.2}$$

where Z_β is the symmetric stable law with characteristic function $\bar\lambda$ given by

$$\bar\lambda(u) = \mathbb{E}\exp(iuZ_\beta) = \exp\left(-\sigma^\beta |u|^\beta\right), \quad u \in \mathbb{R} \tag{1.3}$$

for some constants $\sigma > 0$.

Let $S = (S_k)_{k\in\mathbb{N}}$ be a *random walk* on $\mathbb{Z}$ independent of the random scenery ξ. We suppose that

$$\begin{cases} S_0 = 0, \\ S_n = \sum_{k=1}^n X_k, \quad n \geq 1, \end{cases} \tag{1.4}$$

where $X_i, i \geq 1$ is a stationary Gaussian sequence with mean 0 and correlations $r(i-j) = \mathbb{E}[X_i X_j]$ satisfying

$$\sum_{i=1}^{n}\sum_{j=1}^{n} r(i-j) \sim n^{2H}, \tag{1.5}$$

as $n \to \infty$, with $0 < H < 1$.

We define the *random walk in random scenery* as the process $(Z_n)_{n\geq 0}$ given by

$$Z_n = \sum_{k=0}^{n} \xi_{([S_k])}, \tag{1.6}$$

where $[S_k]$ is the ceiling of S_k. Stated simply, a random walk in random scenery is a cumulative sum process whose summands are drawn from the scenery; the order in which the summands are drawn is determined by the path of the random walk. We extend this definition to non-integer time $s \geq 0$ by linear interpolation

$$Z_s = Z_{[s]} + (s - [s])(Z_{[s]+1} - Z_{[s]}). \tag{1.7}$$

We now describe the limit theorem for the random walk in random scenery established by Wang [14] (in the case $\beta = 2$).
Cumulative sums of the scenery converge in $D(\mathbb{R})$, the space of càd-làg functions:

$$\left(n^{-\frac{1}{\beta}} \sum_{k=0}^{[nx]} \xi_k \right)_{x\in\mathbb{R}} \underset{n\to\infty}{\overset{\mathcal{L}}{\Longrightarrow}} (W(x))_{x\in\mathbb{R}}\,,$$

where W is a bilateral β-stable Lévy process such that $W(0) = 0$, and $W(1)$ and $W(-1)$ are distributed according to Z_β defined in (1.3).
The covariance structure of the sequence X_i given by equation (1.5) implies that $S_n, n \geq 0$ belongs to the domain of attraction of the fractional Brownian motion of Hurst index H, i.e. the following convergence hold in $D([0,\infty))$ (cf. [11].)

$$\frac{1}{n^H}(S_{[nt]})_{t\geq 0} \underset{n\to\infty}{\overset{\mathcal{L}}{\Longrightarrow}} (B_H(t))_{t\geq 0}, \tag{1.8}$$

To describe the limit process known as *fractional Brownian motion in stable scenery* we suppose that B_H and W are two independent processes defined on the same probability space and distributed as above. Let $L_t(x)$ the jointly continuous version of the local time of the process B_H (cf. [1]).

In the case $\beta = 2$ corresponding to the case of a Gaussian scenery, Wang proves the following weak convergence in the space of continuous function $C([0,\infty))$

$$\left(n^{-\delta} Z_{nt}\right)_{t\geq 0} \underset{n\to\infty}{\overset{\mathcal{L}}{\Longrightarrow}} (\Delta(t))_{t\geq 0} \tag{1.9}$$

where $\delta = 1 - H + H\beta^{-1}$ and Δ is the process defined by

$$\Delta(t) = \int_{-\infty}^{+\infty} L_t(x) \mathrm{d}W(x).$$

The limit process Δ is a continuous δ-self-similar stationary increments process.

Our results state the convergence of the so called random-reward scheme to the fBm local time stable fractional motion. We begin with a continuous version and consider $\Delta^{(i)}, i \geq 1$ independent copies of the process Δ.

Theorem 1.1. *The following weak convergence holds in* $C([0,\infty))$:

$$\left(n^{-\frac{1}{\beta}} \sum_{i=1}^{n} \Delta^{(i)}(t) \right)_{t \geq 0} \overset{\mathcal{L}}{\underset{n\to\infty}{\Longrightarrow}} (\Gamma(t))_{t\geq 0}, \tag{1.10}$$

where Γ *is the H-fBm local time stable fractional motion defined by equation* (1.1).

Replacing the stable process in random scenery by a random walk in random scenery, we obtain the random rewards scheme which yields a discrete approximation of the process Γ. Let $\xi^{(i)} = (\xi_x^{(i)})_{x\in\mathbb{Z}},\ i \geq 1$ be independent copies of ξ. Let $S^{(i)} = (S_n^{(i)})_{n\in\mathbb{N}}$ be independent copies of S and also independent of the $\xi^{(i)}, i \geq 1$. Denote by $D_n^{(i)}$ the i-th random walk in random scenery defined by

$$D_n^{(i)}(t) = n^{-\delta} Z_{nt}^{(i)} \tag{1.11}$$

where the definition of $Z_n^{(i)}$ is given by equations (1.6) and (1.7) with ξ and S replaced by the i-th random scenery $\xi^{(i)}$ and the i-th random walk $S^{(i)}$ respectively.

Theorem 1.2. *Let* c_n *be a sequence of integers such that* $\lim c_n = +\infty$. *Then, the following weak convergence holds in* $C([0,\infty))$:

$$\left(c_n^{-\frac{1}{\beta}} \sum_{i=1}^{c_n} D_n^{(i)}(t) \right)_{t \geq 0} \overset{\mathcal{L}}{\underset{n\to\infty}{\Longrightarrow}} (\Gamma(t))_{t\geq 0}. \tag{1.12}$$

2. Sums of stable processes in random scenery, Proof of Theorem 1.1

For $n \geq 1$, let Γ_n the continuous process defined by

$$\Gamma_n(t) = n^{-\frac{1}{\beta}} \sum_{i=1}^{n} \Delta^{(i)}(t), \quad t \geq 0.$$

Theorem 1.1 claims that the sequence Γ_n converges weakly in $C([0,\infty))$. We prove this fact by proving the convergence of the finite dimensional distributions and the tightness of the sequence. Theorem 1.1 is thus a consequence of Propositions 2.1 and 2.2 below.

We first need a Lemma giving the characteristic function of the finite dimensional distribution of Δ:

Lemma 2.1. *For any* $(\theta_1, \cdots, \theta_k) \in \mathbb{R}^k$ *and* $(t_1, \cdots, t_k) \in [0, +\infty)^k$

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{k}\theta_j\Delta(t_j)\right)\right] = \mathbb{E}\left[\exp(-\sigma^\beta X)\right]$$

with

$$X = \int_{\mathbb{R}}\left|\sum_{j=1}^{k}\theta_j L_{t_j}(x)\right|^\beta \mathrm{d}x. \tag{2.1}$$

Proof. This is the analogous of Lemma 5 in Kesten and Spitzer giving the characteristic function of the finite dimensional distribution of the stable Lévy-process in stable scenery, for the fractional Brownian motion in stable scenery. The demonstration is the same replacing the local time of a stable Lévy process by the local time of the fractional Brownian motion. □

Proposition 2.1. *The finite dimensional distributions of* $(\Gamma_n(t))_{t\geq 0}$ *converge weakly as* $n \to \infty$ *to those of* $(\Gamma(t))_{t\geq 0}$.

Proof. Let $(\theta_1, \cdots, \theta_k) \in \mathbb{R}^k$ and $(t_1, \cdots, t_k) \in [0, +\infty)^k$. We compute the characteristic functions

$$\begin{aligned}\mathbb{E}\left[\exp\left(i\sum_{j=1}^{k}\theta_j\Gamma_n(t_j)\right)\right] &= \mathbb{E}\left[\exp\left(in^{-\frac{1}{\beta}}\sum_{j=1}^{k}\theta_j\Delta(t_j)\right)\right]^n \\ &= \mathbb{E}\left[\exp(-n^{-1}\sigma^\beta X)\right]^n.\end{aligned} \tag{2.2}$$

We prove that the following asymptotic holds:

$$\mathbb{E}\left[\exp(-n^{-1}\sigma^\beta X)\right] = 1 - n^{-1}\sigma^\beta\mathbb{E}(X) + o(n^{-1}). \tag{2.3}$$

Note that the integrability of the random variable X follows from the inequality

$$|X| \leq \left(\sum_{j=1}^{k}|\theta_j|\right)^\beta \int_{\mathbb{R}} L_t(x)^\beta \mathrm{d}x,$$

where $t = \max\{t_j\ ,\ 1 \leq j \leq k\}$, and the fact that $\mathbb{E}\int_{\mathbb{R}} L_t(x)^\beta \mathrm{d}x < \infty$ which is proved in [3] Theorem 3.1.
We now prove equation (2.3). To this aim, observe that

$$n\left(\mathbb{E}\left[\exp(-n^{-1}\sigma^\beta X)\right]^n - 1\right) = \mathbb{E}(f_n(X)) \underset{n\to\infty}{\longrightarrow} -\sigma^\beta\mathbb{E}(X),$$

where f_n is defined on $\mathbb{C}$ by $f_n(x) = n(\exp(-n^{-1}\sigma^\beta x) - 1)$. The convergence follows from the dominated convergence Theorem because $f_n(X)$ converges almost surely to $-\sigma^\beta X$ and $|f_n(X)|$ is almost surely bounded from above by $\sigma^\beta |X|$ which is integrable. Finally, equations (2.2) and (2.3) together yield

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^k \theta_j \Gamma_n(t_j)\right)\right] \underset{n\to\infty}{\longrightarrow} \exp\left(-\sigma^\beta \mathbb{E}(X)\right).$$

This proves Proposition 2.1. □

Proposition 2.2. *The sequence of process Γ_n is tight in $C([0,\infty))$.*

Proof. We follow the proof of Proposition 2.2 in [4] and give only the main lines of the proof, the details are to be found in [4]. The difference is that the α-stable Lévy motion Y_t is replaced by the fBm $B_H(t)$ of index H. Hence the local time process of Y is replaced by the local time of B_H and denoted in both context by $L_t(x)$. Furthermore, the self-similarity index of Y is equal to $1/\alpha$ and has to be replaced by H.

The case $\beta = 2$ is straightforward and relies on Itô's isometry: the process Γ_n is square integrable and for all $0 \le t_1 < t_2$

$$\begin{aligned}\mathbb{E}\left[|\Gamma_n(t_2) - \Gamma_n(t_1)|^2\right] &= \mathbb{E}\left[\left|n^{-\frac{1}{2}}\sum_{i=1}^n \Delta^{(i)}(t_2) - \Delta^{(i)}(t_1)\right|^2\right] \\ &= \sigma^2 (t_2 - t_1)^{2-H}\mathbb{E}\left[\int_{\mathbb{R}} L_1(x)^2 \mathrm{d}x\right].\end{aligned}$$

Using Kolmogorov criterion, we deduce that the sequence Γ_n is tight.

In the case $0 < \beta < 2$, the process Γ_n has infinite variance and we use the truncation method. Introduce the Lévy-Ito decomposition of W:

$$W_x = bx + \int_0^x \int_{|u|\le 1} u(\mu - \bar{\mu})(\mathrm{d}u, \mathrm{d}s) + \int_0^x \int_{|u|>1} u\mu(\mathrm{d}u, \mathrm{d}s), \tag{2.4}$$

where b is the drift and μ is a Poisson random measure on $\mathbb{R}\times\mathbb{R}$ with intensity $\bar{\mu}(\mathrm{d}u, \mathrm{d}x) = \lambda(\mathrm{d}u) \otimes \mathrm{d}x$, and λ is the stable Lévy measure on $\mathbb{R}$:

$$\lambda(\mathrm{d}u) = \left(c_- 1_{\{u<0\}} + c_+ 1_{\{u>0\}}\right)\frac{\mathrm{d}u}{|u|^{\beta+1}}, \quad c_-, c_+ \ge 0, c_- + c_+ > 0.$$

For some truncation level $R > 1$, let $W^{(R^-)}$ and $W^{(R^+)}$ be the independent Lévy processes defined by

$$W_x^{(R^-)} = \int_0^x \int_{|u|\le R} u(\mu - \bar{\mu})(\mathrm{d}u, \mathrm{d}s), \quad W_x^{(R^+)} = \int_0^x \int_{|u|>R} u\mu(\mathrm{d}u, \mathrm{d}s).$$

The Lévy-Ito decomposition (2.4) can be rewritten as

$$W_x = b_R x + W_x^{(R^-)} + W_x^{(R^+)},$$

where $b_R = b + \int_{1<|y|\le R} u\lambda(\mathrm{d}u)$ is a drift depending on R. This decomposition of the stable scenery yields the following decomposition of the stable process in random scenery:

$$\Delta(t) = b_R t + \Delta^{(R^-)}(t) + \Delta^{(R^+)}(t),$$

with

$$\Delta^{(R^-)}(t) = \int_{\mathbb{R}} L_t(x)\mathrm{d}W_x^{(R^-)}, \quad \Delta^{(R^+)}(t) = \int_{\mathbb{R}} L_t(x)\mathrm{d}W_x^{(R^+)}.$$

We consider i.i.d. copies Δ^i of Δ, and introduce similarly

$$\Delta^{(i,R^-)}(t) = \int_{\mathbb{R}} L_t^i(x)\mathrm{d}W_x^{(i,R^-)}, \quad \Delta^{(i,R^+)}(t) = \int_{\mathbb{R}} L_t^i(x)\mathrm{d}W_x^{(i,R^+)},$$

where L_t^i and W^i are i.i.d. copies of L_t and W respectively and $W^i = W^{(i,R^-)} + W^{(i,R^+)}$ is the decomposition of the Lévy process W^i defined as $W = W^{(R^-)} + W^{(R^+)}$ above. The following decomposition of Γ_n turns out to be useful:

$$\Gamma_n(t) = n^{1-\frac{1}{\beta}} b_{R_n} t + \Gamma_n^{(R^-)}(t) + \Gamma_n^{(R^+)}(t), \tag{2.5}$$

with

$$\Gamma_n^{(R^-)}(t) = n^{-\frac{1}{\beta}} \sum_{i=1}^{n} \Delta^{(i,R_n^-)}(t), \quad \Gamma_n^{(R^+)}(t) = n^{-\frac{1}{\beta}} \sum_{i=1}^{n} \Delta^{(i,R_n^+)}(t),$$

with truncation level $R_n = Rn^{\frac{1}{\beta}}$. The sequence $n^{1-\frac{1}{\beta}} b_{R_n}$ is known to be bounded (assertion A1 in [4]).

Similarly to equation (23) in [4], the process $\Gamma_n^{(R^-)}(t)$ is square integrable and for any $0 \le t_1 < t_2$,

$$\mathbb{E}\left[(\Gamma_n^{(R^-)}(t_2) - \Gamma_n^{(R^-)}(t_1))^2\right] = \frac{c_- + c_+}{2-\beta} R^{2-\beta}(t_2-t_1)^{2-H}\mathbb{E}\left[\int_{\mathbb{R}} L_1(x)^2\mathrm{d}x\right].$$

Using Kolmogorov criterion, this estimate implies that the sequence of process $\Gamma_n^{(R^-)}$ is tight.

On the other hand, similarly to assertion (A3) in [4], the probability that $\Gamma_n^{(R^+)} \equiv 0$ on $[0,T]$ satisfies

$$\begin{aligned} \mathbb{P}\left(\Gamma_n^{(R^+)} \equiv 0 \text{ on } [0,T]\right) &\ge \left[\mathbb{P}\left(\Delta^{(R_n^+)} \equiv 0 \text{ on } [0,T]\right)\right]^n \\ &\ge \left[1 - 2\frac{c^+ + c^-}{\beta} R_n^{-\beta}\mathbb{E}\left(\sup_{0\le t\le T} |B_H(t)|\right)\right]^n. \end{aligned}$$

and hence

$$\lim_{R\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\Gamma_n^{(R^+)} \equiv 0 \text{ on } [0,T]\right) = 1.$$

These facts imply the tightness of the sequence Γ_n. $\square$

3. Fractional random-reward scheme, Proof of Theorem 1.2

We define the process G_n by

$$G_n(t) = c_n^{-\frac{1}{\beta}} \sum_{i=1}^{c_n} D_n^{(i)}(t), \quad t \geq 0, \tag{3.1}$$

where $D_n^{(i)}$ is the i-th random walk in random scenery properly rescaled and defined by (1.11). Theorem 1.2 states that G_n converges weakly to Γ in $C([0,\infty))$. The key tool in the proof is the local time of the strongly correlated random walk $(S_k)_{k\geq 0}$ (we omit the superscript (i)).

Let $x \in \mathbb{Z}$ and $n \geq 1$. The local time $N_n(x)$ of the random walk $(S_k)_{k\geq 0}$ at point x up to time n is defined by

$$N_n(x) = \sum_{k=0}^{n} 1_{\{[S_k]=x\}}.$$

It represents the amount of time the walk spends in the interval $[x, x+1[$ up to time n. We extend this definition to non-integer time $s \geq 0$ by linear interpolation:

$$N_s(x) = N_{[s]}(x) + (s - [s])(N_{[s]+1}(x) - N_{[s]}(x)).$$

The random walk in random scenery can be written for all $s \geq 0$ as follows:

$$D_n(t) = n^{-\delta} \sum_{x\in\mathbb{Z}} N_{nt}(x)\xi_x, \tag{3.2}$$

where the collection of random variables $\{N_s(x), x \in \mathbb{Z}\}$ and $\{\xi_x, x \in \mathbb{Z}\}$ are independent.

We collect in the next subsection different results about the local times of strongly-correlated random walks that will be of great use in the sequel. Although the results are analogous to the ones in [4] for independent increments random walks, some difficulties arise from the strong correlations of the increments. However in [14], Wang shows how to use the Gaussian structure to get some estimates on the local times of a strongly-correlated random walk.

Then the proof of Theorem 1.2 is quite analogous to the proof of Theorem 2 in [4]. Proposition 3.1 states the convergence of the finite dimensional distribution. The tightness of the sequence is stated in Proposition 3.2. We give the main lines of the proof and omit some details that are to be found in [4].

3.1. Some results about local times

3.1.1. Maximum local time, self intersection local time and range

The maximum local time L_n of the random walk up to time n is defined by

$$L_n = \sup_{x \in \mathbb{Z}} N_n(x).$$

The number of self-intersections V_n of the random walk up to time n is defined by

$$V_n = \sum_{0 \le i,j \le n} 1_{\{[S_i]=[S_j]\}} = \sum_{x \in \mathbb{Z}} N_n(x)^2.$$

The range R_n of the random walk up to time n is defined by

$$R_n = \sum_{x \in \mathbb{Z}} 1_{\{N_n(x) \neq 0\}}.$$

These definitions extend obviously to non-integer time $s \ge 0$.

Our results rely on different estimations of these quantities that we gather in the following Lemma:

Lemma 3.1.

- *The following convergence in probability holds*

$$n^{-\delta} L_n \xrightarrow[n\to\infty]{\mathrm{P}} 0. \tag{3.3}$$

- *For any $p \in [1, +\infty)$, there exists some constant C such that for all $n \ge 1$,*

$$\mathbb{E}\left(V_n^p\right) \le C n^{p(2-H)}. \tag{3.4}$$

- *For any $p \in [1, +\infty)$, there exists some constant C such that for all $n \ge 1$*

$$\mathbb{E}\left(R_n^p\right) \le C n^{pH}. \tag{3.5}$$

Proof.

- We follow the lines of Lemma 4 in Kesten and Spitzer. Let $\varepsilon > 0$, we have,

$$\begin{aligned}\mathbb{P}(n^{-\delta} L_n > \varepsilon) &\le \mathbb{P}(N_n(x) > 0 \text{ for some } |x| > An^H) + \sum_{|x| \le An^H} \mathbb{P}(N_n(x) > n^\delta \varepsilon) \\ &\le \mathbb{P}(\sup_{0 \le k \le n} n^{-H} |[S_k]| > A) + \sum_{|x| \le An^H} \mathbb{E}(N_n(x)^p) n^{-p\delta} \varepsilon^{-p}.\end{aligned}$$

We now use the following estimation from [14] lemma 4.4: there exists some $C > 0$ such that

$$\mathbb{E}(N_n(x)^p) \le C n^{p(1-H)},$$

and hence we have for all $A > 0$ and $\varepsilon > 0$

$$\sum_{|x| \leq An^H} \mathbb{E}(N_n(x)^p) n^{-p\delta} \varepsilon^{-p} \leq 2An^H C n^{p(1-H)} n^{-p\delta} \varepsilon^{-p},$$

and this quantity goes to 0 as $n \to \infty$ if we choose p large enough such that $H + p(1-H) - p\delta = H(1 - p/\beta) < 0$. At last, the term

$$\mathbb{P}(\sup_{0 \leq k \leq n} n^{-H} |[S_k]| > A)$$

converges to $\mathbb{P}(\sup_{0 \leq t \leq 1} |B_H(t)| > A)$ as $n \to \infty$, and this last term goes to zero as $A \to \infty$.

• First notice that we can suppose without restriction that $p \geq 1$ is an integer, because the bound for $p' \geq 1$ is a consequence of the case $p \geq p'$. The number of self-intersections up to time n is bounded from above by

$$V_n \leq \sum_{0 \leq i \leq j \leq n} 2 1_{\{[S_i] = [S_j]\}}.$$

Using Minkowski inequality,

$$\|V_n\|_p \leq 2 \sum_{i=0}^{n} \left\| \sum_{j=i}^{n} 1_{\{[S_i]=[S_j]\}} \right\|_p , \tag{3.6}$$

where $\|X\|_p = \mathbb{E}(|X|^p)^{1/p}$. For fixed i, the stationarity of the random walk's increments implies that the distribution of $\sum_{j=i}^{n} 1_{\{[S_i]=[S_j]\}}$ and $\sum_{j=0}^{n-i} 1_{\{[S_i]=0\}} = N_{n-i}(0)$ are equal. Since $N_{n-i}(0) \leq N_n(0)$, equation (3.6) yields

$$\mathbb{E}(V_n^p) \leq 2^p n^p \mathbb{E}\left(N_n(0)^p\right). \tag{3.7}$$

We now refer to lemma 4.4 in [14] which states that there is some $C > 0$ such that

$$\mathbb{E}\left(N_n(0)^p\right) \leq C n^{p(1-H)}. \tag{3.8}$$

Equations (3.7) and (3.8) together yield equation (3.4).

• We only have to notice that $R_n \leq 1 + 2\sup_{0 \leq k \leq n} |S_k|$ and hence it is enough to prove that $\sup_{0 \leq k \leq n} n^{-H}|S_k|$ is bounded in L^p for all $p \geq 1$. Let $S^n(t)_{t \in [01]}$ be the continuous process defined by

$$S^n(t) = n^{-H} S_{[nt]} + (nt - [nt]) n^{-H} (S_{[nt]+1} - S[nt]).$$

By equation (1.5), the sequence of process S^n converges weakly to B_H in $\mathcal{C}([0,1])$ furnished with the uniform norm $\|.\|_\infty$. Furthermore,

$$\sup_{0 \leq k \leq n} n^{-H} |S_k| = \|S^n\|_\infty.$$

Hence we need to show that $\|S^n\|_\infty$ is bounded in L^p for all $p \geq 1$. Using a concentration result (see e.g. [6, p. 60]), a sequence of Gaussian random variables which is bounded in probability is bounded in all L^p spaces. Since the sequence S_n converges in distribution to B_H, it is bounded in probability, and hence bounded in all L^p spaces. □

3.1.2. Convergence of functionals of local times

The following lemma is an analogous of Lemma 6 in [5] when the random walk in the domain of attraction of a stable Lévy motion is replaced by a random walk in the domain of attraction of a fractional Brownian motion. Note that the case of walks in a L^2 scenery was considered by Wang. Here we generalize Wang's result (Proposition 3.2 in [14]) to the case of a heavy-tailed scenery.

Lemma 3.2. *For all $(\theta_1, \cdots, \theta_k) \in \mathbb{R}^k$, $(t_1, \cdots, t_k) \in [0, +\infty)^k$, $\sigma > 0$, $\beta \in (0, 2]$, the distribution of*

$$X_n = n^{-\delta\beta} \sum_{x \in \mathbb{Z}} \left| \sum_{j=1}^{k} \theta_j N_{nt_j}(x) \right|^{\beta}$$

converges weakly as $n \to \infty$ to X defined by equation (2.1).
Furthermore, X_n is bounded in L^p for all $p \geq 1$.

Proof. Following Kesten and Spitzer [5] and Wang [14], we introduce for small $\tau > 0$ and large N,

$$U(\tau, N, n) = n^{-\beta\delta} \sum_{|x| \leq N\tau n^H} \left| \sum_{j=1}^{k} \theta_j N_{nt_j}(x) \right|^{\beta},$$

$$d(l, n) = n^{-1} \sum_{j=1}^{k} \theta_j \sum_{l\tau n^H \leq y < (l+1)\tau n^H} N_{nt_j}(y),$$

$$V(\tau, N, n) = \tau^{1-\beta} \sum_{|l| \leq N} |d(l, n)|^{\beta}.$$

Then,

(3.9)

$$\begin{aligned} X_n &- U(\tau, N, n) - V(\tau, N, n) \\ &= \sum_{|l| \leq N} \sum_{l\tau n^H \leq x < (l+1)\tau n^H} n^{-\delta\beta} \left\{ \left| \sum_{j=1}^{k} \theta_j N_{nt_j}(x) \right|^{\beta} - n^{\beta}[\tau n^H]^{-\beta} |d(l, n)|^{\beta} \right\} \\ &\quad + \sum_{|l| \leq M} (n^{\beta-\delta\beta}[\tau n^H]^{1-\beta} - \tau^{1-\beta}) |d(l, n)|^{\beta}. \end{aligned}$$

By Lemma 3.1 in [14] $d(l,n)$ converges in distribution to

$$\sum_{j=1}^{k} \theta_j \int_{l\tau}^{(l+1)\tau} L_{t_j}(x) \mathrm{d}x.$$

Since

$$n^{\beta-\delta\beta}[\tau n^H]^{1-\beta} - \tau^{1-\beta} \to 0,$$

the second sum over l in the right hand side of (3.9) tends to zero in probability as $n \to \infty$. We now show that the first sum over l in the right hand side of (3.9) is small in probability when τ is small. We use the following inequality, valid for any $a \geq 0$, $b \geq 0$

$$|a^\beta - b^\beta| \leq \begin{cases} |a-b|^\beta & \text{if} \quad \beta \leq 1 \\ \beta |a-b| (a^{\beta-1} + b^{\beta-1}) & \text{if} \quad \beta < 1, \end{cases}$$

to estimate the sum over x. In the case $\beta \leq 1$,

$$\begin{aligned} &\mathbb{E}\left| \sum_{|l|\leq N} \sum_{l\tau n^H \leq x < (l+1)\tau n^H} n^{-\delta\beta} \left\{ \left| \sum_{j=1}^{k} \theta_j N_{nt_j}(x) \right|^\beta - n^\beta [\tau n^H]^{-\beta} |d(l,n)|^\beta \right\} \right| \\ &\qquad \leq n^{-\delta\beta} \sum_{|l|\leq N} \sum_{l\tau n^H \leq x < (l+1)\tau n^H} \left[\mathbb{E} \left| \sum_{j=1}^{k} \theta_j N_{nt_j}(x) - n[\tau n^H]^{-1} d(l,n) \right|^2 \right]^{\beta/2} \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E} \left| \sum_{j=1}^{k} \theta_j N_{nt_j}(x) - n[\tau n^H]^{-1} d(l,n) \right|^2 \\ &\qquad\qquad \leq [\tau n^H]^{-1} \sum_{i=1}^{k} \theta_i^2 \sum_{j=1}^{k} \sum_{l\tau n^H \leq y < (l+1)\tau n^H} \mathbb{E}|N_{nt_j}(x) - N_{nt_j}(y)|^2. \end{aligned}$$

Now from Lemma 4.6 in [14], there exist C and $r > 0$ such that for large n, large N and small τ (with $A = N\tau$ large enough), and any $|x| \leq A$ and y such that $|x-y| \leq \tau n^H$, the following holds:

$$\mathbb{E}|N_{nt_j}(x) - N_{nt_j}(y)|^2 \leq CA\tau^r n^{2-2H}.$$

Combining these estimates,

$$\begin{aligned} &\mathbb{E}\left| \sum_{|l|\leq N} \sum_{l\tau n^H \leq x < (l+1)\tau n^H} n^{-\delta\beta} \left\{ \left| \sum_{j=1}^{k} \theta_j N_{nt_j}(x) \right|^\beta - n^\beta [\tau n^H]^{-\beta} |d(l,n)|^\beta \right\} \right| \\ &\leq n^{-\delta\beta}(2N+1)[n^H\tau] \left([n^H\tau]^{-1}[n^H\tau] CA\tau^r n^{2-2H} \right)^{\beta/2} \\ &\leq CN\tau^{1+r\beta/2}. \end{aligned}$$

This completes the estimate of (3.9) in the case $\beta \leq 1$.

In the case $\beta > 1$, we have

$$\mathbb{E}\left|\sum_{|l|\leq N}\sum_{l\tau n^H\leq x<(l+1)\tau n^H} n^{-\delta\beta}\left\{\left|\sum_{j=1}^k \theta_j N_{nt_j}(x)\right|^\beta - n^\beta[\tau n^H]^{-\beta}|d(l,n)|^\beta\right\}\right|$$

$$\leq n^{-\delta\beta}\sum_{|l|\leq N}\sum_{l\tau n^H\leq x<(l+1)\tau n^H}\beta\mathbb{E}\left[\left|\sum_{j=1}^k \theta_j N_{nt_j}(x) - n[\tau n^H]^{-1}d(l,n)\right|\right.$$

$$\left.\times\left(\left|\sum_{j=1}^k \theta_j N_{nt_j}(x)\right|^{\beta-1} + |n[\tau n^H]^{-1}d(l,n)|^{\beta-1}\right)\right]$$

$$\leq \beta n^{-\delta\beta}\left[\sum_{|l|\leq N}\sum_{l\tau n^H\leq x<(l+1)\tau n^H}\mathbb{E}\left|\sum_{j=1}^k \theta_j N_{nt_j}(x) - n[\tau n^H]^{-1}d(l,n)\right|^2\right]^{1/2}$$

$$\times\left[\sum_{|l|\leq N}\sum_{l\tau n^H\leq x<(l+1)\tau n^H}\mathbb{E}\left(\left|\sum_{j=1}^k \theta_j N_{nt_j}(x)\right|^{\beta-1} + |n[\tau n^H]^{-1}d(l,n)|^{\beta-1}\right)^2\right]^{1/2}.$$

With the same techniques as above, the first factor is shown to be bounded from above by

$$\beta n^{-\delta\beta}[(2N+1)\tau n^H CA\tau^r n^{2-2H}]^{1/2}.$$

In order to upper-bound the second factor, introduce $T = \sup\{t_j; j = 1...k\}$, and note that $|\sum_{j=1}^k \theta_j N_{nt_j}(x)| \leq CN_{nT}(x)$ and also

$$n[\tau n^H]^{-1}d(l,n) \leq C[\tau n^H]^{-1}\sum_{l\tau n^H\leq y<(l+1)\tau n^H} N_{nT}(y).$$

Using Hölder and Minkowski's inequalities, the second factor is bounded from above by

$$\left[\sum_{|l|\le N}\sum_{l\tau n^H\le x<(l+1)\tau n^H}\mathbb{E}\left(\left|\sum_{j=1}^k\theta_j N_{nt_j}(x)\right|^{\beta-1}+|n[\tau n^H]^{-1}d(l,n)|^{\beta-1}\right)^2\right]^{1/2}$$

$$\le C\left[\sum_{|l|\le N}\sum_{l\tau n^H\le x<(l+1)\tau n^H}\mathbb{E}\left(N_{nT}(x)\right)^{2\beta-2}\right]^{1/2}$$

$$+C\left[\sum_{|l|\le N}\sum_{l\tau n^H\le x<(l+1)\tau n^H}\mathbb{E}\left([\tau n^H]^{-1}\sum_{l\tau n^H\le y<(l+1)\tau n^H}N_{nT}(y)\right)^{2\beta-2}\right]^{1/2}$$

$$\le C\left[\sum_{|l|\le N}\sum_{l\tau n^H\le x<(l+1)\tau n^H}\left(\mathbb{E}N_{nT}(x)^2\right)^{\beta-1}\right]^{1/2}$$

$$+C\left[\sum_{|l|\le N}\sum_{l\tau n^H\le x<(l+1)\tau n^H}[\tau n^H]^{2-2\beta}\left(\sum_{l\tau n^H\le y<(l+1)\tau n^H}\left(\mathbb{E}N_{nT}(y)^2\right)^{1/2}\right)^{2\beta-2}\right]^{1/2}$$

$$\le C\left[(2N+1)\tau n^H n^{(2-2H)(\beta-1)}\right]^{1/2},$$

where the last line follows from Lemma 4.4 in [14] stating that there is some $C>0$ such that

$$\sup_{x\in\mathbb{Z}}\mathbb{E}(N_{nT}(x))^2\le Cn^{2-2H}.$$

Combining these estimates,

$$\mathbb{E}\left|\sum_{|l|\le N}\sum_{l\tau n^H\le x<(l+1)\tau n^H}n^{-\delta\beta}\left\{\left|\sum_{j=1}^k\theta_j N_{nt_j}(x)\right|^{\beta}-n^\beta[\tau n^H]^{-\beta}|d(l,n)|^\beta\right\}\right|$$
$$\le Cn^{-\delta\beta}[(2N+1)\tau n^H\tau^r n^{2-2H}]^{1/2}\left[(2N+1)\tau n^H n^{(2-2H)(\beta-1)}\right]^{1/2}$$
$$\le CN\tau^{1+r/2}.$$

Then the proof of Lemma 3.2 follows from equation (3.9) and from the above estimates as in [5] or [14]: the idea is to show that for large N, n and small τ, $X_n-V(\tau,n,N)\to 0$ in probability and that $V(\tau,n,N)\to X$ in distribution. We omit the details.

Next we prove that $(X_n)_{n\ge1}$ is bounded L^p bound. Let $T=\sup(t_1,\cdots,t_n)$ and $\Theta=\sum_{j=1}^k|\theta_j|$. The random variables $|X_n|$ is bounded above by

$$\Theta^\beta n^{-\delta\beta} \sum_{x\in\mathbb{Z}} N^\beta_{[nT]+1}(x).$$

In the case $\beta = 2$, this quantity is equal to $\Theta^2 n^{H-2} V_{[nT]+1}$, and in this case the L^p bound is a consequence of equation (3.4).
In the case $\beta < 2$, Hölder inequality yields

$$\sum_{x\in\mathbb{Z}} N^\beta_{[nT]+1}(x) \\ \leq \left(\sum_{x\in\mathbb{Z}} 1_{\{N_{[nT]+1}(x)\neq 0\}} \right)^{1-\frac{\beta}{2}} \left(\sum_{x\in\mathbb{Z}} N^2_{[nT]+1}(x) \right)^{\frac{\beta}{2}} = R^{1-\frac{\beta}{2}}_{[nT]+1} V^{\frac{\beta}{2}}_{[nT]+1}.$$

Hence, up to a multiplicative constant, the expectation $\mathbb{E}(|X_n|^p)$ is overestimated by

$$\mathbb{E}\left[\left(n^{-\delta\beta} \sum_{x\in\mathbb{Z}} N^\beta_{[nT]+1}(x) \right)^p \right] \leq \mathbb{E}\left[\left(n^{-H} R_{[nT]+1}\right)^{p(1-\frac{\beta}{2})} \left(n^{H-2} V_{[nT]+1}\right)^{p\frac{\beta}{2}} \right].$$

We now apply Cauchy-Schwartz inequality,

$$\mathbb{E}\left[\left(n^{-\delta\beta} \sum_{x\in\mathbb{Z}} N^\beta_{[nT]+1}(x) \right)^p \right] \\ \leq \mathbb{E}\left[\left(n^{-H} R_{[nT]+1}\right)^{p(2-\beta)} \right]^{\frac{1}{2}} \mathbb{E}\left[\left(n^{H-2} V_{[nT]+1}\right)^{p\beta} \right]^{\frac{1}{2}}.$$

Now the L^p bound follows from equation (3.4) and (3.5) together. □

3.2. Convergence of the finite-dimensional distributions

We study the asymptotic behaviour of the characteristic function of the marginals of G_n. Let λ be the characteristic function of the variables $\xi_k^{(i)}$ defined by

$$\lambda(u) = \mathbb{E}\left(\exp(iu\xi_1^{(1)})\right).$$

Since the random variables $\xi_k^{(i)}$ are in the domain of attraction of Z_β,

$$\lambda(u) = \bar\lambda(u) + o(|u|^\beta), \quad \text{as } u \to 0, \tag{3.10}$$

where $\bar\lambda$ is the characteristic function of Z_β given by equation (1.3).

Proposition 3.1. *The finite dimensional distributions of* $(G_n(t))_{t\geq 0}$ *converge weakly as* $n\to\infty$ *to those of* $(\Gamma(t))_{t\geq 0}$ *defined in equation* (1.9).

Proof. Let $(\theta_1, \cdots, \theta_k) \in \mathbb{R}^k$, $(t_1, \cdots, t_k) \in [0, +\infty)^k$. Computations as in [4] show that the characteristic function of $\Gamma_n(t)$ writes

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{k}\theta_j G_n(t_j)\right)\right] = \left(\mathbb{E}\left[\prod_{x\in\mathbb{Z}}\lambda\left(c_n^{-\frac{1}{\beta}}U_n(x)\right)\right]\right)^{c_n}, \tag{3.11}$$

where

$$U_n(x) = n^{-\delta}\sum_{j=1}^{k}\theta_j N_{nt_j}(x) \quad , \quad x \in \mathbb{Z}.$$

We show that the following asymptotic holds as $n \to \infty$:

$$\mathbb{E}\left[\prod_{x\in\mathbb{Z}}\lambda\left(c_n^{-\frac{1}{\beta}}U_n(x)\right)\right] = \mathbb{E}\left[\prod_{x\in\mathbb{Z}}\lambda\left(c_n^{-\frac{1}{\beta}}U_n(x)\right)\right] + o(c_n^{-1}). \tag{3.12}$$

To see this, note that

$$\begin{aligned}
&c_n\left|\prod_{x\in\mathbb{Z}}\lambda\left(c_n^{-\frac{1}{\beta}}U_n(x)\right) - \prod_{x\in\mathbb{Z}}\bar{\lambda}\left(c_n^{-\frac{1}{\beta}}U_n(x)\right)\right| \\
&\leq c_n\sum_{x\in\mathbb{Z}}\left|\lambda\left(c_n^{-\frac{1}{\beta}}U_n(x)\right) - \bar{\lambda}\left(c_n^{-\frac{1}{\beta}}U_n(x)\right)\right| \\
&\leq \tilde{g}(c_n^{-\frac{1}{\beta}}U_n)\sum_{x\in\mathbb{Z}}|U_n(x)|^{\beta},
\end{aligned} \tag{3.13}$$

with

$$U_n = \sup_{x\in\mathbb{Z}}|U_n(x)|,$$

and $\tilde{g}$ the bounded continuous vanishing at zero function defined by

$$\tilde{g}(u) = \sup_{|v|\leq u}|v|^{-\beta}\left|\lambda(v) - \bar{\lambda}(v)\right|, \quad v \neq 0.$$

(The properties of $\tilde{g}$ follow from equation (3.10).) From Lemma 3.1, U_n converge in probability to 0 as $n \to \infty$. Since $\tilde{g}$ is bounded continuous and vanishes at 0, $\tilde{g}(c_n^{-\frac{1}{\beta}}U_n)$ converges also in probability to 0 and is bounded in L^∞. From Lemma 3.2, $\sum_{x\in\mathbb{Z}}|U_n(x)|^\beta$ converges in distribution and is bounded in L^p. As a consequence, the right hand side of (3.13) converges to zero in probability and is bounded in L^p, and hence its expectation has limit 0. This proves equation (3.12).

We now prove the following estimation

$$\mathbb{E}\left[\prod_{x\in\mathbb{Z}}\bar{\lambda}\left(c_n^{-\frac{1}{\beta}}U_n(x)\right)\right] = 1 - c_n^{-1}\sigma^\beta\mathbb{E}\left[X\right] + o(c_n^{-1}), \tag{3.14}$$

where X is defined in Lemma 3.2. To see this, recall the definition of the random variable X_n from Lemma 3.2 and of the characteristic function $\bar{\lambda}$ from equation (1.3). With these notations, equation (3.14) is equivalent to

$$\lim_{n\to+\infty} \mathbb{E}\left(f_n(X_n)\right) = \sigma^\beta \mathbb{E}(X),$$

where f_n is the function defined on $\mathbb{C}$ by

$$f_n(x) = c_n\left(1 - \exp(-c_n^{-1}\sigma^\beta x)\right).$$

It is easy to verify that the sequence of functions f_n converges uniformly on compact sets to the function $x \mapsto \sigma^\beta x$. Furthermore Lemma 3.2 states that the sequence $(X_n)_{n\geq 1}$ converges in distribution to X when $n \to \infty$. Using the diagonal mapping Theorem (Theorem 5.5 of [2]), we prove the weak convergence of the sequence of random variables $f_n(X_n)$ to $\sigma^\beta X$. Furthermore, using Lemma 3.2, $|f_n(X_n)| \leq |X_n|$ is bounded in L^p for any $p \geq 1$. Hence $\mathbb{E}(f_n(X_n))$ has limit $\sigma^\beta \mathbb{E}(X)$ and equation (3.14) is proved.

Finally, combining equations (3.11), (3.12) and (3.14) we prove easily that

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^k \theta_j G_n(t_j)\right)\right] = \left(1 - c_n^{-1}\ \sigma^\beta \mathbb{E}(X) + o(c_n^{-1})\right)^{c_n} \underset{n\to\infty}{\longrightarrow} \exp(-\sigma^\beta \mathbb{E}(X))$$

and Proposition 3.1 is proved. □

3.3. Tightness

Proposition 3.2. *The family of processes $(G_n(t))_{t\geq 0}$ is tight in $C([0,\infty))$.*

Proof. As in the continuous case, we prove the tightness using truncations in order to deal with finite variance processes. We decompose the scenery $(\xi_x^{(i)})_{x\in\mathbb{Z}, i\geq 1}$ into two parts

$$\xi_x^{(i)} = \bar{\xi}_{a,x}^{(i)} + \hat{\xi}_{a,x}^{(i)},$$

where $(\bar{\xi}_{a,x}^{(i)})$ denote the i-th truncated scenery defined by

$$\bar{\xi}_{a,x}^{(i)} = \xi_x^{(i)} 1_{\{|\xi_x^{(i)}|\leq a\}},$$

and $\hat{\xi}_{a,x}^{(i)}$ the remainder scenery

$$\hat{\xi}_{a,x}^{(i)} = \xi_x^{(i)} 1_{\{|\xi_x^{(i)}|> a\}}.$$

We recall the following estimates from Lemma 3.3 in [4]: there exists some $C > 0$ such that

$$|\mathbb{E}\left(\bar{\xi}_{a,x}^{(i)}\right)| \leq Ca^{1-\beta}\ ,\ \mathbb{E}\left(|\bar{\xi}_{a,x}^{(i)}|^2\right) \leq Ca^{2-\beta},\ \ \mathbb{P}\left(\hat{\xi}_{a,x}^{(i)} \neq 0\right) \leq Ca^{-\beta}. \tag{3.15}$$

For $a > 0$, we use truncations with $a_n = an^{\frac{H}{\beta}} c_n^{\frac{1}{\beta}}$ and write

$$G_n(t) = \bar{\Gamma}_{n,a}(t) + \hat{\Gamma}_{n,a}(t), \tag{3.16}$$

where

$$\bar{\Gamma}_{n,a}(t) = n^{-\delta} c_n^{-\frac{1}{\beta}} \sum_{i=1}^{c_n} \sum_{x \in \mathbb{Z}} N_{nt}^{(i)}(x) \bar{\xi}_{a_n,x}^{(i)},$$
$$\hat{\Gamma}_{n,a}(t) = n^{-\delta} c_n^{-\frac{1}{\beta}} \sum_{i=1}^{c_n} \sum_{x \in \mathbb{Z}} N_{nt}^{(i)}(x) \hat{\xi}_{a_n,x}^{(i)}.$$

Now, with the same techniques as in the proof of Proposition 3.2 in [4], we compute:

$$\begin{aligned}
\mathbb{P}\left(\sup_{t \in [0,T]} |\hat{\Gamma}_{n,a}(t)| = 0\right) &\geq \left(\mathbb{E}\left[\left(\mathbb{P}(\hat{\xi}_{a_n,0}^{(1)} = 0)\right)^{R_{[nT]+1}}\right]\right)^{c_n} \\
&\geq \left(\mathbb{E}\left[\left(1 - Ca_n^{-\beta}\right)^{R_{[nT]+1}}\right]\right)^{c_n} \\
&\geq \left(1 + \log(1 - Ca_n^{-\beta}) \mathbb{E}(R_{[nT]+1})\right)^{c_n}.
\end{aligned}$$

Using the asymptotic for a_n and Lemma 3.1 to estimate the range, the above inequality implies

$$\lim_{a \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(\sup_{t \in [0,T]} |\hat{\Gamma}_{n,a}(t)| > 0\right) = 0. \tag{3.17}$$

On the other hand, the variance of the truncated process $\Gamma_{n,a}$ is overestimated by

$$\begin{aligned}
&\mathbb{E}\left[\left| \ \bar{\Gamma}_{n,a}(t_2) - \bar{\Gamma}_{n,a}(t_1) \ \right|^2\right] \\
&\leq n^{-2\delta} c_n^{-\frac{2}{\beta}} c_n(c_n - 1) \left[\sum_{x \in \mathbb{Z}} \mathbb{E}(N_{nt_2}^{(1)}(x) - N_{nt_1}^{(1)}(x))\right]^2 \left[\mathbb{E}|\bar{\xi}_{a_n,0}^{(1)}|\right]^2 \\
&+ n^{-2\delta} c_n^{-\frac{2}{\beta}} c_n \sum_{x \neq y \in \mathbb{Z}} \mathbb{E}\left[(N_{nt_2}^{(1)}(x) - N_{nt_1}^{(1)}(x))(N_{nt_2}^{(1)}(y) - N_{nt_1}^{(1)}(y))\right] \left[\mathbb{E}|\bar{\xi}_{a_n,0}^{(1)}||\right]^2 \\
&+ n^{-2\delta} c_n^{-\frac{2}{\beta}} c_n \sum_{x \in \mathbb{Z}} \mathbb{E}\left[(N_{nt_2}^{(1)}(x) - N_{nt_1}^{(1)}(x))^2\right] \mathbb{E}\left[|\bar{\xi}_{a_n,0}^{(1)}|^2\right].
\end{aligned}$$

Using equation (3.15) and the following estimations,

$$\mathbb{E}\left[\sum_{x\in\mathbb{Z}}(N^{(1)}_{nt_2}(x) - N^{(1)}_{nt_1}(x))\right] = n(t_2 - t_1),$$

$$\mathbb{E}\left[\sum_{x\neq y\in\mathbb{Z}}(N^{(1)}_{nt_2}(x) - N^{(1)}_{nt_1}(x))(N^{(1)}_{nt_2}(y) - N^{(1)}_{nt_1}(y))\right]$$

$$= n^2(t_2 - t_1)^2 - \mathbb{E}\left[\sum_{x\in\mathbb{Z}}(N^{(1)}_{nt_2}(x) - N^{(1)}_{nt_1}(x))^2\right],$$

$$\mathbb{E}\left[\sum_{x\in\mathbb{Z}}(N^{(i)}_{nt_2}(x) - N^{(i)}_{nt_1}(x))^2\right] \leq \mathbb{E}(V_{[nt_2]-[nt_1]+1}) \leq C([nt_2] - [nt_1] + 1)^{2-H},$$

we prove that there exists some C such that if $|t_2 - t_1| \geq \frac{1}{n}$, then

$$\mathbb{E}\left[\left|\ \bar{\Gamma}_{n,a}(t_2) - \bar{\Gamma}_{n,a}(t_1)\ \right|^2\right] \leq C|t_2 - t_1|^{2-H}.$$

In the case $|t_2 - t_1| \leq 1/n$, we can see that

$$\mathbb{E}\left[\sum_{x\in\mathbb{Z}}(N^{(1)}_{nt_2}(x) - N^{(1)}_{nt_1}(x))^2\right] \leq 2(nt_2 - nt_1)^2,$$

since in the sum, at most two terms are not zero and those terms are bounded by $(nt_2 - nt_1)^2$. Using theorem 12.3 in Billingsley, these estimates prove the tightness of the family of processes $\left(\bar{\Gamma}_{n,a}(t)\right)_{t\geq 0}$. This together with equations (3.17) and (3.16) implies the tightness of the sequence G_n, and hence Proposition 3.2. □

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