J. Math. Kyoto Univ.
1-2 (1962) 255-269.

On the imbedding of the Schwarzschild space-time III.

By

Tamehiro FUJITANI, Mineo IKEDA and Makoto MATSUMOTO

(Received Dec. 4, 1961)

In parts I and II of this series [4], we developed a systematic discussion of imbeddings of the Schwarzschild space-time into a pseudo-euclidean space of six dimensions. We have already obtained the stationary imbeddings of three types (i), (ii) and (iii) in the first paper, which will be called *elliptic*, *hyperbolic* and *parabolic* respectively in the present paper. In this paper, first of all, we shall discuss these imbeddings in detail and illustrate by figures. It should seem difficult to find explicit functions of imbedding except for the stationary cases, and the remainder of this paper will be devoted to considerations of various imbeddings of the Schwarzschild space-time.

At the end of this series we wish to express our sincere thanks to Dr. Y. Akizuki and Dr. S. Sasaki for their continued encouragements. Our thanks go also to the members of the seminar for the differential geometry in Kyoto University and the members of the Research Institute for Theoretical Physics, Hiroshima University who have kindly helped us by frequent discussions.

§ 1. The equation of (t, r)-geodesics.

The equations of a geodesic in the Schwarzschild space-time V^4 are given by

$$\frac{d^2t}{du^2}+\frac{2m}{\gamma r^2}\frac{dt}{du}\frac{dr}{du}=0,$$

$$
\begin{aligned}
&\frac{d^2r}{du^2}+\frac{m\gamma}{r^2}\left(\frac{dt}{du}\right)^2-\frac{m}{\gamma r^2}\left(\frac{dr}{du}\right)^2-\gamma r\left[\left(\frac{d\theta}{du}\right)^2+\sin^2\theta\left(\frac{d\varphi}{du}\right)^2\right]=0\,,\\
(1.1)\quad &\frac{d^2\theta}{du^2}+\frac{2}{r}\frac{dr}{du}\frac{d\theta}{du}-\sin\theta\cos\theta\left(\frac{d\varphi}{du}\right)^2=0\,,\\
&\frac{d^2\varphi}{du^2}+\frac{2}{r}\frac{dr}{du}\frac{d\varphi}{du}+2\cot\theta\frac{d\theta}{du}\frac{d\varphi}{du}=0\,,
\end{aligned}
$$

where the parameter u is a special one [8, p. 7], and we have further the condition $g_{ij}dx^i dx^j=\varepsilon k^2du^2$, namely

$$
(1.2)\qquad \gamma(dt)^2-\frac{1}{\gamma}(dr)^2-r^2[(d\theta)^2+\sin^2\theta(d\varphi)^2]=\varepsilon k^2du^2\,,
$$

where the k is a constant and the ε the indicator for a timelike or spacelike geodesic; the geodesic is a null geodesic if $k=0$.

We consider the two-dimensional subspace S in the V^4 such that the coordinates θ and φ are constant. A null geodesic of the V^4, which is contained in the above S, is given from (1.1) and (1.2) as follows.

$$
\begin{aligned}
&\frac{d^2t}{du^2}+\frac{2m}{\gamma r^2}\frac{dt}{du}\frac{dr}{du}=0\,,\\
&\frac{d^2r}{du^2}+\frac{m\gamma}{r^2}\left(\frac{dt}{du}\right)^2-\frac{m}{\gamma r^2}\left(\frac{dr}{du}\right)^2=0\,,\\
&\gamma^2(dt)^2-(dr)^2=0\,.
\end{aligned}
$$

These equations are immediately integrated and we have the equation of such a null geodesic. If we choose a direction of the parameter t suitably, the equation is written

$$
(1.3)\qquad t=r+2m\log(r-2m)+t_0\,,
$$

where the t_0 is a constant. For some constant t_0 we shall call the geodesic the (t, r)-geodesic and denote by G_{t_0}. It should be remarked that the parameter t can take all real numbers as the variable r runs from $2m+0$ to $+\infty$.

§ 2. The elliptic imbedding.

In this section we shall treat the imbedding (i) of part I, which was called to be of elliptic type, and was the same essentially as the one obtained by E. Kasner [6]. By this imbedding the surface S of the V^4 is realised in the pseudo-euclidean 3-space as follows.

$$x = \sqrt{\gamma}\sin t\,, \quad y = \sqrt{\gamma}\cos t\,, \quad z = f(r)\,, \tag{2.1}$$

where x, y and z are rectangular coordinates, and we took $k=1$ and $2m=1$, so that $\gamma=1-r^{-1}$. The function $f(r)$ is determined by

$$\left(\frac{df}{dr}\right)^2 = \frac{4r^3+1}{4r^3(r-1)}\,, \qquad \lim_{r=1+0} f(r) = 0\,. \tag{2.2}$$

It follows from (2.1) that

$$x^2+y^2 = \gamma\,, \tag{2.3}$$

$$\frac{x}{y} = \tan t\,. \tag{2.4}$$

We illustrate the xy-, yz- and zx-projection of the surface S and draw the projections of t-line ($r=$const.) C_r and r-line ($r=$ const.) C_t. It is clear that the surface S is a surface of revolution and the axis of rotation is the z-axis. The meridian on the zx-plane is

$$x = \sqrt{\gamma}\,, \quad z = f(r)\,. \tag{2.5}$$

This plane curve is shown as OAA′ in Fig. 1. The equation (2.3) shows that a curve C_r is a parallel, whose radius $\sqrt{\gamma}$ starts from 0 and tends to 1. On the other hand we see from (2.4) that a curve C_t is a meridian. For $t=\pi/2$ we have the curve OAA′ on the zx-plane, and along the curve we have

$$\lim_{x=0}\frac{dz}{dx} = \lim_{r=1+0}\sqrt{4r^3+1} = \sqrt{5}\,.$$

Next we consider a (t, r)-geodesic G_{t_0}, which will be written with bold strokes in Fig. 1 and 2. For $r=1+0$ the xy-projection of a G_{t_0} turns round and round about the origin 0 and tends to 0 as the logarithmic spiral. For various values of t_0 a curve G_{t_0} will be obtained from a fixed G_0 by rotating the whole about the z-axis by the angle t_0.

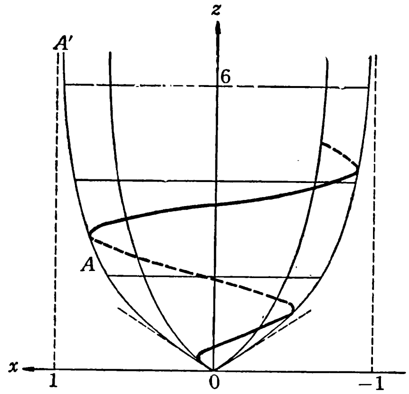

Fig. 1

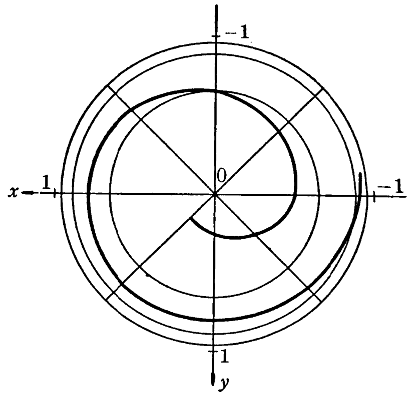

Fig. 2

As a result we can say that the singularity $r=1+0$ of the metric of the V^4 is represented as a single point, namely the origin 0, if we deal with the elliptic imbedding.

§ 3. The hyperbolic imbedding.

We noticed in part I that the imbedding obtained by C. Fronsdal [3] was a special case of our hyperbolic imbedding. By this imbedding the surface S is realised by

$$(3.1) \qquad x = 2\sqrt{\gamma}\sinh\frac{t}{2}, \quad y = 2\sqrt{\gamma}\cosh\frac{t}{2}, \quad z = g(r),$$

where the function $g(r)$ is

$$(3.2) \qquad \left(\frac{dg}{dr}\right)^2 = \frac{r^2+r+1}{r^3}, \qquad \lim_{r=1+0} g(r) = 0.$$

Following C. Fronsdal we took $k=2$ and $2m=1$ again. It follows from (3.1) that

$$(3.3) \qquad y^2-x^2 = 4\gamma, \qquad y>0,$$

$$(3.4) \qquad \frac{x}{y} = \tanh\frac{t}{2}.$$

We first consider a t-line C_r, and see from (3.3) that each C_r is a half of a hyperbola on the plane $z=g(r)$ and its xy-projection has the common asymptotics $y=\pm x$ (Fig. 5). The vertices of those hyperbolas move along the curve OCC′ (Fig. 4) as the parameter r varies. The y of the curve OCC′ starts from 0 and approaches to 2.

Next we treat a r-line C_t. It follows from (3.4) that the xy-projection of a C_t is a half of the straight line issuing from the origin (Fig. 5). Along a C_t we get

$$\frac{dz}{dx}=\frac{\sqrt{\gamma r(r^2+r+1)}}{\sinh\frac{t}{2}},\quad \frac{dz}{dy}=\frac{\sqrt{\gamma r(r^2+r+1)}}{\cosh\frac{t}{2}},$$

and hence the zx-projection contacts with the x-axis except when $t=0$, and the yz-projection with the x-axis for all t.

Now we consider a (t, r)-geodesic G_{t_0}, the equation of which is given by (1.3). Along this curve we have easily

$$\lim_{r=1+0} y = -\lim_{r=1+0} x = \exp\left(-\frac{1+t_0}{2}\right),$$

from which it follows that each G_{t_0} tends to the point $A_{t_0}\left(-\exp\left(-\frac{1+t_0}{2}\right), \exp\left(-\frac{1+t_0}{2}\right)\right)$ for $r=1+0$ (Fig. 5). Thus we get the pictures of those G_{t_0}, which are written with bold strokes on each coordinate plane (Fig. 3, 4, 5). The broken lines $A_{t_0}T_{t_0}$

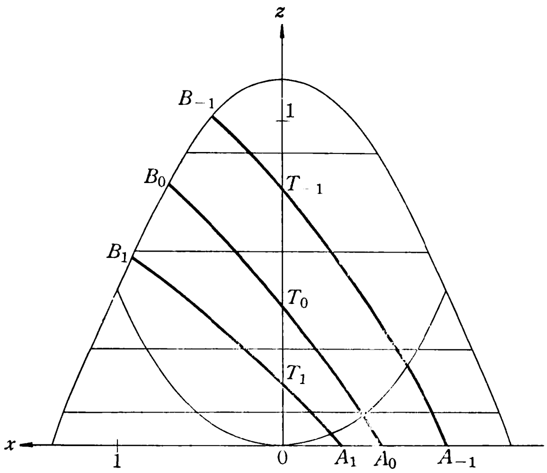

Fig. 3

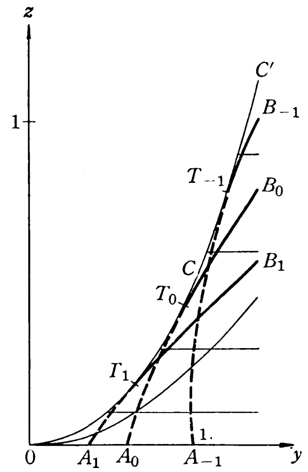

Fig. 4

on the yz-plane are used to denote that the x is negative on the arcs.

Consequently we can say that the singularity of the metric of the V^4 is distributed densely on the half of the straight line $y=-x$, if we are concerned only with the hyperbolic imbedding.

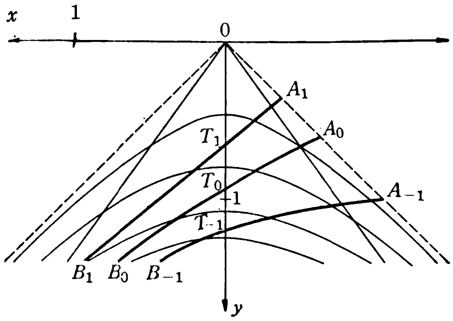

Fig. 5

§ 4. The parabolic imbedding.

The parabolic imbedding was found first by authors in part I, which was given by

$$z_1 = \frac{t^2-1}{2}\sqrt{\gamma}+h(r), \qquad z_2 = t\sqrt{\gamma},$$

$$z_3 = \frac{t^2+1}{2}\sqrt{\gamma}+h(r), \qquad z_4 = r\sin\theta\sin\varphi,$$

$$z_5 = r\sin\theta\cos\varphi \quad , \qquad z_6 = r\cos\theta.$$

For the purpose of examining the behavior of the surface S in this imbedding, it had better to introduce new orthogonal coordinates x, y and z instead of z_1, z_2 and z_3 as follows.

$$\sqrt{2}\,z_1 = x-z, \quad z_1 = y, \quad \sqrt{2}\,z_3 = x+z.$$

Then the surface S is represented

$$x = t^2\sqrt{\frac{\gamma}{2}}+\sqrt{2}\,h(r), \quad y = t\sqrt{\gamma}, \quad z = \sqrt{\frac{\gamma}{2}}, \tag{4.1}$$

where the function $h(r)$ is determined by

$$\frac{dh}{dr} = \frac{r}{\sqrt{\gamma}}, \qquad \lim_{r=1+0} h(r) = 0. \tag{4.2}$$

It follows from (4.1) that

$$y^2 = \sqrt{2\gamma}(x-\sqrt{2}\,h(r)), \tag{4.3}$$

$$\frac{y}{z} = \sqrt{2}\,t, \tag{4.4}$$

We shall proceed in similar manner with the previous sections, and consider first a t-line C_r. By means of (4.3) we see that each C_r is a parabola on the plane $z=\sqrt{\gamma/2}$ whose vertex is the point $(\sqrt{2}\,h,\ 0,\ \sqrt{\gamma/2})$ as in Fig. 6. The locus of the vertices are shown as OCC′ in Fig. 7. Along this curve we have $\lim\limits_{r=1+0}(dx/dz)=4$.

The curve OCC′ is a r-line for $t=0$, and we see from (4.4) that the yz-projection of a general r-line C_t is a half of a straight line issuing from the origin (Fig. 8). Along a C_t we have

$$\lim_{r=1+0}\frac{dx}{dz}=t^2+4\,.$$

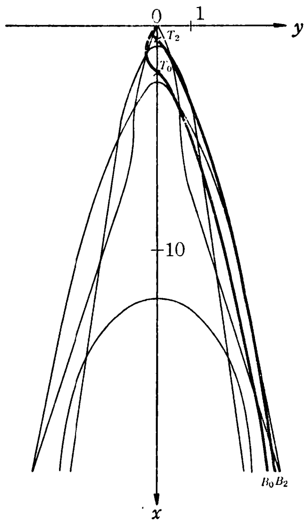

Fig. 6

Finally we consider a (t, r)-geodesic G_{t_0}. Making use of (1.3) we have $\lim\limits_{r=1+0}x=\lim\limits_{r=1+0}y=\lim\limits_{r=1+0}z=0$. The pictures of G_{t_0} are written in Fig. 6, 7 and 8 with bold strokes, and the broken lines on the zx-plane show that $y<0$ at those parts of curves. On the other hand the broken lines on the xy-plane mean that those parts slip under the lower part of the surface S.

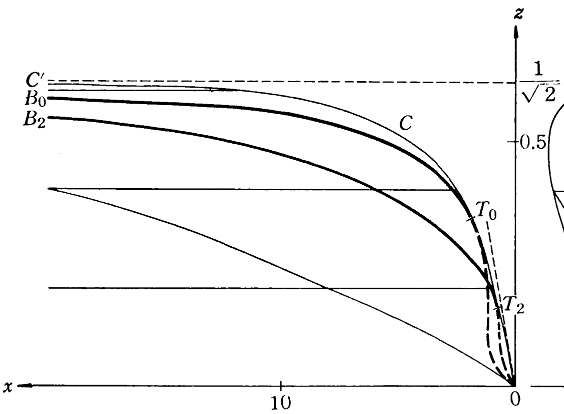

Fig. 7

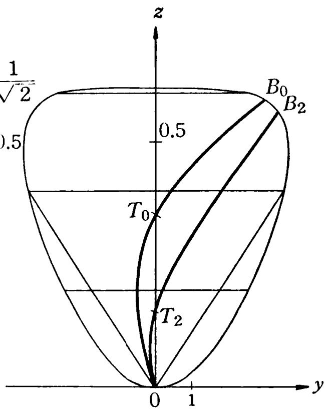

Fig. 8

Thus we may say that the singularity is then the single point 0 as for the parabolic imbedding.

§ 5. Impossibility of a certain imbedding of the V^4 in E^6.

We recall the process in part II. We considered there the hypersurface $V^3(t)$ of the Schwarzschild space-time V^4 which was defined by $t=$const., and showed that the $V^3(t)$ was imbedding in an euclidean 4-space. Making use of this imbedding we saw that the V^4 was thought of as a hypersurface of the space V^5 of five dimensions, the metric being

$$ds^2 = \gamma dt^2 - \frac{2m}{\gamma r} dr^2 - (dz^4)^2 - (dz^5)^2 - (dz^6)^2 . \tag{5.1}$$

Therefore our imbedding problem of the V^4 was reduced to the one of the V^2 of two dimensions, whose metric was

$$ds^2 = \gamma dt^2 - \frac{2m}{\gamma r} dr^2 . \tag{5.2}$$

We concluded part II by a remark that the stationary imbeddings corresponded to the special case for $b_{01}=0$, where b_{ij} $(i, j=0, 1)$ were second fundamental quantities of the V^2 when it was looked upon as a surface of a pseudo-euclidean 3-space. The geometrical meaning of $b_{01}=0$ is that t-lines and r-lines on the V^2 are lines of curvature.

Though our imbedding problem of the V^4 was reduced to the simpler one, it should seem rather difficult to obtain imbedding functions of the V^2 explicitly except for the stationary case. However, from the geometrical and perhaps physical points of view we are interested in a special imbedding of the V^4 such that a certain family of geodesics of the V^4 is represented as a family of straight lines in E^6, similar to the case for the empty universe of De Sitter [8, pp. 261-264]. But we shall prove a negative result as follows.

Theorem 1. *There exists no imbedding of the Schwarzschild space-time V^4 in a pseudo-euclidean 6-space E^6 such that the V^4 contains a straight line of the E^6.*

Proof. We saw already that the V^4 was imbedded in the V^5 of the metric (5.1), and the imbedding functions were given by

$$(5.3)\qquad \begin{cases} y^0 = x^0, \quad y^1 = x^1, \quad y^2 = x^1 \sin x^2 \sin x^3, \\ y^3 = x^1 \sin x^2 \cos x^3, \quad y^4 = x^1 \cos x^2, \end{cases}$$

where we putted $(t, r, \theta, \varphi)=(x^0, x^1, x^2, x^3)$ and $(t, r, z_4, z_5, z_6)=(y^0, y^1, y^2, y^3, y^4)$. Hence if $C: y^a=y^a(u)$, $a=0, 1, 2, 3, 4$, is a curve of the V^5 drawn on the V^4, we have

$$(5.4)\qquad \begin{cases} \dfrac{dy^a}{du} = \dfrac{\partial y^a}{\partial x^i}\dfrac{dx^i}{du}, & a, b, c = 0, 1, 2, 3, 4 \\ & i, j = 0, 1, 2, 3, \\ \dfrac{d^2y^a}{du^2}+\bar{\Gamma}^a_{bc}\dfrac{dy^b}{du}\dfrac{dy^c}{du} = Y^a_{ij}\dfrac{dx^i}{du}\dfrac{dx^j}{du}+\Big(\dfrac{d^2x^i}{du^2}+\Gamma^i_{jk}\dfrac{dx^j}{du}\dfrac{dx^k}{du}\Big)\dfrac{\partial y^a}{\partial x^i}, \end{cases}$$

where (Y^a_{ij}) is so-called the Euler-Schouten's tensor of the V^4 defined by

$$Y^a_{ij} = \frac{\partial^2 y^a}{\partial x^i \partial x^j}+\bar{\Gamma}^a_{bc}\frac{\partial y^b}{\partial x^i}\frac{\partial y^c}{\partial x^j}-\Gamma^k_{ij}\frac{\partial y^a}{\partial x^k},$$

and the Γ^k_{ij} and $\bar{\Gamma}^a_{bc}$ are Christoffel's symbols of the second kind of the V^4 and V^5 respectively. Therefore if the C is a geodesic of the V^4, then the C satisfies (1.1), and in terms of the coordinates (t, r, y^2, y^3, y^4) of the V^5 the C is written in the form

$$(5.5)\qquad \begin{cases} \dfrac{d^2t}{du^2}+\dfrac{2m}{\gamma r^2}\dfrac{dt}{du}\dfrac{dr}{du} = 0, \\ \dfrac{d^2r}{du^2}+\dfrac{m\gamma}{r^2}\Big(\dfrac{dt}{du}\Big)^2-\dfrac{m}{\gamma r^2}\Big(\dfrac{dr}{du}\Big)^2-\gamma r\Big[\Big(\dfrac{d\theta}{du}\Big)^2+\sin^2\theta\Big(\dfrac{d\varphi}{du}\Big)^2\Big] = 0, \\ \dfrac{d^2y^2}{du^2}+\sin\theta\sin\varphi A(u) = 0, \qquad \dfrac{d^2y^3}{du^2}+\sin\theta\cos\varphi A(u) = 0, \\ \dfrac{d^2y^4}{du^2}+\cos\theta A(u) = 0, \end{cases}$$

where by definition

$$A(u) = \frac{m\gamma}{r^2}\Big(\frac{dt}{du}\Big)^2-\frac{r^2}{\gamma m}\Big(\frac{dr}{du}\Big)^2+2m\Big[\Big(\frac{d\theta}{du}\Big)^2+\sin^2\theta\Big(\frac{d\varphi}{du}\Big)^2\Big].$$

Next, the V^5 is imbedded in a pseudo-euclidean 6-space E^6, and the imbedding is given as the result of the one of the V^2, the metric being (5.2), in a pseudo-euclidean 3-space E^3. Let $y(t, r)$ be an imbedding vector of the V^2, which is a solution of the system of equations II (21), (22), (23) and (24), in which (b_{00}, b_{01}, b_{11}) is a system of solutions of the Gauss equation II (19) and the Codazzi equations II (20). If we express $y=(z^1(t, r), z^2(t, z), z^3(t, r))$, then the imbedding functions of the V^5 in E^6 are given by

$$(5.6)\qquad \begin{cases} z^1 = z^1(t, r), \quad z^2 = z^2(t, r), \quad z^3 = z^3(t, r), \\ z^4 = y^2, \quad z^5 = y^3, \quad z^6 = y^4. \end{cases}$$

Then the curve C is looked upon as a curve in the E^6, along which we have

$$(5.7)\qquad \begin{cases} \dfrac{dz^\alpha}{du} = \dfrac{\partial z^\alpha}{\partial y^a}\dfrac{dy^a}{du}, & \alpha = 1, 2, 3, 4, 5, 6, \\ & a, b, c = 0, 1, 2, 3, 4, \\ \dfrac{d^2z^a}{du^2} = Z^\alpha_{ab}\dfrac{dy^a}{du}\dfrac{dy^b}{du} + \left(\dfrac{d^2y^a}{du^2} + \overline{\Gamma}^a_{bc}\dfrac{dy^b}{du}\dfrac{dy^c}{du}\right)\dfrac{\partial z^\alpha}{\partial y^a}, \end{cases}$$

where (Z^α_{ab}) is the Euler-Schouten's tensor of the V^5 such as

$$Z^\alpha_{ab} = \frac{\partial^2 z^\alpha}{\partial y^a \partial y^b} - \overline{\Gamma}^c_{ab}\frac{\partial z^\alpha}{\partial y^c}.$$

The second equations of (5.7) are written in virtue of (5.5)

$$\begin{aligned} \frac{d^2z^\alpha}{du^2} &= Z^\alpha_{ab}\frac{dy^a}{du}\frac{dy^b}{du} + Y^a_{ij}\frac{dx^i}{du}\frac{dx^j}{du}\frac{\partial z^\alpha}{\partial y^a} \\ &\quad + \left(\frac{d^2x^i}{du^2} + \Gamma^i_{jk}\frac{dx^j}{du}\frac{dx^k}{du}\right)\frac{\partial y^a}{\partial x^i}\frac{\partial z^\alpha}{\partial y^a}. \end{aligned}$$

Since the C is a geodesic, the term in the parenthesis vanishes, and it follows from (5.5) that we get

$$(5.8)\qquad \begin{cases} \dfrac{d^2z^\alpha}{du^2} = \left(\dfrac{\partial^2 z^\alpha}{\partial t^2} - \dfrac{m\gamma}{r^2}\dfrac{\partial z^\alpha}{\partial r}\right)\left(\dfrac{dt}{du}\right)^2 + 2\left(\dfrac{\partial^2 z^\alpha}{\partial t \partial r} - \dfrac{m}{\gamma r^2}\dfrac{\partial z^\alpha}{\partial t}\right)\dfrac{dt}{du}\dfrac{dr}{du} \\ \qquad + \left(\dfrac{\partial^2 z^\alpha}{\partial r^2} + \dfrac{m}{\gamma r^2}\dfrac{\partial z^\alpha}{\partial r}\right)\left(\dfrac{dr}{du}\right)^2 + \gamma r\left[\left(\dfrac{d\theta}{du}\right)^2 + \sin^2\theta\left(\dfrac{d\varphi}{du}\right)^2\right]\dfrac{\partial z^\alpha}{\partial r}, \\ \qquad \alpha = 1, 2, 3, \\ \dfrac{d^2z^4}{du^2} + \sin\theta\sin\varphi A(u) = 0, \quad \dfrac{d^2z^5}{du^2} + \sin\theta\cos\varphi A(u) = 0, \\ \dfrac{d^2z^6}{du^2} + \cos\theta A(u) = 0. \end{cases}$$

Making use of the equations II (21), three terms in the parenthesis in the above equations are written in the forms

$$(5.9)\qquad \begin{cases} \dfrac{\partial^2 z^\alpha}{\partial t^2}-\dfrac{m\gamma}{r^2}\dfrac{\partial z^\alpha}{\partial r} = \dfrac{\gamma^2}{2r}\dfrac{\partial z^\alpha}{\partial r}+b_{00}m^\alpha\,, \\ \dfrac{\partial^2 z^\alpha}{\partial t\partial r}-\dfrac{m}{\gamma r^2}\dfrac{\partial z^\alpha}{\partial t} = b_{01}m^\alpha\,, \qquad\qquad \alpha = 1,2,3\,, \\ \dfrac{\partial^2 z^\alpha}{\partial r^2}+\dfrac{m}{\gamma r^2}\dfrac{\partial z^\alpha}{\partial r} = -\dfrac{1}{2r}\dfrac{\partial z^\alpha}{\partial r}+b_{11}m^\alpha\,. \end{cases}$$

Now, supposing that the geodesic C of the V^4 be represented as a straight line of the E^6, and then $d^2z^\alpha/du^2=0$ and hence we obtain first $A(u)=0$ from (5.8). This fact and (1.2) give us

$$(5.10)\qquad \begin{cases} \gamma dt^2-\dfrac{1}{\gamma}dr^2 = \dfrac{2}{3}\varepsilon k^2du^2\,, \\ d\theta^2+\sin^2\theta\, d\varphi^2 = -\dfrac{1}{3r^2}\varepsilon k^2 du^2\,. \end{cases}$$

Substituting from (5.9) and (5.10) in (5.8), we have

$$(5.11)\qquad b_{00}dt^2+2b_{01}dtdr+b_{11}dr^2 = 0\,.$$

On the other hand, the equations (1.1) of a geodesic are easily integrated once by means of (1.2), and we have

$$(5.12)\qquad \begin{cases} \dfrac{dt}{du} = \dfrac{a}{\gamma}\,, \quad \left(\dfrac{dr}{du}\right)^2 = a^2+\dfrac{\gamma}{r^2}b-\varepsilon k^2\gamma\,, \\ \left(\dfrac{d\theta}{du}\right)^2 = -\dfrac{b}{r^4}-\dfrac{c^2}{r^4\sin^2\theta}\,, \quad \dfrac{d\varphi}{du} = \dfrac{c}{r^2\sin^2\theta}\,, \end{cases}$$

where a, b and c are constants. Substituting from (5.12) in (5.10) the condition $b=(1/3)\varepsilon k^2r^2$ is derived, and hence $k=0$ or $r=r_0$ (const.) because of $b=$const. If $k=0$, the curve C is a null geodesic. On the other hand, if $r=r_0$ ($>2m$), we have from (5.12)

$$\left(\frac{d\theta}{du}\right)^2 = -\frac{1}{3r_0^2}\varepsilon k^2-\frac{c^2}{r_0^4\sin^2\theta}\,,$$

$$\left(\frac{dr}{du}\right)^2 = 0 = a^2-\frac{2}{3}\varepsilon k^2\left(1-\frac{2m}{r_0}\right).$$

The second equation gives $\varepsilon=1$ and we have $k=c=0$, because we should consider only a real geodesic. Thus we have $k=0$ in this case also. Consequently k, b and c vanish, and (5.12) are written

$$\frac{dt}{du}=\frac{a}{\gamma},\quad \frac{dr}{du}=\eta a,\qquad \eta=\pm 1,\quad a\neq 0,$$
$$\frac{d\theta}{du}=\frac{d\varphi}{du}=0.$$

Substitution of the above equations in (5.11) leads to

$$(5.13)\qquad b_{00}+2b_{01}\eta\gamma+b_{11}\gamma^2=0.$$

The second fundamental quantities b_{ij} in (5.13) are not arbitrary but must be determined by II (19) and (20). From the algebraic equations (5.13) and II (19) we deduce that

$$b_{00}=(\eta' b-\eta b_{01})\gamma,$$
$$b_{11}=(-\eta' b-\eta b_{01})\frac{1}{\gamma},\qquad b=\sqrt{\frac{3m}{2r^3}},$$

and, substituting from these in II (20), we have $(b/2r)-db/dr=0$, which is clearly not satisfied. Consequently we establish the theorem 1.

Now we consider generally a null geodesic C of the V^4. Then the equations (5.12) ($k=0$) are satisfied, and for the C we have

$$A(u)=-\frac{3m}{r^4}b,$$
$$Y^0_{ij}\frac{dx^i}{du}\frac{dx^j}{du}=0,\quad Y^1_{ij}\frac{dx^i}{du}\frac{dx^j}{du}=-\frac{3\gamma}{2r^3}b,$$
$$Y^2_{ij}\frac{dx^i}{du}\frac{dx^j}{du}=\frac{3m\sin\theta\sin\varphi}{r^4}b,\quad Y^3_{ij}\frac{dx^i}{du}\frac{dx^j}{du}=\frac{3m\sin\theta\cos\varphi}{r^4}b,$$
$$Y^4_{ij}\frac{dx^i}{du}\frac{dx^j}{du}=\frac{3m\cos\theta}{r^4}b.$$

Therefore it is easily seen that a null geodesic C of the V^4 is a null geodesic of the V^5 as well if, and only if, the constant b vanishes. Such a real geodesic C is then given by

$$(5.14)\qquad \begin{cases}\dfrac{dt}{du}=\dfrac{a}{\gamma},\quad \dfrac{dr}{du}=\eta a,\qquad \eta=\pm 1,\quad a\neq 0,\\[2mm] \dfrac{d\theta}{du}=\dfrac{d\varphi}{du}=0.\end{cases}$$

It follows from (5.8), (5.9) and $A(u)=0$ that the geodesic is expressed in the E^6 as follows.

$$(5.15)\qquad \begin{cases} \dfrac{d^2z^\alpha}{du^2} = (b_{00}+2b_{01}\eta\gamma+b_{11}\gamma^2)\dfrac{a^2}{r^2}m^\alpha\,, & \alpha = 1, 2, 3\,, \\ z^\lambda = a^\lambda u+b^\lambda\,, \qquad \lambda = 4, 5, 6\,, \end{cases}$$

where a^λ and b^λ are constants. The result seems worthy to state as a theorem.

Theorem 2. *There exist null geodesices of the Schwarzschild space-time V^4 which are also null geodesics even in the V^5, the metric of which is* (5.1). *In terms of the coordinates (t, r, η, φ) of the V^4 those geodesics are given by* (5.14). *Furthermore, in terms of the coordinates $(z^1, z^2, z^3, z^4, z^5, z^6)$ of a pseudo-euclidean* 6*-space enveloping the V^4, those geodesics satisfy the equations* (5.15), *where b_{ij} $(i, j=0, 1)$ are a system of solutions of II* (19) *and* (20).

§ 6. Impossibility of an imbedding of the V^4 in S^5.

It is well known that a Riemannian space V^n can not be imbedded in a pseudo-euclidean space E^{n+1} of $(n+1)$ dimensions, if the Ricci tensor $R_{ij}=R_i{}^k{}_{\cdot jk}$ is equal to zero [5], [2, p. 200]. Therefore the Schwarzschild space-time V^4 is not realised as a hypersurface of E^5. We should also pay attention to the paper by C. B. Allendoerfer [1], in which he studied the necessary and sufficient condition that an Einstein space V^n of non-vanishing scalar curvature can be imbedded in a pseudo-euclidean $(n+1)$-space. Further we are interested in the paper by H. Takeno [9], who investigated an imbedding of a spherically symmetric space-time in a pseudo-euclidean 5-space.

One of the authors developed in 1953 the theory of imbedding of a Riemannian space V^n in a space S^{n+1} of constant curvature K [7], in which he gave a system of algebraic equations by means of which the curvature K is determined only from quantities of the V^n. If we apply the result, we can discuss an imbedding of a V^n in a S^{n+1} for the case where the Ricci tensor R_{ij} vanishes or is equal to $(R/n)g_{ij}$. In this final section we shall prove a negative result as follows.

Theorem 3. *There exists no Riemannian space of constant curvature in which the Schwarzschild space-time is imbedded as a hypersurface.*

Proof. If a space S^{n+1} of the constant curvature K can envelope a given V^n, the curvature K has to be found as a solution of the system of equations [7, (4.8)]

$$A_{i_1i_2|j_1j_2|k_1k_2k_3k_4}\cdot K-2R_{i_1i_2|j_1j_2|k_1k_2k_3k_4}=0, \tag{6.1}$$

in which components of the tensors A and R are defined in terms of the fundamental tensor g_{ij} and the curvature tensor R_{ijkl} of the V^n as follows.

$$\begin{aligned} A_{i_1i_2|j_1j_2|k_1k_2k_3k_3} &= (R_{i_1j_1a_1b_1}g_{i_2a_2}g_{j_2b_2}-R_{i_1j_2a_1b_2}g_{i_2a_2}g_{j_1b_1}\\ &\quad +R_{i_2j_2a_2b_2}g_{i_1a_1}g_{j_1b_1}-R_{i_2j_1a_2b_1}g_{i_1a_1}g_{j_2b_2})\delta^{a_1a_2b_1b_2}_{k_1k_2k_3k_4},\\ R_{i_1i_2|j_1j_2|k_1k_2k_3k_4} &= \frac{1}{4}(R_{i_1j_1a_1b_1}R_{i_2j_2a_2b_2}-R_{i_1j_2a_1b_2}R_{i_2j_1a_2b_1})\delta^{a_1a_2b_1b_2}_{k_1k_2k_3k_4}, \end{aligned}$$

where δ's are the generalised Kronecke's deltas. The equations (6.1) themselves are somewhat complicated formally, but if the V^n is an Einstein space, namely $R_{ij}=(R/n)g_{ij}$, then we can reduce a simpler system of equations from (6.1). That is, contracting (6.1) by $g^{i_2k_3}g^{j_2k_4}$ and making use of the equations $R_{ij}=(R/n)g_{ij}$, we have

$$\begin{aligned} &\left[\frac{R}{2}(g_{ik}g_{jl}-g_{il}g_{jk})-6R_{ijkl}\right]K \\ &\quad =\frac{R^2}{16}(g_{ik}g_{jl}-g_{il}g_{jk})-\frac{R}{2}R_{ijkl}-R^a_{i\cdot kb}R^b_{j\cdot la}\\ &\qquad\qquad +R^a_{i\cdot lb}R^b_{j\cdot ka}+R^b_{a\cdot ij}R^a_{b\cdot kl}, \end{aligned} \tag{6.2}$$

where we changed indices and restricted already to the case $n=4$.

Now we return to the consideration of the Schwarzschild space-time, so that the scalar curvature R vanishes, and it is easily seen that the above equations have a solution

$$K=-\frac{m}{2r^3}. \tag{6.3}$$

However this K is not constant unfortunately, and thus we have the theorem 3.

Though our attempt to imbed the V^4 in a S^5 has gone to waste, we obtain a by-product that the curvature tensor of the V^4 is decomposed as follows.

$$R_{ijkl}=K(g_{ik}g_{jl}-g_{il}g_{jk})-(b_{ik}b_{jl}-b_{il}b_{jk}),\quad i,j,k,l=0,1,2,3, \tag{6.4}$$

where, referring to the coordinates (t, r, θ, φ), components of the tensor b are defined by

$$\begin{cases} b_{00}=-\sqrt{\dfrac{3m}{2r^3}}\gamma\,, \quad b_{11}=\sqrt{\dfrac{3m}{2r^3}}\dfrac{1}{\gamma}\,, \\ b_{22}=-\sqrt{\dfrac{3m}{2r^3}}r^2\,, \quad b_{33}=b_{22}\sin^2\theta\,, \qquad b_{ij}=0\,,\quad i\neq j\,. \end{cases} \tag{6.5}$$

The equations (6.4) are of the same form as the Gauss equations of the V^4 provided that the V^4 be imbedded in a S^5 [2, p. 211]. It should be noted that the tensor b_{ij} is proportional to the fundamental tensor g_{ij} of the V^4 to within algebraic sign.

University of Osaka Prefecture,
Research Institute for Theoretical Physics, Hiroshima University,
and Institute of Mathematics, Kyoto University.

REFERENCES

[1] C. B. Allendoerfer: Einstein spaces of class one, Bull. Amer. Math. Soc., vol. 43, 1937, pp. 265–270.

[2] L. P. Eisenhart: Riemannian geometry, Princeton, 1949.

[3] C. Fronsdal: Completion and embedding of the Schwarzschild solution, Phys. Rev., vol. 116, 1959, pp. 778–781.

[4] T. Fujitani, M. Ikeda and M. Matsumoto: On the imbedding of the Schwarzschild space-time I, II, This Journal, vol. 1, 1961, pp. 43–61, 63–70.

[5] E. Kasner: The impossibility of Einstein fields immersed in flat space of five dimensions, Amer. J. Math., vol. 43, 1921, pp. 126–129.

[6] E. Kasner: Finite representation of the solar gravitational field in flat space of six dimensions, Amer. J. Math., vol. 43, 1921, pp. 130–133.

[7] M. Matsumoto: Local imbedding of Riemann spaces, Memoirs of Univ. Kyoto, vol. 28, 1953, pp. 179–207.

[8] J. L. Synge: Relativity: The general theory, Amsterdam, 1960.

[9] H. Takeno: Theory of the spherically symmetric space-time III, J. Sci. Hiroshima Univ., vol. 16, 1952, pp. 291–298.