MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXIX, Mathematics No. 3, 1955.

Note on the solutions of a system of differential equations

By

Taro YOSHIZAWA

(Received October 20, 1954)

The system of ordinary differential equations has long been studied and the properties of its solutions have been made clear gradually. Several authors have given remarkable results concerning the boundary value problem, the existence of a periodic solution, the boundedness of solutions and those for a differential equation of the second order. Recently we have also obtained some results concerning the boundedness of solutions, the existence of a periodic solution and the stability ([15], [16], [17], [18], [19]).

Here in case of the differential equation having discontinuities with regard to the independent variable and the like, we will consider a system of differential equations (cf. [2], [5]) and a differential equation of the second order (cf. [12], [21]) of Carathéodory's type. In this paper the integral is *of Lebesgue sense.* In the first half we will discuss the boundary value problem of a differential equation of the second order and in the second half we will discuss the boundedness and the others of solutions of a system of differential equations.

1. Consider a system of ordinary differential equations,

$$\frac{dy}{dx}=F(x, y), \tag{1}$$

where y denotes an n-dimensional vector and $F(x, y)$ is a given vector field, finite and defined in the domain

$$\varDelta: \quad a \leqq x \leqq b, \quad -\infty < y_i < +\infty \quad (i=1, 2, \cdots, n).$$

Moreover we suppose that $F(x, y)$ is *measurable* with regard to x for every fixed y and is a *continuous* function of y for every fixed

x. And we suppose that there exists a summable function $M(x)$ for $a \leqq x \leqq b$ such as

$$|F(x, y)| \leqq M(x).$$

The definition of the solution of this differential equation is the same with that given in [2] and [5].

Then by Carathéodory's existence theorem ([2], [5]) we see that there exists at least a solution which issues to the right from a point (a, y_0) (y_0 being given arbitrarily). In this case every solution is continuable to the whole interval $a \leqq x \leqq b$. The continuation of a solution may be interpreted in several ways; but here we consider it as a part of a solution defined beforehand in the whole interval $a \leqq x \leqq b$. Now a solution $y=y(x)$ defined for $a \leqq x < X$ is continuable up to $x=X$. Since the equation (1) has a solution going to the right from $(X, y(X))$ and defined in $X \leqq x \leqq b$, this solution, combined with the above solution $y=y(x)$, defines a solution $y=Y(x)$ in $a \leqq x \leqq b$. Remark that $y=y(x)$ is therefore a part of $y=Y(x)$. Hence every solution is continuable to the whole interval.

Let E be the set of all the points lying on all the solution curves for $a \leqq x \leqq b$, starting from (a, y_0), then E is a bounded closed set; it is clear since we have

$$y(x) = y_0 + \int_a^x F(x, y(x))\,dx$$

and $|F(x, y)| \leqq M(x)$. Therefore we can verify the following lemma.

Lemma 1. *Given an arbitrary positive number ε, if for a suitable positive number δ, the distance of a point $(a, \bar{y}_0)$ to (a, y_0) be smaller than δ, then the distances of all the points $(x, \varphi(x))$ $(a \leqq x \leqq b)$ on a solution curve $y=\varphi(x)$ of* (1) *issuing to the right from $(a, \bar{y}_0)$ to a solution curve lying in E are smaller than ε.*

Proof. Suppose that for a given $\varepsilon > 0$, there is no such δ. Let $(a, \bar{y}_\nu)$ $(\nu=1, 2, \cdots)$ be a sequence of points tending to (a, y_0) for $\nu \to \infty$ and let $y=y_\nu(x)$ $(\nu=1, 2, \cdots)$ be the solutions of (1) issuing to the right from $(a, \bar{y}_\nu)$ respectively. Then how we choose a solution curve lying in E, it would not happen that the distances of all the points on the arcs of the former to the latter are smaller than ε. Then since we have

$$y_\nu(x) = \bar{y}_\nu + \int_a^x F(x, y_\nu(x))\,dx \tag{2}$$

and $|F(x, y)| \leqq M(x)$, the sequence of functions $\{y_\nu(x)\}$ is uniformly bounded and equicontinuous. Hence we can select an uniformly convergent sequence for $a \leqq x \leqq b$. Now we denote its suffix by ν again and let $y(x)$ be its limit function. Then since we have

$$y(x) = y_0 + \int_a^x F(x, y(x))\, dx$$

for $\nu \to \infty$ in (2), $y = y(x)$ is a solution of (1) passing through (a, y_0). Therefore the point $(x, y(x))$ should belong to E. But if ν be sufficiently great, the distance between $y = y_\nu(x)$ and $y = y(x)$ is smaller than ε. This contradicts that the distance of $y = y_\nu(x)$ to E is greater than ε. Hence if we choose a δ suitably, the distance of $(x, \varphi(x))$ to a solution curve lying in E must be smaller than ε.

By this lemma and the *generalized* Kneser's theorem (see p. 21 in [4]), we obtain the following lemma in the same way as in the case where $F(x, y)$ is continuous with regard to (x, y).

Lemma 2. *Given a set of points A on the hyperplane $x = a$, consider all the solution curves of* (1), *going to the right from each point of A. Let B be their section cut by the hyperplane $x = b$. If A be connected, then B is so also.*

2. Now we consider a differential equation of the second order,

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right), \tag{3}$$

where $f(x, y, y')$, defined in the domain

$$D: \quad a \leqq x \leqq b, \quad -\infty < y < +\infty, \quad -\infty < y' < +\infty,$$

is finite and measurable with regard to x and continuous with regard to (y, y'). We suppose there is a summable function $M(x)$ such that, in D,

$$|f(x, y, y')| \leqq M(x).$$

The existence of a solution $y = y(x)$ of (3) such that $y(a) = A$ and $y(b) = B$, A and B being arbitrary, has been discussed by Scorza Dragoni, Zwirner and others. Zwirner [20] has proved its existence by the successive approximation; but here we will show it briefly by the elementary Okamura's method (§ 23 in [9]) according to the facts which we mentioned in § 1.

Instead of (3), consider a system

$$\frac{dy}{dx}=y', \qquad \frac{dy'}{dx}=f(x, y, y'). \tag{4}$$

For a solution $y=y(x)$ of (4) issuing to the right from $x=a$, it holds that

$$|y'(x)|\leqq|y'(a)|+\int_a^b M(x)\,dx,$$

as long as it is continuable. Hence we have $|y'(x)|\leqq L+\int_a^b M(x)dx$ provided $y'(a)$ is bounded by L. For such solutions, we consider instead of (4)

$$\frac{dy}{dx}=g(y'), \ \frac{dy'}{dx}=f(x, y, y'),$$

where $g(y')$ is a continuous and bounded function in $-\infty<y'<+\infty$ and $g(y')=y'$ if $|y'|\leqq L+\int_a^b M(x)\,dx$. Here L may be arbitrarily great, but it is a fixed positive constant. Then by § 1, every solution $y=y(x)$ issuing from $x=a$ is continuable to $x=b$ and by Lemma 2, the set of $y(b)$ of all solutions satisfying $y(a)=A$ and $l\leqq y'(a)\leqq m$ becomes an interval, where l and m are two numbers such as $l<m$. On the other hand, since we have

$$y(b)=y(a)+y'(a)(b-a)+\int_a^b\left[\int_a^x f(t, y(t), y'(t))\,dt\right]dx \tag{5}$$

and $|f(x, y, y')|\leqq M(x)$, if we choose l and m suitably, we can have $y(b)<B$ for the solution $y=y(x)$ such as $y'(a)=l$ and $y(b)>B$ for the solution $y=y(x)$ such as $y'(a)=m$ respectively. Therefore we can prove the existence of a solution satisfying $y(b)=B$.

3. In the case where the right-hand side $f(x, y, y')$ of the differential equation (3) is continuous with regard to (x, y, y'), the boundary value problem has been discussed by Tonelli, Nagumo [7], [8], Okamura (§ 23 in [9]), Scorza Dragoni [11] and the others and very remarkable results have been obtained. Nagumo and Okamura have limited the region of y by utilizing two functions such that $\underline{\omega}''(x)\geqq f(x, \underline{\omega}(x), \underline{\omega}'(x))$ and $\bar{\omega}''(x)\leqq f(x, \bar{\omega}(x), \bar{\omega}'(x))$. On the other hand for the boundary value problem in the case where the right-hand side of (3) is measurable with regard to x and continuous with regard to (y, y'), Scorza Dragoni and Zwirner have settled some hypotheses as the generalization of this idea (see [12], [21]). Now we will deduce the existence of a solution in the boundary value problem, by the extension of the region of y by Scorza Dragoni's idea and by reducing to such a solution stated in § 2.

For this purpose, we give the *definition.* We say that a function $\varphi(x, y)$ (of course, y is an n-dimensional vector) is *absolutely continuous on x uniformly at a point* (x_0, y_0) when for a certain positive number ρ and for any given positive number ε, we can determine a positive number δ such that we have

$$\sum_{k=1}^{m}|\varphi(x_k', y_k)-\varphi(x_k, y_k)|<\varepsilon,$$

provided

$$\sum_{k=1}^{m}|x_k'-x_k|<\delta \quad [x_0-\rho\leqq x_1\leqq x_1'\leqq x_2\leqq x_2'\leqq\cdots\leqq x_m\leqq x_m'\leqq x_0+\rho],$$

where y_k is an arbitrary value satisfying $|y_k-y_0|\leqq\rho$ and δ depends only on ε and is independent of the choise of x_k, x_k', y_k and of m. We will say briefly the function under this definition has *the property a.c.u.*. If such a function $\varphi(x, y)$ satisfies locally the Lipschitz condition with regard to y and $y(x)$ be an absolutely continuous function, then $\varphi(x, y(x))$ is also an absolutely continuous function in a neighborhood of x. For at first we have

$$\begin{aligned}
&\sum_{k=1}^{m}|\varphi(x_k', y(x_k'))-\varphi(x_k, y(x_k))|\\
&\leqq\sum_{k=1}^{m}|\varphi(x_k', y(x_k'))-\varphi(x_k, y(x_k'))|+\sum_{k=1}^{m}|\varphi(x_k, y(x_k'))-\varphi(x_k, y(x_k))|\\
&\leqq\sum_{k=1}^{m}|\varphi(x_k', y(x_k'))-\varphi(x_k, y(x_k'))|+L\sum_{k=1}^{m}|y(x_k')-y(x_k)|\,;
\end{aligned}$$

then since $\varphi(x, y)$ has the property a.c.u. and $y(x)$ is absolutely continuous, we have for $\sum_{k=1}^{m}|x_k'-x_k|<\delta$

$$<\varepsilon+L\varepsilon=(1+L)\varepsilon\,;$$

i.e., $\varphi(x, y(x))$ is absolutely continuous.

For an example, take the function $\varphi(x, y)$ considered in the necessary and sufficient condition of the uniqueness (see p. 231 in [5]). It satisfies

$$|\varphi(x, y)-\varphi(\bar{x}, \bar{y})|\leqq L|y-\bar{y}|+\left|\int_x^{\bar{x}}M(x)\,dx\right|\,;$$

this function fulfils the above stated conditions. Hence if $y(x)$ is an absolutely continuous function, $\varphi(x, y(x))$ is so also.

Moreover we explain here a condition which will be used later. Now consider a system of differential equations defined in a suitable domain

$$\frac{dy}{dx}=F(x, y).$$

Let a function $\varphi(x, y)$ be continuous with regard to (x, y), have the property a.c.u. and satisfy locally the Lipschitz condition with regard to y. If it satisfies almost everywhere with regard to x (*from now on we denote it briefly by a. e.*)

$$\varliminf_{t\to 0}\frac{1}{t}\{\varphi(x+t,\ y+tF(x, y))-\varphi(x, y)\}\geqq 0, \tag{6}$$

then $\varphi(x, y(x))$ is absolutely continuous and we have

$$\lim_{t\to 0}\frac{1}{t}\{\varphi(x+t,\ y(x+t))-\varphi(x, y(x))\}\geqq 0 \quad a.e.,$$

where $y=y(x)$ is a solution of the system. Therefore $\varphi(x, y)$ is a non-decreasing function of x along this solution. If (6) be replaced by $\varlimsup_{t\to 0}\frac{1}{t}\{\varphi(x+t, y+tF(x, y))-\varphi(x, y)\}\leqq 0$ *a.e.*, $\varphi(x, y)$ is non-increasing. Of course, if $F(x, y)$ is continuous with regard to (x, y), $\varphi(x, y)$ requires no such additional condition (see p. 232 Remark in [5]).

4. Now we consider the differential equation of the second order

$$\frac{d^2y}{dx^2}=f\left(x, y, \frac{dy}{dx}\right) \tag{7}$$

and the domain

$$\mathfrak{D}:\quad a\leqq x\leqq b,\quad \underline{\omega}(x)\leqq y\leqq \bar{\omega}(x).$$

We assume that $f(x, y, y')$ is *measurable* with regard to x and *continuous* with regard to (y, y') in the domain

$$\mathfrak{D}^*:\quad (x, y)\in\mathfrak{D},\quad -\infty<y'<+\infty$$

and that we have

$$|f(x, y, y')|\leqq M(x) \tag{8}$$

for $|y'| \leqq L$ (L being arbitrary), while $M(x)$ is a summable function for $a \leqq x \leqq b$ which may depend on L. For this differential equation we will discuss the boundary value problem.

As the boundary value problem, there are cases where one end is fixed or two ends are fixed and the remaining cases. At first we state some hypotheses necessary to these cases.

a) $\underline{\omega}'(x)$ and $\bar{\omega}'(x)$ are continuous in $a \leqq x \leqq b$.

b) For every y', $f(x, \underline{\omega}(x), y')$ and $f(x, \bar{\omega}(x), y')$ are two measurable functions of x.

c) $\underline{\omega}'(x) - \int_a^x f(u, \underline{\omega}(u), \underline{\omega}'(u))\,du$ and $\int_a^x f(u, \bar{\omega}(u), \bar{\omega}'(u))\,du - \bar{\omega}'(x)$ are non-decreasing in $a \leqq x \leqq b$.

d) We can find a positive number δ such that for $|\underline{\omega}'(x) - y'| < \delta$ and $|\bar{\omega}'(x) - y'| < \delta$, we have, almost everywhere in $a \leqq x \leqq b$

$$|f(x, \underline{\omega}(x), \underline{\omega}'(x)) - f(x, \underline{\omega}(x), y')| \leqq \lambda(x)\,\mu(\underline{\omega}'(x) - y'),$$

$$|f(x, \bar{\omega}(x), \bar{\omega}'(x)) - f(x, \bar{\omega}(x), y')| \leqq \lambda(x)\,\mu(\bar{\omega}'(x) - y')$$

respectively, where $\lambda(x)$ is summable in $a \leqq x \leqq b$, $0 < \lambda(x) < +\infty$ and $\mu(u)$ is positive for $0 < |u| < \delta$, $\mu \to 0$ when $u \to 0$ and $\mu(0) = 0$.

These are the assumptions imposed on $\underline{\omega}(x)$ and $\bar{\omega}(x)$, while the functions next given are to limit the region of y'. Namely we suppose that $\varphi_1(x, y, y')$, $\psi_1(x, y, y')$ and $\varphi_2(x, y, y')$, $\psi_2(x, y, y')$ are positive and continuous in the domains

$$a \leqq x \leqq b,\ \underline{\omega}(x) \leqq y \leqq \bar{\omega}(x),\ y' \geqq K$$

(constant $K > 0$ may be however great)

and

$$a \leqq x \leqq b,\ \underline{\omega}(x) \leqq y \leqq \bar{\omega}(x),\ y' \leqq -K$$

respectively and they tend to zero uniformly for (x, y) for $y' \to \pm\infty$ respectively and that in the interior of their respective domains, they satisfy locally the Lipschitz condition with regard to (y, y') and they have *the property a.c.u.*, and finally we have

$$\underline{\lim}_{t \to 0} \frac{1}{t}\{\varphi_i(x+t,\ y+ty',\ y'+tf) - \varphi_i(x, y, y')\} \geqq 0 \ a.e. \tag{9}$$

and

$$\overline{\lim}_{t \to 0} \frac{1}{t}\{\psi_i(x+t, y+ty', y'+tf) - \psi_i(x, y, y')\} \leqq 0 \quad a.e. \tag{10}$$

$(i=1, 2)$ respectively.

5. We have the following theorem for the left end fixed.

Theorem 1. *We add to the hypotheses in* § 4 *one more assumption that* $\underline{\omega}(a) = \bar{\omega}(a)$. *Now if the above stated functions* $\varphi_1(x, y, y')$ *and* $\varphi_2(x, y, y')$ *exist, then it gives in* $\mathfrak{D}$ *at least a solution of* (7) *which passes through the point* $(a, \underline{\omega}(a))$ *and an arbitrary point in* $\mathfrak{D}$.

Proof. We can assume that the x-coordinate of the second point in $\mathfrak{D}$ be $x=b$. Since we may assume K great, we suppose that $K>|\bar{\omega}'(x)|$ and $K>|\underline{\omega}'(x)|$. Now take a number M such as $M>K$ and as

$$\min \varphi_1(x, y, K) > \max \varphi_1(x, y, M)$$

and

$$\min \varphi_2(x, y, -K) > \max \varphi_2(x, y, -M)$$

for (x, y) of $\mathfrak{D}$. This is possible, for these functions tend to zero uniformly when $y'\to\pm\infty$ respectively. Now put

$$g(x, y, y') = \begin{cases} f(x, y, M) & (y'>M) \\ f(x, y, y') & (-M \leqq y' \leqq M) \\ f(x, y, -M) & (y'<-M) \end{cases}$$

and consider the following function $f^*(x, y, y')$:

$$f^*(x, y, y') = \begin{cases} g(x, \bar{\omega}(x), y') + \dfrac{y-\bar{\omega}(x)}{y-\bar{\omega}(x)+1}\lambda(x) & (y>\bar{\omega}(x)) \\ g(x, y, y') & (\underline{\omega}(x) \leqq y \leqq \bar{\omega}(x)) \\ g(x, \underline{\omega}(x), y') - \dfrac{\underline{\omega}(x)-y}{\underline{\omega}(x)-y+1}\lambda(x) & (y<\underline{\omega}(x)), \end{cases}$$

where $\lambda(x)$ is that in the condition d) in § 4.

Consider a differential equation

$$y''=f^*(x, y, y') .$$

Here $f^*(x, y, y')$ is defined in $[a \leqq x \leqq b, \ -\infty<y<+\infty, \ -\infty<y' <+\infty]$, measurable with regard to x and continuous with regard to (y, y') and in this domain we have

$$|f^*(x, y, y')| \leqq M(x)+\lambda(x)=\overline{M}(x),$$

where $\overline{M}(x)$ becomes a summable function. Therefore $y''=f^*(x, y, y')$

has a solution $y=y(x)$ which passes through the point $(a, \underline{\omega}(a))$ and an arbitrary point in $\mathfrak{D}$, since this equation satisfies the conditions in § 2. Now we show that this solution $y(x)$ satisfies the inequality $\underline{\omega}(x) \leqq y(x) \leqq \bar{\omega}(x)$. We may assume that $\lambda(x) \geqq 1$.

Now if it be $y(x) < \underline{\omega}(x)$ at some point, there exist two points x_1 and x_2 such that $z(x_1)=z(x_2)=0$ and $z(x)<0$ for $x_1<x<x_2$, where $z(x)=y(x)-\underline{\omega}(x)$. On the other hand we have for $x_1 \leqq x \leqq x_2$

$$z'(x)=y'(x)-\underline{\omega}'(x)=y'(x_1)+\int_{x_1}^{x} f^*(u, y(u), y'(u))\,du-\underline{\omega}'(x)$$

$$=y'(x_1)-\underline{\omega}'(x)+\int_{x_1}^{x}\left\{g(u, \underline{\omega}(u), y'(u))-\lambda(u)\frac{\underline{\omega}(u)-y(u)}{1+\underline{\omega}(u)-y(u)}\right\}du$$

$$=y'(x_1)-\underline{\omega}'(x)+\int_{x_1}^{x} g(u, \underline{\omega}(u), \underline{\omega}'(u))\,du$$

$$+\int_{x_1}^{x}\left\{\lambda(u)\frac{y(u)-\underline{\omega}(u)}{1+\underline{\omega}(u)-y(u)}+g(u, \underline{\omega}(u), y'(u))-g(u, \underline{\omega}(u), \underline{\omega}'(u))\right\}du$$

$$=y'(x_1)-\int_{a}^{x_1} g(u, \underline{\omega}(u), \underline{\omega}'(u))\,du+\int_{a}^{x} g(u, \underline{\omega}(u), \underline{\omega}'(u))\,du-\underline{\omega}'(x)$$

$$+\int_{x_1}^{x}\left\{\lambda(u)\frac{y(u)-\underline{\omega}(u)}{1+\underline{\omega}(u)-y(u)}+g(u, \underline{\omega}(u), y'(u))-g(u, \underline{\omega}(u), \underline{\omega}'(u))\right\}du.$$

At the same time, there exists ξ such as $x_1<\xi<x_2$ and $z'(\xi)=0$ and if we take $\delta>0$ suitably, in its neighborhood $|x-\xi| \leqq \delta$ we can have

$$\frac{y(x)-\underline{\omega}(x)}{1+\underline{\omega}(x)-y(x)}<-\varepsilon,\quad \mu(y'(x)-\underline{\omega}'(x))<\frac{\varepsilon}{2}, \tag{11}$$

where ε is a certain positive number. Then in the neighborhood of ξ we have

$$\frac{z'(\xi+h)-z'(\xi)}{h}$$

$$=\frac{1}{h}\left\{\int_{a}^{\xi+h} g(u, \underline{\omega}(u), \underline{\omega}'(u))\,du-\underline{\omega}'(\xi+h)\right.$$

$$\left.-\left[\int_{a}^{\xi} g(u, \underline{\omega}(u), \underline{\omega}'(u))\,du-\underline{\omega}'(\xi)\right]\right\}$$

$$+\frac{1}{h}\int_{\xi}^{\xi+h}\left\{\lambda(u)\frac{y(u)-\underline{\omega}(u)}{1+\underline{\omega}(u)-y(u)}\right.$$

$$+g(u, \underline{\omega}(u), y'(u)) - g(u, \underline{\omega}(u), \underline{\omega}'(u))\Big\} du.$$

Now we can assume that $|y'(x)| < M$, for if δ be sufficiently small, the value of $y'(x)$ will be sufficiently near to that of $\underline{\omega}'(x)$ by the continuity of $z'(x)$. Hence we have finally

$$\frac{z'(\xi+h) - z'(\xi)}{h}$$

$$= \frac{1}{h}\Big\{\Big[\int_a^{\xi+h} f(u, \underline{\omega}(u), \underline{\omega}'(u))du - \underline{\omega}'(\xi+h)$$

$$-\Big[\int_a^{\xi} f(u, \underline{\omega}(u), \underline{\omega}'(u))du - \underline{\omega}'(\xi)\Big]\Big\}$$

$$+\frac{1}{h}\int_\xi^{\xi+h}\Big\{\lambda(u)\frac{y(u) - \underline{\omega}(u)}{1+\underline{\omega}(u) - y(u)}$$

$$+f(u, \underline{\omega}(u), y'(u)) - f(u, \underline{\omega}(u), \underline{\omega}'(u))\Big\} du.$$

And yet the first term of the right-hand side is not positive by the condition c) in § 4 and the second term is, by (11),

$$< -\frac{\varepsilon}{2h}\int_\xi^{\xi+h}\lambda(u)du \quad (|h| < \delta).$$

Since $\lambda(x) \geqq 1$, we have ultimately

$$\frac{z'(\xi+h) - z'(\xi)}{h} < -\frac{\varepsilon}{2}.$$

From this we see that the four derivates of $z'(x)$ at $x=\xi$ are all negative. Since this contradicts Scorza Dragoni's lemma (p. 268 in [12]), we must have $y(x) \geqq \underline{\omega}(x)$. In the same way we can prove that $y(x) \leqq \bar{\omega}(x)$. We have $\underline{\omega}(x) \leqq y(x) \leqq \bar{\omega}(x)$ after all. Therefore if this $y(x)$ satisfies $|y'(x)| < M$, this becomes a required solution of (7), for $f^*(x, y, y')$ coincides with $f(x, y, y')$ in $[(x, y) \in \mathfrak{D}, |y'| \leqq M]$.

Since $\underline{\omega}(x) \leqq y(x) \leqq \bar{\omega}(x)$, $y'(a)$ satisfies the inequality $|y'(a)| < K$. Now if we suppose that at some x, say x_1, we have $y'(x_1) \geqq M$, there exist two values of x, x_2 and x_3 say, such that $a < x_2 < x_3 \leqq x_1$, $y'(x_2) = K$, $y'(x_3) = M$ and $K < y'(x) < M$ for $x_2 < x < x_3$. Then consider the function $\varphi_1(x, y(x), y'(x))$. Since $\underline{\omega}(x) < y(x) < \bar{\omega}(x)$ for $x_2 < x < x_3$ (for we have $K > |\bar{\omega}'(x)|$ and $K > |\underline{\omega}'(x)|$), this function is by (9) non-decreasing. This contradicts the choise of M. Therefore we have $y'(x) < M$ and in the same way we can prove that $y'(x) > -M$ by the aid of $\varphi_2(x, y, y')$. This completes the proof of the theorem.

In the case where the right end is fixed, we can obtain an analogous theorem by assuming $\underline{\omega}(b) = \bar{\omega}(b)$ and the existence of $\psi_i(x, y, y')$ $(i=1, 2)$.

The case where two ends are fixed is obtained as a collorary of the case of Tonelli's type. In the case of Tonelli's type we can extend the theorem (pp. 154-156 Lemma 1 in [14]) which has been obtained by Okamura when $f(x, y, y')$ is a continuous function of (x, y, y'), but we do not state the theorem *purposely* to avoid the repetition. As a collorary of this theorem, if we put $\psi(x) = \underline{\omega}(x)$ by using the symbol in [14], we shall have

Theorem 2. *Under the hypotheses in* §4, *if there exist* $\psi_1(x, y, y')$ *and* $\varphi_2(x, y, y')$, *we have in* $\mathfrak{D}$ *at least a solution of* (7), $y=y(x)$, *which satisfies*

$$y(a) = \underline{\omega}(a) \text{ and } y(b) = \underline{\omega}(b).$$

Scorza Dragoni [12] has verified the existence of a solution satisfying $y(a) = \alpha$ and $y(b) = \beta$, where $\underline{\omega}(a) \leqq \alpha \leqq \bar{\omega}(a)$ and $\underline{\omega}(b) \leqq \beta \leqq \bar{\omega}(b)$, under the condition $|f(x, y, y')| \leqq \vartheta(y') + \chi(x)$, $\vartheta(u) > 0$ being continuous in $-\infty < u < +\infty$ and $\chi(x) \geqq 0$ summable for $a \leqq x \leqq b$; moreover

$$\int_0^{+\infty} \frac{u}{\vartheta(u)} du = \int_0^{-\infty} \frac{u}{\vartheta(u)} du = +\infty$$

and

$$\left| \frac{u}{\vartheta(u)} \right| < k \quad (k = \text{const.}).$$

The last condition is not necessary when $\chi(x) \equiv 0$. For this case, *using the symbols in* [14], it is clear that we may take $\Phi_i(x, y, y')$ and $\Psi_i(x, y, y')$ $(i=1, 2)$ as follows:

$$\Phi_1(x, y, y') = e^{k\int_a^x \chi(t)dt + y - \int_0^{y'} \frac{u}{\vartheta(u)} du} \qquad (y' \geqq 0),$$

$$\Phi_2(x, y, y') = e^{k\int_a^x \chi(t)dt - y - \int_0^{y'} \frac{u}{\vartheta(u)} du} \qquad (y' \leqq 0),$$

$$\Psi_1(x, y, y') = e^{-k\int_a^x \chi(t)dt - y - \int_0^{y'} \frac{u}{\vartheta(u)} du} \qquad (y' \geqq 0),$$

$$\Psi_2(x, y, y')=e^{-k\int_a^x \chi(t)dt+y-\int_0^{y'}\frac{u}{\vartheta(u)}du} \qquad (y'\leqq 0).$$

Also when $|f(x, y, y')|\leqq\vartheta(y')\chi(x)$, where $\vartheta(y')$ is positive continuous, $\chi(x)$ is summable for $a\leqq x\leqq b$ and

$$\int^{+\infty}\frac{du}{\vartheta(u)}=\infty, \quad \int^{-\infty}\frac{du}{\vartheta(u)}=-\infty,$$

we may put as follows; namely

$$\Phi_1=e^{\int_a^x \chi(t)dt-\int_K^{y'}\frac{du}{\vartheta(u)}} \qquad (y'\geqq K),$$

$$\Phi_2=e^{\int_a^x \chi(t)dt+\int_{-K}^{y'}\frac{du}{\vartheta(u)}} \qquad (y'\leqq -K),$$

$$\Psi_1=e^{-\int_a^x \chi(t)dt-\int_K^{y'}\frac{du}{\vartheta(u)}} \qquad (y'\geqq K),$$

$$\Psi_2=e^{-\int_a^x \chi(t)dt+\int_{-K}^{y'}\frac{du}{\vartheta(u)}} \qquad (y'\leqq -K).$$

But from the theorem of Tonelli's type, we cannot prove the existence of a solution of the differential equation

$$y''=\frac{yy'^3}{\sqrt{x}} \tag{12}$$

and the others, while if we use the following theorem in which we take particularly $\varphi_1(x, y, y')$ and $\varphi_2(x, y, y')$ in Theorem 1, we can prove the existence of a solution of (12) passing through the point $(0, 0)$ and an arbitrary point in the domain

$$0\leqq x\leqq 1, \quad x(x-1)\leqq y\leqq 0$$

by putting $\bar{\omega}(x)\equiv 0$ and $\underline{\omega}(x)=x(x-1)$.

Theorem 3. *Let $f(x, y, y')$, $\underline{\omega}(x)$ and $\bar{\omega}(x)$ be the same as in* §4 *and suppose that $\underline{\omega}(a)=\bar{\omega}(a)$. Then if we have*

$$\begin{cases} f(x, y, y')\leqq 0 \text{ a.e., provided } y'>K \\ f(x, y, y')\geqq 0 \text{ a.e., provided } y'<-K, \end{cases} \tag{13}$$

there exists in $\mathfrak{D}$ at least a solution of (7) *passing through the point $(a, \underline{\omega}(a))$ and an arbitrary point in $\mathfrak{D}$.*

Proof. Since this is a case where the left end is fixed, it is sufficient to prove that there exist $\varphi_i(x, y, y')$ $(i=1, 2)$ in Theorem

1. Now if we put $\varphi_1(x, y, y')=e^{-y'^2}$, the inequality (9) which shall be satisfied by this $\varphi_1(x, y, y')$ becomes

$$-2y'f(x, y, y')\geqq 0 \quad \text{a.e.}.$$

Hence it is sufficient that $f(x, y, y')\leqq 0$ a.e. when $y'>K$. In the same way, we can see that we may have $f(x, y, y')\geqq 0$ a.e. when $y'<-K$ by putting $\varphi_2(x, y, y')=e^{-y'^2}$. Therefore we obtain (13).

In the similar way, for the case where two ends are fixed we have

Theorem 4. *$f(x, y, y')$, $\underline{\omega}(x)$ and $\bar{\omega}(x)$ are the same as in §4. If $f(x, y, y')\geqq 0$ a.e. when $|y'|>K$, there exists in $\mathfrak{D}$ at least a solution $y=y(x)$ of (7) satisfying*

$$y(a)=\underline{\omega}(a) \quad \text{and} \quad y(b)=\underline{\omega}(b).$$

Putting these together, we obtain

Theorem 5. *Let $f(x, y, y')$, $\underline{\omega}(x)$ and $\bar{\omega}(x)$ be the same as before and suppose that $\underline{\omega}(a)=\bar{\omega}(a)$ and $\underline{\omega}(b)=\bar{\omega}(b)$. If $f(x, y, y')$ has the definite sign almost everywhere with regard to x when $y'>K$ and $y'<-K$ (K may be sufficiently great) respectively, there exists in $\mathfrak{D}$ at least a solution $y=y(x)$ of (7) satisfying*

$$y(a)=\underline{\omega}(a)[=\bar{\omega}(a)] \quad \text{and} \quad y(b)=\underline{\omega}(b)[=\bar{\omega}(b)].$$

As before, we obtain various criteria by taking $\varphi_i(x, y, y')$ and $\psi_i(x, y, y')$ $(i=1, 2)$ suitably.

6. This time we consider a system of differential equations,

$$\frac{dx}{dt}=F(t, x), \tag{14}$$

where x denotes an n-dimensional vector and $F(t, x)$ is a given vector field, finite and defined in the domain

$$D: \quad 0\leqq t<\infty, \quad -\infty<x_i<+\infty \quad (i=1, 2,\cdots, n).$$

Moreover we suppose that $F(t, x)$ is a *measurable* function of t when x is fixed and it is a *continuous* function of x when t is fixed, and that, if $|x|\leqq L$ (L arbitrary), we have

$$|F(t, x)|\leqq M(t),$$

where $M(t)$ is summable for $0\leqq t\leqq T$ (T arbitrary) and of course, it may depend on L. We shall call the solution $x(t)$ of (14) *periodic* with period ω, if it satisfies

$$x(t+\omega)=x(t). \tag{15}$$

By Carathéodory's existence theorem, (14) has at least a solution which passes through an arbitrary point in D and is defined for a suitable interval of t. Then in the same way as in [17], we have the following boundedness theorem (see pp. 153-154 Theorem 1 in [17]).

Theorem 6. *Let R_0 be a positive constant which may be sufficiently great and D^* be the domain such as*

$$0\leqq t<\infty,\ |x|\geqq R_0.$$

Suppose that there exists a continuous function $\varphi(t, x)$ satisfying the following conditions in D^*; namely

1° *for any positive number $R(\geqq R_0)$, there exists a positive constant $G(R)$ such that*

$$\varphi(t, x)\geqq G(R)>0$$

for $|x|=R$,

2° *$\varphi(t, x)$ tends to zero uniformly for $|x|\to\infty$,*

3° *$\varphi(t, x)$ satisfies locally the Lipschitz condition with regard to x and in the interior of D^* $\varphi(t, x)$ has the property a.c.u. and we have*

$$\varliminf_{h\to 0}\frac{1}{h}\{\varphi(t+h,\ x+hF(t, x))-\varphi(t, x)\}\geqq 0 \text{ a.e..} \tag{16}$$

Then given an arbitrary positive number α, we can find a positive number $\beta(>\alpha)$ such that, for any solution $x=x(t)$ of (14) satisfying

$$|x(t_0)|\leqq\alpha \tag{17}$$

at an arbitrary t_0 $(\geqq 0)$, we have for $t\geqq t_0$

$$|x(t)|<\beta. \tag{18}$$

Proof. Let us assume that $\alpha>R_0$, for this case alone is worth to consider. We can choose β so large that

$$G(\alpha)>\varphi(t, x) \tag{19}$$

when $|x|=\beta$. Since by the condition 3°, $\varphi(t, x(t))$ is a non-decreasing function of t, we can see from (19) that the above stated β is the desired.

Theorem 6 is valid for an arbitrary t_0. But for any solution $x(t)$ of (14) satisfying $|x(0)|\leqq\alpha$ only, such a sufficient condition as $|x(t)|<\beta$ for $t\geqq 0$ may be obtained similarly, if we assume that

$\varphi(t, x)$ exists in D and the condition 1° of Theorem 6 be replaced by the existence of $G(R)$ satisfying for $|x| \leqq R$

$$\varphi(0, x) \geqq G(R) > 0. \tag{20}$$

For example, we suppose that the right-hand side of (14) satisfies for $r \geqq R_0$

$$|F(t, x)| \leqq \chi(t) \vartheta(r),$$

where $r = \sqrt{x_1^2 + \cdots + x_n^2}$ and $\vartheta(r)$ is positive, continuous for $r \geqq R_0$ and is subjected to the condition

$$\int_{R_0}^{\infty} \frac{dr}{\vartheta(r)} = \infty ;$$

while $\chi(t)$ is summable in $[0, t]$ (t arbitrary) and $\int_0^t \chi(t)\,dt < K$ (K, a certain positive constant independent of t). Then putting in $r \geqq R_0$

$$\varphi(t, x) = e^{n\int_0^t \chi(t)dt - \int_{R_0}^r \frac{dr}{\vartheta(r)}},$$

we can verify the boundedness of solutions.

By the aid of Theorem 6 and the following theorem, we can prove the *ultimate boundedness* of a solution such as $x = x_0$ (x_0 arbitrary) for $t = t_0$.

Theorem 7. *We suppose that the conclusion of Theorem 6 is true and that there exists a function $\psi(t, x)$ satisfying the following conditions in D^*:*

1° *$\psi(t, x)$ is positive, continuous in D^*,*
2° *when for any positive constant K ($K > R_0$), we have $|x| \leqq K$, then $\psi(t, x)$ tends to zero uniformly for $t \to \infty$,*
3° *$\psi(t, x)$ satisfies locally the Lipschitz condition with regard to x and in the interior of D^*, it has the property a.c.u. and we have*

$$\varliminf_{h \to 0} \frac{1}{h} \{\psi(t+h, x+hF) - \psi(t, x)\} \geqq 0 \quad a.e.. \tag{21}$$

Then for any solution $x = x(t)$ of (14) *for which we have $x(t_0) = x_0$ at $t = t_0$ and $|x_0| \leqq \gamma$ (γ arbitrary), we have, at some value of t, say T ($\geqq t_0$), $|x(T)| \leqq R_0$, where t_0 is arbitrary.*

Proof. Of course, the theorem is true when $\gamma \leqq R_0$. Hence we assume that $\gamma > R_0$. Let D' be the domain such as $t_0 \leqq t < \infty$, $|x| \leqq \gamma^*$ (γ^* is what corresponds to γ by Theorem 6) and E be the domain such as $t_0 \leqq t < \infty$, $|x| < R_0$. Observing in $D' - E$, the inequality (21) shows that $\psi(t, x(t))$ does not decrease along the solution. Hence we can easily prove this theorem.

For instance, let $\varphi(t, x)$ be the same with that in Theorem 6 and be bounded in D^* (if we take a sufficiently great R_0 again, $\varphi(t, x)$ becomes bounded by the condition 2° in Theorem 6) and we replace (16) by the following condition:

When $|x| \leqq K$ (K arbitrary), we have in the interior of D^*

$$\varliminf_{h \to 0} \frac{1}{h} \{\varphi(t+h, x+hF) - \varphi(t, x)\} \geqq \varepsilon(K) > 0 \ \ a.e., \tag{22}$$

where $\varepsilon(K)$ may be arbitrarily small, but it is a fixed positive number.

And considering a function $\varphi(t, x)e^{-Nt}$ $(N > 0)$ in the domain $D' - E$, we take this function for $\psi(t, x)$. Then this function satisfies the conditions in Theorem 7. As to the property a.c.u., we can prove it as follows: Since $\varphi(t, x)$ is bounded, if we choose L suitably, we shall have

$$\begin{aligned} &\sum_{k=1}^{m} |\varphi(t_k', x_k) e^{-Nt_k'} - \varphi(t_k, x_k) e^{-Nt_k}| \\ &\leqq \sum_{k=1}^{m} \varphi(t_k', x_k) |e^{-Nt_k'} - e^{-Nt_k}| + \sum_{k=1}^{m} e^{-Nt_k} |\varphi(t_k', x_k) - \varphi(t_k, x_k)| \\ &\leqq L \sum_{k=1}^{m} |e^{-Nt_k'} - e^{-Nt_k}| + \sum_{k=1}^{m} |\varphi(t_k', x_k) - \varphi(t_k, x_k)|. \end{aligned}$$

On the other hand, by the fact that $\varphi(t, x)$ has the property a.c.u. and the absolute continuity of e^{-Nt}, we have

$$< L\varepsilon + \varepsilon = (L+1)\varepsilon$$

when $\sum_{k=1}^{m} |t_k' - t_k| < \delta$. Therefore we can see that $\varphi(t, x)e^{-Nt}$ has the property a.c.u.. And the proof of the inequality (21) is quite same as in [15] (see pp. 135-136 Proof of Lemma 2).

By Theorems 6 and 7, the ultimate boundedness of solutions is proved; so we have

Theorem 8. *If the same assumptions as those in Theorems 6 and 7 hold good, there exists a positive constant B (independent of the particular solution considered) such that any solution $x = x(t)$ of* (14) *satisfies ultimately*

$$|x(t)| < B$$

(*cf.* p. 136 Theorem 1 and p. 137 Remark 2 in [15]).

7. In § 6 we have obtained a sufficient condition in order that $|x(t)|<\beta$ for $t_0 \leqq t$ if $|x(t_0)| \leqq \alpha$. When for any solution of (14) such as $x(t_0)=x_0$ (x_0 arbitrary), there is a positive constant β depending only on x_0 and we have always $|x(t)|<\beta$ for $t \geqq t_0$ as long as the solution exists, then by the existence theorem a solution such as $x=x_0$ at $t=t_0$ is defined in $t_0 \leqq t<\tau$ (τ be a suitable constant) and we can see that it exists really in $t_0 \leqq t \leqq \tau$ in the same way with $F(t, x)$ continuous. In the present case we consider the following function $F^*(t, x)$: It coincides with $F(t, x)$ when $|x| \leqq \beta$ and if $|x|>\beta$, its values at the points on a straight line joining $x=0$ and a point on the hypersphere $|x|=\beta$ at every t are equal to the value of $F(t, x)$ at the intersection point of this line and the sphere. Then $F^*(t, x)$ becomes defined in $0 \leqq t<\infty$, $|x|<\infty$, measurable with regard to t and continuous with regard to x and besides there exists a summable function $M(t)$ such as

$$|F^*(t, x)| \leqq M(t)$$

in $0 \leqq t \leqq T$ (T be arbitrary), $|x|<\infty$. Now consider the differential equation

$$\frac{dx}{dt}=F^*(t, x). \tag{23}$$

Then there exists a solution defined in $\tau \leqq t \leqq T$ passing through $(\tau, x(\tau))$. And this solution does not attain to $|x|=\beta$; for if so, the equation (14) has a solution such as $|x(t)|=\beta$ at some point. Therefore this becomes a solution of (14), since in $|x| \leqq \beta$ we have $F^*(t, x)=F(t, x)$. From this we see that a solution defined in $t_0 \leqq t<\tau$ is a part of a solution defined in $t_0 \leqq t \leqq T (T \geqq \tau)$. It is so for every solution. Hence it is sufficient to observe only the solutions defined in $t_0 \leqq t \leqq T$. Representing this fact geometrically, we have

Lemma 3. *In the differential equation* (14) *defined in* $t_0 \leqq t \leqq T$, $|x|<\infty$, *if for any solution such as* $x=x_0$ *at* $t=t_0$, *there exists a positive constant* β *depending only on* x_0 *and we have always* $|x(t)|<\beta$ *for* $t \geqq t_0$ *as long as the solution exists, then it is continuable to* $t=T$.

In this case we can see by the aid of (23) that the section of all the solutions starting from any point P, cut by $t=T$ is also a *bounded closed set.* Hence considering the distance between a point Q on the t-axis such as $t_P \leqq t_Q$ and the section by $t=t_Q$, in the same way as in [18] (namely considering the distance between the infinity point and solutions by the change of the metric), we have the following theorem analogous to Theorem 3 in [18] (pp. 296-298).

Theorem 9. *In order that, for any solution of* (14), $x=x(t)$, *passing through any point* P *in* D, *there exists* $\alpha(P)$ *such as* $|x(t)| < \alpha(P)$ *for* $t_P \leqq t < \infty$, *it is necessary and sufficient that there exists a positive function* $\varphi(t, x)$ *of* (t, x) *satisfying the following conditions in* D:

1° $\varphi(t, x)$ *tends to zero uniformly for* t, *when* $|x| \to \infty$,

2° *for any solution of* (14), $x=x(t)$, *the function* $\varphi(t, x(t))$ *is a non-decreasing function of* t.

Moreover for the differential equation

$$\begin{cases} \dfrac{dx}{dt}=f(t, x, y) \\ \dfrac{dy}{dt}=g(t, x, y), \end{cases}$$

the method stated in [17] by which at first we prove the boundedness for x or y and next we prove the boundedness for the other (pp. 155–157 Theorem 2 in [17]) may be applied to the present case; only it is necessary that $\varphi(t, x, y)$ and $\psi(t, x, y)$ satisfy such conditions as those in § 3.

Theorem 9 is stated about any solution starting from an arbitrary point. But when the solution starting from any point is unique, a necessary and sufficient condition in order that a bounded solution exists (as the mentioned in Massera's theorem (p. 460 in [6])) is also obtained by modifying $\varphi(t, x)$ only as follows; i.e., in Theorem 9, we have

$$\varphi(t, x) \geqq 0$$

and there exists an $\bar{x}$ so as $\varphi(0, \bar{x}) > 0$. Conversely from this we may regard Theorem 9 as a natural consequence.

8. Now we assume that every solution of (14) is *unique for the Cauchy-problem.* A necessary and sufficient condition for that has been obtained in [5]. And besides we assume that the result in Theorem 6 holds good; i.e., every solution is *bounded.* Then considering in $t_0 \leqq t \leqq \tau$, if $x=x(t)$ be the solution such as $x=x_0$ at $t=t_0$, then we have $|x(t)-\bar{x}(t)|<\varepsilon$ (ε: however small) in $t_0 \leqq t \leqq \tau$ for the solution $\bar{x}(t)$ starting from a sufficiently near point of x_0 at $t=t_0$. This may be proved as follows. Now we assume that, even if we take any neighborhood of x_0 at $t=t_0$, there is a solution $\bar{x}(t)$ starting from there such as $|\bar{x}(t)-x(t)|=\varepsilon$ at some t in $t_0 \leqq$

$t \leqq \tau$. Namely we assume that for the solution $x = x^{(j)}(t)$ starting from $x_0^{(j)}$ converging to x_0 at $t = t_0$, we have $|x^{(j)}(t_j) - x(t_j)| = \varepsilon$ for some t_j. If $|x_0^{(j)}| \leqq \alpha$, there exists β such as $|x^{(j)}(t)| < \beta$ and hence all such solutions arrive at $t = \tau$ and besides they are uniformly bounded. For $t_0 \leqq t \leqq \tau$, we have

$$x^{(j)}(t) = x_0^{(j)} + \int_{t_0}^{t} F(t, x^{(j)}(t))\,dt. \tag{24}$$

Since then there exists a summable function $M(t)$ such as $|F(t, x)| \leqq M(t)$, $x^{(j)}(t)$ $(j = 1, 2, \cdots)$ are equicontinuous. Therefore we can select an uniformly convergent sequence $x^{(k)}(t)$. Their limit function $X(t)$ becomes the solution passing through (t_0, x_0) by (24), but this coincides with $x(t)$ (by the uniqueness). Then we have for a sufficiently great k

$$|x(t) - x^{(k)}(t)| < \varepsilon$$

and this is contradictory to the assumption. Therefore if we take a suitable neighborhood of x_0, the solutions starting from there lie in ε-neighborhood of $x(t)$. That is to say, they are continuous with regard to the initial values.

What we have stated above is the case where the solution is unique to the right. Also the case to the left is the same. Hence if the solution is unique to the right and to the left, the *transformation* of the point P in $t = t_0$ into the point on the same solution in $t = \tau$ is a *topological mapping*. Therefore when $n = 2$ and $F(t, x)$ is periodic with regard to t, then Massera's existence theorem of a periodic solution is true (Theorem 2 in [6]), for the above transformation is sense-preserving. Hence when $n = 2$, we can deduce an existence theorem of a periodic solution from the boundedness of solutions of (14).

9. Reuter [10] has discussed a non-linear differential equation

$$\ddot{x} + f(\dot{x}) + g(x) = p(t),$$

where $f(\dot{x})$, $g(x)$ and $p(t)$ are continuous, and he has obtained an ultimate boundedness theorem which we have proved in [15] as an example by our method. In p. 127 of Mathematical Reviews Vol. 15 it is reported that de Castro [3] has discussed

$$\ddot{x} + f(x, \dot{x}, t)\dot{x} + g(x) = e(t)\ ;$$

but the author has yet no chance to read its details. Now in order to consider a case where the forces have *discontinuities* in regard to the time and the other cases, we will discuss now the differential equation having the type considered above. Thence we will deduce a simple sufficient condition for the *ultimate boundedness.* Now let the differential equation be

$$\ddot{x}+f(x, \dot{x}, t)+g(x)=0, \tag{25}$$

where $f(x, \dot{x}, t)$ includes the term $p(t)$ or $e(t)$ as before. Instead of (25) we consider the system

$$\dot{x}=y,\ \dot{y}=-f(x, y, t)-g(x). \tag{26}$$

We assume that $g(x)$ is continuous, while $f(x, y, t)$ is a measurable function of t and is continuous with regard to (x, y), moreover that we have $|f(x, y, t)|\leqq M(t)$ when $\sqrt{x^2+y^2}\leqq L$ (L arbitrary), where $M(t)$ is a summable function, and

$$\mathit{sgn}\ y\cdot f(x, y, t)\geqq\varepsilon \text{ a.e., provided } |y|\geqq b$$

(ε *may be however small, but fixed, positive, constant*),

$$|f(x, y, t)|\leqq A \text{ a.e., provided } |y|\leqq b$$

and

$$\mathit{sgn}\ x\cdot g(x)>A(b+1)+\eta, \text{ provided } |x|\geqq a$$

(η *may be however small, but fixed, positive, constant*).

Then we can prove the ultimate boundedness of solutions of (25) by defining the function $\varphi(t, x, y)$ as follows: choosing a' so large as $\dfrac{2b}{a'}<\varepsilon$,

$$\begin{array}{ll}
e^{-G(x)-\frac{y^2}{2}} & \text{for } |x|<\infty,\ y\geqq b,\\
e^{-G(x)-\frac{y^2}{2}-y+b} & \text{for } x\geqq a',\ |y|\leqq b,\\
e^{-G(x)-\frac{y^2}{2}+2b} & \text{for } x\geqq a',\ y\leqq -b,\\
e^{-G(x)-\frac{y^2}{2}+\frac{2b}{a'}x} & \text{for } |x|\leqq a',\ y\leqq -b,\\
e^{-G(x)-\frac{y^2}{2}-2b} & \text{for } x\leqq -a',\ y\leqq -b,\\
e^{-G(x)-\frac{y^2}{2}+y-b} & \text{for } x\leqq -a',\ |y|\leqq b,
\end{array}$$

where $G(x)=\int_0^x g(x)\,dx$.

Therefore it follows that if $f(x, y, t)$ is a periodic function of t and the solution of (25) is unique, then it has at least a *periodic* *solution.*

And Cahen [1] is reported to have discussed about

$$\ddot{x}+f(\dot{x})+\dot{x}b(x)+r(x)=0\,;$$

yet the author does not know his result. Now generalizing his equation, we will consider the differential equation

$$\ddot{x}+f(x)\dot{x}+h(\dot{x},t)+g(x)=p(t)\,;$$

or rather letting $p(t)$ into the term $h(\dot{x},t)$, we will consider

$$\ddot{x}+f(x)\dot{x}+h(\dot{x},t)+g(x)=0, \tag{27}$$

which we deal as the system:

$$\dot{x}=y,\ \dot{y}=-f(x)y-h(y,t)-g(x),$$

where $f(x)$ and $g(x)$ are continuous and $h(y,t)$ is of the same category as that we are observing in this paper. Now we suppose that $f(x)\geqq 0$ *for* $|x|<\infty$, $h(y,t)\geqq\varepsilon>0$ *a.e. for* $y\geqq b$, $h(y,t)\leqq 0$ *a.e. for* $y\leqq -b$, $|h(y,t)|\leqq A$ *a.e. for* $|y|\leqq b$ *and* $g(x)\geqq A(b+1)$ *for* $x\geqq a$, $g(x)\leqq -A(b+1)$ *for* $x\leqq -a$, where we may take as $b\geqq 1$.

In this case we can verify the *boundedness* of solutions of (27) by choosing a' so large that we have $\dfrac{4b}{a'}\leqq\varepsilon$ and defining the function $\varphi(t,x,y)$ as follows:

$$\begin{array}{ll}
e^{u(x,y)-F(x)-2b} & \textit{for } x\geqq a',\ y\geqq b, \\
e^{u(x,y)-F(x)-y-b} & \textit{for } x\geqq a',\ |y|\leqq b, \\
e^{u(x,y)-F(x)} & \textit{for } x\geqq 0,\ y\leqq -b, \\
e^{u(x,y)+F(x)} & \textit{for } x\leqq 0,\ y\leqq -b, \\
e^{u(x,y)+F(x)+y+b} & \textit{for } x\leqq -a',\ |y|\leqq b, \\
e^{u(x,y)+F(x)+2b} & \textit{for } x\leqq 0,\ y\geqq b, \\
e^{u(x,y)-F(x)-\frac{4b}{a'}x+2b} & \textit{for } 0\leqq x\leqq a',\ y\geqq b,
\end{array}$$

where $G(x)=\int_0^x g(x)\,dx$, $F(x)=\int_0^x f(x)\,dx$ and $u(x,y)=-G(x)-\dfrac{y^2}{2}$.

Moreover when in the differential equation

$$\ddot{x}+f(x)\dot{x}+g(x,t)=p(t)\,;$$

$p(t)$ is measurable and $G(x,t)=\int_0^x g(x,t)\,dx$ has the property a.c.u., then reasoning by aid of $e^{-G(x,t)-\frac{y^2}{2}}$, we can see the boundedness of

x and y by the method of Theorem 2 in [17] under certain conditions for $f(x)$, $g(x, t)$ and $p(t)$ (e.g., conditions in [13]).

10. In this section we will obtain a sufficient condition in order that $x(t)\to 0$ as $t\to+\infty$ for the solution of the differential equation (14). Now we assume that all the solutions of (14) are bounded, i.e., that for any solution such as $|x(0)|\leqq\alpha$ at $t=0$, we have $|x(t)|<\beta$ for $0\leqq t<\infty$, where α is arbitrary and β is a positive constant depending upon α. Namely we consider a case where Theorem 6 or Theorem 9 holds good.

Lemma 4. *Let Δ_β be the domain such as*

$$0\leqq t<\infty,\ |x|\leqq\beta$$

for every β and Δ_δ be the domain such as

$$0\leqq t<\infty,\ |x|<\delta$$

for every $\delta>0$ (δ may be small). Suppose that there exists a continuous function $\varphi_{\beta\delta}(t, x)\equiv\varphi(t, x)$ in $\Delta_\beta-\Delta_\delta$ satisfying the following conditions:

1° *$\varphi(t, x)$ is positive in $\Delta_\beta-\Delta_\delta$,*

2° *$\varphi(t, x)$ tends to infinity (or to zero) uniformly for x when $t\to\infty$,*

3° *$\varphi(t, x)$ satisfies locally the Lipschitz condition with regard to x and in the interior of $\Delta_\beta-\Delta_\delta$ it has the property a.c.u. and we have*

$$\overline{\lim_{h\to 0}}\ \frac{1}{h}\{\varphi(t+h, x+hF(t, x))-\varphi(t, x)\}\leqq 0\ \text{a.e.} \tag{28}$$

$$\left(\text{or } \underline{\lim_{h\to 0}}\ \frac{1}{h}\ \{\varphi(t+h, x+hF)-\varphi(t, x)\}\geqq 0\ \text{a.e.}\right).$$

Then for any solution $x=x(t)$ of (14) *such as $x(0)=x_0$ and $|x_0|>\delta$, we have at a certain T*

$$|x(T)|\leqq\delta.$$

For, assume that some solution such as $|x_0|>\delta$ satisfy $|x(t)|>\delta$ for $0\leqq t<\infty$. Then to this solution, there exists some β such that $|x(t)|<\beta$ for $0\leqq t<\infty$. Therefore using $\varphi_{\beta\delta}(t, x)$ corresponding to this β and δ, the lemma may be proved by the absurdity.

Lemma 5. *For every $\varepsilon>0$, let Δ_ε be the domain such as*

$$0\leqq t<\infty,\ |x|\leqq\varepsilon.$$

Suppose that there exists a continuous function $\psi(t, x)$ in Δ_ε satisfying *the following conditions :*

1° $\psi(t, x)=0$ *for* $|x|=0$,

2° *there exists a positive constant* λ *such that* $\psi(t, x) \geqq \lambda$ *when* $|x|=\varepsilon$ (λ *may depend on* ε),

3° $\psi(t, x)$ *satisfies the Lipschitz condition with regard to* x *and for a positive constant* K, *and in the interior of* Δ_ε *it has the property a.c.u. and we have*

$$\overline{\lim_{h\to 0}} \frac{1}{h}\{\psi(t+h, x+hF(t, x))-\psi(t, x)\} \leqq 0 \text{ a.e.}.$$

Then for any solution $x=x(t)$ *of* (14), *there exists a positive number* δ *such that if* $|x(t_0)| \leqq \delta$ (t_0 *arbitrary*), *then we have* $|x(t)|<\varepsilon$ *for* $t_0 \leqq t<\infty$.

δ may be taken as follows:

$$\delta<\min(\lambda/K, \varepsilon)$$

(cf. the proof of Lemma 3 in [16]).

Therefore for any solution $x=x(t)$ such as $|x(0)| \leqq \alpha$ (α arbitrary) at $t=0$, if we have $|x(t)|<\beta$ for $0 \leqq t<\infty$, then we can obtain the following theorem by the aid of Lemmas 4 and 5; namely

Theorem 10. *Suppose that the assumptions in Lemmas 4 and 5 hold good. Then for any solution* $x=x(t)$ *of* (14) *issuing to the right from* $t=0$, *we have* $x(t)\to 0$ *as* $t\to\infty$.

For example, let us consider the system

$$\frac{dx}{dt}=Ax+f(t, x),$$

where A is a given constant matrix (a_{ij}) and $\sum_{i,j} a_{ij}x_i x_j$ is negative definite. We assume that $f(t, x)$ satisfies $|f(t, x)| \leqq g(t)|x|$, where $g(t)$ is summable and $\int_0^t g(t)\,dt<K$. In this case we can see $x(t)\to 0$ $(t\to\infty)$ by considering

$$\varphi(t, x)=e^{-2n\int_0^t g(t)dt+Nt}\cdot|x|^2$$

and

$$\psi(t, x)=e^{-2n\int_0^t g(t)dt}\cdot|x|^2$$

(since there is a constant $\kappa>0$ such as $\sum_{i,j} a_{ij}x_i x_j \leqq -\kappa$ for $\delta \leqq |x| \leqq \beta$, (28) holds for a suitable N).

In conclusion we notice that it is possible to discuss about the convergence theorem [16] and the stability of solutions [19] in similar ways.

The case where $F(t, x)$ is a continuous function of (t, x) is a special one of our research; but in that case the conditions on φ and ψ may be modified slightly. Finally remark sure that *a.e.* *regards to each independent variable.*

REFERENCES

[1] Cahen; "*Systèmes électromécaniques non linéaires*", Rev. Gén. Electricité 62(1953), pp. 277-293.

[2] Carathéodory; *Vorlesungen über reelle Funktionen*, 2te Anf.

[3] de Castro; "*Sopra l'equazione differenziale delle oscillazioni non-lineari*", Rivista Mat. Univ. Parma 4(1953), pp. 133–143.

[4] Hayashi; "*On the solutions of the system of ordinary differential equations*", These Memoirs, Vol. 28(1953), pp. 19–25.

[5] Hayashi and Yoshizawa; "*New treatise of solutions of a system of ordinary differential equations and its application to the uniqueness theorems*", These Memoirs Vol. 26(1951), pp. 225–233.

[6] Massera; "*The existence of periodic solutions of systems of differential equations*", Duke Math. Journal, Vol. 17(1950), pp. 457–475.

[7] Nagumo; "*Über die Differentialgleichung* $y''=f(x, y, y')$", Proc. of Physico-Math. Soc. of Japan, series 3, Vol. 19(1937), pp. 861–866.

[8] Nagumo; "*Über das Randwertproblem der nicht linearen gewöhnlichen Differentialgleichungen zweiter Ordnung*", Proc. of Physico-Math. Soc. of Japan, series 3, Vol. 24(1942), pp. 845–851.

[9] Okamura; *Introduction to the ordinary differential equation* (in Japanese).

[10] Reuter; "*Boundedness theorems for non-linear differential equations of the second order* (II)", The Journal of the London Math. Soc., Vol. 27 (1952), pp. 48–58.

[11] Scorza Dragoni; "*Su un problema di valori ai limiti per le equazioni differenziali ordinarie del secondo ordine*", Rend. Sem. Mat. R. Univ. di Roma, serie 4, Vol. 2(1938), pp. 177–215.

[12] Scorza Dragoni; "*Elementi uniti di transformazioni funzionali e problemi di valori ai limiti*", ibid., pp. 255–275.

[13] Yamaguti; "*On some properties of the non-linear differential equations of the parametric excitation*", These Memoirs, Vol. 28(1953), pp. 87–96.

[14] Yoshizawa; "*Note on the non-increasing solutions of* $y''=f(x, y, y')$", These Memoirs, Vol. 27(1952), pp. 153–162.

[15] Yoshizawa; "*On the non-linear differential equation*", These Memoirs, Vol. 28 (1953), pp. 133–141.

[16] Yoshizawa; "*On the convergence of solutions of the non-linear differential equation*", *ibid.*, pp. 143–151.

[17] Yoshizawa; "*Note on the existence theorem of a periodic solution of the non-linear differential equation*", *ibid.*, pp. 153–159.

[18] Yoshizawa; "*Note on the boundedness of solutions of a system of differential equations*", *ibid.*, pp. 293–298.

[19] Yoshizawa; "*On the stability of solutions of a system of differential equations*", These Memoirs Vol. 29(1955), pp. 27–33.

[20] Zwirner; "*Un'osservazione su un problema ai limiti per le eqazioni differenziali*", Boll. Un. Mat. Ital., 1(1939), pp. 334-336.

[21] Zwirner; "*Su un problema di valori al contorno per equazioni differenziali del secondo ordine*", Rend. Sem. Mat. Univ. Roma, serie 4, Vol. 3(1939), pp. 57-70.