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# A remark on the theory of the base*

By

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In connection with the *theorem of the base*[1] we can propose the following problem: Let $V^n$ be a normal variety in a projective space. *Can we find a positive integer m depending on V such that every subvariety $C^{n-1}$ of $V^n$ is linearly equivalent to a "reducible variety", when the degree of C is greatar than m?* This seems to be very plausible, but actually it is not true. In fact we shall show later that on an Abelian variety $A^n$ we can find a subvariety $C^{n-1}$ of degree greater than any preassigned positive integer $m$ such that every positive divisor $X$ on $A$, which is "numerically" equivalent to $C$,[2] is irreducible at least when $A$ is not "trivial" in a certain sense. We can give another expression of this statement: We know in general[3] that the divisor class group modulo numerical equivalence of any normal variety is a free Abelian group with a finite set of generators. Let us say that a divisor class is *positive* if it contains a positive divisor. They form a semigroup in the group of all divisor classes. If we remember a result of Chow-van der Waerden, our previous statement is shown to be a consequence of the *non-existence of a finite set of generators of this semi-group.*

1. *Abelian Varieties.*—We take a square *real* matrix $J$ satisfying $J^2=-1_{2n}$. Here $1_{2n}$ is the unit matrix of degree $2n$. Such a matrix converts a real vector space $\boldsymbol{R}^{2n}$ into a complex vector space $\boldsymbol{C}^n$ by the definition $(\alpha+i\beta)\cdot x=\alpha x+\beta Jx$. Here $\alpha$, $\beta$ are real numbers and $x$ is an element of $\boldsymbol{R}^{2n}$. If we consider the factor group of this $\boldsymbol{C}^n$ by its discrete subgroup of "integral points", we get a complex torus $A^n$. We know[4] that $A^n$ is an *Abelian variety* if and only if there exists a skew-symmetric *integral* matrix $E$ of degree $2n$ such that $EJ$ is symmetric and positive definite. We say that $A$ is a *simple* Abelian variety, if $A$ does not contain any Abelian subvariety except the trivial ones. We conclude from an

elemetary part of the theory of complex tori that $A$ is simple if and only if a skew-symmetric integral matrix $E$ such that $EJ$ is symmetric can not be positive semi-definite without being positive definite.

Now we can attach to every divisor $X^{n-1}$ of $A^n$ a skew-symmetric integral matrix $E(X)$ in the following way: We denote by $e_i$ a unit vector in $R^{2n}$ whose $i$-th component is one. It is clear that the image in $A$ of a two-cell in $R^{2n}$ with an oriented boundary $x_0 \to x_0+e_i \to x_0+e_i+e_j \to x_0+e_j \to x_0$ defines a two cycle $\Gamma_{ij}$ of $A$, whose homology class is independent of the point $x_0$ of $R^{2n}$. Moreover $n(2n-1)$ cycles $\Gamma_{ij}$ for $i<j$ form a fundamental base of two cycles. Therefore if we define $n(2n-1)$ integrs $e_{ij}(X)$ by intersection numbers $I(X, \Gamma_{ij})$, they determine a skew-symmetric integral matrix $E(X)$ of degree $2n$ in an obvious manner. It is clear that the mapping $X \to E(X)$ is a homomorphism of the group of all divisors on $A$ into the module of skew-symmetric integral matrices of degree $2n$ such that the kernel is the group of divisors which are homologous to zero over rationals. On the other hand we know in general[5] that this group is the same as the one which is defined by numerical equivalence. Therefore the rank $\rho$ of the module of $E(X)$ is equal to the *Picard number* of $A$. We can also determine explicitly the image of the above mapping in terms of our matrix $J$. A skew-symmetric integral matrix $E$ lies in the image if and only if $EJ$ is symmetric. Moreover $E$ corresponds to a positive class if and only if $EJ$ is positive semi-definite. These two assertions are substantially equivalent to the existence theorem of thêta functions.[6]

In the following we shall assume that $A$ is simple.

If we consider a real vector space $R^\rho$, its discrete subgroup of integral points can be identified with the module of matrices $E(X)$. On the other hand the totality $\Delta$ of skew-symmetric *real* matrices $E$ such that $EJ$ is positive definite is an open subset of of $R^\rho$ satisfying the following properties: (i) If $E$ is in $\Delta$, and if $\alpha$ is a positive real number, $\alpha E$ is also in $\Delta$. (ii) If $E_1$ and $E_2$ are in $\Delta$, then $E_1+E_2$ is in $\Delta$. A point set in $R^\rho$ satisfying the property (i) is called a *cone*. If it also satisfies (ii), it is called a *convex cone*. Thus our $\Delta$ is an open convex cone in $R^\rho$. Moreover the semi-group of integral points in $\Delta$ is the same as the semi-group of positive divisor classes. We shall now state the following elementary lemma:

LEMMA. *Any non-empty open cone in $\boldsymbol{R}^{\rho}$ contains at least one integral point.*

It is now a simple matter to prove the following:

THEOREM 1. *If $A^n$ is a simple Abelian variety whose Picard number $\rho$ is at least equal to two, the semi-group of positive divisor classes modulo numerical equivalence has no finite set of generators.*

*Proof*: Assume semi-group has a finite set of generators $E_1, E_2, \cdots, E_r$. Consider the smallest convex cone $\Delta'$ in $\boldsymbol{R}^{\rho}$ containing them. Since $E_1, E_2, \cdots, E_r$ are points of $\Delta$, we see that $\Delta'$ is contained in $\Delta$. Since the dimension of the boundary of $\Delta$ is equal to $\rho-1$, if $\rho$ is at least equal to two, $\Delta-\Delta'$ is a non-empty open cone in $\boldsymbol{R}^{\rho}$. Hence $\Delta-\Delta'$ contains at least one integral point according to the previous lemma. However this contradicts to the difinition of $\Delta'$.

We can make this theorem slightly better by using the following general theorem:

THEOREM 2. *If $A$ and $B$ are isogenous Abelian varieties such that the semi-group of one of them has a finite set of generators, then so is the other.*

*Proof*: Since $A$ and $B$ are isogenous, there are two homomorphisms $\alpha$ and $\beta$ from $A$ onto $B$ and from $B$ onto $A$ such that $\beta\alpha=d$, i.e., $d$-times the identity automorphism of $A$. Since the situation is symmetric, we may assume that the semi-group of $B$ has a finite set of generators. Let $Y_1, \cdots, Y_N$ be positive divisors on $B$ which represent these generators, and define $X_i$ by $\alpha^{-1}(Y_i)$ for $i=1, \cdots, N$. If $X$ is any positive divisor on $A$, we can find $N$ non-negative integers $c_i$ such that $\beta^{-1}(X)$ is numerically equivalent to $\sum_{i=1}^{N} c_i Y_i$. Therefore $d^2X$ is numerically equivalent to $\sum_{i=1}^{N} c_i X_i$ with the same $c_i$. Now let $X_j^*$ be a positive divisor on $A$ such that $d^2X_j^*$ is numerically equivalent to a certain integer combination $\sum_{i=1}^{N} c_i' X_i$ of $X_i$ satisfying the conditions $d^2>c'\geq 0$ for $i=1, \cdots, N$. If we apply the mapping $E(\ )$ to $X_j^*$ and $X_i$, we conclude that the number of inequivalent $X_j^*$ is at most equal to $d^{2N}$. Since the $c_i$ are non-negative integers, we can write them as $c_i=c_i'+d^2c_i''$, $d^2>c_i'\geq 0$, $c_i''\geq 0$ for $i=1, \cdots, N$. Then by applying the mapping $E(\ )$ again to $X$ and $X_i$, we conclude that $X-\sum_{i=1}^{N} c_i'' X_i$ is numerically equivalent to one of the $X_j^*$. In other words the classes of $X_i$ and $X_j^*$ form a set of generators of the semi-group.

Now if the semi-group of an Abelian variety has no finite set of generators, any product of this variety with another Abelian variety inherits the same property. Therefore the above theorems imply the following corollary:

COROLLARY. *If an Abelian variety $A^n$ contains at least one simple Abelian variety with $\rho \geq 2$, the semi-group of A has no finite set of generators.*

Finally we add also a simple remark about the " equation " of the convex cone $\Delta$. Let $E$ be a point of $\boldsymbol{R}^\rho$ with coordinates $x_1, x_2, \cdots, x_\rho$. Then we can find a homogeneous polynomial $P(X)$ of degree $n$ such that $\det(E) = P(x)^2$. It is clear that $\Delta$ is one of cones into which $\boldsymbol{R}^\rho$ is divided by a real hypersurface with the equation $P(X)=0$. This polynomial $P(X)$ has also the following meaning.[7] If our $E$ corresponds to a divisor $X^{n-1}$ of $A^n$, the $n$-fold intersection number $I(X, X, \cdots, X)$ is equal to $n! P(x)$.

2. *A General Theorem.*—Let $V^n$ be a normal variety in a projective space. We say that $V$ has the *property* $(F)$, if the semi-group of positive divisor classes has a finite set of generators. We say also that $V$ has the *property* $(Z)$, if we can find a positive integer $m$ such that every subvariety $C^{n-1}$ of $V^n$ of degree greater than $m$ is numerically equivalent to a reducible variety. It is clear that the property $(F)$ implies the property $(Z)$. We shall prove the converse. Assume that $V$ has the property $(Z)$ but not the property $(F)$. The set of positive divisors of $V$ of degree not greater than $m$ forms a finite set of maximal algebraic families by a result of Chow-van der Waerden.[8] Since the numerical equivalence is " broader " than the algebraic equivalence, the set of classes corresponding to such divisors is finite. Therefore this set can not generate the whole semi-group, which brings a contradiction. It would be interesting to investigate whether the property $(F)$ is birationally invariant or not.

3. *Examples.*—First of all our theorem 1 has no meaning unless we can prove the existence of a simple Abelian variety $A^n$ whose Picard number $\rho$ is at least equal to two. We can construct many such varieties and the following may be one of the simplest: Consider a hyperelliptic curve with an equation $y^2 = 1 - x^5$. The ring of endomorphisms of its Jacobian variety $J^2$ is the " principal order " of the cyclotomic field of the fifth root of unity. Therefore $J^2$ is simple and $\rho = 2$.

In the next place we take as $V^2$ a quadratic transform of a projective plane. Every divisor $X$ on $V$ is numerically equivalent to a divisor of the form $aD+bS$. Here $D$ is the transform of a straight line on the original plane not passing through the center of the quadratic transformation and $S$ is the image of the center. It follows from the theorem of Riemann-Roch for surfaces that $D-S$ is linearly equivalent to a positive divisor $C$. Moreover we can show that every divisor $X$ on $V$ is numerically equivalent to a divisor of the form $aC+bS$. Moreover the class of $X$ is positive if and only if both $a$ and $b$ are non-negative and $a+b>0$. It must be remarked that the classes of $D$ and $S$ do not form a set of generators of the semi-group. We note also that the one half of the domain $I(X, X)>0$, which is restricted by deg $(X)>0$, is properly contained in the angular domain of our semi-group. Such a case does not happen, if $V^2$ is an Abelian variety.

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NOTES

1. O. Zariski, *Algebraic Surfaces* (New York: Chelsea, 1948), pp. 91-95. A. Néron, " Problèmes arithmétiques et géométriques rattaches á la notion de rang d'une courbe algébrique dans un corps," *Bull. Soc. Math. de France*, 80, 101-166, 1952.
2. Two divisors $X$ and $Y$ on a variety are called numerically equivalent, if their integral multiples $mX$ and $mY$ are algebraically equivalent for a certain $m\neq 0$.
3. Néron, *op. cit.*
4. This theorem is usually ascribed to Lefschetz.
5. J. Igusa, " On the Picard varieties attached to algebraic varieties," *Amer. J. of Math.*, 74, 1-22, 1952.
6. Cf. A. Weil, *Théorèmes fondamentaux de la théorie des fonctions thêta* (" Séminaire Bourbaki " [Paris: Mimeographed, 1949]).
7. S. Lefschetz, *L'analysis situs et la géométrie algébrique* (Paris: Borel series, 1924), pp. 114-116.
8. W. L. Chow and B. L. van der Waerden, " Ueber zugeordnete Formen und algebraische Systeme von algebraischen Mannigfaltigkeiten," *Math. Ann.*, 113, 692-704, 1937.