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Resultants and universal coverings

By

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Abstract

We construct the universal coverings of spaces of self-holomorphic maps on the complex projective space $\mathbb{C}\mathrm{P}^n$ by using the resultants, and we study their homotopy types.

1. Introduction

Let $j : S^2 = \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^m$ be the inclusion map given by $j([x : y]) = [x : y : 0 : \cdots : 0]$. If $1 \leq m \leq n$ and $f : \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ is a continuous map, the corresponding integer of the homotopy class of $f \circ j$ in $\pi_2(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}$ is called the *degree* of f. Let $\mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ denote the space of all continuous maps $f : \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ of degree d, and let $\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ be the subspace consisting of all based maps $f \in \mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ such that $f(\mathbf{e}_m) = \mathbf{e}_n$, where $\mathbf{e}_k = [1 : 0 : \cdots : 0] \in \mathbb{C}\mathrm{P}^k$ is a base point of $\mathbb{C}\mathrm{P}^k$ $(k = m, n)$. Similarly, $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \subset \mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ (resp. $\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \subset \mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$) be the corresponding the subspace of all (resp. based) holomorphic maps $f : \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ of degree d. Remark that $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) = \emptyset$ if $d < 0$, and that any holomorphic map $f : \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ of degree 0 is a constant map. So we always assume that $d \geq 1$.

When $m \geq 2$, we also consider the subspaces $H_d(m, n) \subset \mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ and $F_d(m, n) \subset \mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ defined by

$$(1.1) \qquad \begin{cases} H_d(m,n) = \{f \in \mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) : f \circ i' = \psi_d^{m-1,n}\}, \\ F_d(m,n) = \{f \in \mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) : f \circ i' = \psi_d^{m-1,n}\}, \end{cases}$$

where $i' : \mathbb{C}\mathrm{P}^{m-1} \to \mathbb{C}\mathrm{P}^m$ denotes the inclusion given by $i'([x_0 : \cdots : x_{m-1}]) = [x_0 : \cdots : x_{m-1} : 0]$ and $\psi_d^{m,n} \in \mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ is the based holomorphic map defined by $\psi_d^{m,n}([x_0 : x_1 : \cdots : x_m]) = [x_0^d : x_1^d : \cdots : x_m^d : 0 : \cdots : 0]$. It is known that there is a homotopy equivalence $F_d(m, n) \simeq \Omega^{2m}\mathbb{C}\mathrm{P}^n$ ([9], [12]).

The principal motivation of this paper derives from the work of G. Segal [13] and J. Mostovoy [10], in which they show that the following Atiyah-Jones-Segal type homotopy (or homology) stability result holds.

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Theorem 1.1 (G. Segal, [13]; J. Mostovoy, [10]). *Let $1 \leq m \leq n$ be integers and let*

$$\begin{cases} i_d : \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \\ j_d : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \\ i'_d : H_d(m,n) \to F_d(m,n) \simeq \Omega^{2m}\mathbb{C}\mathrm{P}^n \end{cases}$$

be the corresponding inclusion maps.

(i) *If $m = 1$, the inclusions i_d and j_d are homotopy equivalences up to dimension $(2n-1)d$.*

(ii) *If $m \geq 2$, the inclusions i_d, j_d and i'_d are homotopy equivalences through dimension $D(d;m,n)$ when $m < n$ and homology equivalences through dimension $D(d:m,n)$ when $m = n$, where $\lfloor x \rfloor$ denotes the integer part of a real number x and $D(d;m,n)$ is the number given by*

$$D(d;m,n) = (2n-2m+1)\left(\lfloor \frac{d+1}{2} \rfloor + 1\right) - 1.$$

Remark. A map $f : X \to Y$ is called *a homotopy equivalence up to dimension D* if the induced homomorphism $f_* : \pi_k(X) \to \pi_k(Y)$ is bijective when $k < D$ and surjective when $k = D$. Analogously, it is called *a homotopy equivalence through dimension D* (*resp. a homology equivalence through dimension D*) if $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X,\mathbb{Z}) \to H_k(Y,\mathbb{Z})$) is an isomorphism for any $k \leq D$.

If we recall several Atiyah-Jones-Segal type Theorems (c.f. [1], [2], [6], [13]), we may expect that the inclusions i_d, j_d, and i'_d may be homotopy equivalences through dimension $D(d;m,n)$ for $m = n \geq 2$, and we would like to consider this problem. From now on, for $m = n$, we write

$$(1.2)\quad \begin{cases} \mathrm{Hol}_d(n) = \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n), \quad \mathrm{Hol}_d^*(n) = \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n), \\ \mathrm{Map}_d(n) = \mathrm{Map}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n), \quad \mathrm{Map}_d^*(n) = \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n), \\ H_d(n) = H_d(n,n) \ \text{ and } \ F_d(n) = F_d(n,n) \simeq \Omega^{2n}\mathbb{C}\mathrm{P}^n. \end{cases}$$

In order to settle the homotopy stability problem it seems necessary to understand the universal covering spaces $\widetilde{H_d(n)}$, $\widetilde{\mathrm{Hol}_d^*(n)}$ and $\widetilde{\mathrm{Hol}_d(n)}$, where $\widetilde{X}$ denotes the universal covering of a connected space X.

Let z_k $(k = 0,1,2,\cdots,n)$ be complex variables, let $\mathcal{H}_d(n)$ denote the space consisting of all homogenous polynomials $g \in \mathbb{C}[z_0,\cdots,z_n]$ of degree d, and let $X_d(n) \subset \mathcal{H}_d(n)^{n+1}$ be the subspace consisting of all $(n+1)$-tuples $(f_0,\cdots,f_n) \in \mathcal{H}_d(n)^{n+1}$ such that the polynomials $f_0, f_1, \cdots, f_n$ have no common root except $\mathbf{0}_{n+1} = (0,\cdots,0) \in \mathbb{C}^{n+1}$.

For $(f_0,\cdots,f_n) \in \mathcal{H}_d(n)^{n+1}$, let $R(f_0,\cdots,f_n) \in \mathbb{C}$ denote *the resultant for the forms of several variables* of homogenous polynomials $(f_0,\cdots,f_n)$ defined as in [7] (see Section 2 in detail). It is known that $(f_0,\cdots,f_n) \in X_d(n)$ if and only if $R(f_0,\cdots,f_n) \neq 0$ for $(f_0,\cdots,f_n) \in \mathcal{H}_d(n)^{n+1}$ ([7]) , and we can identify

$$X_d(n) = \big\{(f_0, \cdots, f_n) \in \mathcal{H}_d(n)^{n+1} : R(f_0, \cdots, f_n) \neq 0\big\}. \tag{1.3}$$

Define the free right $\mathbb{C}^*$-action on $X_d(n)$ by

$$(f_0, \cdots, f_n) \cdot \alpha = (\alpha f_0, \cdots, \alpha f_n) \tag{1.4}$$

for $((f_0, \cdots, f_n), \alpha) \in X_d(n) \times \mathbb{C}^*$. Because any holomorphic map $f \in \mathrm{Hol}_d(n)$ is represented as $f = [f_0 : \cdots : f_n]$ for some $(f_0, \cdots, f_n) \in X_d(n)$ (c.f. [9], [10]), we can easily see that there is a homeomorphism

$$\mathrm{Hol}_d(n) \cong X_d(n)/\mathbb{C}^*. \tag{1.5}$$

If $f \in \mathrm{Hol}_d^*(n)$, since $f(\mathbf{e}_n) = \mathbf{e}_n$, it is represented as $f = [f_0 : \cdots : f_n]$ such that $(f_0, \cdots, f_n) \in Y_d(n)$, where $Y_d(n) \subset X_d(n)$ denotes the subspace consisting of all $(n+1)$-tuples $(f_0, \cdots, f_n) \in X_d(n)$ such that the coefficient of z_0^d of f_0 is 1 and 0 in the other polynomials f_k $(1 \leq k \leq n)$.

For each integer $0 \leq k \leq n$, define the subspace $W_k(d) \subset \mathbb{C}[z_0, \cdots, z_n]$ by

$$W_k(d) = \begin{cases} \big\{z_k^d + z_n g : g \in \mathcal{H}_{d-1}(n)\big\} & \text{if } k \neq n \\ \big\{z_n g : g \in \mathcal{H}_{d-1}(n)\big\} & \text{if } k = n \end{cases}$$

and consider the space $V_d(n) = W_0(d) \times W_1(d) \times \cdots \times W_n(d) \subset \mathbb{C}[z_0, \cdots, z_n]^{n+1}$. If $f \in H_d(n)$, it is represented as $f = [f_0 : \cdots : f_n]$ such that $(f_0, \cdots, f_n) \in X_d(n) \cap V_d(n)$, and it is easy to see that there are homeomorphisms

$$\mathrm{Hol}_d^*(n) \cong Y_d(n) \quad \text{and} \quad H_d(n) \cong Z_d(n), \tag{1.6}$$

where we write $Z_d(n) = X_d(n) \cap V_d(n)$.

We also denote by $HF_d(n)$ and $HF_d^*(n)$ the homotopy fibers of the inclusions $j_d : \mathrm{Hol}_d(n) \to \mathrm{Map}_d(n)$ and $i_d : \mathrm{Hol}_d^*(n) \to \mathrm{Map}_d^*(n)$, respectively. Remark that there is a homotopy equivalence $HF_d^*(n) \simeq HF_d(n)$ (see Lemma 5.1). Then the main results of this paper are stated as follows.

Theorem 1.2.
(i) *There exists a homeomorphism* $\widetilde{\mathrm{Hol}_d(n)} \cong R^{-1}(1)$.
(ii) *There are homotopy equivalences*

$$\widetilde{\mathrm{Hol}_d^*(n)} \simeq R_1^{-1}(1) \quad \text{and} \quad \widetilde{H_d(n)} \simeq R_2^{-1}(1).$$

Here, $R^{-1}(1)$, $R_1^{-1}(1)$ *and* $R_2^{-1}(1)$ *denote the subspaces of* $X_d(n)$ *given by*

$$\begin{cases} R^{-1}(1) = \big\{(f_0, \cdots, f_n) \in X_d(n) : R(f_0, \cdots, f_n) = 1\big\}, \\ R_1^{-1}(1) = \big\{(f_0, \cdots, f_n) \in Y_d(n) : R(f_0, \cdots, f_n) = 1\big\}, \\ R_2^{-1}(1) = \big\{(f_0, \cdots, f_n) \in Z_d(n) : R(f_0, \cdots, f_n) = 1\big\}. \end{cases} \tag{1.7}$$

Although we know the fundamental group actions on the universal coverings $\widetilde{\mathrm{Hol}_d(n)}$, $\widetilde{\mathrm{Hol}^*_d(n)}$ and $\widetilde{H_d(n)}$, we cannot determine whether they are nilpotent actions or not. If these inclusions are homotopy equivalences through dimension $D(d;n,n)$, $HF_d(n)$ and $HF^*_d(n)$ must be $\lfloor \frac{d+1}{2} \rfloor$-connected. Although we cannot prove this statement, we can show the weaker one as follows.

Theorem 1.3. *$HF^*_d(n)$ and $HF_d(n)$ are simply connected.*

This paper is organized as follows. In Section 2, we construct the universal covering of $\mathrm{Hol}_d(n)$ geometrically by using the resultant for the forms of several variables. In Section 3 and 4, we also construct the universal coverings of $\mathrm{Hol}^*_d(n)$ and $H_d(n)$ by using this resultant, and finally in Section 5, we give the proof of Theorem 1.3.

2. Resultants and the space $\widetilde{\mathrm{Hol}_d(n)}$

First, recall about resultants. For each $I = (i_0, \cdots, i_n) \in \mathbb{Z}^{n+1}_{\geq 0}$, we write $|I| = \sum_{k=0}^n i_k$ and $z^I = z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$. We denote by $\mathcal{I}(d)$ the set

$$\mathcal{I}(d) = \{I = (i_0, \cdots, i_n) \in \mathbb{Z}^{n+1}_{\geq 0} : |I| = d\}.$$

If $(f_0, f_1, \cdots, f_n) \in \mathcal{H}_{d_0}(n) \times \mathcal{H}_{d_1}(n) \times \cdots \times \mathcal{H}_{d_n}(n)$, each homogenous polynomial f_k of degree d_k can be written as $f_k = \sum_{I \in \mathcal{I}(d_k)} c_{I,k} z^I$ ($c_{I,k} \in \mathbb{C}$). Then for each such possible pair of indices (I,k) with $I \in \mathcal{I}(d_k)$ and $0 \leq k \leq n$, we introduce a variable $Z_{I,k}$. Then for a polynomial $P \in \mathbb{C}[Z_{I,k} : I \in \mathcal{I}(d_k),\ 0 \leq k \leq n]$, let $P(f_0, \cdots, f_n)$ denote the complex number obtained by replacing variable $Z_{I,k}$ in P with the corresponding coefficient $c_{I,k}$.

Lemma 2.1 ([7], [[4]; Chap. 3, Theorem 2.3, Theorem 3.1])**.** *For each $(n+1)$-tuple $J = (d_0, \cdots, d_n)$ of positive integers, there exists a unique irreducible homogenous polynomial $R_J \in \mathbb{Z}[Z_{I,k} : I \in \mathcal{I}(d_k), 0 \leq k \leq n]$ of degree $\sum_{k=0}^n d_0 \cdots d_{k-1} d_{k+1} \cdots d_n$ which satisfies the following three conditions*:

(i) *R_J is an irreducible polynomial even in $\mathbb{C}[Z_{I,k} : I \in \mathcal{I}(d_k), 0 \leq k \leq n]$.*
(ii) *$R_J(z_0^{d_0}, z_1^{d_1}, \cdots, z_n^{d_n}) = 1$.*
(iii) *If $(f_0, \cdots, f_n) \in \mathcal{H}_{d_0}(n) \times \cdots \times \mathcal{H}_{d_n}(n)$,*

$$R_J(f_0, \cdots, f_{k-1}, \lambda f_k, f_{k+1}, \cdots, f_n) = \lambda^{d_0 \cdots d_{k-1} d_{k+1} \cdots d_n} R_J(f_0, \cdots, f_k, \cdots, f_n)$$

for any $\lambda \in \mathbb{C}^*$, and the equation $f_0 = f_1 = \cdots = f_n = 0$ has no solution except $\mathbf{0}_{n+1} \in \mathbb{C}^{n+1}$ if and only if $R_J(f_0, \cdots, f_n) \neq 0$.

Remark. In general, the polynomial R_J can be regarded as the generalization of the determinant (c.f. [4], [7]). To see this, consider the case $d_0 = d_1 = \cdots = d_n = 1$. If $(f_0, \cdots, f_n) \in \mathcal{H}_1(n)^{n+1}$, each f_k can be written as $f_k = \sum_{j=0}^n c_{j,k} z_k$ ($c_{j,k} \in \mathbb{C}$). If $Z_{j,k}$ denotes the corresponding variable to $c_{j,k}$ and set $J = (1, 1, \cdots, 1)$, R_J can be written as $R_J = \det(Z_{j,k})$ and $R_J(f_0, \cdots, f_n) = \det(c_{j,k})$.

From now on, we always assume that $d_0 = d_1 = \cdots = d_n = d \geq 1$, and we write

$$R = R_J = R_{(d,d,\cdots,d)} \quad \text{for } J = (d,d,\cdots,d). \tag{2.1}$$

Because $R(f_0,\cdots,f_n) \neq 0$ for any $(f_0,\cdots,f_n) \in X_d(n)$, R can be regarded as the map $R : X_d(n) \to \mathbb{C}^*$.

Let $G_{d,n}$ be the subgroup of $\mathbb{C}^*$ defined by $G_{d,n} = \{g \in \mathbb{C}^* : g^{(n+1)d^n} = 1\} \cong \mathbb{Z}/(n+1)d^n$, and consider the space $\mathbb{C}^* \times_{G_{d,n}} R^{-1}(1)$, where we identify $[g\beta,(f_0,\cdots,f_n)] = [\beta,(gf_0,\cdots,gf_n)]$ in $\mathbb{C}^* \times_{G_{d,n}} R^{-1}(1)$ if $g \in G_{d,n}$ and $((\beta,(f_0,\cdots,f_n)) \in \mathbb{C}^* \times R^{-1}(1)$.

Define the map $\varphi_d : \mathbb{C}^* \times_{G_{d,n}} R^{-1}(1) \to \mathbb{C}^*$ by $\varphi_d([\beta,f]) = \beta^{(n+1)d^n}$ for $[\beta,f] \in \mathbb{C}^* \times_{G_{d,n}} R^{-1}(1)$. Because R is a homogenous polynomial of degree $(n+1)d^n$ and it satisfies the equality

$$R(\lambda f_0,\cdots,\lambda f_n) = \lambda^{(n+1)d^n} R(f_0,\cdots,f_n) \tag{2.2}$$

for $((f_0,\cdots,f_n),\lambda) \in X_d(n) \times \mathbb{C}^*$, this implies the following result.

Lemma 2.2 (c.f. [13, Proposition 6.1])**.**
(i) *There exists a $\mathbb{C}^*$-equivariant homeomorphism*

$$\Phi_d : X_d(n) \xrightarrow{\cong} \mathbb{C}^* \times_{G_{d,n}} R^{-1}(1)$$

such that $\varphi_d \circ \Phi_d = R : X_d(n) \to \mathbb{C}^*$.

(ii) *The map $R : X_d(n) \to \mathbb{C}^*$ is a fiber bundle with non-singular fibers and structure group $G_{d,n} \cong \mathbb{Z}/(n+1)d^n$.*

(iii) *The monodromy $T : R^{-1}(1) \to R^{-1}(1)$ (i.e. the action of the generator of the structure group) is given by $T(f_0,f_1,\cdots,f_n) = (\xi f_0,\xi f_1,\cdots,\xi f_n)$, where ξ is a primitive root of unity of order $(n+1)d^n$.*

Proof. (i) Let $f = (f_0,\cdots,f_n) \in X_d(n)$ be an element, and let $\alpha_k \in \mathbb{C}^*$ $(k = 1,2)$ be two complex numbers such that $\alpha_1^{(n+1)d^n} = \alpha_2^{(n+1)d^n} = R(f)$. Consider the element $F(\alpha_k) = (\alpha_k,(\frac{f_0}{\alpha_k},\cdots,\frac{f_n}{\alpha_k})) \in \mathbb{C}^* \times R^{-1}(1)$ $(k = 1,2)$. In this case, since there exists some element $g \in G_{d,n}$ such that $\alpha_2 = g\alpha_1$, $[F(\alpha_1)] = [F(\alpha_2)]$ in $\mathbb{C}^* \times_{G_{d,n}} R^{-1}(1)$. So define the map $\Phi_d : X_d(n) \to \mathbb{C}^* \times_{G_{d,n}} R^{-1}(1)$ by $\Phi_d(f) = [\alpha,(\frac{f_0}{\alpha},\cdots,\frac{f_n}{\alpha})] = [\alpha,\frac{f}{\alpha}]$ for $f = (f_0\cdots,f_n) \in X_d(n)$ if $\alpha^{(n+1)d^n} = R(f)$. Next, let $[\beta,f] \in \mathbb{C}^* \times_{G_{d,n}} R^{-1}(1)$ be any element such that $(\beta,f) = ((f_0,\cdots,f_n),\beta) \in \mathbb{C}^* \times X_d(n)$. If $[\beta,f] = [\beta_1,h]$ $(\beta,\beta_1 \in \mathbb{C}^*$, $f,h \in R^{-1}(1))$, there exists some $g \in G_{d,n}$ such that $(\beta_1,h) = (g^{-1}\cdot\beta, g\cdot f)$. Hence, $\beta_1 \cdot h = \beta \cdot f$ and the element $\beta \cdot f = (\beta f_0,\cdots,\beta f_n) \in X_d(n)$ does not depend on the choice of the representative (β,f). So one can define the map $G_d : \mathbb{C}^* \times_{G_{d,n}} R^{-1}(1) \to X_d(n)$ by $G_d([\beta,f]) = \beta \cdot f = (\beta f_0,\cdots,\beta f_n)$.

If $[\beta,f] \in \mathbb{C}^* \times_{G_{d,n}} R^{-1}(1)$, because $R(f) = 1$, $R(\beta \cdot f) = \beta^{(n+1)d^n} R(f) = \beta^{(n+1)d^n}$. Hence, $\Phi_d \circ G_d([\beta,f]) = \Phi_d(\beta \cdot f) = [\beta,\frac{\beta f}{\beta}] = [\beta,f]$, and we have $\Phi_d \circ G_d = \mathrm{id}$. An analogous computation also shows that $G_d \circ \Phi_d = \mathrm{id}$ and so that Φ_d is a homeomorphism.

Furthermore, if $(f,\beta)\in X_d(n)\times\mathbb{C}^*$ with $R(f)=\alpha^{(n+1)d^n}$ $(\alpha\in\mathbb{C}^*)$, because $R(\beta\cdot f)=\beta^{(n+1)d^n}R(f)=(\beta\alpha)^{(n+1)d^n}$, $\Phi_d(\beta\cdot f)=[\beta\alpha,\frac{\beta f}{\beta\alpha}]=[\beta\alpha,\frac{f}{\alpha}]=\beta\cdot[\alpha,\frac{f}{\alpha}]=\beta\cdot\Phi_d(f)$. Hence, Φ_d is a $\mathbb{C}^*$-equivariant map. Because a similar computation shows that G_d is also a $\mathbb{C}^*$-equivariant map, Φ_d is a $\mathbb{C}^*$-equivariant homeomorphism.

If $f\in X_d(n)$ and $R(f)=\alpha^{(n+1)d^n}$, $\varphi_d\circ\Phi_d(f)=r([\alpha,\frac{f}{\alpha}])=\alpha^{(n+1)d^n}=R(f)$. Hence, $\varphi_d\circ\Phi_d=R$ and the assertion (i) is proved.

(ii) It follows from (i) that we may identify R with the map φ_d. So it suffices to prove the local triviality for the map φ_d.

We write $D=(n+1)d^n$, and let $\beta\in\mathbb{C}^*$ be any element. From now on, we choose the fixed constant $\theta_0\in\mathbb{R}$ such that $\beta=|\beta|\exp(\sqrt{-1}\theta_0)$, and set $a_0=|\beta|^{1/D}\exp(\frac{\sqrt{-1}\theta_0}{D})$. Then because $\{\alpha\in\mathbb{C}^*:\alpha^D=\beta\}=\{ga_0:g\in G_{d,n}\}$, we note that

$$\begin{aligned}\varphi_d^{-1}(\beta)&=\{[ga_0,f]:g\in G_{d,n},f\in R^{-1}(1)\}=\{[a_0,gf]:g\in G_{d,n},f\in R^{-1}(1)\}\\&=\{[a_0,f]:f\in R^{-1}(1)\}\cong R^{-1}(1).\end{aligned}$$

Let $\phi(r,\theta)$ denote the function $\phi(r,\theta)=r\exp(\sqrt{-1}\theta)$ $(r>0,\ \theta\in\mathbb{R})$, and let U be a sufficiently small connected open neighborhood U of β such that $\phi|U$ is injective. For example, let U be the open set given by

$$U=\left\{\phi(r,\theta):\frac{3|\beta|}{4}<r<\frac{5|\beta|}{4},-\frac{\pi}{100}<\theta-\theta_0<\frac{\pi}{100}\right\}\subset\mathbb{C}^*.$$

If we remark the above isomorphism, we can see that the map $h:U\times R^{-1}(1)\to\varphi_d^{-1}(U)$ given by by $h(\phi(r,\theta),f)=[\phi(r^{1/D},\theta/D),f]$ is a homeomorphism. Furthermore, if $q_1:U\times R^{-1}(1)\to U$ denotes the first projection, clearly the equality $\varphi_d\circ h=q_1$ holds. Hence, the local triviality is proved.

(iii) The assertion (iii) easily follows from the proof of (i). □

By using Lemma 2.2, we have the fibration sequence

$$R^{-1}(1)\xrightarrow{\subset}X_d(n)\xrightarrow{R}\mathbb{C}^*. \tag{2.3}$$

We also recall from [14, Appendix] that there is a fibration sequence

$$\mathrm{Hol}_d^*(n)\xrightarrow{\subset}\mathrm{Hol}_d(n)\xrightarrow{ev}\mathbb{C}\mathrm{P}^n, \tag{2.4}$$

where the map ev is given by $ev(f)=f(\mathbf{e}_n)$ for $f\in\mathrm{Hol}_d(n)$.

Lemma 2.3.

(i) $\pi_1(X_d(n))\cong\mathbb{Z}$.

(ii) *There is a homotopy equivalence* $\widetilde{X_d(n)}\simeq R^{-1}(1)$, *and the map* $R:X_d(n)\to\mathbb{C}^*\simeq K(\mathbb{Z},1)$ *represents the generator of the based homotopy set* $[X_d(n),K(\mathbb{Z},1)]\cong H^1(X_d(n),\mathbb{Z})\cong\mathbb{Z}$, *where* $\widetilde{X_d(n)}$ *denotes the universal covering of* $X_d(n)$.

Proof. (i) Let $\tilde{\mathbf{e}}_n = (1, 0, 0, \cdots, 0) \in \mathbb{C}^{n+1}$ and define the map $\tilde{ev} : X_d(n) \to \mathbb{C}^{n+1} \setminus \{\mathbf{0}_{n+1}\} \simeq S^{2n+1}$ by $\tilde{ev}(f_0, \cdots, f_n) = (f_0(\tilde{\mathbf{e}}_n), \cdots, f_n(\tilde{\mathbf{e}}_n))$ for $(f_0, \cdots, f_n) \in X_d(n)$. We also remark that there is a $\mathbb{C}^*$-principal bundle $\mathbb{C}^* \to X_d(n) \xrightarrow{\pi} \mathrm{Hol}_d(n) \cong X_d(n)/\mathbb{C}^*$, because (1.4) is a free action and the local triviality is satisfied. Then if $\gamma_n : S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is a Hopf fibering, it is easy to see that $ev \circ \pi = \gamma_n \circ \tilde{ev}$. Hence, if F_0 denotes the homotopy fiber of the map $\tilde{ev}$, it follows from [3, Lemma 2.1] that we have the homotopy commutative diagram

$$\begin{array}{ccccc}
* & \longrightarrow & F_0 & \xrightarrow[\simeq]{} & \mathrm{Hol}_d^*(n) \\
\downarrow & & \downarrow & & \cap\downarrow \\
\mathbb{C}^* & \longrightarrow & X_d(n) & \xrightarrow{\pi} & \mathrm{Hol}_d(n) \\
\| & & \tilde{ev}\downarrow & & ev\downarrow \\
\mathbb{C}^* & \longrightarrow & S^{2n+1} & \xrightarrow{\gamma_n} & \mathbb{C}\mathrm{P}^n
\end{array}$$

such that all horizontal and vertical sequences are fibration sequences. Hence, there is a homotopy equivalence $F_0 \simeq \mathrm{Hol}_d^*(n)$ and we have the fibration sequence (up to homotopy equivalence)

$$\mathrm{Hol}_d^*(n) \longrightarrow X_d(n) \xrightarrow{\tilde{ev}} S^{2n+1}. \tag{2.5}$$

Since S^{2n+1} is 2-connected and $\pi_1(\mathrm{Hol}_d^*(n)) \cong \mathbb{Z}$ ([14]), there is a isomorphism $\pi_1(X_d(n)) \cong \mathbb{Z}$.

(ii) Since $R^{-1}(1)$ is connected, by using the homotopy exact sequence induced from the fibration (2.3), $R_* : \pi_1(X_d(n)) \to \pi_1(\mathbb{C}^*) = \mathbb{Z}$ is surjective. However, because $\pi_1(X_d(n)) = \mathbb{Z}$, R_* is an isomorphism and $R^{-1}(1)$ is simply connected. Hence, there is a homotopy equivalence $\widetilde{X_d(n)} \simeq R^{-1}(1)$ and $R : X_d(n) \to \mathbb{C}^* \simeq K(\mathbb{Z}, 1)$ represents the generator of $[X_d(n), K(\mathbb{Z}, 1)] \cong H^1(X_d(n), \mathbb{Z}) \cong \mathbb{Z}$. $\square$

Lemma 2.4. *If $f = (f_0, \cdots, f_n) \in X_d(n)$, $f_k \neq 0$ for any $0 \leq k \leq n$.*

Proof. If $f_k = 0$ for some k, the holomorphic map $g = [f_0 : \cdots : f_n] = \pi(f) \in \mathrm{Hol}_d(n)$ satisfies the condition $f(\mathbb{C}\mathrm{P}^n) \subset \mathbb{C}\mathrm{P}^{n-1}$. Hence, $g^* = 0$ on $H^{2n}(\mathbb{C}\mathrm{P}^n, \mathbb{Z})$. However, because the degree of g is $d \geq 1$, the degree of g^* on $H^{2n}(\mathbb{C}\mathrm{P}^n, \mathbb{Z})$ is $d^n \neq 0$, which is a contradiction. $\square$

Theorem 2.1. *There is a homeomorphism $\widetilde{\mathrm{Hol}_d(n)} \cong R^{-1}(1)$.*

Proof. By using (1.5) and Lemma 2.2, there is a homeomorphism

$$\mathrm{Hol}_d(n) \cong X_d(n)/\mathbb{C}^* \cong (\mathbb{C}^* \times_{G_{d,n}} R^{-1}(1))/\mathbb{C}^* \cong G_{d,n} \backslash R^{-1}(1).$$

Since $\mathbb{C}^*$ acts on $X_d(n)$ freely, the subgroup $G_{d,n}$ also acts on $R^{-1}(1)$ freely. Hence, we have the covering space sequence $G_{d,n} \to R^{-1}(1) \to \mathrm{Hol}_d(n)$.

However, because $\pi_1(\mathrm{Hol}_d(n)) \cong \mathbb{Z}/(n+1)d^n \cong G_{d,n}$ and $R^{-1}(1)$ is connected, $R^{-1}(1)$ is simply connected and there is a homeomorphism $\widetilde{\mathrm{Hol}_d(n)} \cong R^{-1}(1)$. □

Corollary 2.1. *There is a homotopy equivalence* $\widetilde{X_d(n)} \simeq \widetilde{\mathrm{Hol}_d(n)}$.

3. The space $\widetilde{\mathrm{Hol}_d^*(n)}$

As in (1.6), we identify $\mathrm{Hol}_d^*(n) = Y_d(n)$ and consider the map $R_1 : \mathrm{Hol}_d^*(n) = Y_d(n) \to \mathbb{C}^*$ defined by the restriction $R_1 = R|Y_d(n)$. If we recall that $(f_0, \lambda f_1, \lambda f_2, \cdots, \lambda f_n) \in \mathrm{Hol}_d^*(n)$ and the equality

$$R_1(f_0, \lambda f_1, \lambda f_2, \cdots, \lambda f_n) = \lambda^{nd^n} R_1(f_0, \cdots, f_n) \tag{3.1}$$

holds for any $((f_0, \cdots, f_n), \lambda) \in \mathrm{Hol}_d^*(n) \times \mathbb{C}^*$, by using a complete analogous proof of Lemma 2.2 one can show the following result.

Lemma 3.1.
(i) *There exists a* $\mathbb{C}^*$*-equivariant homeomorphism*

$$\Psi_d : \mathrm{Hol}_d^*(n) \xrightarrow{\cong} \mathbb{C}^* \times_{G^*_{d,n}} R_1^{-1}(1)$$

such that $\psi_d \circ \Psi_d = R_1 : \mathbb{C}^* \times_{G^*_{d,n}} R_1^{-1}(1) \to \mathbb{C}^*$*, where* $G^*_{d,n} = \{g \in \mathbb{C}^* : g^{nd^n} = 1\} \cong \mathbb{Z}/nd^n$*. and the map* $\psi_d : \mathbb{C}^* \times_{G^*_{d,n}} R_1^{-1}(1) \to \mathbb{C}^*$ *is given by* $\psi_d([\beta, f]) = \beta^{nd^n}$.
(ii) *The map* $R_1 : \mathrm{Hol}_d^*(n) \to \mathbb{C}^*$ *is a fiber bundle with non-singular fibers and structure group* $G^*_{d,n}$.
(iii) *The monodromy* $T_1 : R_1^{-1}(1) \to R_1^{-1}(1)$ *is given by*

$$T_1(f_0, f_1, \cdots, f_n) = (f_0, \xi_1 f_1, \xi_1 f_2, \cdots, \xi_1 f_n),$$

where ξ_1 *is a primitive root of unity of order* nd^n.

Hence, we have the fibration sequence

$$R_1^{-1}(1) \xrightarrow{\subset} \mathrm{Hol}_d^*(n) \xrightarrow{R_1} \mathbb{C}^*. \tag{3.2}$$

Theorem 3.1. *There is a homotopy equivalence* $\widetilde{\mathrm{Hol}_d^*(n)} \simeq R_1^{-1}(1)$ *and there is a fibration sequence* $\widetilde{\mathrm{Hol}_d^*(n)} \to \widetilde{\mathrm{Hol}_d(n)} \to S^{2n+1}$.

Proof. By using the fibration sequences (2.3) and (3.2), we obtain the homotopy commutative diagram

$$\begin{tikzcd}
R_1^{-1}(1) \arrow[r] \arrow[d] & R^{-1}(1) \arrow[r] \arrow[d] & S^{2n+1} \arrow[d, equal] \\
\mathrm{Hol}_d^*(n) \arrow[r, "\subset"] \arrow[d, "R_1"] & X_d(n) \arrow[r, "\tilde{ev}"] \arrow[d, "R"] & S^{2n+1} \arrow[d] \\
\mathbb{C}^* \arrow[r, "="] & \mathbb{C}^* \arrow[r] & *
\end{tikzcd}$$

where all horizontal and vertical sequences are fibration sequences.

If we consider the fibration sequence $R_1^{-1}(1) \to R^{-1}(1) \to S^{2n+1}$, because S^{2n+1} is 2-connected and $R^{-1}(1)$ is simply connected, $R_1^{-1}(1)$ is simply connected. Then, because $R_1^{-1}(1)$ is connected, by using the homotopy exact sequence induced from the fibration sequence $R_1^{-1}(1) \to \mathrm{Hol}_d^*(n) \xrightarrow{R_1} \mathbb{C}^*$, $R_{1*} : \pi_1(\mathrm{Hol}_d^*(n)) \xrightarrow{\cong} \pi_1(\mathbb{C}^*)$ is an isomorphism. Hence, there is a homotopy equivalence $\widetilde{\mathrm{Hol}_d^*(n)} \simeq R_1^{-1}(1)$. Moreover, because $\widetilde{\mathrm{Hol}_d(n)} \simeq R^{-1}(1)$, the homotopy fibration sequence $R_1^{-1}(1) \to R^{-1}(1) \to S^{2n+1}$ reduces to the desired homotopy fibration sequence. □

Remark. It is known that there is a homotopy equivalence $\widetilde{\mathrm{Hol}_d(1)} \simeq \widetilde{\mathrm{Hol}_d^*(1)} \times S^3$ ([5], [11]). Hence, the homotopy fibration sequence given in Theorem 3.1 is trivial if $n = 1$.

Since $(f_0, \alpha f_1, \alpha f_2, \cdots, \alpha f_n) \in \mathrm{Hol}_d^*(n)$ for any $(f, \alpha) = ((f_0, \cdots, f_n), \alpha) \in \mathrm{Hol}_d^*(n) \times \mathbb{C}^*$, we can define the right $\mathbb{C}^*$-action on $\mathrm{Hol}_d^*(n)$ by

$$(f_0, \cdots, f_n) \cdot \alpha = (f_0, \alpha f_1, \alpha f_2, \cdots, \alpha f_n) \tag{3.3}$$

for $((f_0, \cdots, f_n), \alpha) \in \mathrm{Hol}_d^*(n) \times \mathbb{C}^*$. By using Lemma 2.4, we can easily see that (3.3) is a free $\mathbb{C}^*$-action.

Proposition 3.1. *$\pi_1(\mathrm{Hol}_d^*(n)/\mathbb{C}^*) \cong \mathbb{Z}/nd^n$ and there is a homeomorphism $\widetilde{\mathrm{Hol}_d^*(n)/\mathbb{C}^*} \cong R_1^{-1}(1)$, where $\widetilde{\mathrm{Hol}_d^*(n)/\mathbb{C}^*}$ denotes the universal covering of the orbit space $\mathrm{Hol}_d^*(n)/\mathbb{C}^*$.*

Proof. It follows from Lemma 3.1 that there is a homeomorphism

$$\mathrm{Hol}_d^*(n)/\mathbb{C}^* \cong (\mathbb{C}^* \times_{G_{d,n}^*} R_1^{-1}(1))/\mathbb{C}^* \cong G_{d,n}^* \backslash R_1^{-1}(1).$$

Since the group $G_{d,n}^*$ acts on $R_1^{-1}(1)$ freely, there is a covering space sequence $G_{d,n}^* \to R_1^{-1}(1) \to \mathrm{Hol}_d^*(n)/\mathbb{C}^*$. However, since $R_1^{-1}(1)$ is simply connected, $\pi_1(\mathrm{Hol}_d^*(n)/\mathbb{C}^*) \cong G_{d,n}^* \cong \mathbb{Z}/nd^n$ and there is a homeomorphism $\widetilde{\mathrm{Hol}_d^*(n)/\mathbb{C}^*} \cong R_1^{-1}(1)$. □

Corollary 3.1. *There is a homotopy equivalence $\widetilde{\mathrm{Hol}_d^*(n)} \simeq \widetilde{\mathrm{Hol}_d^*(n)/\mathbb{C}^*}$.*

4. The space $\widetilde{H_d(n)}$

In this section, we construct the universal covering $\widetilde{H_d(n)}$ explicitly. For this purpose, we identify $H_d(n) = Z_d(n)$ and consider the map $R_2 : H_d(n) \to \mathbb{C}^*$ defined by the restriction $R_2 = R|H_d(n)$.

Since $(f_0, \cdots, f_{n-1}, \lambda f_n) \in H_d(n)$ and the equality

$$R_2(f_0, \cdots, f_{n-1}, \lambda f_n) = \lambda^{d^n} R_2(f_0, \cdots, f_{n-1}, f_n) \tag{4.1}$$

holds for any $((f_0, \cdots, f_n), \lambda) \in H_d(n) \times \mathbb{C}^*$, by using a complete analogous proof of Lemma 2.2 one can show the following result.

Lemma 4.1.
(i) *There is a* $\mathbb{C}^*$*-equivariant homeomorphism*

$$f_d : H_d(n) \xrightarrow{\cong} \mathbb{C}^* \times_{H_{d,n}} R_2^{-1}(1)$$

such that $r_d \circ f_d = R_2 : H_d(n) \to \mathbb{C}^*$*, where* $H_{d,n} = \{g \in \mathbb{C}^* : g^{d^n} = 1\} \cong \mathbb{Z}/d^n$ *and the map* $r_d : \mathbb{C}^* \times_{H_{d,n}} R_2^{-1}(1) \to \mathbb{C}^*$ *is given by* $r_d([\beta, f]) = \beta^{d^n}$*.*

(ii) *The map* $R_2 : H_d(n) \to \mathbb{C}^*$ *is a fiber bundle with non-singular fibers and structure group* $H_{d,n}$*.*

(iii) *The monodromy* $T_2 : R_2^{-1}(1) \to R_2^{-1}(1)$ *is given by*

$$T_2(f_0, f_1, \cdots, f_n) = (f_0, \cdots, f_{n-1}, \xi_2 f_n),$$

where ξ_2 *is a primitive root of unity of order* d^n*.*

Let $j'_d : H_d(n) \to \mathrm{Hol}_d^*(n)$ denote the inclusion.

Theorem 4.1. *If* $n \geq 2$*,* $j'_{d*} : \pi_1(H_d(n)) \xrightarrow{\cong} \pi_1(\mathrm{Hol}_d^*(n)) = \mathbb{Z}$ *is an isomorphism.*

Proof. From now on, we identify $\mathrm{Hol}_d^*(n) = Y_d(n)$ and $H_d(n) = Z_d(n)$ as in (1.6). If $(f_0, f_1) \in \mathrm{Hol}_d^*(1) \subset \mathbb{C}[z_0, z_1]^2$, it can be written as

$$f_0 = f_0(z_0, z_1) = z_0^d + z_1 g_0(z_0, z_1), \quad f_1 = f_1(z_0, z_1) = z_1 g_1(z_0, z_1)$$

for some homogenous polynomial $g_k = g_k(z_0, z_1) \in \mathbb{C}[z_0, z_1]$ $(k = 0, 1)$. Then, if we change $z_1 \mapsto z_n$ in f_0 and f_1, we can easily see that the element

$$\begin{aligned}\varphi(f_0, f_1) &= (f_0(z_0, z_n), z_1^d, z_2^d, \cdots, z_{n-1}^d, f_1(z_0, z_n)) \\ &= (z_0^d + z_n g_0(z_0, z_n), z_1^d, z_2^d, \cdots, z_{n-1}^d, z_n g_1(z_0, z_n))\end{aligned}$$

is contained in $H_d(n)$. So define the subspace $G_d(n) \subset H_d(n)$ by

$$G_d(n) = \left\{\varphi(f_0, f_1) : (f_0, f_1) \in \mathrm{Hol}_d^*(1)\right\} \cong \mathrm{Hol}_d^*(1).$$

Next, consider the subspace $G'_d(n) \subset H_d(n)$ defined by

$$G'_d = \left\{(f_0, \epsilon_1 z_1^d, \cdots, \epsilon_{n-1} z_{n-1}^d, f_1) : f_0, f_1 \in \mathbb{C}[z_0, \cdots, z_n], \epsilon_k \in \mathbb{C}^*\right\} \cap H_d(n).$$

Consider the subspaces $G_d(n) \subset G'_d(n) \subset H_d(n)$. Since $n \geq 2$, the complement of $G_d(n)$ in $G'_d(n)$ and that of $G'_d(n)$ in $H_d(n)$ are of codimension 1. So the complement of $G_d(n)$ in $H_d(n)$ is of codimension 2, and the inclusion $j''_d : G_d(n) \to H_d(n)$ induces an epimorphism $j''_{d*} : \pi_1(G_d(n)) \to \pi_1(H_d(n))$. However, because $\pi_1(G_d(n)) \cong \pi_1(\mathrm{Hol}_d^*(1)) \cong \mathbb{Z}$ by [13], there is an isomorphism $\pi_1(H_d(n)) \cong \mathbb{Z}/l$ for some integer $l \geq 0$.

Next, because $G_d(n) \subset H_d(n) \subset \mathrm{Hol}_d^*(n)$, the complement of $G_d(n)$ in $\mathrm{Hol}_d^*(n)$ is codimension > 2 and the inclusion $j'_d \circ j''_d : G_d(n) \to \mathrm{Hol}_d^*(n)$ also induces an epimorphism $j'_d \circ j''_{d*} : \pi_1(G_d(n)) \to \pi_1(\mathrm{Hol}_d^*(n))$. Hence, by using

$\pi_1(G_d(n)) = \pi_1(\mathrm{Hol}_d^*(n)) = \mathbb{Z}$ ([14]), $j'_{d*} \circ j''_{d*} : \pi_1(G_d(n)) \xrightarrow{\cong} \pi_1(\mathrm{Hol}_d^*(n))$ is an isomorphism. So that if we recall the composite of homomorphisms

$$\mathbb{Z} = \pi_1(G_d(n)) \xrightarrow{j''_{d*}} \pi_1(H_d(n)) \xrightarrow{j'_{d*}} \pi_1(\mathrm{Hol}_d^*(n)) = \mathbb{Z}$$

and recall that $\pi_1(H_d(n)) = \mathbb{Z}/l$, we have $l = 0$ and $j'_{d*} : \pi_1(H_d(n)) \xrightarrow{\cong} \pi_1(\mathrm{Hol}_d^*(n)) = \mathbb{Z}$ is an isomorphism. □

Since $(f_0, \cdots, f_{n-1}, \alpha f_n) \in H_d(n)$ for any $((f_0, \cdots, f_n), \alpha) \in H_d(n) \times \mathbb{C}^*$, if we identify $H_d(n) = Z_d(n)$ as in (1.6), we can define the right $\mathbb{C}^*$-action on $H_d(n)$ by

$$(f_0, \cdots, f_n) \cdot \alpha = (f_0, \cdots, f_{n-1}, \alpha f_n) \tag{4.2}$$

for $((f_0, \cdots, f_n), \alpha) \in H_d(n) \times \mathbb{C}^*$. It is easy to see that the action (4.2) is free by using Lemma 2.4. Similarly, consider the right $\mathrm{GL}_n(\mathbb{C})$ action on $\mathrm{Hol}_d^*(n)$ given by the matrix multiplication

$$(f_0, f_1, \cdots, f_n) \cdot A = (f_0, f_1, \cdots, f_n) \begin{pmatrix} 1 & \mathbf{0}_n \\ {}^t\mathbf{0}_n & A \end{pmatrix} \tag{4.3}$$

for $((f_0, f_1, \cdots, f_n), A) \in \mathrm{Hol}_d^*(n) \times \mathrm{GL}_n(\mathbb{C})$. By using Lemma 2.4, we can see that the above right $\mathrm{GL}_n(\mathbb{C})$-action on $\mathrm{Hol}_d^*(n)$ is free, and we obtain the following commutative diagram of fibration sequences

$$\begin{tikzcd} \mathbb{C}^* \arrow[r, "i''_d"] \arrow[d, "\hat{j}_d"', "\cap"] & H_d(n) \arrow[r] \arrow[d, "j'_d"', "\cap"] & H_d(n)/\mathbb{C}^* \arrow[d, "q_d"'] \\ \mathrm{GL}_n(\mathbb{C}) \arrow[r] & \mathrm{Hol}_d^*(n) \arrow[r] & \mathrm{Hol}_d^*(n)/\mathrm{GL}_n(\mathbb{C}) \end{tikzcd} \tag{4.4}$$

where the natural inclusions $i''_d : \mathbb{C}^* \to H_d(n)$ and $\hat{j}_d : \mathbb{C}^* \to \mathrm{GL}_n(\mathbb{C})$ are defined by

$$\begin{cases} i''_d(\alpha) = (z_0^d, \cdots, z_n^d) \cdot \alpha = (z_0^d, z_1^d, \cdots, z_{n-1}^d, \alpha z_n^d), \\ \hat{j}_d(\alpha) = \begin{pmatrix} E_n & 0 \\ 0 & \alpha \end{pmatrix} \qquad (E_n : \ (n \times n) \text{ identity matrix}). \end{cases}$$

Lemma 4.2. *$\pi_1(H_d(n)/\mathbb{C}^*) \cong \mathbb{Z}/d^n$.*

Proof. Consider the commutative diagram of exact sequences induced from (4.3):

$$\begin{tikzcd} \pi_1(\mathbb{C}^*) \arrow[r, "i''_{d*}"] \arrow[d, "\hat{j}_{d*}"', "\cong"] & \pi_1(H_d(n)) \arrow[r] \arrow[d, "j'_{d*}"', "\cong"] & \pi_1(H_d(n)/\mathbb{C}^*) \arrow[r] \arrow[d, "q_{d*}"'] & 0 \\ \pi_1(\mathrm{GL}_n(\mathbb{C})) \arrow[r] & \pi_1(\mathrm{Hol}_d^*(n)) \arrow[r] & \pi_1(\mathrm{Hol}_d^*(n)/\mathrm{GL}_n(\mathbb{C})) \arrow[r] & 0 \end{tikzcd}$$

Since $\hat{j}_{d*}$ and j'_{d*} are isomorphisms by Theorem 4.1, q_{d*} is so. However, because there is an isomorphism $\pi_1(\mathrm{Hol}_d^*(n)/\mathrm{GL}_n(\mathbb{C})) \cong \mathbb{Z}/d^n$ by [14], we have an isomorphism $\pi_1(H_d(n)/\mathbb{C}^*) \cong \mathbb{Z}/d^n$. □

Theorem 4.2. *There is a homotopy equivalence* $\widetilde{H_d(n)} \simeq R_2^{-1}(1)$.

Proof. It follows from Lemma 4.1 that there is a fibration sequence

$$R_2^{-1}(1) \xrightarrow{\subset} H_d(n) \xrightarrow{R_2} \mathbb{C}^*. \tag{4.5}$$

If $\mu_0 : \mathbb{C}^* \to \mathbb{C}^*$ denotes the map given by $\mu_0(\alpha) = \alpha^{d^n}$ for $\alpha \in \mathbb{C}^*$, it is the d^n-fold covering projection. Furthermore, for $\alpha \in \mathbb{C}^*$, by using Lemma 2.1,

$$R_2 \circ i''_d(\alpha) = R(z_0^d, \cdots, z_{n-1}^d, \alpha z_n^d) = \alpha^{d^n} R(z_0^d, \cdots, z_n^d) = \alpha^{d^n} = \mu_0(\alpha).$$

Hence, $R_2 \circ i''_d = \mu_0$ and it follows from [[3], Lemma 2.1] that there is a homotopy commutative diagram

$$\begin{array}{ccccc}
\mathbb{Z}/d^n & \longrightarrow & R_2^{-1}(1) & \longrightarrow & H_d(n)/\mathbb{C}^* \\
\downarrow & & \cap\downarrow & & \| \\
\mathbb{C}^* & \xrightarrow{i''_d} & H_d(n) & \longrightarrow & H_d(n)/\mathbb{C}^* \\
\mu_0\downarrow & & R_2\downarrow & & \downarrow \\
\mathbb{C}^* & \xrightarrow{=} & \mathbb{C}^* & \longrightarrow & *
\end{array}$$

where all horizontal and vertical sequences are fibration sequences.

Consider the homotopy fibration sequence $\mathbb{Z}/d^n \to R_2^{-1}(1) \to H_d(n)/\mathbb{C}^*$. Since $\pi_1(H_d(n)/\mathbb{C}^*) \cong \mathbb{Z}/d^n$ (by Lemma 4.2) and $R_2^{-1}(1)$ is connected, $R_2^{-1}(1)$ is simply connected. Hence, by using (4.5) we also obtain a homotopy equivalence $\widetilde{H_d(n)} \simeq R_2^{-1}(1)$. □

Corollary 4.1.

(i) *There is a homeomorphism* $\widetilde{H_d(n)/\mathbb{C}^*} \cong R_2^{-1}(1)$, *where* $\widetilde{H_d(n)/\mathbb{C}^*}$ *denotes the universal covering of the orbit space* $H_d(n)/\mathbb{C}^*$.

(ii) *There is a homotopy equivalence* $\widetilde{H_d(n)} \simeq \widetilde{H_d(n)/\mathbb{C}^*}$.

Proof. Since the assertion (ii) easily follows from (i) and Theorem 4.2, it remains to show (i). It follows from Lemma 4.1 that there is a homeomorphism

$$H_d(n)/\mathbb{C}^* \cong (\mathbb{C}^* \times_{H_{d,n}} R_2^{-1}(1))/\mathbb{C}^* \cong H_{d,n}\backslash R_2^{-1}(1).$$

By using Lemma 2.4, we can see that the group $H_{d,n}$ acts on $R_2^{-1}(1)$ freely. Hence, there is a covering space sequence $H_{d,n} \to R_2^{-1}(1) \to H_d(n)/\mathbb{C}^*$. Since $\pi_1(H_d(n)/\mathbb{C}^*) \cong \mathbb{Z}/d^n \cong H_{d,n}$ (by Lemma 4.2) and $R_2^{-1}(1)$ is connected, there is a homeomorphism $\widetilde{H_d(n)/\mathbb{C}^*} \cong R_2^{-1}(1)$. □

Proof of Theorem 1.2. The assertion follows from Theorem 2.1, Theorem 3.1 and Theorem 4.2. □

5. Homotopy fibers

In this section we give the proof of Theorem 1.3.

Lemma 5.1. *There is a homotopy equivalence $HF_d^*(n) \simeq HF_d(n)$.*

Proof. Consider the evaluation map $e : \mathrm{Map}_d(n) \to \mathbb{C}\mathrm{P}^n$ given by $e(f) = f(\mathbf{e}_n)$. Then it follows from the fibration sequence (2.3) and [3, Lemma 2.1] that there is a commutative diagram

$$\begin{array}{ccccc}
HF_d^*(n) & \xrightarrow[\simeq]{} & HF_d(n) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hol}_d^*(n) & \xrightarrow{\subset} & \mathrm{Hol}_d(n) & \xrightarrow{ev} & \mathbb{C}\mathrm{P}^n \\
{\scriptstyle i_d}\downarrow & & {\scriptstyle j_d}\downarrow\cap & & \| \\
\mathrm{Map}_d^*(n) & \xrightarrow{\subset} & \mathrm{Map}_d(n) & \xrightarrow{e} & \mathbb{C}\mathrm{P}^n
\end{array}$$

such that all horizontal and vertical sequences are fibration sequences. Then the assertion easily follows from the diagram chasing. $\square$

Proof of Theorem 1.3. It suffices to show that HF_d^* is simply connected. If $d = 1$, the assertion follows from Theorem 1.1, and assume $d \geq 2$. Because $i_{d*} : \pi_1(\mathrm{Hol}_d^*(n)) \xrightarrow{\cong} \pi_1(\mathrm{Map}_d^*(n))$ is bijective by [14], it is sufficient to show that i_d induces a surjection on π_2.

Let $i'' : \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n$ denote the inclusion given by $i''([x_0 : \cdots : x_{n-1}] = [x_0 : \cdots : x_{n-1} : 0]$, and define the restriction map $r' : \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^{n-1}, \mathbb{C}\mathrm{P}^n)$ by $r'(f) = f \circ i''$. Then we have the fibration sequence

$$F_d(n) \xrightarrow{j'} \mathrm{Map}_d^*(n) \xrightarrow{r'} \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^{n-1}, \mathbb{C}\mathrm{P}^n). \tag{5.1}$$

Define the map $g_d'' : \Omega^{2n}\mathbb{C}\mathrm{P}^n \to F_d(n)$ by

$$g_d''(\varphi) = \nabla \circ (\varphi_d^{n,n} \vee \varphi) \circ \mu' : \mathbb{C}\mathrm{P}^n \xrightarrow{\mu'} \mathbb{C}\mathrm{P}^n \vee S^{2n} \xrightarrow{\varphi_d^{n,n} \vee \varphi} \mathbb{C}\mathrm{P}^n \vee \mathbb{C}\mathrm{P}^n \xrightarrow{\nabla} \mathbb{C}\mathrm{P}^n$$

for $\varphi \in \Omega^{2n}\mathbb{C}\mathrm{P}^n$, where $\nabla : \mathbb{C}\mathrm{P}^n \vee \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ is a folding map, and $\mu' : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n \vee S^{2n}$ denotes the co-action map obtained by collapsing the hemisphere of $2n$-cell e^{2n} in the mapping cone $\mathbb{C}\mathrm{P}^n = \mathbb{C}\mathrm{P}^{n-1} \cup_{\gamma_{n-1}} e^{2n}$. Note that $g_d'' : \Omega_0^{2n}\mathbb{C}\mathrm{P}^n \xrightarrow{\simeq} F_d(n)$ is a homotopy equivalence ([9]). Let $\epsilon_d : \mathrm{Hol}_d^*(1) \to H_d(n)$ be the inclusion given by $\epsilon_d(f, g)) = (f, g, z_2^d, \cdots, z_n^d)$, where we identify $\mathrm{Hol}_d^*(1)$ with the space consisting of all pair $(f, g) \in \mathbb{C}[z_0, z_1]^2$ of homogenous polynomials of the same degree d with no common root except $\mathbf{0}_2 = (0, 0) \in \mathbb{C}^2$ such that the coefficient of z_0^d of f is 1 and that of g is 0. It is routine to check that the following diagram is homotopy commutative

$$
\begin{tikzcd}
\mathrm{Hol}_d^*(1) \arrow[r, "\epsilon_d"] \arrow[d, "i"', "\cap"] & \mathrm{H}_d(n) \arrow[r, "j''", "\subset"'] \arrow[d, "i''_d"', "\cap"] & \mathrm{Hol}_d^*(n) \arrow[d, "i_d"', "\cap"] \\
\Omega_d^2\mathbb{C}\mathrm{P}^1 & F_d(n) \arrow[r, "j'", "\subset"'] & \mathrm{Map}_d^*(n) \\
\Omega_0^2\mathbb{C}\mathrm{P}^1 \arrow[r, "\epsilon"] \arrow[u, "{*[d]}", "\simeq"'] & \Omega^{2n}\mathbb{C}\mathrm{P}^n \arrow[u, "g''_d", "\simeq"'] & \\
\Omega^2 S^3 \arrow[r, "\Omega^2 E^{2n-2}"] \arrow[u, "\Omega^2\gamma_1", "\simeq"'] & \Omega^{2n}S^{2n+1} \arrow[u, "\Omega^{2n}\gamma_n", "\simeq"'] &
\end{tikzcd}
\tag{5.2}
$$

where $E^{2n-2} : S^3 \to \Omega^{2n-2}S^{2n+1}$ denotes the $(2n-2)$-fold suspension, $*[d]$ is the d-times loop sum with the identity map on S^2, $i : \mathrm{Hol}_d^*(1) \to \Omega_d^2\mathbb{C}\mathrm{P}^1$ is an inclusion and the map ϵ is given by

$$\epsilon(f)(x \wedge s_2 \wedge s_3 \wedge \cdots \wedge s_n) = [f(x) : s_2 : \cdots : s_n]$$

for $(f, x) \in \Omega_0^2\mathbb{C}\mathrm{P}^1 \times S^2$ and $s_j \in S^1$ $(j = 2, 3, \cdots, n)$.

Since $\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^{n-1}, \mathbb{C}\mathrm{P}^n)$ is 2-connected ([9]), the map j' induces a surjection on π_2. By Theorem 1.1, $i_* : \pi_2(\mathrm{Hol}_d^*(1)) \to \pi_2(\Omega_d^2\mathbb{C}\mathrm{P}^1)$ is an isomorphism if $d \geq 3$ and an epimorphism if $d = 2$. Because $\Omega^2 E_*^{2n-2} : \pi_2(\Omega^2 S^3) \xrightarrow{\cong} \pi_2(\Omega^{2n}S^{2n+1})$ is an isomorphism, by applying π_2 to the diagram (5.2), we see that $i_{d*} : \pi_2(\mathrm{Hol}_d^*(n)) \to \pi_2(\mathrm{Map}_d^*(n))$ is also a surjection. $\square$

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