MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXIX, Mathematics No. 3, 1955.

Generalization of Lichnerowicz's theorem for a completely harmonic Riemannian space

By

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(Recieved October, 10, 1955)

André Lichnerowicz concerned with a completely harmonic Riemannian space H^n, whose fundamental form is positive-definite [I], and proved the

Theorem (L). *If a Riemannian space H^n is completely harmonic and is imbedded isometrically as a hypersurface in a space S^{n+1} of constant curvature K_0, then H^n is of constant curvature.*

Theorem (L) was based on the following two theorems.

I. If H^n is such that the equation

$$(f'(0))^2 \geqq -\tfrac{5}{2}k(n-1)f''(0) \tag{1}$$

holds for a number $k>1$, then H^n is locally euclidean.

II. Any Einstein space E^n, which is not of constant curvature and is imbedded as a hypersurface in a space S^{n+1} of constant curvature, is a product of spaces of constant curvature.

The second of the above theorems proved by A. Fialkow even as the fundamental forms of E^n and S^{n+1} are not positive-definite [**2**], under the condition that E^n is a *proper* hypersurface of S_{n+1}. On the other hand, the first of them does not always hold for H^n, the fundamental form being of indefinite sign. Hence Theorem (L) was proved under the restriction that the fundamental form of H^n is positive-definite.

In the following we shall show that Theorem (L) holds equally well without the above restriction. We suppose throughout the paper that the dimension n of H^n is greater than three, because it is well-known that any completely harmonic Riemannian space of dimension two or three is of constant curvature [**3**].

We consider a completely harmonic Riemannian space H^n ($n\geqq 4$), imbedded in a S^{n+1} of constant curvature K_0. Let N^a($a=$

$1,\cdots,n+1)$ be the unit normal vector of H^n and $e=\pm 1$ the indicator of the normal, that is, $g_{\alpha\beta}N^\alpha N^\beta=e$, where $g_{\alpha\beta}$ is the fundamental tensor of S^{n+1}. The Gauss equation is

$$(2)\qquad R_{hijk}=K_0(g_{hj}g_{ik}-g_{hk}g_{ij})+e(H_{hj}H_{ik}-H_{hk}H_{ij}),$$

where H_{ij} are coefficients of the second fundamental form of H^n as a hypersurface of S^{n+1}. Since H^n is completely harmonic, we have [**4**]

$$(3)\qquad R_{i\cdot ja}^{\cdot a\cdot}=\frac{R}{n}g_{ij},$$

$$(4)\qquad \begin{aligned}&(R_{h\cdot ib}^{\cdot a}+R_{i\cdot hb}^{\cdot a})(R_{j\cdot ka}^{\cdot b}+R_{k\cdot ja}^{\cdot b})\\ &+(R_{h\cdot jb}^{\cdot a}+R_{j\cdot hb}^{\cdot a})(R_{k\cdot ia}^{\cdot b}+R_{i\cdot ka}^{\cdot b})\\ &+(R_{h\cdot kb}^{\cdot a}+R_{k\cdot hb}^{\cdot a})(R_{i\cdot ja}^{\cdot b}+R_{j\cdot ia}^{\cdot b})\\ &=4K_2(g_{hi}g_{jk}+g_{hj}g_{ki}+g_{hk}g_{ij}),\end{aligned}$$

where R is the salar curvature (constant) of H^n and K_2 is a constant [**3**] as follows.

$$K_2=\frac{1}{n(n+2)}\left(\frac{R^2}{n}+R^{abcb}R_{abcd}+R^{abcd}R_{adcb}\right).$$

The equations (3) and (4) are equivalent to the facts that the following equations are satisfied for any choice of vector u^i.

$$(3')\qquad \operatorname{tr}\Gamma=\frac{R}{n}u,$$

$$(4')\qquad \operatorname{tr}\Gamma^2=K_2u^2,$$

where we put $u=g_{ij}u^iu^j$ and Γ is the matrix (Γ_j^i), the elements Γ_j^i [4] be defined by $\Gamma_j^i=R_{a\cdot bj}^{\cdot i}u^au^b$. Now we put $u_H=H_{ij}u^iu^j$, $H_j^i=g^{ia}H_{aj}$ and $H=g^{ij}H_{ij}$. Then it follows from (2)

$$(5)\qquad \Gamma_j^i=K_0(u\delta_j^i-u^iu_j)+e(u_HH_j^i-H_a^iH_{jb}u^au^b).$$

Thus the equation (3′) is written in the form

$$(n-1)K_0u+e(u_HH-H_{ai}H_j^au^iu^j)=\frac{R}{n}u.$$

Since the above equation is satisfied for any vector u^i, then we have

$$(6)\qquad H_{ai}H_j^a=HH_{ij}+Lg_{ij},$$

where by definition $L=e((n-1)K_0-R/n)$. The equation (6) gives

$$g^{ai}g^{bj}H_{ab}H_{ij}=H^2+nL. \tag{7}$$

Next, making use of (5), (6) and (7), the equation (4′) is written as follows.

$$((n-1)K_0^2+L^2-2eLK_0)u^2+(n-2)Lu_H^2=K_2u^2,$$

from which we obtain

$$((n-1)K_0^2+L^2-2eLK_0-K_2)(g_{hi}g_{jk}+g_{hj}g_{ki}+g_{hk}g_{ij})$$
$$+(n-2)L(H_{hi}H_{jk}+H_{hj}H_{jk}+H_{hk}H_{ij})=0.$$

Contracting this equation by g^{hi} and making use of (6), we have

$$[(n+2)((n-1)K_0^2-2eLK_0-K_2)+(3n-2)L^2]g_{jk} \tag{8}$$
$$+3(n-2)LHH_{jk}=0.$$

If $LH\neq 0$, it follows from (8) that there exists a scalar p such that $H_{jk}=p\cdot g_{jk}$, and hence we see from (2) that H^n is of constant curvature.

Next we suppose $L=0$. In this case we have from (6)

$$H_{ai}H_j^a=HH_{ij}. \tag{9}$$

Since H^n is a proper hypersurface of S^{n+1}, we may refer to the coördinates x^i such that the coördinate lines coincide with the lines of curvature, and then we have $g_{ii}=e_i$, $g_{ij}=0$, $H_{ii}=e_i\rho_i$ and $H_{ij}=0$ $(i\neq j)$; where $e_i=\pm 1$ are the indicators of the lines of curvature and ρ_i are the characteristic roots (real) of the determinant equation $|H_{ij}-\rho g_{ij}|=0$. If we take $i=j$ in (9), then we obtain

$$\rho_i(\rho_1+\cdots+\rho_n-\rho_i)=0 \qquad (i=1,\cdots,n). \tag{9′}$$

If $n-1$ of ρ_i are equal to zero, then the rank of the matrix (H_{ij}) is less than two, so that $H_{hj}H_{ik}-H_{hk}H_{ij}=0$. Therefore the equation (2) shows that H^n is of constant curvature. Next we suppose $\rho_1\rho_2\neq 0$. If we take $i=1$ and 2 in (9′), then we have $\rho_2+\rho_3+\cdots+\rho_n=0$, $\rho_1+\rho_3+\cdots+\rho_n=0$, from which $\rho_1=\rho_2=\rho$. Accordingly we obtain

$$\rho+\rho_3+\cdots+\rho_n=0. \tag{10}$$

If we take $i=p>2$ in (9′) then $\rho_p(\rho_p-\rho)=0$. Hence we have $\rho_p=0$ or ρ, so that $\rho=0$ from (10), contradicting to hypothesis.

Finally we suppose $H=0$. We refer again to the above coördi nates x^i and then $H=0$ is written as follows.

$$\rho_1+\cdots+\rho_n=0. \tag{11}$$

It follows from (6) that $H_{\alpha i}H_j^\alpha=Lg_{ij}$, that is, $\rho_i^2=L(i=1, \cdots, n)$, so that we have $L\geqq 0$. If $L=0$, then all of ρ_i vanish and H^n is of constant curvature. If $L=l^2(l>0)$, then the equation (11) shows that the dimensional number n be even $(n=2r)$ and $\rho_1=\cdots=\rho_r=+l$, $\rho_{r+1}=\cdots=\rho_{2r}=-l$. Consequently, by means of the theorem (6·2) of Fialkow's paper [**2**], H^n is a product of spaces S^r and S'^r, both of which are of constant curvature K, given by

$$K=e(\pm l)^2+K_0=eL+K_0, \tag{12}$$

making use of the equation (6·13) of the paper [**2**]. Now, the similar process which was used in Lichnerowicz's paper [**1**] to obtain $f''(0)=0$ from the fact that H^n is decomposable, can be applicable to the equation (4) and gives $L_2=0$. On the other hand, we have $el(-l)+K_0=K_0-eL=0$, making use of the equation (6·20) of the paper [**2**]. We obtain from (8)

$$(n+2)\left((n-1)K_0^2-2eLK_0\right)+(3n-2)L^2=0,$$

in virtue of $H=K_2=0$. Combining this equation with $L=eK_0$, we have $K_0=L=0$, so that $K=0$ from (12). Therefore both of spaces S^r and S'^r are locally flat, and hence H^n is so.

We have finally proved Theorem (L), under the restriction that H^n is a *proper* hypersurface of S^{n+1}.

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