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Partially hypoelliptic and partially elliptic systems of differential operators with constant coefficients

By

Shigetake MATSUURA

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§ 1. Introduction.

In a recent work [4], L. Garding and B. Malgrange introduced certain new types of partial differential operators that were called partially hypoelliptic, partially elliptic, etc. And they characterized them completely by the shapes of the complex zeros of their corresponding characteristic polynomials.

The aim of our present paper is just to extend the results of Gårding and Malgrange [4] to the case of systems (generally overdetermined) of differential operators. This problem of generalization was one of the problems posed in [4]. It can also be considered as a partial generalization of Hörmander [5] and our previous work [8].

§ 2. Notations and preliminaries.

Let $\boldsymbol{C}$ be the complex number field and $\boldsymbol{C}[X, Y]$ be the polynomial ring over $\boldsymbol{C}$ in $m+n$ variables $X=(X_1, \cdots, X_m)$ and $Y=(Y_1, \cdots, Y_n)$. And we consider a matrix with p rows and q columns

$$(1) \qquad P(X, Y) = \begin{pmatrix} P_{11}(X, Y) \cdots P_{1q}(X, Y) \\ \cdots\cdots\cdots\cdots\cdots \\ P_{p1}(X, Y) \cdots P_{pq}(X, Y) \end{pmatrix}$$

with coefficients in $\boldsymbol{C}[X, Y]$. Putting $P_{jk}(X, Y)=0$ $(j\geq p;\ k=1, 2, \cdots, q)$ if necessary, we can always assume that $p\geq q$.

We denote by $\mathfrak{a}$ the ideal of $C[X, Y]$ generated by all the (q, q)-minors of the matrix (1), and call it *the ideal attached to the matrix* (1).

Replacing the variables X and Y by the differential operators $\frac{1}{i}\frac{\partial}{\partial x}=\left(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_m}\right)$ and $\frac{1}{i}\frac{\partial}{\partial y}=\left(\frac{1}{i}\frac{\partial}{\partial y_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial y_n}\right)$ respectively in (1), we get a matrix of differential operators of general type $P\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)$ in the $m+n$ real variable $x=(x_1, \cdots, x_m)$ and $y=(y_1, \cdots, y_n)$. Here i denotes the imaginary unit.

Now consider the following system of differential equations

$$P\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)U = 0. \tag{2}$$

Or, more generally,

$$P\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)U \in \prod_{1}^{p} \mathcal{H}_\Omega, \tag{2'}$$

where Ω denotes an open set in $(m+n)$-space $\boldsymbol{R}^{m+n}$ and $\mathcal{H}_\Omega$ a linear subspace of $\mathcal{D}'_\Omega$ (the space of distributions in Ω)[1] which is closed under the operations of partial differentiations; and

$$U = \begin{pmatrix} u_1 \\ \vdots \\ u_q \end{pmatrix}$$

is an unknown vector function whose components are in $\mathcal{D}'_\Omega$.

It is easy to see that the equation

$$Q\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)u_k = 0. \tag{3}$$

or

$$Q\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)u_k \in \mathcal{H}_\Omega \tag{3'}$$

holds respectively for any k $(1 \leq k \leq q)$ and for any $Q(X, Y) \in \mathfrak{a}$. In fact, by eliminating all the unknown functions other than u_k in the equation (2) (resp. in (2′)), it is seen that (3) (resp. (3′)) holds when Q is one of the (q, q)-minors of the matrix (1); and

1) We use the notations of L. Schwartz [10].

since these minors generate the ideal $\mathfrak{a}$, (3) (resp. (3′)) is true for any $Q(X, Y)$ in $\mathfrak{a}$.

We decompose also the $(m+n)$-dimensional complex affine space $\boldsymbol{C}^{m+n}$ into two factors: $\boldsymbol{C}^{m+n}=\boldsymbol{C}^m\times\boldsymbol{C}^n$; and the current coordinates shall be written in the form (ξ, η) with $\xi=(\xi_1, \cdots, \xi_m)$, $\eta=(\eta_1, \cdots, \eta_n)$. When $\mathfrak{a}$ is the ideal attached to the matrix $P(X, Y)$ and V is the algebraic variety defined by $\mathfrak{a}$, we say also that V is *the variety attached to the matrix* $P(X, Y)$.

By definition of the variety V, the rank of the matrix $P(\xi, \eta)$ is less than q if (ξ, η) is in V. Therefore there exists a non zero vector $C=C(\xi, \eta)$ such that

$$P(\xi, \eta)C = 0 \tag{4}$$

for $(\xi, \eta)\in V$. This implies that

$$U(x, y) = e^{i<x, \xi>+i<y, \eta>}\cdot C \tag{5}$$

is a non zero solution of the equation (2), since

$$P\Big(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\Big)U = e^{i<x, \xi>+i<y, \eta>}P(\xi, \eta)C = 0\,.$$

§ 3. Partially hypoelliptic systems.

First let us recall the definition of partial regularity (see [4]).

Definition 1. Let Ω be an open set in $\boldsymbol{R}^m\times\boldsymbol{R}^n$ and $f(x, y)\in\mathcal{D}'_\Omega$ be a distribution. We say that f is *regular in* x if, for every pair of open sets $A\subseteq\boldsymbol{R}^m$, $B\subseteq\boldsymbol{R}^n$, $A\times B\subseteq\Omega$, and for any $\varphi\in\mathcal{D}_B$, the distribution in x

$$g(x) = \int f(x, y)\varphi(y)\,dy$$

is a regular function, i. e. an indefinitely continuously differentiable function.

Definition 2. A system of differential operators $P\Big(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\Big)$ is called *hypoelliptic in* x if, for an open set Ω, the components of every solution $U\in\prod_1^q\mathcal{D}'_\Omega$ of the equation (2) are regular in x.

Theorem 1. *For a system of differential operators* $P\left(\frac{1}{i}\frac{\partial}{\partial x},\ \frac{1}{i}\frac{\partial}{\partial y}\right)$ *to be hypoelliptic in x, it is necessary and sufficient that the variety V attached to $P(X, Y)$ satisfies the following condition*:

(PH) *When $(\xi, \eta)\in V$. $\Re\xi$ is bounded if $\Im\xi$ and $|\eta|$ are bounded.*[2)]

Proof. *Necessity of* (PH). Let $A\times B\subseteq\Omega$ be the product of open cubes in $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$. We denote by $\mathcal{U}$ the space of all the solutions $U(x, y)$ of the equation (2) that are definined and continuous in Ω with the topology of uniform convergence on every compact set in Ω. Since this topology is stronger than the induced topology from $\prod_1^q\mathcal{D}'_\Omega$, $\mathcal{U}$ becomes a Fréchet space. Now let us consider, for any $\varphi\in\mathcal{D}_B$, the mapping

$$U\rightarrow G(x)=\int U(x, y)\varphi(y)\,dy$$

which carries $\mathcal{U}$ into $\prod_1^q\mathcal{E}_A$ according to the hypothesis of hypoellipticity in x. Since the mapping is clearly a closed linear mapping and since $\mathcal{U}$ and $\prod_1^q\mathcal{E}_A$ are Fréchet spaces, it must be continuous by the closed graph theorem ([2], p. 37). Hence, for any compact set $K\subseteq A$, there exists a compact set $L\subseteq\Omega$ and a constant $c(\varphi)$ such that we have

$$\max_K\left|\frac{\partial g_j}{\partial x_k}\right|\leq c(\varphi)\sum_{s=1}^q\max_L|u_s(x, y)| \tag{6}$$

$$(k=1, 2, \cdots, m\,;\ j=1, 2, \cdots, q)$$

for all $U\in\mathcal{U}$, with

$$U=\begin{pmatrix}u_1\\ \vdots\\ u_q\end{pmatrix}\quad\text{and}\quad G=\begin{pmatrix}g_1\\ \vdots\\ g_q\end{pmatrix}.$$

Now let $(\xi, \eta)\in V$ and let $\Im\xi$ and $|\eta|$ be bounded. Assume

2) $\Re$ stands for "the real part of", $\Im$ for "the imaginary part of".

specifically that

$$(7) \qquad |\Im\xi| \leq M \quad \text{and} \quad |\eta| \leq M.$$

We are to show that $\Re\xi$ is bounded. Since the rank of $P(\xi, \eta)$ is less than q, there exist solutions (5) of the equation (2) (see §1) with suitable constante $C=C(\xi, \eta)$

$$C = \begin{pmatrix} c_1 \\ \vdots \\ c_q \end{pmatrix}.$$

Here we can suppose that

$$(8) \qquad \max_j |c_j| = 1$$

where the suffix j for which $c_j=1$ holds might depend on $(\xi, \eta) \in V$.

Let us apply the inequality (6) to these solutions. For these U,

$$\begin{aligned} G(x) &= \int e^{i\langle x, \xi\rangle + i\langle y, \eta\rangle} \cdot C \cdot \varphi(y)\, dy \\ &= e^{i\langle x, \xi\rangle} \phi(\eta) C. \end{aligned}$$

Therefore

$$g_j(x) = e^{i\langle x, \xi\rangle} \phi(\eta) c_j(\xi, \eta)$$

and (6) becomes

$$(9) \qquad \begin{aligned} &(\max_K e^{-\langle x, \Im\xi\rangle}) |\phi(\eta)| \, |\xi_k| \, |c_j(\xi, \eta)| \\ &\leq c(\varphi) \sum_{s=1}^{q} |c_s(\xi, \eta)| (\max_L e^{-\langle x, \Im\xi\rangle - \langle y, \Im\eta\rangle}). \end{aligned}$$

Since $\phi(\eta)$ is a non zero entire function (if $\varphi \neq 0$) defined by

$$\phi(\eta) = \int e^{i\langle y, \eta\rangle} \varphi(y)\, dy,$$

we can suppose (by a suitable modification if necessary)[3] that

$$(10) \qquad |\phi(\eta)| \geq d \quad \text{if} \quad |\eta| \leq M$$

with a positive constant d. Now choose a suffix $j=j(\xi, \eta)$ such that $|c_j|=1$, and add the the inequalities (9) with k running from 1 to m. Then, by (7), (8) and (10), we get

3) This modification is easily done by some linear transformation of coordinates and by multiplying a certain exponential factor.

$$(\max_{K} e^{-M|x|}) \sum_{k=1}^{m} |\xi_k| \leq \frac{mq}{d} c(\varphi)(\max_{L} e^{M(|x+y|)}) . \tag{11}$$

From this inequality we can easily see that $\Re\xi$ is bounded since $\Im\xi$ is bounded.

Sufficiency of (PH). We know already that (PH) is a sufficient condition for the hypoellipticity in x in the case of a single operator. (see [4], 3, Theorem 1). But since the equation (3) holds for any $Q \in \mathfrak{a}$, for the proof, it is enough to prove the following

Lemma 1. *If* (PH) *holds for* V, *there exists a polynomial* $Q \in \mathfrak{a}$ *such that* (PH) *holds for the hypersurface* V_Q *defined by* $Q(X, Y)=0$.

Let us prove this lemma. Consider first the homomorphism $\boldsymbol{C}[X, Y] \rightarrow \boldsymbol{C}[X, Y, Z]$ which carries each polynomial $F(X, Y)$ into $F(X, Y, Z)$ by the following fomula

$$F(X, Y, Z) = F\left(X, \frac{-iZ+Y}{2}\right),$$

where Z is a new variable: $Z=(Z_1, \cdots, Z_n)$. And let $\tilde{\mathfrak{a}}$ be the ideal of $\boldsymbol{C}[X, Y, Z]$ generated by the totality of images of polynomials in $\mathfrak{a}$. Consider the variety $\tilde{V}$ in $\boldsymbol{C}^{m+n+n}$ defined by $\tilde{\mathfrak{a}}$ and take its intersection by the linear variety defined by $iZ+Y=0$. We denote by $\hat{V}$ this intersection. Then it is clear that $\hat{V}$ is of the form

$$\hat{V} = \{(\xi, \eta, i\eta) \mid (\xi, \eta) \in V\} . \tag{12}$$

Let $\hat{\mathfrak{a}}$ be the ideal determined by $\hat{V}$. Now, according to *Lech's theorem* (see [7]), we can find a polynomial $L \in \hat{\mathfrak{a}}$ such that

$$d(r, \hat{V}_L) \geq c \cdot d(r, \hat{V}) \tag{13}$$

holds for any *real* point r of $\boldsymbol{C}^{m+n+n}$ with a positive constant c independent of r. (Generally, for a point p and for a S, $d(p, S)$ denotes the distance between p and S, i.e. $d(p, S) = \inf_{p' \in S} |p-p'|$.)

Now put

$$R(X, Y) = L(X, Y, iY) .$$

From (12) it is clear that R vanishes on the variety V. Therefore, there exists a positive integer s such that $Q=R^s$ belongs to the ideal $\mathfrak{a}$. (see [13] p. 6). Let us prove that this $Q(X, Y)$ satisfies the requirement of the lemma, i. e.

$$V_Q = \{(\xi, \eta) \mid L(\xi, \eta, i\eta) = 0\} \tag{14}$$

satisfies the condition (PH).

Let (ξ, η) be in V_Q and assume that $\Im\xi$ and $|\eta|$ are bounded. Specifically, we put

$$|\Im\xi| \leq M \quad \text{and} \quad |\eta| \leq M. \tag{15}$$

We are to show that $\Re\xi$ is bounded.
Let us apply the inequality (13) to the real points $r=(\Re\xi, \Re\eta, -\Im\eta)$. Since $\hat{V}$ is a closed set, $d(r, \hat{V})$ is attained by a point $(\xi', \eta', i\eta')$ in $\hat{V}$. Hence we have that

$$\begin{aligned} 2M &\geq |\Im\xi| + |\eta| \\ &\geq d((\Re\xi, \Re\eta, -\Im\eta), (\xi, \eta, i\eta)) \\ &\geq d(r, \hat{V}_L) \geq c \cdot d(r, \hat{V}) \\ &= c \cdot d((\Re\xi, \Re\eta, -\Im\eta), (\xi', \eta', i\eta)) \\ &= c \cdot \{|\Re\xi - \Re\xi'|^2 + |\Im\xi'|^2 + |\Re\eta - \Re\eta'|^2 + |\Im\eta'|^2 \\ &\quad + |\Im\eta - \Im\eta'|^2 + |\Re\eta'|^2\}^{1/2}. \end{aligned} \tag{16}$$

Thus, in particular, we have that

$$|\Im\xi'| \leq \frac{2M}{c}, \quad |\Re\eta'| \leq \frac{2M}{c} \quad \text{and} \quad |\Im\eta'| \leq \frac{2M}{c}. \tag{17}$$

From these inequalities, we see that $\Re\xi'$ is bounded since (ξ', η') is in V which satisfies the condition (PH). But since from (16) we see that

$$|\Re\xi - \Re\xi'| \leq \frac{2M}{c},$$

$\Re\xi$ should be bounded. This completes the proof.

Theorem 2. (Inhomogeneous equation). *Let* $P\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)$ *be a system of differential operators which is hypoelliptic in* x. *And let* F *be a vector whose components are distributions defined and* *regular in x in an open set Ω of $\boldsymbol{R}^{m+n}$. Then, all the solutions $U \in \prod_1^q \mathcal{D}'_\Omega$ of the equation*

$$P\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)U = F$$

are regular in x.

Proof. Let $\mathcal{H}_\Omega$ be the space of all the distributions in Ω which are regular in x. Since $\mathcal{H}_\Omega$ is closed under the operations of partial differentiations as is easily seen, (3′) holds for any $Q \in \mathfrak{a}$. By Lemma 1, we can take as Q a polynomial which is hypoelliptic in x. Since the theorem is true when $P(X, Y)$ is a single polynomial (see [4] 3., Theorem 1, Remark 2.), we see from (3′) that the theorem is true for general systems also.

§ 4. Partially elliptic systems.

Definition 3. Let Ω be an open set in $\boldsymbol{R}^m \times \boldsymbol{R}^n$ and $f(x, y) \in \mathcal{D}'_\Omega$ be a distribution. We say that f is *analytic in x* if, for any pair of open sets $A \subseteq \boldsymbol{R}^m$, $B \subseteq \boldsymbol{R}^n$, $A \times B \subseteq \Omega$, and for any $\varphi \in \mathcal{D}_B$ the distribution in x

$$g(x) = \int f(x, y)\varphi(y)\,dy$$

is an analytic function.

Definition 4. A system of differential operators $P\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)$ is called *elliptic in x* if, for an open set $\Omega \subseteq \boldsymbol{R}^m \times \boldsymbol{R}^n$, every solution $U \in \prod_1^q \mathcal{D}'_\Omega$ of the equation (2) is analytic in x.

Theorem 3. *For a system of differential operators $P\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)$ to be elliptic in x, it is necessary and sufficient that the variety V attached to the matrix $P(X, Y)$ satisfies the following condition:*

(PE) *There exists a positive constant M such that the inequality*

$$|\Re\xi| \leq M(1 + |\Im\xi| + |\Im\eta|)$$

holds when $(\xi, \eta) \in V$.

Proof. *Necessity of* (PE). The necessity of the condition can be proved quite analoguously to in the case of a single operator. We need only replace single exponential solutions by the solution vectors (5) of the equation (2) as we did in the proof of Theorem 1. (See the proof of [4] 4., Theorem 2.)

Sufficiency of (PE). Since we know already that (PE) is a sufficient condition for a single operator to be elliptic in x and since the equation (3) holds for any $Q \in \mathfrak{a}$, for the proof, it is enough to prove the following

Lemma 2. *If* (PE) *holds for* V, *there exists a polynomial* $Q \in \mathfrak{a}$ *such that* (PE) *holds for the hypersurface* V_Q *defined by* $Q(X, Y)=0$.

Let us prove the lemma. According to Lech's theorem [7], we can find a polynomial Q in $\mathfrak{a}$ such that

$$d(r, V_Q) \geq c \cdot d(r, V) \tag{18}$$

holds for any real point r of $\boldsymbol{C}^{m+n}$ with a positive constant c independent of r. Let us show that this polynomial Q satisfies the requirement of the lemma. We are to show that there exists a constant A such that

$$|\Re\xi| \leq A(1+|\Im\xi|+|\Im\eta|) \tag{19}$$

holds when $(\xi, \eta) \in V_Q$. Let (ξ, η) be in V_Q and apply the inequality (18) to the real points $r=(\Re\xi, \Re\eta)$. Since V is a closed set, $d(r, V)$ is attained by a point (ξ', η') in V. Hence we have that

$$\begin{aligned} |\Im\xi|+|\Im\eta| &\geq d((\Re\xi, \Re\eta), (\xi, \eta)) \\ &\geq d(r, V_Q) \geq c \cdot d(r, V) \\ &= c \cdot d((\Re\xi, \Re\eta), (\xi', \eta')) \\ &= c\{|\Re\xi-\Re\xi'|^2+|\Im\xi'|^2+|\Re\eta-\Re\eta'|^2+|\Im\eta'|^2\}^{1/2}. \end{aligned} \tag{20}$$

From this, we have in particular that, with new constants A_1 and A_2,

$$|\Re\xi-\Re\xi'| \leq A_1(1+|\Im\xi|+|\Im\eta|) \tag{21}$$

$$(1+|\Im\xi'|+|\Im\eta'|) \leq A_2(1+|\Im\xi|+|\Im\eta|). \tag{21'}$$

But since V satisfies the condition (PE), we have that

$$|\Re\xi'|\leq M(1+|\Im\xi'|+|\Im\eta'|). \tag{22}$$

From (21), (21′) and (22), we have that

$$|\Re\xi|\leq A(1+|\Im\xi|+|\Im\eta|) \tag{23}$$

with $A=A_1+MA_2$. This completes the proof.

Theorem 4. (Inhomogeneous equation). *Let* $P\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)$ *be a system of differential operators which is elliptic in x. And let F be a vector whose components are distributions defined and analytic in x in an open set* Ω *of* $\boldsymbol{R}^{m+n}$. *Then all the solutions of the equation*

$$P\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)U = F$$

are analytic in x.

Proof. Let $\mathcal{H}_\Omega$ be the space of all the distributions in Ω which are analytic in x. Since $\mathcal{H}_\Omega$ is closed under the operations of partial differentiations as is easily seen, (3′) holds for any $Q\in\mathfrak{a}$. By Lemma 2, we can take as Q a polynomial which is elliptic in x. Since the theorem is true when $P(X, Y)$ is a single polynomial (see [4] 4. Remark after the proof of Theorem 2), we see from (3′) that the theorem is true for general systems also.

§ 5. Conditionally elliptic systems.

Definition 5. A system of differential operators $P\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)$ is called *conditionally elliptic in x* if, for an open set Ω, the components of every solution $U\in\prod_1^q\mathcal{D}'_\Omega$ of the equation (2) that are analytic in y are analytic in x and y.

Theorem 5. *For a system of differential operators* $P\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)$ *to be conditionally elliptic in x, it is necessary and sufficient that the variety V attached to P(X, Y) satisfies the following condition:*

(CE) *There exists a positive constant M such that the inequality*

$$|\Re\xi| \leq M(1+|\Im\xi|+|\eta|)$$

hold when $(\xi, \eta) \in V$.

Proof. Though we may proceed as in the proof of Theorem 1, we employ here a different method which does not use Lech's theorem.

First, we embed canonically the affine space $\boldsymbol{C}^{m+n}$ in the complex projective space $\boldsymbol{P}_{m+n}(\boldsymbol{C})$ of $m+n$ dimensions. A point of $\boldsymbol{C}^{m+n}$ with coordinates $(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_n)$ is identified with the point of $\boldsymbol{P}_{m+n}(\boldsymbol{C})$ with homogeneous coordinates $(1, \xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_n)$. And let us denotes by $\mathfrak{a}^*$ the homogeneous ideal constructed canonically from $\mathfrak{a}$ (see [9]), and let V^* be the projective variety defined by $\mathfrak{a}^*$. A point in V^* but not in V is called a *point at infinity* of V.

We notice that the following lemma holds.

Lemma 3. *The condition* (CE) *on a variety V can be restated as follows*:

(CE′) *The variety V has no point at infinity with homogeneous coordinates of the form* $(0, \xi_1, \cdots, \xi_m, 0, \cdots, 0)$ *with* $\xi = (\xi_1, \cdots, \xi_m)$ *being a non-vanishing real vector.*

First let us prove that (CE) implies (CE′). Suppose the contrary and assume that V has a real point at infinity of the form $(0, \xi, 0)$. Then there exists a curve on V depending on a real parameter s with $|s|$ large

$$\zeta(s) = s\cdot(\xi, 0)+(\xi'(s), \eta'(s)) \in V \tag{24}$$

with condition: $\lim_{s\to\infty} \frac{1}{s} (\xi'(s), \eta'(s))=0$ (see [8]. Theorem 1). Substituting (24) into the inequality (CE), we get

$$|s\xi+\Re\xi'(s)| \leq M(1+|\Im\xi'(s)|+|\eta'(s)|).$$

Dividing by s and letting $s\to\infty$, we get $|\xi|=0$. This contradicts the fact that ξ is a non vanishing vector.

Secondly, let us prove that (CE′) implies (CE). Since $\mathfrak{a}^*$ is a homogeneous ideal, there exists a basis of $\mathfrak{a}^*$ consisting of homogeneous polynomials $F_1, \cdots, F_h$. Let d_j be the degree of F_j and d be the least common multiple of $d_1, \cdots, d_h$. And put $G_j = F_j^{d/d_j}$. Let us consider the polynomial defined by $L = \sum_{j=1}^{h} G_j \bar{G}_j$, where $\bar{G}_j$ is the polynomial with coefficients which are complex conjugates of those of G_j. It is clear that the real points at infinity of the hyperplane defined by $L=0$ are just those of V^*. (H is well-defined since L is homogeneous.) Now we put

$$Q(X, Y) = L(1, X, Y). \tag{25}$$

It is clear, by its construction, that the (affine) hypersurface V_Q defined by $Q=0$ satisfies the condition (CE′) and that $Q \in \mathfrak{a}$. Since V_Q contains V, it is enough, for the proof of Lemma 3, to prove the following

Lemma 4. *V_Q satisfies the condition* (ČE).

Let d be the degree of the polynomial Q and let $Q = Q_d + Q_{d-1} + \cdots + Q_0$ be the decomposition of Q into homogeneous parts, the degrees being indicated by the subscripts. Since Q satisfies the condition (CE′), $Q_d(\xi, 0) \neq 0$ for any non-vanishing real ξ. Therefore $Q_d(\xi, \eta)$ has a positive lower bound in some complex neighbourhood of the set defined by the conditions that $|\xi| = 1$ (ξ: real), $\eta = 0$. Thus there exists come positive ε and c such that we have

$$|Q_d(\zeta)| \geq c \quad \text{if} \quad |\zeta| = 1 \quad \text{and} \quad |\Im\xi| + |\eta| \leq \varepsilon |\Re\xi|$$

where $\zeta = (\xi, \eta)$. Since Q_d is homogeneous this gives

$$|Q_d(\zeta)| \geq c|\zeta|^d \quad \text{if} \quad |\Im\xi| + |\eta| \leq \varepsilon |\Re\xi|.$$

Estimating the lower order terms in an obvious fashion we get with another constant c_1

$$\begin{aligned} &|Q(\zeta)| \geq c|\zeta|^d - c_1(|\zeta|^{d-1} + \cdots + 1) \\ &\text{if} \quad |\Im\xi| + |\eta| \leq \varepsilon |\Re\xi|. \end{aligned}$$

Hence $Q(\zeta) \neq 0$ if $|\zeta| \geq c_2$ (c_2 being a large constant) and if $|\Im\xi| + |\eta| \leq \varepsilon |\Re\xi|$. Thus $Q(\xi, \eta) \neq 0$ if $|\Re\xi| \geq M(1 + |\Im\xi| + |\eta|)$ with

$M = \max(\varepsilon^{-1}, c_2)$. Or, $|\Re\xi| \leq M(1 + |\Im\xi| + |\eta|)$ if $(\xi, \eta) \in V_Q$. This completes the proof of Lemma 4.

Now let us return to the proof of the theorem.

Sufficiency of (CE). Since we know already that (CE) is a sufficient condition for a single operator to be conditionally elliptic in x (see [4] 5. Theorem 1) and since the equation (3) holds for any $Q \in \mathfrak{a}$, it is enough, for the proof, to notice that the polynomial $Q \in \mathfrak{a}$ defined in (25) has the property that the hypersurface defined by $Q(X, Y) = 0$ satisfies the condition (CE) (see Lemma 4).

Necessity of (CE). Although we may proceed as in [4] only replacing single exponential solutions by the solution vectors (5) of the equation (2), let us utilize here the null solutions for general systems of differential operators constructed in [8].

Let $P\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)$ be a system of differential operators which is conditionally elliptic in x, and suppose that the variety V attached to $P(X, Y)$ doesn't satisfy the condition (CE). Since by Lemma 3, (CE) is equivalent to (CE′), there exists a point at infinity of V of the form $(0, \xi, 0)$ with ξ a non-vanishing real vector. Therefore we have a non zero C^∞ solution

$$U(x, y) = \begin{pmatrix} u_1(,x, y) \\ \vdots \\ u_q(x, y) \end{pmatrix}$$

of the equation (2) of the form (see [8]).

$$u_k(x, y) = \int_{i\tau-\infty}^{i\tau+\infty} C_k(s) e^{i\langle x, s\xi + \tilde{\xi}(s)\rangle} e^{i\langle y, \tilde{\eta}(s)\rangle} e^{-\left(\frac{s}{i}\right)^{\rho'}} ds \tag{26}$$
$$(k = 1, 2, \cdots, q),$$

where the integration is taken in the complex s-plane, with conditions:

(i) For all k, $\quad |C_k(s)| \leq 1$.

(ii) There exists a positive constant ρ such that

$$0 < \rho < \rho' < 1 \quad \text{and} \quad \lim_{s\to\infty} \frac{\tilde{\xi}(s)}{s^\rho} = \lim_{s\to\infty} \frac{\tilde{\eta}(s)}{s^\rho} = 0. \tag{27}$$

(iii) $\quad U(x, y) = 0 \quad \text{if} \quad \langle x, \xi\rangle = x_1\xi_1 + \cdots + x_m\xi_m > 0$.

Because of (27), the integral (26) converges uniformly even when we allow the variable y to take any complex values. Therefore $u_k(x, y)$ is analytic in y[4] for each fixed x. And it is easy to show, by the Cauchy integral, that $u_k(x, y)$ is analytic in y in the sense of Definition 3. But since $U(x, y)$ is non-trivial and vanishes in an open half space according to (iii), there existe some k for which $u_k(x, y)$ is not analytic in both variables x and y. This contradicts the assumption that $P\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right)$ is conditionally elliptic in x. This completes the proof of the theorem.

REMARKS. (1) An arbitrary basis of $\mathfrak{a}$ cannot always play the role played by that of the homogeneous ideal $\mathfrak{a}^*$ (see [9]).

(2) The Lech's polynomial [4] also can play the same role as that of Q defined in (25) (see [8] Corollary to Theorem 2).

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4) Analytic function in y in the ordinary sense.