On Cartan connexions and their torsions

By

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Introduction

In recent years, the general theory of connexions on differentiable fibre bundles has been developed by many authors. The purpose of this paper is to investigate on the structure of Cartan connexion and some facts related with it. One of the useful notions introduced in § 1 is the tensorial form on a principal fibre bundle. In § 2, we define the basic tensorial form of the soldered structure of bundle, and making its use we give expositions of Cartan connexions. The last section is concerned with the torsion forms of Cartan connexions. I think that the underlying principle of the tensor calculus for general Cartan connexion has been made clear through these debates.

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§ 1. Preliminaries and notations

1. Throughout this paper we shall denote by $T(X)$ and by $T_x(X)$ the tangent vector bundle over any differentiable manifold X and the tangent vector space of X at a point $x \epsilon X$ respectively. Any differentiable mapping φ of X into another differentiable manifold X' induces a linear mapping $\varphi^* : T_x(X) \rightarrow T_{x'}(X')$, where $x' = \varphi(x)$.

Let $\mathfrak{B}(M, Y, G, \pi)$ be a differentiable fibre bundle, where M, Y, G, and π denote respectively the base space, the fibre, the structure

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1), 2) Cf. [10] pp. 37~39. The number in the bracket refers to the bibliography at the end of the paper.

group, and the projection. We assume that M and Y are differentiable manifolds and that G is a Lie group of transformations which operate effectively and differentiably on Y. The associated principal bundle $\mathfrak{B}''$ of $\mathfrak{B}$ can be regared as the set of all *admissible maps*[1]. A tangent vector $v \epsilon T(\mathfrak{B})$ is said to be *vertical*, if it is tangent to a fibre Y_x over $x \epsilon M$. The *principal map*[2] $\chi : \mathfrak{B}'' \times Y \to \mathfrak{B}$ is defined by

$$\chi(b, y) = by \quad \text{for} \quad b \epsilon \mathfrak{B}'', y \epsilon Y.$$

In particular, for the associated principal bundle $\mathfrak{B}''$, the principal map $\chi'' : \mathfrak{B}'' \times G \to \mathfrak{B}''$ is given by

$$\chi''(b, s) = bs \quad \text{for} \quad b \epsilon \mathfrak{B}'', s \epsilon G.$$

If we set

$$\rho(s)b = bs \quad \text{for} \quad s \epsilon G,$$

the homeomorphism $\rho(s) : \mathfrak{B}'' \to \mathfrak{B}''$ which transforms each fibre G_x of $\mathfrak{B}''$ onto itself is called a *right translation* of $\mathfrak{B}''$, and its induced mapping $\rho^*(s) : T(\mathfrak{B}'') \to T(\mathfrak{B}'')$ transforms each vertical vector of $T(\mathfrak{B}'')$ on such a vector.

DEFINITION 1·1. Let $\mathfrak{B}''(M, G, G)$ be a differentiable principal fibre bundle over a differentiable manifold M with Lie group G, and let $\rho(s)$ denote the right translation of $\mathfrak{B}''$ corresponding to $s \epsilon G$. Let (r^*, R) be any differentiable representation of G on a vector space R; that is, to each $s \epsilon G$, corresponds a linear automorphism $r^*(s)$ of R such that $r^*(s_1 s_2^{-1}) = r^*(s_1) r^*(s_2)^{-1}$. *A k-from θ on $\mathfrak{B}''$ is called a tensorial k-form of type (r^*, R), if it satisfies the following conditions:*

    (i) *θ is a k-form on $\mathfrak{B}''$ with values in R.*[3]

    (ii) *$\theta(t_1, \cdots, t_k) = 0$, if t_1 is vertical.*

    (iii) *$\theta_\rho^*(s) = r^*(s^{-1}) \theta$ for any $s \epsilon G$.*

In the case $k = 0$, a differentiable mapping $\theta : \mathfrak{B}'' \to R$ is called a tensor of type (r^*, R), if it satisfies the relation:

$$\theta_\rho(s) = r^*(s^{-1}) \theta \quad \text{for any} \quad s \epsilon G.$$

2. Let us consider a Lie group $\hat{G}$ of transformations which operate differentiably, transitively, and effectively on a differentiable manifold F; namely F is a homogeneous space $\hat{G}/G$, where G is

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    3) Cf. [12].

the closed subgroup of $\hat{G}$ leaving invariant a point $o \epsilon F$. We shall denote the transformation on F corresponding to each $\hat{s} \epsilon \hat{G}$ by $\alpha_0(\hat{s})$: *i.e.*

$$\alpha_0(\hat{s})y = \hat{s}y \quad \text{for all} \quad y \epsilon F.$$

Let $\alpha(\hat{s})$ denote the inner automorphism of $\hat{G}$ corresponding to each $\hat{s} \epsilon \hat{G}$: *i.e.*

$$\alpha(\hat{s})y = \hat{s}y\hat{s}^{-1} \quad \text{for all} \quad y \epsilon \hat{G}.$$

The Lie algebra $\hat{L}$ of $\hat{G}$ can be identified with the tangent vector space $T_\varepsilon(\hat{G})$ at the neutral element $\varepsilon \epsilon \hat{G}$. The Lie algebra L of G can be identified with $T_\varepsilon(G)$ being a linear subspace of $\hat{L}$. Moreover the tangent space $E = T_o(F)$ at the point $o \epsilon F$ can be identified with the vector space $\hat{L}/L$. Since $\alpha(\hat{s})$ for $\hat{s} \epsilon \hat{G}$ leaves invariant the neutral element ε, it induces an automorphism $\alpha^*(\hat{s})$ of $\hat{L} = T_\varepsilon(G)$ which is an element of the so-called *linear adjoint group* of $\hat{G}$; and since $\alpha_0(s)$ for $s \epsilon G$ leaves invariant the point $o \epsilon F$, it induces an automorphism $\alpha_0^*(s)$ of $E = T_o(F)$ which is an elemant of the so-called *linear isotropy group* of G. Let p denote the canonical projection: $\hat{G} \to F = \hat{G}/G$. We have $p\alpha(s) = \alpha_0(s)p$ for $s \epsilon G$. Then, p induces the projection $p^* : L \to E = \hat{L}/L$, and we have $p^*\alpha^*(s) = \alpha_0^*(s)p^*$ for $s \epsilon G$. The subspace L in $\hat{L}$ is invariant under $\alpha^*(s)$ for $s \epsilon G$. Hence, $\alpha^*(s)$ induces an automorphism of $E = \hat{L}/L$ which coincides with $\alpha_0^*(s)$. Throughout this paper, we shall denote by $(\alpha^*, \hat{L})$ or (α^*, L) the linear adjoint representations of $\hat{G}$ or G and by (α_0^*, E) the linear isotropy representation of G.

3. Take a base $(e_1, \cdots, e_n, e_{n+1}, \cdots, e_r)$ of the Lie algebra $\hat{L}$ such that the set $(e_{n+1}, \cdots, e_r)$ forms a base of L. We have then the equations of structure:

$$(1 \cdot 1) \qquad [e_A, e_B] = \sum_{K=1}^{r} C_{AB}^{K} e_K \quad (A, B = 1, \cdots, r),$$

in which

$$(1 \cdot 2) \qquad C_{\alpha\beta}^{i} = 0 \quad (i = 1, \cdots, n; \alpha, \beta = n+1, \cdots, r).$$

We shall use, throughout the paper, the following ranges of indices:

$$A, B, C, \cdots = 1, 2, \cdots, r,$$

$$i, j, k, \cdots = 1, 2, \cdots, n,$$

$$\alpha, \beta, \gamma, \cdots = n+1, n+2, \cdots, r.$$

By a change of base of $\hat{L}$, we shall mean only that of the following type:

$$(1\cdot3) \qquad \bar{e}_A = \sum_B a_A^B e_B, \qquad a_\alpha^i = 0, \qquad |a_j^i| \neq 0, \qquad |a_\beta^\alpha| \neq 0.$$

Denoting by $\bar{C}_{AB}^K$ the structure constants with respect to the new base, we get the relations:

$$(1\cdot4) \qquad \sum_h a_h^i \bar{C}_{\alpha j}^h = \sum_{\beta,k} C_{\beta k}^i a_\alpha^\beta a_j^k.$$

Identifying E with $\hat{L}/L$, we can regard the set $(e_1, \cdots, e_n)$ as a base of E. Denote by $\hat{L}^*$, L^*, E^* the dual spaces of $\hat{L}$, L, E respectively. For a base $(e_1, \cdots, e_n, e_{n+1}, \cdots, e_r)$ of $\hat{L}$, its dual base $(e^1, \cdots, e^n, e^{n+1}, \cdots, e^r)$ can be finded in $\hat{L}^*$, and the sets $(e^{n+1}, \cdots, e^r)$, $(e^1, \cdots, e^n)$ can be regarded as the bases of L^*, E^* respectively. According to the relation $(1\cdot4)$, we see that the element $\sum_{\alpha,j,k} C_{\alpha j}^k e^\alpha \otimes e^j \otimes e_k$ belonging to the space $L^* \otimes E^* \otimes E$ does not depend on the choice of the base, where $\otimes$ denotes the tensor product. Taking a base of $\hat{L}$, we can express an elemat $\alpha^*(\hat{s})$ of the linear adjoint group of $\hat{G}$ by a matrix $\|\alpha_B^A(\hat{s})\|$:

$$(1\cdot5) \qquad \alpha^*(\hat{s})x = \sum_{A,B} \alpha_B^A(\hat{s}) x^B e_A \qquad \text{for all} \qquad x = \sum_A x^A e_A \in \hat{L}.$$

From the well-known relation

$$(1\cdot6) \qquad [\alpha^*(\hat{s})x, \alpha^*(\hat{s})y] = \alpha^*(\hat{s})[x, y]$$

for $\hat{s} \in \hat{G}$ and $x, y \in \hat{L}$, it follows that

$$(1\cdot7) \qquad \sum_L \alpha_L^A(\hat{s}) C_{BC}^L = \sum_{J,K} C_{JK}^A \alpha_B^J(\hat{s}) \alpha_C^K(\hat{s}) \qquad \text{[4]} \quad \text{for} \quad \hat{s} \in \hat{G}.$$

Above all, since $\alpha_\alpha^j(s) = 0$ for $s \in G$, we have

$$(1\cdot8) \qquad \sum_h \alpha_h^i(s) C_{\alpha j}^h = \sum_{\beta,k} C_{\beta k}^i \alpha_\alpha^\beta(s) \alpha_j^k(s) \qquad \text{for} \quad s \in G.$$

An element $\alpha_0^*(s)$ of the linear isotropy group of G is given by the matrix $\|\alpha_j^i(s)\|$, as the base $(e_1, \cdots, e_n)$ of E is taken: *i.e.* from $(1\cdot5)$ it follows that

$$(1\cdot9) \qquad \alpha^*(s)x \equiv \sum_{i,j} \alpha_j^i(s) x^j e_i \qquad (mod \ L).$$

4. We put $V = E^* \otimes L$ and $W = (E^* \wedge E^*) \otimes E$. Referred to

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4) Cf. [7], p·3.

a base $(e_1, \cdots, e_n, e_{n+1}, \cdots, e_r)$, each element $\hat{\xi} \epsilon V$ can be written as

$$\xi = \sum_{\alpha, j} \xi_j^{\alpha} e^j \otimes e_{\alpha},$$

and each element $\eta \epsilon W$ can be expressed by

$$\eta = \sum_{i, j, k} \eta_{jk}^i e^j \wedge e^k \otimes e_i$$

with

$$\eta_{jk}^i + \eta_{kj}^i = 0.$$

DEFINITION 1·2. *Let* $(e_1, \cdots, e_n, e_{n+1}, \cdots, e_r)$ *be a base of* $\hat{L}$. *The linear map* $\Phi : V \to W$ *is defined by*

$$(1 \cdot 10) \qquad \Phi \left(\sum_{\alpha, k} \xi_k^{\alpha} e^k \otimes e_{\alpha} \right) = \sum_{i, j, k} \eta_{jk}^i e^j \wedge e^k \otimes e_i.$$

with

$$(1 \cdot 11) \qquad \eta_{jk}^i = \sum_{\alpha} (C_{\alpha j}^i \xi_k^{\alpha} - C_{\alpha k}^i \xi_j^{\alpha}).$$

The representations (α^*, L), (α_0^*, E) naturally induce the representations on the spaces V and W which we denote by (α_1^*, V) and (α_2^*, W) respectively. According to the relation $(1 \cdot 4)$ and $(1 \cdot 8)$, we obtain immediately the following proposition.

PROPOSITION 1·1 *The linear map* $\Phi : V \to W$ *does not depend on the choice of the base, and it holds that*

$$\Phi \alpha_1^*(s) = \alpha_2^*(s) \Phi \qquad \text{for any} \qquad s \epsilon G.$$

§2. The soldered fibre bundle and the Cartan connexion

5. In this and following sections we assume that a differentiable fibre bundle $\mathfrak{B}(M, F, \hat{G})$ over a differentiable manifold M fulfills the following conditions:

(i) *The fibre* F *is the homogeneous space* $\hat{G}/G$ *which we have introduced in the preceding section.*

(ii) *There exists a differentiable cross-section* $f : M \to \mathfrak{B}$.

(iii) *The dimensions of* F *and* M *are equal.*

From the condition (ii), it follows that the associated principal bundle $\hat{\mathfrak{B}}(M, \hat{G}, \hat{G})$ of $\mathfrak{B}$ is equivalent to a principal bundle $\mathfrak{B}''(M, G, G)$ whose group is the subgroup G leaving invariant a point $o \epsilon F$, and $\mathfrak{B}''$ is the submanifold of $\hat{\mathfrak{B}}$ such that $\chi(\mathfrak{B}'', o) = f(M)$, where χ denotes the prinicipal map of $\mathfrak{B}$. Identifying any point

$x \epsilon M$ with the point $f(x) \epsilon \mathfrak{B}$, we can suppose the base space M to be embedded in $\mathfrak{B}$.

Let $\mathfrak{X}(M)$ be the submanifold of $T(\mathfrak{B})$ consisting of all vertical vectors at any point of $f(M)$. Then $\mathfrak{X}(M)$ becomes an associated bundle of $\mathfrak{B}^0$ with fibre $E = T_0(F)$ and with the linear isotropy group as its structure group. Each fibre $\mathfrak{X}_x(M)$ of $\mathfrak{X}(M)$ over $x \epsilon M$ is the space of all vertical vectors of $T(\mathfrak{B})$ at $f(x)$.

DEFINITION 2·1. *In the above circumstances, the bundle $\mathfrak{B}^0$ is said to be soldered with $\mathfrak{B}$, if there is given an identification $\iota : T(M) \to \mathfrak{X}(M)$; namely, ι is a bundle homeomorphism : $T(M) \to \mathfrak{X}(M)$, and is an isomorphism between $T_x(M)$ and $\mathfrak{X}_x(M)$ for each $x \epsilon M$.*

It is clear that the condition (iii) is necessary for existence of a soldered structurs.

Let $(\iota, \mathfrak{B}^0)$ be a soldered bundle with $\mathfrak{B}$ and let $\chi(b)$ denote the admissible map : $F \to F_x$ of $\mathfrak{B}$ corresponding to $b \epsilon \mathfrak{B}^0$:

$$\chi(b) y = \chi(b, y) = by \quad \text{for} \quad y \epsilon F.$$

We denote by π the projection of $\mathfrak{B}^0$. Since $\chi(b)$ maps the point $o \epsilon F$ to the point $f(\pi b) \epsilon \mathfrak{B}$, it induces an onto isomorphism $\chi^*(b) : E \to \mathfrak{X}_{\pi b}(M)$. Recall the right translation $\rho(s)$ of $\mathfrak{B}^0$ and the transformation $\alpha_0(s)$ on F corresponding to $s \epsilon G$:

$$\rho(s) b = \chi^0(b, s) = bs \quad \text{for} \quad b \epsilon \mathfrak{B}^0,$$

$$\alpha_0(s) y = sy \quad \text{for} \quad y \epsilon F.$$

Obviously it holds that

$$\chi(\rho(s) b) = \chi(b) \alpha_0(s),$$

and hence we have the formula :

(2·1) $$\chi^*(\rho(s) b) = \chi^*(b) \alpha_0^*(s).$$

DEFINITION 2·2. *In the above circumstance, the basic form ω_0 of a soldered bundle $(\iota, \mathfrak{B}^0)$ with $\mathfrak{B}$ is defined by*

(2·2) $$\omega_0(t) = \chi^{*-1}(b) \iota \pi^* t,$$

for any tangent vector $t \epsilon T_b (\mathfrak{B}^0)$.

PROPOSITION 2·1. *The basic form ω_0 of a soldered bundle $(\iota, \mathfrak{B}^0)$ with $\mathfrak{B}$ satisfies the following conditions :*

  (i) *ω_0 is a tensorial 1-form on $\mathfrak{B}^0$ of type (α_0^*, E).*
  (ii) *If $\omega_0(t) = 0$, then t is vertical.*

Proof. By the definition, ω_0 becomes a 1-from on $\mathfrak{B}^0$ with values in E. For a vertical vector t of $T(\mathfrak{B}^0)$, we have $\omega_0(t) = 0$ because $\pi^* t = 0$. Since $\rho(s)$ transforms each fibre of $\mathfrak{B}^0$ onto itself, we see $\pi^* \rho^*(s) = \pi^*$ for any $s \epsilon G$. If $t \epsilon T_b(\mathfrak{B}^0)$, then $\rho^*(s) t \epsilon T_{\rho(s)b}(\mathfrak{B}^0)$. On the other hand, from (2·1) it follows that

$$\chi^{*-1}(\rho(s)b) = \alpha_0^*(s^{-1}) \chi^{*-1}(b).$$

Therefore

$$\omega_0 \rho^*(s) = \chi^{*-1}(\rho(s)b) \iota \pi^* \rho^*(s)$$
$$= \alpha_0^*(s^{-1}) \chi^{*-1}(b) \iota \pi^* = \alpha_0^*(s^{-1}) \omega_0.$$

This proves that ω_0 is a tensorial 1-form of type (α_0^*, E). Moreover, since $\chi^*(b)$ and ι are one-to-one mappings, $\omega_0(t) = 0$ implies $\pi^*(t) = 0$. This means that t is vertical.

We shall show conversely that a soldered structure of bundle is completely determined by its basic form.

PROPOSITION 2·2. *Suppose that the bundle $\mathfrak{B}$ fulfills the three conditions introduced at the beginning of this section. If there is given on the bundle $\mathfrak{B}^0$ a 1-form ω_0 having the properties* (i) *and* (ii) *in the Proposition* 2·1, *there exists one and only one soldered structure* $(\iota, \mathfrak{B}^0)$ *with $\mathfrak{B}$ having ω_0 as its basic form.*

Proof. For any vector $v \epsilon T_x(M)$ there exists a vector $t \epsilon T_b(\mathfrak{B}^0)$ such that $\pi^* t = v$. Let us show that the element $\chi^*(b) \omega_0(t) \epsilon \mathfrak{T}_x(M)$ does not depend on the choice of t. Let t and t' be two vectors of $T(\mathfrak{B}^0)$ such that $\pi^* t = \pi^* t' = v$. If $t \epsilon T_b(\mathfrak{B}^0)$ and $t' \epsilon T_{b'}(\mathfrak{B}^0)$, there exists a unique right translation $\rho(s)$ of $\mathfrak{B}^0$ such that $\rho(s)b = b'$, because $\pi b = \pi b' = x$. Since $\pi^* \rho^*(s) = \pi^*$, we find $\pi^*(t' - \rho^*(s)t) = \pi^* t' - \pi^* t = 0$; that is, the vector $t' - \rho^*(s)t \epsilon T_{b'}(\mathfrak{B}^0)$ is vertical. Making use of the property (i), we have $\omega_0(t' - \rho^*(s)t) = 0$ and $\omega_0(\rho^*(s)t) = \alpha_0^*(s^{-1}) \omega_0(t)$. It follows that $\omega_0(t') = \alpha_0^*(s^{-1}) \omega_0(t)$. On the other hand the formula (2·1) shows that $\chi^*(b') = \chi^*(b) \alpha_0^*(s)$. Consequently we have

$$\chi^*(b') \omega_0(t') = \chi^*(b) \omega_0(t).$$

Accordingly, we can define a mapping $\iota : T(M) \rightarrow \mathfrak{T}(M)$ by

(2·3) $\qquad \iota v = \chi^*(b) \omega_0(t) \quad$ for $\quad v \epsilon T(M)$,

where $t \epsilon \pi^{*-1}v$ and $t \epsilon T_b(\mathfrak{B}^0)$. If $v \neq 0$, then t is not vertical; and, from the propery (ii), it follows that $\omega_0(t) \neq 0$, and so $\iota v \neq 0$. This

means that ι is univalrnt. Since $dim\ T_x(M) = dim \mathfrak{T}_x(M)$, ι becomes an onto isomorphism between $T_x(M)$ and $\mathfrak{T}_x(M)$ for each $x \in M$. The desired soldered structure $(\iota, \mathfrak{B}^0)$ is thus defined. Our process having constructed the identification ι assures that there exists one and only one bundle map ι satisfying the relation: $\omega_0 = \chi^{*-1}(b)\iota\pi^*$, when a form ω_0 having the two properties is given. The Proposition 2·2 has been proved.

REMARK. Observing $dim\ T_x(M) = dim\ E$, we can show easily that the condition (ii) in the Proposetion 2·1 is equivalent to the condition :

(ii)' ω_0 *maps* $T_b(\mathfrak{B}^0)$ *onto* E *for each* $b \in \mathfrak{B}^0$.

6. Let $\hat{\mathfrak{B}}(M, \hat{G}, \hat{G})$ be a differentiable principal bundle. We denote by $\hat{\chi}(\hat{b})$ the admissible map of $\hat{\mathfrak{B}}$ corresponding to $\hat{b} \in \hat{\mathfrak{B}}$, and by $(\alpha^*, \hat{L})$ the linear adjoint representation of $\hat{G}$. A *connexion* on the principal bundle $\hat{\mathfrak{B}}$ can be defined by a 1-form $\hat{\omega}$ satisfying the following conditions :[5]

(i) $\hat{\omega}$ *is a* 1-*form on* $\hat{\mathfrak{B}}$ *with values in the Lie algebra* $\hat{L}$.

(ii) *If a vector* $\hat{t} \in T_{\hat{b}}(\hat{\mathfrak{B}})$ *is vertical, then* $\hat{\omega}(\hat{t}) = \hat{\chi}^{*-1}(\hat{b})\hat{t}$.

(iii) *For any right translation* $\rho(\hat{s})$ *of* $\hat{\mathfrak{B}}$, *it holds that*
$$\hat{\omega}\rho^*(\hat{s}) = \alpha^*(\hat{s}^{-1})\ \hat{\omega}.$$

We shall call the form $\hat{\omega}$ the *Pfaffian form of connexion on* $\hat{\mathfrak{B}}$, or, merely the *connexion on* $\hat{\mathfrak{B}}$ for the sake of simplicity.

In general, the *bracket product*[6] of forms on a differentiable manifold X with valus in a Lie algebra A is defined by

$$[\theta, \varphi](t_1 \cdots, t_{k+h})$$
$$= \sum_\sigma \frac{\varepsilon(\sigma)}{(k+h)!} [\theta(t_{\sigma(1)}, \cdots, t_{\sigma(k)}), \varphi(t_{\sigma(k+1)}, \cdots, t_{\sigma(k+h)})],$$

where k, h are degrees of forms θ, φ respectively, $t_1, \cdots, t_{k+h} \in T(X)$ $x \in X$, the summation is extended over all permutations σ of the set $\{1, 2, \cdots, k+h\}$, and $\varepsilon(\sigma)$ is the sign of σ. Then, we have the relations :

$$[\varphi, \theta] = (-1)^{kh-1}[\theta, \varphi],$$
$$(-1)^{h(l+l)}[\theta,[\varphi, \psi]] + (-1)^{l(k+h)}[\varphi, [\psi, \theta]] + (-1)^{k(l+h)}[\psi, [\theta, \varphi]] = 0,$$
$$d[\theta, \varphi] = [d\theta, \varphi] + (-1)^k[\theta, d\varphi],$$

5) Cf. [6], [1].

6) Cf. [12].

where k, h, l are degrees of θ, φ, ψ respectively.

The curvature form $\hat{\Omega}$ of a connexion $\hat{\omega}$ on $\hat{\mathfrak{B}}$ is given by the *equation of structure*[7]

$$(2\cdot4) \qquad\qquad d\hat{\omega} = -\tfrac{1}{2}[\hat{\omega}, \hat{\omega}] + \hat{\Omega},$$

and *Bianchi's identity*[8] is written as

$$(2\cdot5) \qquad\qquad d\hat{\Omega} = [\hat{\Omega}, \hat{\omega}].$$

The following proposition is well-kown.

PROPOSITION 2·3.[9] *The curvature from $\hat{\Omega}$ is a tensorial 2-form on $\hat{\mathfrak{B}}$ of type* $(\alpha^*, \hat{L})$.

DEFINITION 2·3. Let $\mathfrak{B}^0(M, G, G)$ be a soldered bundle with $\mathfrak{B}(M, F, \hat{G})$, and let $\hat{\mathfrak{B}}(M, \hat{G}, \hat{G})$ denote the associated principal bundle of $\mathfrak{B}$. *A connexion $\hat{\omega}$ on $\hat{\mathfrak{B}}$ is said to be a Cartan connexion of type F on $\hat{\mathfrak{B}}$ with respect to the soldered structure, if the restriction of $p^*\hat{\omega}$ on $\mathfrak{B}^0$ coincides with the basic form ω_0 of the soldered structure,* where $p: \hat{G} \to F = \hat{G}/G$ denotes the canonical projecton.

PROPOSITION 2·4.[10] *Suppose that the bundle $\mathfrak{B}$ fulfills the three conditions introduced at the beginning of § 2. A Cartan connexion $\hat{\omega}$ of type F on $\hat{\mathfrak{B}}$ can be defined, if there is given on $\mathfrak{B}^0$ a 1-form ω satisfying the following conditions:*

(i) *ω is a 1-form on $\mathfrak{B}^0$ with values in $\hat{L}$.*

(ii) *If a vector $t \in T_b(\mathfrak{B}^0)$ is vertical, then $\omega(t) = \chi^{0*-1}(b)t$.*

(iii) *For any right translation $\rho(s)$ of $\mathfrak{B}^0$, it holds that*
    $\omega\rho^*(s) = \alpha^*(s^{-1})\omega.$

(iv) *If $\omega(t) = 0$, then $t = 0$.*

In this case, ω is the restriction of $\hat{\omega}$ on $\mathfrak{B}^0$.

We shall call ω the *restricted Pfaffian form of the Cartan connexion on $\mathfrak{B}^0$*, or, merely the *Cartan connexion on $\mathfrak{B}^0$* for the sake of simplicity.

Proof. Since any element $\hat{b} \in \hat{\mathfrak{B}}$ can be written as $\hat{b} = \rho(\hat{s})b$ where $\hat{s} \in \hat{G}$ and $b \in \mathfrak{B}^0$, any vector $\hat{t} \in T_{\hat{b}}(\hat{\mathfrak{B}})$ can be given by $\hat{t} = \rho^*(\hat{s})t + t_0$, where $t \in T_b(\mathfrak{B}^0)$, and t_0 is a vertical vector of $T_{\hat{b}}(\hat{\mathfrak{B}})$. Extend ω to a form $\hat{\omega}$ over the whole $\hat{\mathfrak{B}}$ by

7) Cf. [1].
8) Cf. [12].
9) Cf. [1].
10) Cf. [6], p. 43.

(2·6) $\hat{\omega}(\hat{t}) = \alpha^*(\hat{s}^{-1})\omega(t) + \hat{\chi}^{*-1}(b) t_0.$

Since the conditions (ii) and (iii) assure that $\hat{\omega}(\hat{t})$ does not depend on the choice of elements $\hat{s}$, t, and t_0 such that $\hat{t} = \rho^*(\hat{s})t + t_0$, the definition (2·6) of $\hat{\omega}$ has a sense, and $\hat{\omega}$ becomes a connexion on $\hat{\mathfrak{B}}$. Moreover, it is easy to see that the form $\omega_0 = p^*\omega$ satisfies the conditions in the Proposition 2·1; and so, by the Proposition 2·2, the seldered structure $(\iota, \mathfrak{B}^0)$ with $\mathfrak{B}$ can be determined. Our proposition has been thus proved.

Taking account of the condition (iv) and observing that *dim* $T_b(\mathfrak{B}^0) = dim \hat{L}$, we have the following:

PROPOSITION 2·5.[11] *Let ω be a Cartan connexion on $\mathfrak{B}^0$. Then ω is an onto isomorphism between $T_b(\mathfrak{B}^0)$ and $\hat{L}$ for each $b \in \mathfrak{B}^0$. It follows that the tangent vector bundle $T(\mathfrak{B}^0)$ over $\mathfrak{B}^0$ is equivalent to a product bundle $\mathfrak{B}^0 \times \hat{L}$.*

Accordingly, we can define the absolute parallelism on $T(\mathfrak{B}^0)$.

DEFINITION 2·4. Let ω be a Cartan connexion on $\mathfrak{B}^0$. Two vectors t_1, $t_2 \in T(\mathfrak{B}^0)$ are said to be *parallel* with respect to the Cartan connexion, if $\omega(t_1) = \omega(t_2)$. A tangent vector field χ on $\mathfrak{B}^0$ is called a *parallel field* if $\omega(\chi) = constant$.

PROPOSITION 2·6. *If χ is a parallel field, then so is $\rho^*(s)\chi$, where $\rho(s)$ denotes a right translation of $\mathfrak{B}^0$.*

The homogeneous space F is said to be *reductive*,[12] if there exists in $\hat{L}$ a linear subspace complementary with L and invariant under $\alpha^*(s)$ for all $s \in G$. Then this linear subspace can be idetified with E, and the vector space $\hat{L}$ is decomposed by the direct sum: $\hat{L} = E + L$. And, for any $s \in G$, the restriction of $\alpha^*(s)$ on E is nothing but $\alpha_0^*(s)$.

PROPOSITION 2·7. *Assume that F is reductive and a decomposition $\hat{L} = E + L$ is given. Let $p_1^*: \hat{L} \to L$ be the projection with respect to the given decomposition. Then p_1^* gives the one-to-one correspondence between the set of all Cartan connexion of type F on $\hat{\mathfrak{B}}$ and the set of all connexion on $\mathfrak{B}^0$ having a soldered structure with $\mathfrak{B}$.*

Proof. Let ω be a Cartan connexion on $\mathfrak{B}^0$. It is obvious that $p^*\omega$ and $p_1^*\omega$ define respectively a soldered structure of $\mathfrak{B}^0$ with $\mathfrak{B}$ and a connexion on $\mathfrak{B}^0$. Conversely, if the basic form ω_0 of a soldered structure of $\mathfrak{B}^0$ with $\mathfrak{B}$ and a connexion ω_1 on $\mathfrak{B}^0$ are

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11) Cf. [6], p. 43.
12) Cf. [8].

given, then $\omega = \omega_0 + \omega_1$ being a form on $\mathfrak{B}^0$ with values in $\hat{L}$ defines a Cartan connexion of $\mathfrak{B}^0$.

DEFINITION 2·5. *A Cartan connexion ω on $\mathfrak{B}^0$ is said to be reductive, if the homogeneous space F is reductive and a decompodition $\hat{L} = E + L$ is given.*

The proposition 2·7 asserts that a reductive Cartan connexion ω can be decomposed as $\omega = \omega_0 + \omega_1$, where $\omega_0 = p^*\omega$ is the basic form of the soldered structure and $\omega_1 = p_1{}^*\omega$ is a connexion on $\mathfrak{B}^0$.

7. As to the existence of Cartan connexions, we have known the following proposition.

PROPOSITION 2·8.[13] *Let $\mathfrak{B}^0$ be a soldered bundle with $\mathfrak{B}$. Then, there exists a Cartan connexion of type F on $\hat{\mathfrak{B}}$ with respect to the soldered structure.*

Let ω and ω' be two Cartan connexions on a same soldered bundle $\mathfrak{B}^0$. Setting $\theta = \omega' - \omega$, we have $p^*\theta = 0$; and if a vector $t \epsilon T(\mathfrak{B}^0)$ is vertical, then $\theta(t) = 0$. Hence θ is a tensorial 1-form on $\mathfrak{B}^0$ of type (α^*, L). We have thus the following proposition:

PROPOTITION 2·9. *Let ω be a Cartan connexion on a soldered bundle $\mathfrak{B}^0$. Define the transformation τ^* on the set of all 1-forms on $\mathfrak{B}^0$ with values in $\hat{L}$ by*

$$\tau^*\psi = \psi - \omega.$$

Then τ^* gives the one-to-one correspondence between the set of all Cartan connexions on the soldered bundle $\mathfrak{B}^0$ and the set of all tensorial 1-forms on $\mathfrak{B}^0$ of type (α^*, L).

§ 3. The torsion of the Cartan connexion

8. Let $\hat{\Omega}$ be curvature form of a Cartan connexion $\hat{\omega}$ of type F on $\hat{\mathfrak{B}}$. We call the restriction of $\hat{\Omega}$ on $\mathfrak{B}^0$ the *restricted curvature form* of the Cartan connexion and denote it by Ω. Then Ω is given by the equation

(3·1) $$d\omega = -\tfrac{1}{2}[\omega, \omega] + \Omega,$$

where ω denotes the Cartan connexion on $\mathfrak{B}^0$.

DEFINITION 3·1. *Let Ω be the restricted curvature form of a Cartan connexion. The torsion form Ω_0 of the Cartan connexion is*

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13) Cf. [6], p. 43.

defined by

(3·2) $\Omega_0 = p^*\Omega.$

Since $\hat{\Omega}$ is a tensorial 2-form on $\hat{\mathfrak{B}}$ of type $(\alpha^*, \hat{L})$, we have:

PROPOSITION 3·1.   *The torsion form Ω_0 is a tensorial 2-form of type (α_0^*, E).*

A Cartan connexion on $\mathfrak{B}^0$ is said to be *without torsion*, if its torsion form vanishes.

PROPOSITION 3·2.   *If and only if a Cartan connexion on $\mathfrak{B}^0$ is without torsion, its restricted curvature form becomes a tensorial 2-form on $\mathfrak{B}^0$ of type (α^*, L).*

Take a base $(e_1, \cdots, e_n, e_{n+1}, \cdots, e_r)$ of $\hat{L}$ such that $(e_{n+1}, \cdots, e_r)$ and $(e_1, \cdots, e_n)$ become bases of L and E respectively. A Cartan connexion ω and its restricted curvature form Ω can be expressed by

$$\omega = \sum_A \omega^A \otimes e_A \quad \text{and} \quad \Omega = \sum_A \Omega^A \otimes e_A,$$

where ω^A and Ω^A are forms on $\mathfrak{B}^0$ with real values. Then, the basic form ω_0 of the soldered structure and the torsion form Ω_0 are written as

$$\omega_0 = \sum_i \omega^i \otimes e_i \quad \text{and} \quad \Omega_0 = \sum_i \Omega^i \otimes e_i.$$

Using the condition (ii) in the Proposition 2·1, we can show that the forms $\omega^1, \cdots, \omega^n$ are linearly independent; and taking account of the Propofition 3·1 we can set

(3·3) $\Omega^i = \sum_{j,k} S_{jk}^i \omega^j \wedge \omega^k,$

where S_{jk}^i are functions on $\mathfrak{B}^0$ with real values satisfying the relation

(3·4) $S_{jk}^i + S_{kj}^i = 0.$

If we put

(3·5) $S = \sum_{i,j,k} S_{jk}^i e^j \wedge e^k \otimes e_i,$

then S becomes a tensor on $\mathfrak{B}^0$ of type (α_2^*, W). We shall call S the *torsion tensor* of the Cartan connexion. The equation (3·1) is now written as

$$d\omega^A = \frac{1}{2} \sum_{B,C} C_{BC}^A \omega^B \wedge \omega^C + \Omega^A,$$

and hence we have

(3·6) $\qquad \Omega^i = d\omega^i - \frac{1}{2} \sum_{j,k} C^i_{jk}\omega^j \wedge \omega^k - \sum_{\alpha,k} C^i_{\alpha k}\omega^\alpha \wedge \omega^k.$

Let ω' be another Cartan connexion on $\mathfrak{B}^0$ having the common soldered structure with ω. For the curvature form, the torsion form, their components referred to the base, etc. of ω', we shall use the same notation with *primes* as those of ω. It follows that $\omega'^i = \omega^i$. Setting $\theta^\alpha = \omega'^\alpha - \omega^\alpha$, we have $\theta = \omega' - \omega = \sum_\alpha \theta^\alpha \otimes e_\alpha$. Since θ is a tensorial 1-form on $\mathfrak{B}^0$ of type (α^*, L), we can set

(3·7) $\qquad \theta^\alpha = \sum_j \Gamma^\alpha_j \omega^j,$

where Γ^α_j are functions on $\mathfrak{B}^0$ with real values. Setting

(3·8) $\qquad \Gamma = \sum_{\alpha,j} \Gamma^\alpha_j e^j \otimes e_\alpha,$

we see Γ is a tensor on $\mathfrak{B}^0$ of type (α_1^*, V) and that the Cartan connexion ω' is determined when the tensor Γ is assigned. From (3·6) we obtain the relation:

(3·9) $\qquad \Omega'^i - \Omega^i = - \sum_{\alpha,k} C^i_{\alpha k} \theta^\alpha \wedge \omega^k.$

Since $\omega^1, \cdots, \omega^n$ are linearly independent, this relation is translated to the equation:

$$2(S'^i_{jk} - S^i_{jk}) = \sum_\alpha (C^i_{\alpha j}\Gamma'^\alpha_k - C^i_{\alpha k}\Gamma'^\alpha_j),$$

that is,

(3·10) $\qquad 2(S' - S) = \Phi\Gamma,$

where Φ is the linear map: $V \to W$ introduced in the Definition 1·2. Finally we get the following:

PROPOSITION 3·3. *Take a fixed Cartan connexion on* $\mathfrak{B}^0$, *and let* $\bar{S}$ *denote its torsion tensor. Then, any Cartan connexion on the soldered bundle* $\mathfrak{B}^0$ *is defined, when a tensor* Γ *of type* (α_1^*, V) *is given; and its torsion tensor* S *is obtained from the formula:*

(3·11) $\qquad S = \frac{1}{2}\Phi\Gamma + \bar{S}.$

9. The exterior derivative of 1-form φ is given by the well-knwon formula:[14]

(3·12) $\qquad d\varphi(\mathfrak{x}, \mathfrak{y}) = \frac{1}{2}(\mathfrak{x}\varphi(\mathfrak{y}) - \mathfrak{y}\varphi(\mathfrak{x}) - \varphi([\mathfrak{x}, \mathfrak{y}])),$

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14) Cf. [1], [9].

where $\mathfrak{x}$, $\mathfrak{y}$ denote vector fields.

Take parallel fields $\mathfrak{x}$, $\mathfrak{y}$ with respect to a Cartan connexion ω. Since $\omega(\mathfrak{x})$ and $\omega(\mathfrak{y})$ are constant, $\mathfrak{x}\omega(\mathfrak{y})=\mathfrak{y}\omega(\mathfrak{x})=0$. From (3·1) and (3·12), it follows that

$$(3·13) \qquad \varOmega(\mathfrak{x}, \mathfrak{y}) = \tfrac{1}{2}([\omega(\mathfrak{x}), \omega(\mathfrak{y})] - \omega([\mathfrak{x}, \mathfrak{y}])).$$

Accordingly, as regards the torsion form $\varOmega_0$ and the basic form ω_0 of the soldered structure, we have

$$(3·14) \qquad \varOmega_0(\mathfrak{x}, \mathfrak{y}) = \tfrac{1}{2}(p^*[\omega(\mathfrak{x}), \omega(\mathfrak{y})] - \omega_0([\mathfrak{x}, \mathfrak{y}])).$$

Since $[\omega(\mathfrak{x}), \omega(\mathfrak{y})]$ is constant we obtain the following propositions.

PROPOSITION 3·4   *Let* $\mathfrak{x}$, $\mathfrak{y}$ *be parallel fields with respect to a Cartan connexion. Then* $[\mathfrak{x}, \mathfrak{y}]$ *becomes a parallel field if and only if* $\varOmega(\mathfrak{x}, \mathfrak{y})$ *is constant.*

PROPOSITION 3·5.   *Assume that the curvature form of a Cartan connexion vanishes. If* $\mathfrak{x}$, $\mathfrak{y}$ *are parallel field, then so is* $[\mathfrak{x}, \mathfrak{y}]$.

PROPOSITION 3·6.   *Assume that a Cartan connexion is without torsion. If* $\mathfrak{x}$, $\mathfrak{y}$ *are parallel fields, then* $\omega_0([\mathfrak{x}, \mathfrak{y}])$ *is constant, where* ω_0 *denotes the basic form of the soldered structure.*

10.   We are going to consider the case that the Cartan connexion is reductive.

DEFINITION 3·2.   *Let* $\omega = \omega_0 + \omega_1$, *be a reductive Cartan connexion on* $\mathfrak{B}^0$. *The torsion form* θ *of the connexion* ω_1 *is defined by*

$$(3·15) \qquad \theta = d\omega_0 + [\omega_1, \omega_0]^{15)}$$

PROPOSITION 3·7.   *Let* $\omega = \omega_0 + \omega_1$, *be a reductive Cartan connexion on* $\mathfrak{B}^0$. *The torsion form* θ *of the connexion* ω_1 *is a tensorial 2-form on* $\mathfrak{B}^0$ *of type* (α_0^*, E), *and is given by the relation*

$$(3·16) \qquad \varOmega_0 = \theta + \tfrac{1}{2}p^*[\omega_0, \omega_0],$$

where $\varOmega_0$ *denotes the torsion form of the Cartan connexion* ω.

Proof.   Since the linear subspace E in $\hat{L}$ is invariant under $\alpha^*(s)$ for $s\epsilon G$, $[\omega_1, \omega_0]$ is a 2-form on $\mathfrak{B}^0$ with values in E; therefore, so is θ. We have

$$\varOmega = d(\omega_0 + \omega_1) + \tfrac{1}{2}[\omega_0 + \omega_1, \omega_0 + \omega_1] = \theta + \tfrac{1}{2}[\omega_0, \omega_0] + \varOmega_1,$$

where $\varOmega_1$ is the curvature form of the connexion ω_1. Since $p^*\theta = \theta$

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15) θ is the so-called *covariant derivative* of ω_0 with respect to the connexion ω_1. Cf. [4], [1].

· and $p^*\Omega_1 = 0$, we obtain the relation :

$$\Omega_0 = \theta + \tfrac{1}{2}p^*[\omega_0,\ \omega_0]$$

Accordingly, in order to prove that θ is a tensorial 2-form of type (α^*_0, E), it is sufficient to show that so is $p^*[\omega_0, \omega_0]$. In this case, we have

$$[\omega_0,\ \omega_0](t_1,\ t_2) = [\omega_0(t_1),\ \omega_0(t_2)],$$

$$\omega_0 \rho^*(s) = \alpha_0^*(s^{-1})\,\omega_0 = \alpha^*(s^{-1})\,\omega_0 \quad \text{for} \quad s \epsilon G\,;$$

and from (1·6), it follows that $[\alpha^*(s^{-1})\,\omega_0,\ \alpha^*(s^{-1})\,\omega_0] = \alpha^*(s^{-1})[\omega_0,\ \omega_0]$. Hence, $[\omega_0,\ \omega_0]\rho^*(s) = \alpha^*(s^{-1})[\omega_0,\ \omega_0]$; and so

$$p^*[\omega_0,\ \omega_0]\rho^*(s) = \alpha_0^*(s^{-1})\,p^*[\omega_0,\ \omega_0].$$

Moreover, if t_1 is vertical, $[\omega_0,\ \omega_0](t_1,\ t_2) = 0$ because $\omega_0(t_1) = 0$. This proves that $p^*[\omega_0,\ \omega_0]$ is a tensorial 2-form of type (α_0^*, E).

PROPOSITION 3·8. *If the homogeneous space F is symmetric, then the torsion form Ω_0 of the Cartan connexion $\omega_0 + \omega_1$ coincides with the torsion form θ of the connexion ω_1.*

Proof. Since F is a symmetric space, $[a, b] \epsilon L$ provided $a, b \epsilon E$. Hence we have $p^*[\omega_0,\ \omega_0] = 0$, and from (3·16) it follows that $\Omega_0 = \theta$.

Let $\omega = \omega_0 + \omega_1$, be a reductive Cartan connexion. The subset $\omega^{-1}(E)$ in $T(\mathfrak{B}'')$ constitutes the set of all *horizontal vectors*[16] of the connexion ω_1, because $\omega_1(t) = 0$ if and only if $\omega(t) \epsilon E$. A parallel field $\mathfrak{x}$ with respect to the Cartan connexion ω will be called a *horizontal parallel field, if $\omega(\mathfrak{x})\ (= const.) \epsilon E$.*

PROPOSITION 3·9. *Let $\omega = \omega_0 + \omega_1$ be a reductive Cartan connexion. The connexion ω_1 is without torsion: i.e. $\theta = 0$, if and only if every vector field $[\mathfrak{x}, \mathfrak{y}]$ for horizontal parallel field $\mathfrak{x}, \mathfrak{y}$ is vertical.*

Proof. For any $t \epsilon T(\mathfrak{B}'')$, we can take a unique horizontal parallel field $\mathfrak{x}$ such that $\omega(\mathfrak{x}) = \omega_0(t)$. It follows that there exist horizontal parallel fields $\mathfrak{x}_1, \mathfrak{x}_2$ such that $\theta(\mathfrak{x}_1(b),\ \mathfrak{x}_2(b)) = \theta(t_1, t_2)$ for arbitrary $t_1, t_2 \epsilon T_b(\mathfrak{B}'')$.

Suppose $\mathfrak{x}, \mathfrak{y}$ to be horizontal parallel fields. Then $\omega(\mathfrak{x}) = \omega_0(\mathfrak{x}),\ \omega(\mathfrak{y}) = \omega_0(\mathfrak{y})$. Hence, from (3·13) we have

$$(\Omega - \tfrac{1}{2}[\omega_0,\ \omega_0])\,(\mathfrak{x}, \mathfrak{y}) = -\tfrac{1}{2}\omega([\mathfrak{x}, \mathfrak{y}]).$$

Applying the projection p^* to the both sides and employing (3·16),

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16) Cf. [1].

we get the relation

$$\theta(\mathfrak{x}, \mathfrak{y}) = -\tfrac{1}{2}\omega_0([\mathfrak{x}, \mathfrak{y}]).$$

This implies that $\theta(\mathfrak{x}, \mathfrak{y}) = 0$ if and only if $[\mathfrak{x}, \mathfrak{y}]$ is vertical. The Proposition 3·9 has been proved.

Taking the base $(e_1, \cdots, e_n, e_{n+1}, \cdots, e_r)$, we can set

$$\theta = \sum_{i,j,k} T^i_{jk}\omega^j \wedge \omega^k \otimes e_i$$

with

$$T^i_{jk} + T^i_{kj} = 0.$$

If we put

$$T = \sum_{i,j,k} T^j_{jk}e^j \wedge e^k \otimes e_i,$$

then T becomes a tensor $\mathfrak{B}''$ of type $(\alpha^*{}_2, W)$, and is called the *torsion tensor* of the connexion ω_1. By the Proposition 3·3 and the relation (3·16), the following proposition is obvions.

PROPOSITION 3·10. *Assume that there is given a fixed reductive Cartan connexion $\bar{\omega} = \omega_0 + \bar{\omega}_1$ on $\mathfrak{B}''$ and let $\bar{T}$ denote the torsion tensor of $\bar{\omega}_1$. Then, for any reductive Cartan connexion $\omega = \omega_0 + \omega_1$ on $\mathfrak{B}''$, the torsion tensor T of ω_1 is obtained from the formula*

(3·17) $$T = \tfrac{1}{2}\Phi\Gamma' + \bar{T}.$$

We can deduce directly a formula for the torsion form analogous to Bianchi's identity.

PROPOSITION 3·11. *Let $\omega = \omega_0 + \omega_1$ be a reductive Cartan connexion. Denote by Ω_1 and θ respectively the curvature form and the torsion form of the connexion ω_1. Then it holds that*

(3·18) $$d\theta + [\omega_1, \theta] = [\Omega_1, \omega_0].$$

In general, if φ is a k-form on $\mathfrak{B}''$ with values in $\hat{L}$, we have

(3·19) $$d(d\varphi + [\omega_1, \varphi]) + [\omega_1, d\varphi + [\omega_1, \varphi]] = [\Omega_1, \varphi].$$

The formula (3·18) means that the covariant derivative of θ with respect to the connexion ω_1 is equal to the tensorial 3-form $[\Omega_1, \omega_0]$ of type (α_0^*, E), while Bianchi's identity

(3·20) $$d\Omega_1 + [\omega_1, \Omega_1] = 0$$

for the connexion ω_1 shows that the covariant derivative of Ω_1 vanishes.

11. At the end we give some remarks on the covariant derivative which was used in preceding articles of this paper.

When a connexion ω is given on a principal bundle $\mathfrak{B}^0(M, G)$, each tangent space $T_b(\mathfrak{B}^0)$ is decomposed in the direct sum of the horizontal space $H_b(\mathfrak{B}^0)$ and the vertrical space $V_b(\mathfrak{B}^0)$; and for any $b \in \mathfrak{B}^0$ the natural projections

$$h: \quad T_b(\mathfrak{B}^0) \to H_b(\mathfrak{B}^0) \quad \text{and} \quad v: \quad T_b(\mathfrak{B}^0) \to V_b(\mathfrak{B}^0)$$

are defined.

The covariant derivative of a p-from θ on $\mathfrak{B}^0$ is defined by

$$(3 \cdot 21) \qquad\qquad D\theta = d\theta h^{\,17)} .$$

However, when the form θ is of special type, another covariant derivative may be defined.

Let (r^*, R) be a representation of G and $(\bar{r}^*, R)$ be its induced representation of the Lie algebra L of G. For any $a \in L$, $\bar{r}^*(a)$ is an endomorphism of the vector space R. Let θ be a p-form on $\mathfrak{B}^0$ with values in R satisfying the following condition:

(C) $\qquad \theta_{\rho^*}(s) = r^*(s^{-1})\theta \qquad$ *for any right translation* $\rho(s)$.

Then the covariant derivative of another kind is defined by

$$(3 \cdot 22) \qquad\qquad D'\theta = d\theta + \bar{r}^*(\omega)\theta^{\,18)} ,$$

where $\bar{r}^*(\omega)\theta$ is a $(p+1)$-form given by

$$(3 \cdot 23) \qquad \bar{r}^*(\omega)\theta(t_1, \cdots, t_{p+1})$$

$$= \sum_{i=1}^{p+1} \frac{(-1)^{i-1}}{p+1} \bar{r}^*(\omega(t_i))\theta(t_1, \cdots, \hat{t}_i, \cdots, t_{p+1})$$

for $t_1, \cdots, t_{p+1} \in T_b(\mathfrak{B}^0)$.

It can be proved that $D'\theta$ also satisfies the condition (C).[19] In the case that (r^*, R) is the linear adjoint representation (α^*, L) of G we have $\bar{r}^*(\omega)\theta = [\omega, \theta]$.

PROPOSITION 3·12. *If a p-form θ on $\mathfrak{B}^0$ with values in R satisfies the condition (C), then $d\theta h$ is a tensorial p-from of type (r^*, R).*

Proof. Since the right translation $\rho^*(s)$ preserves the decomposition $T_b(\mathfrak{B}^0) = H_b(\mathfrak{B}^0) + V_b(\mathfrak{B}^0)$, we have $\rho^*(s)h = h\rho^*(s)$. It follows that, if a form φ on $\mathfrak{B}^0$ with values in R satisfies the condition (C), φh becomes a tensorial form. According to the relation

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17) .Cf. [1]
18) Cf. [4]
19) Cf. [4]

$\omega h = 0$, we have

$$(3 \cdot 24) \qquad d\theta h = (d\theta + \bar{r}^*(\omega)\,\theta)\,h \ ;$$

and since $d\theta + \bar{r}^*(\omega)\,\theta$ satisfies the condition (C), $d\theta h$ is a tensorial form.

PROPOSITION 3·13. *If θ is a tensorial p-form of type (r^*, R), then*

$$(3 \cdot 25) \qquad d\theta h = d\theta + \bar{r}^*(\omega)\,\theta \ .$$

Proof. To any element $a \epsilon L$, corresponds a unique vertical vector field $\mathfrak{z}$ on $\mathfrak{B}^0$ such that $\omega(\mathfrak{z}) = a$ for all $b \epsilon \mathfrak{B}^0$. We denote this correspondence by q and set $q(L) = \mathfrak{Q}$. Then $\mathfrak{Q}$ becomes an algebra consisting of vertical vector fields of $\mathfrak{B}^0$ and q is an isomorphism of L onto $\mathfrak{Q}$.

Since $d\theta h = (d\theta + \bar{r}^*(\omega)\,\theta)\,h$, in order to prove our proposition it is sufficient to show that

$$(d\theta + \bar{r}^*(\omega)\,\theta)\,(t_1, \cdots, t_{p+1}) = 0 \text{ provided } t_1 \text{ is vertical.}$$

We take vector fields $\mathfrak{x}_i (i = 1, \cdots, p+1)$ such that $\mathfrak{x}_i(b) = t_i$ and $\mathfrak{x}_1 \epsilon \mathfrak{Q}$. By the well-known formula[20]

$$d\theta(\mathfrak{x}_1, \cdots, \mathfrak{x}_{p+1}) = \sum_{i=1}^{p+1} \frac{(-1)^{i-1}}{p+1} \mathfrak{x}_i(\theta(\mathfrak{x}_1, \cdots, \hat{\mathfrak{x}}_i, \cdots, \mathfrak{x}_{p+1}))$$

$$+ \sum_{i<j} \frac{(-1)^{i+j}}{p+1} \theta([\mathfrak{x}_i, \mathfrak{x}_j], \mathfrak{x}_1, \cdots, \hat{\mathfrak{x}}_i, \cdots, \hat{\mathfrak{x}}_j, \cdots, \mathfrak{x}_{p+1}) \ .$$

Taking into account that $\theta(\mathfrak{y}_1, \cdots, \mathfrak{y}_p) = 0$ provided one of $\mathfrak{y}_i$'s is vertical, we have

$$(3 \cdot 26) \quad d\theta(\mathfrak{x}_1, \cdots, \mathfrak{x}_{p+1}) = -\frac{1}{p+1} \mathfrak{x}_1(\theta(\mathfrak{x}_2, \cdots, \mathfrak{x}_{p+1}))$$

$$+ \sum_{j=2}^{p+1} \frac{(-1)^{j+1}}{p+1} \theta([\mathfrak{x}_1, \mathfrak{x}_j], \mathfrak{x}_2, \cdots, \hat{\mathfrak{x}}_j, \cdots, \mathfrak{x}_{p+1}) \ .$$

Moreover we have

$$\bar{r}^*(\omega)\,\theta(\mathfrak{x}_1, \cdots, \mathfrak{x}_{p+1}) = \frac{1}{p+1} \bar{r}^*(\omega(\mathfrak{x}_1))\,\theta(\mathfrak{x}_2, \cdots, \mathfrak{x}_{p+1}) \ .$$

The field $\mathfrak{x}_1 \epsilon \mathfrak{Q}$ generates the 1-parameter group $\rho^*(g)$ of right translations, where g is the 1-parameter subgroup of G generated by the element $\omega(\mathfrak{x}_1) \epsilon L$.

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20) Cf. [1], [9]

According to the relation $r^*(g^{-1})\theta = \theta\rho^*(g)$, we have

$$r^*(-\omega(\mathfrak{x}_1))\theta = L(\mathfrak{x}_1)\theta,$$

where $L(\mathfrak{x}_1)$ denote the Lie derivation with respect to the field $\mathfrak{x}_1$. Hence, it holds that

$$(3\cdot27) \quad -\bar{r}^*(\omega)\theta(\mathfrak{x}_1, \cdots, \mathfrak{x}_{p+1}) = \frac{1}{p+1}(L(\mathfrak{x}_1)\theta)(\mathfrak{x}_2, \cdots, \mathfrak{x}_{p+1}).$$

On the other hand, by a formula[21] for the Lie derivative, we have

$$(3\cdot28) \quad (L(\mathfrak{x}_1)\theta)(\mathfrak{x}_2, \cdots, \mathfrak{x}_{p+1}) = \mathfrak{x}_1(\theta(\mathfrak{x}_2, \cdots, \mathfrak{x}_{p+1}))$$

$$+ \sum_{j=2}^{p+1}(-1)^{j+1}\theta([\mathfrak{x}_1 \mathfrak{x}_j], \mathfrak{x}_2, \cdots, \hat{\mathfrak{x}}_j, \cdots, \mathfrak{x}_{p+1}).$$

From $(3\cdot26)$, $(3\cdot27)$ and $(3\cdot28)$ it follows that

$$(d\theta + \bar{r}^*(\omega)\theta)(\mathfrak{x}_1, \cdots, \mathfrak{x}_{p+1}) = 0.$$

Accordingly, if θ is a tensorial form, so is $d\theta + \bar{r}^*(\omega)\theta$, and we have

$$d\theta + \bar{r}^*(\omega)\theta = (d\theta + \bar{r}^*(\omega)\theta)h = d\theta h.$$

Our proposition has been thus proved.

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21) Cf. [9]