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On the Riemann's relation on open Riemann surfaces

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1. F being an arbitrary open Riemann surface, we consider an exhaustion $\{F_n\}$ $(n=1, 2, \cdots)$ of F by regular regions satisfying the conditions:

i) for each n, F_n is a domain in F whose boundary Γ_n consists of a finite number of closed analytic curves in F,

ii) for each n, $\bar{F}_n = F_n \cup \Gamma_n \subset F_{n+1}$,

iii) $\bigcup_{n=1}^{\infty} F_n = F$

and

iv) for each n, any connected component of $F - F_n$ is non-compact in F.

Then there exists a canonical homology basis $A_1, B_1, \cdots, A_{k(n)}, B_{k(n)}, \cdots$ such that $A_1, B_1, \cdots, A_{k(n)}, B_{k(n)}$ form a canonical homology basis of F_n (mod ∂F_n) and $A_i \times B_j = \delta_{ij}$, $A_i \times A_j = B_i \times B_j = 0$[1] (Ahlfors [1], Ahlfors-Sario [2]).

We denote by Γ_h the class of all square integrable harmonic differentials defined on F. The relation which expresses the inner product (ω, σ^*) for two differentials ω, $\sigma \in \Gamma_h$ (or subclass of Γ_h) in terms of periods of ω, σ is called the *Riemann's bilinear relation*, where σ^* denotes the conjugate differential to σ. Some conditions which insure the validity of the Riemann's bilinear relation are found by some suthors (Ahlfors [1], Pfruger [3], [4], Kusunoki

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1) We note, throughout this paper, the intersection number of two cycles A, B is taken such that $A \times B$ has the positive sign when A crosses B from right to left as in [2]. Hence it has the opposite sign to that in [1].

[5], Accola [6]). In this paper we shall give some metric criteria which insure the validity of the Riemann's bilinear relation.

2. Let $F_n^{(i)}$ $(i=1, 2, \cdots, m(n))$ be components of $F_{n+1}-\bar{F}_n$. The boundary of $F_n^{(i)}$ consists of closed analytic curves contained in $\Gamma_n \cup \Gamma_{n+1}$. We denote by $\alpha_n^{(i)}$ the part of the boundary of $F_n^{(i)}$ on Γ_n and by $\beta_n^{(i)}$ that on Γ_{n+1}. Let $u_n^{(i)}(p)$ be a harmonic function in $F_n^{(i)}$ which vanishes on $\alpha_n^{(i)}$ and is equal to $\mu_n^{(i)}$ on $\beta_n^{(i)}$ having a conjugate harmonic function $v_n^{(i)}(p)$ which has the variation 2π on $\beta_n^{(i)}$, that is,

$$\int_{\beta_n^{(i)}} dv_n^{(i)} = 2\pi$$

where the integral is taken in the positive sense with respect to $F_n^{(i)}$. The quantity $\mu_n^{(i)}$ is called *the harmonic modulus of the domain* $F_n^{(i)}$. If we choose an additive constant of $v_n^{(i)}(p)$ suitably, the function $u_n^{(i)}(p)+iv_n^{(i)}(p)$ maps conformally $F_n^{(i)}$ with a finite number of slits onto a slit rectangle $0<u_n^{(i)}<\mu_n^{(i)}$, $0<v_n^{(i)}<2\pi$. Similarly, the harmonic modulus of the open set $F_{n+1}-\bar{F}_n$ is defined as follows. Let $u_n(p)$ be the harmonic function in $F_{n+1}-\bar{F}_n$ which is equal to zero on Γ_n and to μ_n on Γ_{n+1}, and its conjugate harmonic $v_n(p)$ has the variation 2π on Γ_{n+1}, that is,

$$\int_{\Gamma_{n+1}} dv_n = 2\pi\,.$$

The quantity μ_n is *the harmonic modulus of the open set* $F_{n+1}-\bar{F}_n$. If we choose adequately an additive constant of $v_n(p)$, the function $u_n(p)+iv_n(p)$ maps conformally $F_n^{(i)}$ with a finite number of slits onto a slit rectangle $0<u_n<\mu_n$, $b_i<v_n<a_i+b_i$, where a_i and b_i are constants satisfying the following conditions

$$a_i = 2\pi\frac{\mu_n}{\mu_n^{(i)}}\,, \quad \sum_{i=1}^{m} a_i = 2\pi$$

and

$$b_1 = 0\,, \quad b_i = \sum_{k=1}^{i-1} a_k \quad (1<i\leqslant m)\,.$$

The function $u_n(p)+iv_n(p)$ maps conformally $F_{n-1}-\bar{F}_n$ with a finite number of slits onto a slit rectangle $0<u_n<\mu_n$, $0<v_n<2\pi$. The function $u(p)+iv(p)$ defined by $u_n(p)+iv_n(p)+\sum_{j=1}^{n-1}\mu_j$ for each $F_{n+1}-\bar{F}_n$ $(n=1, 2, \cdots)$ maps $F-\bar{F}_1$ with at most an enumerable number of suitable slits onto a strip domain $0<u<R=\sum_{j=1}^{\infty}\mu_j$, $0<v<2\pi$ with at most an enumerable number of slits one to one and conformally. This strip domain thus obtained is the *graph of F associated with the exhaustion* $\{F_n\}$ *in Noshiro's sense* (Noshiro [7], Kuroda [8]).

3. Let us consider an open Riemann surface F and its exhaustion $\{F_n\}$, and we shall construct the graph $0<u<R$, $0<v<2\pi$ of F associated with this exhaustion. For any r $(0\leqq r<R)$, the locus γ of points of F satisfying $u(p)=r$ consists of a finite number of closed analytic curves $\gamma_r^{(i)}$ $(i=1, 2, \cdots, m(r))$. Let $\omega_i=a_idx+b_idy$ $(i=1, 2)$ be two square integrable harmonic differentials. We consider the following integral on the level curve $\gamma_r^{(i)}$

$$L_i(r)=\int_{\gamma_r^{(i)}}|\omega_1|\int_{\gamma_r^{(i)}}|\omega_2|$$

and put

$$L(r)=\sum_{i=1}^{m}L_i(r)\,,$$

Further, when $\sum_{j=1}^{n-1}\mu_j\leqq r<\sum_{j=1}^{n}\mu_j$, we put

$$\Lambda(r)=\max_{1\leqq i\leqq m}\int_{\gamma_r^{(i)}}dv=\max_{1\leqq i\leqq m}\int_{\gamma_r^{(i)}}dv_n\,.$$

Then we obtain the following

LEMMA 1. *If the integral* $\int_0^R\frac{dr}{\Lambda(r)}$ *is divergent, then there exists a sequence* $\{\gamma_n\}$ $(n=1, 2, \cdots)$ *of level curves* γ_n; $u(p)=r_n$ *tending to the ideal boundary of* F *such that*

$$\lim_{n\to\infty}L(r_n)=0\,.$$

Proof. When $\sum_{j=1}^{n-1}\mu_j \leqslant r < \sum_{j=1}^{n}\mu_j$, by the Schwarz's inequality, we have

$$\begin{aligned}
L_i(r) &= \int_{\gamma_r^{(i)}} |\omega_1| \int_{\gamma_r^{(i)}} |\omega_2| = \int_{\gamma_r^{(i)}} |b_1|\,dv_n \int_{\gamma_r^{(i)}} |b_2|\,dv_n \\
&\leqslant \int_{\gamma_r^{(i)}} dv_n \Big(\int_{\gamma_r^{(i)}} |b_1|^2 dv_n\Big)^{1/2} \Big(\int_{\gamma_r^{(i)}} |b_2|^2 dv_n\Big)^{1/2} \\
&\leqslant \Lambda(r) \Big(\int_{\gamma_r^{(i)}} |b_1|^2 dv_n\Big)^{1/2} \Big(\int_{\gamma_r^{(i)}} |b_2|^2 dv_n\Big)^{1/2}.
\end{aligned}$$

Summing up from $i=1$ to $i=m(r)$, we obtain

$$\begin{aligned}
L(r) &\leqslant \Lambda(r) \sum_{i=1}^{m} \Big(\int_{\gamma_r^{(i)}} |b_1|^2 dv_n\Big)^{1/2} \Big(\int_{\gamma_r^{(i)}} |b_2|^2 dv_n\Big)^{1/2} \\
&\leq \Lambda(r) \Big(\sum_{i=1}^{m} \int_{\gamma_r^{(i)}} |b_1|^2 dv_n\Big)^{1/2} \Big(\sum_{i=1}^{m} \int_{\gamma_r^{(i)}} |b_2|^2 dv_n\Big)^{1/2} \\
&= \Lambda(r) \Big(\int_0^{2\pi} |b_1|^2 dv\Big)^{1/2} \Big(\int_0^{2\pi} |b_2|^2 dv\Big)^{1/2}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\int_{\sum_{j=1}^{n-1}\mu_j}^{\sum_{j=1}^{n}\mu_j} \frac{L(r)}{\Lambda(r)}\,dr &= \int_{\sum_{j=1}^{n-1}\mu_j}^{\sum_{j=1}^{n}\mu_j} \Big\{\Big(\int_0^{2\pi} |b_1|^2 dv\Big)^{1/2} \Big(\int_0^{2\pi} |b_2|^2 dv\Big)^{1/2}\Big\}\,dr \\
&\leq \Big(\int_{\sum_{j=1}^{n-1}\mu_j}^{\sum_{j=1}^{n}\mu_j} \int_0^{2\pi} |b_1|^2 dv\,du\Big)^{1/2} \Big(\int_{\sum_{j=1}^{n-1}\mu_j}^{\sum_{j=1}^{n}\mu_j} \int_0^{2\pi} |b_2|^2 dv\,du\Big)^{1/2} \\
&\leq \Big(\int_{\sum_{j=1}^{n-1}\mu_j}^{\sum_{j=1}^{n}\mu_j} \int_0^{2\pi} (|a_1|^2 + |b_1|^2)\,dv\,du\Big)^{1/2} \\
&\quad\times \Big(\int_{\sum_{j=1}^{n-1}\mu_j}^{\sum_{j=1}^{n}\mu_j} \int_0^{2\pi} (|a_2|^2 + |b_2|^2)\,dv\,du\Big)^{1/2} \\
&= \|\omega_1\|_{F_{n+1}-F_n} \|\omega_2\|_{F_{n+1}-F_n}.
\end{aligned}$$

Consequently, we have

$$\int_0^R \frac{L(r)}{\Lambda(r)}\,dr \leqslant \sum_{n=1}^{\infty} ||\omega_1||_{F_{n+1}-F_n}||\omega_2||_{F_{n+1}-F_n}$$

$$\leqslant \left(\sum_{n=1}^{\infty} ||\omega_1||^2_{F_{n+1}-F_n}\right)^{1/2} \left(\sum_{n=1}^{\infty} ||\omega_2||^2_{F_{n+1}-F_n}\right)^{1/2}$$

$$\leqslant ||\omega_1||\ ||\omega_2|| < \infty\ .$$

Since the integral $\int_0^R \frac{dr}{\Lambda(r)}$ is divergent by our assumption, we obtain $\varliminf_{r\to R} L(r)=0$. Therefore, we obtained the above-mentioned result.

Since

$$\int_{\sum_{j=1}^{n-1}\mu_j}^{\sum_{j=1}^{n}\mu_j} \frac{1}{\max_i \int_{\gamma_r^{(i)}} dv}\,dr = \mu_n \cdot \min_i \frac{\mu_n^{(i)}}{2\pi\mu_n} = \frac{1}{2\pi} \min_i \mu_n^{(i)}\ ,$$

we have

$$2\pi \int_0^r \frac{dr}{\Lambda(r)} \geq 2\pi \int_0^{\sum_{j=1}^{n}\mu_j} \frac{dr}{\Lambda(r)} \geq \sum_{j=1}^{n} (\min_i \mu_j^{(i)})$$

for any r satisfying $\sum_{j=1}^{n}\mu_j \leqslant r < \sum_{j=1}^{n+1}\mu_j$. Thus we can say that *the result of lemma* 1 *hold, if* $\sum_{n=1}^{\infty}(\min_i \mu_n^{(i)})$ *is divergent.*

Next, we suppose that the exhaustion $\{F_n\}$ is canonical, that is, each contour $\Gamma_n^{(i)}$ $(i=1, 2, \cdots, m(n))$ of Γ_n is a dividing cycle. Let $D_n^{(i)}$ $(i=1, 2, \cdots, m)$ be annuli each of which includes a contour $\Gamma_n^{(i)}$ and are disjoint each other. We put $D_n=\bigcup_{i=1}^{m} D_n^{(i)}$ and assume that D_n $(n=1, 2, \cdots)$ are disjoint each other. We constructe the graph of $\bigcup_{n=1}^{\infty} D_n$ associated with the sequence $\{D_n\}$ of open sets D_n and denote the harmonic modulus of $D_n^{(i)}(D_n)$ by $\nu_n^{(i)}(\nu_n)$. Also we denote the function which maps $\bigcup_{n=1}^{\infty} D_n$ onto the strip domain $0<u<R=\sum_{n=1}^{\infty}\nu_n$, $0<v<2\pi$ by $u(p)+iv(p)$. When $\sum_{j=1}^{n-1}\nu_j \leq r < \sum_{j=1}^{n}\nu_j$, we put

$$\Lambda_0(r) = \max_i \int_{\gamma_r^{(i)}} dv\ .$$

We point out that in this case each component of the level curve is a dividing cycle. Then, by the same way as we did in the proof of lemma 1, we have

LEMMA 2. *If the integral* $\int_0^R \frac{dr}{\Lambda_0(r)}$ *is divergent, then there exists a sequence of level curves tending to ideal boundary of* F *such that each component of the curves is a dividing cycle and* $\lim_{n\to\infty} L(r_n)=0$.

In the same way as the remark in lemma 1, we can conclude that *the result of lemma 2 holds, if* $\sum_{n=1}^{\infty}(\min \nu_n^{(i)})$ *is divergent.*

4. Let us denote by Γ_{hse} (Γ_{ase}) the class of semi-exact harmonic (analytic) differentials in Γ_h and by Γ_{he} the class of exact harmonic differentials in Γ_h and further by Γ_{h0} the orthogonal complement in Γ_h of Γ_{he}^*. Then $\Gamma_{h0}\subset\Gamma_{hse}$. Now let c be a cycle, then there exists a harmonic differential $\sigma(c)$ so that $\int_c \omega=(\omega, \sigma(c)^*)$ for $\omega\in\Gamma_h$. Such a $\sigma(c)$ is unique, real, of class Γ_{h0}. If c and c' are two cycles, then $(\sigma(c'), \sigma(c)^*)$ is an integer, that is, the intersection number $c'\times c$ of c' and c (Ahlfors-Sario [2]).

LEMMA 3. *Suppose* $\overline{\Omega}$ *is a compact bordered surface and* ω *and* σ *are in* $\Gamma_{hse}(\overline{\Omega})$. *Let* $\{A_i, B_i\}$ $(i=1, 2, \cdots, k)$ *b ea canonical homology basis of* Ω (mod $\partial\Omega$). *Then*

$$(\omega, \sigma^*) = \sum_{i=1}^{k}\left(\int_{A_i}\omega\int_{B_i}\bar{\sigma}-\int_{A_i}\bar{\sigma}\int_{B_i}\omega\right)-\int_{\partial\Omega}u\bar{\sigma},$$

where $u(p)$ *is a function defined separately on each contour of* $\partial\Omega$. *If* α *is a contour of* $\partial\Omega$, *then* $u(p)=\int_{p_0}^{p}\omega$ *where* p_0 *is a fixed point on* α *and the integration is in the positive sence of* α.

Proof. Let $a_i=\int_{A_i}\omega$ and $b_i=\int_{B_i}\omega$. Let $\omega'=\sum_{i=1}^{k}(b_i\sigma(A_i)-a_i\sigma(B_i))$, then ω' has the same periods as ω and ω' belongs to $\Gamma_{h0}(\Omega)$. Since $\omega-\omega'$ has no periods, we have $\omega-\omega'=du$, where u is a harmonic function. By the Green's formula we have

$$(\omega-\omega', \sigma^*) = (du, \sigma^*) = -\int_{\partial\Omega}u\bar{\sigma}.$$

Therefore

$$(\omega, \sigma^*) = (\omega', \sigma^*) - \int_{\partial\Omega} u\bar{\sigma}$$
$$= \sum_{i=1}^{k}\left(\int_{A_i}\omega\int_{B_i}\bar{\sigma} - \int_{A_i}\bar{\sigma}\int_{B_i}\omega\right) - \int_{\partial\Omega} u\bar{\sigma}\,.$$

THEOREM I. *If the integral* $\int_0^R \frac{dr}{\Lambda_0(r)}$ *is divergent for a canonical exhaustion, then for a corresponding canonical homology basis the Riemann's bilinear relation*

$$(1) \qquad (\omega, \sigma^*) = \lim_{n'\to\infty}\sum_{k=1}^{p(n')}\left(\int_{A_k}\omega\int_{B_k}\bar{\sigma} - \int_{A_k}\bar{\sigma}\int_{B_k}\omega\right)$$

holds for two differentials $\omega, \sigma \in \Gamma_{hse}$.

Proof. We shall take the sequence $\{\gamma_n\}$ of level curves satisfying lemma 2 for two differentials ω and σ. Since each component $\gamma_n^{(i)}$ $(i=1, 2, \cdots, m)$ of γ_n is a dividing cycle, if $\gamma_n \subset D_{n'}$, we may suppose that $F_{n'}$ and the relatively compact domain Ω_n bounded by level curve γ_n have the same homology basis $A_1, B_1, \cdots, A_{p(n')}, B_{p(n')}$. By the application of lemma 3 to Ω_n, we have

$$(\omega, \sigma^*)_{\Omega_n} = \sum_{k=1}^{p(n')}\left(\int_{A_k}\omega\int_{B_k}\bar{\sigma} - \int_{A_k}\bar{\sigma}\int_{B_k}\omega\right) - \int_{\partial\Omega_n} u\bar{\sigma}\,.$$

Since $\sigma \in \Gamma_{hse}$, we have $\int_{\gamma_n^{(i)}}\bar{\sigma} = 0$. Hence for a fixed point $p_0 \in \gamma_n^{(i)}$

$$\left|\int_{\gamma_n^{(i)}} u\bar{\sigma}\right| = \left|\int_{\gamma_n^{(i)}} (u(p) - u(p_0))\bar{\sigma}\right| \leqslant \int_{\gamma_n^{(i)}}|\omega|\int_{\gamma_n^{(i)}}|\sigma|,$$

therefore

$$\left|\int_{\partial\Omega_n} u\bar{\sigma}\right| \leqslant \sum_{i=1}^{m}\int_{\gamma_n^{(i)}}|\omega|\int_{\gamma_n^{(i)}}|\sigma| \to 0 \ (n\to\infty)\,.$$

Thus the proof is completed.

By the remark in 3, we get the following

COROLLARY. *If* $\sum_{n=1}^{\infty}(\min_i \nu_n^{(i)})$ *is divergent, the Riemann's bilinear* *relation* (1) *holds for two differentials* ω, $\sigma \in \Gamma_{hse}$.

Thus we know that on such a surface every $\omega \in \Gamma_{ase}$ is determined uniquely by its A-periods.

By the definition of the modulus we have

$$\frac{1}{\nu_n} = \frac{1}{\nu_n^{(1)}} + \cdots\cdots + \frac{1}{\nu_n^{(m)}},$$

hence $\nu_n \leqslant \min_i \nu_n^{(i)}$. If $\sum_{n=1}^{\infty} \nu_n$ is divergent, F belongs to O_G and so $\Gamma_{hse} = \Gamma_h$. Thus we have

COROLLARY (Kusunoki [5]). *If* $\sum_{n=1}^{\infty} \nu_n$ *is divergent, then the Riemann's bilinear relation* (1) *holds for two* ω, $\sigma \in \Gamma_h$.

5. Next we choose annuli $R_n^{(i)}$ $(i=1, 2, \cdots, m)$ in canonical region F_n so that $\Gamma_n^{(i)} \subset \bar{R}_n^{(i)}$, $R_n^{(i)} \cap R_n^{(j)} = \phi$ $(i \neq j)$. Let $R_n = \bigcup_{i=1}^{m} R_n^{(i)}$ and $\mu(R_n)$ and $\mu(R_n^{(i)})$ be the harmonic modulis of R_n and $R_n^{(i)}$, respectively.

Define μ_{F_n} to be the supremum of $\mu(R_n)$ as R_n ranges over all possible choices. Accola [6] has given the following sufficient condition for the validity of the Riemann's bilinear relation:

If $\mu_{F_n} \geq M > 0$ *for* $n \to \infty$ (M; constant), *then*

$$(\omega, \sigma^*) = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\int_{A_k} \omega \int_{B_k} \bar{\sigma} - \int_{A_k} \bar{\sigma} \int_{B_k} \omega \right)$$

holds for $\sigma \in \Gamma_{hse}$ *and for all* $\omega \in \Gamma_{h0}$.

We shall remark that the above sufficient condition can be extended to the following form:

If $\sup_{R_n} (\min_i \mu(R_n^{(i)})) \geq M > 0$ *for* $n \to \infty$, *then the bilinear relation holds for* $\sigma \in \Gamma_{hse}$ *and for* $\omega \in \Gamma_{h0}$.

This can be proved, with a slight modification, by the same way as in [6] and so we shall omit its proof.

In [6], Accola has constructed a Riemann surface for which the bilinear relation holds. His example is the symmetric hyperelliptic Riemann surface. Let $\{a_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive number such that $a_k \to \infty$ $(k \to \infty)$. Denote by α_n the segment between a_{2n-1} and a_{2n}. Cut the plane along the slits. Take two copies of slits plane π_+ and π_- and cross along the slits in usual way. The surface thus obtained is of infinite genus and parabolic. We exhaust it by the portion F_n lying over the open disk with center at zero and radius a_{2n}. Let R_n be the ring domain lying above the annulus $a_{2n-1}<|z|<a_{2n}$. Then the harmonic modulus of R_n is $\frac{1}{4\pi}\log\frac{a_{2n}}{a_{2n-1}}$. According to Accola's condition, for the validity of thc bilinear relation it needs that $\frac{a_{2n}}{a_{2n-1}}>\rho>1$ for a subquence a_n's. But, according to the corollary to theorem 1, we know that, for the validity of the bilinear relation, it is sufficient to hold $\prod_{n=1}^{\infty}\frac{a_{2n}}{a_{2n-1}}=\infty$.

6. Ahlfors [1] has constructed a canonical homology basis with respect to an exhaustion $\{F_n\}$ of F such that the cycles on ∂F_n are *weakly* homologous to a linear combination of only A-cycles and if the index n of ∂F_n is large, each of index of corresponding A-cycle is large. In following we shall use such a canonical homology basis.

Now let $\{F_n\}$ be an exhaustion of F by regular regions and for each n, $\Gamma_n(t_j)$ be a set of finite number of level curves; $u(p)=t_j$ $\left(\sum_{k=1}^{n-1}\mu_k=t_1<t_2<\cdots<t_j<\cdots<t_\nu=\sum_{k=1}^{n}\mu_k\right)$ such that at least one critical point of $u(p)$ is contained in $\Gamma_n(t_j)$ $(j\neq 1,\ \nu)$, where $u(p)$ is the function defined in **2**. We shall consider the relatively compact regions bounded by $\Gamma_n(t_j)$ $(n=1, 2, \cdots, j=1, 2, \cdots, \nu(n))$, then we may suppose that those regions construct an exhaustion $\{\Omega_{nj}\}$. Let us introduce a canonical homology basis with respect to this exhaustion, then the region bounded by $\Gamma_n(t)$ $(t_i\leqslant t<t_{i+1})$ has the same canonical homology basis as that of the region bounded by $\Gamma_n(t_i)$ (cf. Ahlfors [1], Hilfssatz 5). For such a canonical homology basis we have the following

THEOREM II. *If the integral* $\int_0^R\frac{dr}{\Lambda(r)}$ *is divergent for an exhaustion* $\{F_n\}$, *then there exist an exhaustion and a corresponding canonical homology basis such that the Riemann's bilinear relation*

(2) $$(\omega, \sigma^*) = \sum_k \left(\int_{A_k} \omega \int_{B_k} \bar{\sigma} - \int_{A_k} \bar{\sigma} \int_{B_k} \omega\right) \quad \text{(a finite sum)}$$

holds for two $\omega, \sigma \in \Gamma_{hse}$ *having only a finite number of non-vanishing A-periods.*

Proof. We consider the relatively compact subregion Ω_n which are bounded by the level curves which were constructed in lemma 1. ω and σ have only a finite number of non-vanishing A-periods, hence also have vanishing periods on each contour of $\partial\Omega_m$ for sufficientlt large m. Therefore ω and σ belong to $\Gamma_{hse}(\overline{\Omega}_m)$, because any dividing cycles in Ω_m are homologous to a linear combination of cycles on $\partial\Omega_m$. Let $\alpha_m^{(j)}$ $(j=1, 2, \cdots, l(m))$ be contours of $\partial\Omega_m$. Since $\int_{\alpha_m^{(j)}} \bar{\sigma}=0$, We have anologously in theorem I

$$\left|\int_{\alpha_m^{(j)}} u\bar{\sigma}\right| \leqslant \int_{\alpha_m^{(j)}} |\omega| \int_{\alpha_m^{(j)}} |\sigma| .$$

Hence

$$\left|\int_{\partial\Omega_m} u\bar{\sigma}\right| \leq \sum_{j=1}^{l} \int_{\alpha_m^{(j)}} |\omega| \int_{\alpha_m^{(j)}} |\sigma| \rightarrow 0 \quad (m \rightarrow \infty)$$

Thus the proof is completed.

COROLLARY. *If* $\sum_{n=1}^{\infty} (\min_i \mu_n^{(i)})$ *is divergent for an exhaustion* $\{F_n\}$, *then the Riemann's bilinear relation* (2) *holds.*

For such a canonical homology basis, on such surface every $\omega \in \Gamma_{ase}$ is determined uniquely by its A-periods. Thus we have

COROLLARY. (Sario [9]). *If the integral* $\int_0^R \frac{dr}{\Lambda(r)}$ *is divergent, then Riemann surface belongs to* O_{AD}.

Since $\min_i \mu_n^{(i)} \geq \mu_n$, if $\sum_{n=1}^{\infty} \mu_n = \infty$, then theorem II holds. If F belongs to O_G, then there exists an regular exhausion such that $\sum_{n=1}^{\infty} \mu_n = \infty$ (Noshiro [7]), hence we have the following

COROLLARY (Ahlfors [1]). *If F belongs to O_G, then there exist an exhaustion and the corresponding canonical homology basis such that*

$$(\omega,\ \sigma^*) = \sum_k \left(\int_{A_k} \omega \int_{B_k} \bar{\sigma} - \int_{A_k} \bar{\sigma} \int_{B_k} \omega \right) \quad \text{(a finite sum)}.$$

holds for two $\omega,\ \sigma \in \Gamma_h$ *having only a finite number of non-vanishing A-periods.*

7. Let G be any region on F whose relative boundary c consists of at most an enumerable number of analytic curves, compact or non-compact and clusters nowhere in F. If there exists no non-constant, single valued, analytic function $f(p)$ which has the finite Dirichlet integral over G and its real part vanishes continuously at every point of c, then G is called the subregion of the class SO_{AD}. We now suppose that the integral $\int_0^R \frac{dr}{\Lambda(r)}$ is divergent for an exhausion $\{F_n\}$ of F by regular regions. We consider the subset G_r of G;

$$G_r = G \cap \{p : u(p) \leq r \ (0 \leq r < R)\}$$

where $u(p)$ is the function defined in **2**. If some components of $G - \bar{G}_r$ are relatively compact, we consider the union of these components and G_r. For simplicity, we denote it by G_r again. Let $f(p) = U(p) + iV(p)$ be a single valued analytic function in G whose real part $U(p)$ vanishes at every point of the relative boundary c of G. Then two differentials dU and dV belong to $\Gamma_{he}(\bar{G})$ and dU vanishes along c. Thus we have by lemma 3 and $U(p)=0$ $(p \in c)$

$$||dU||^2_{G_r} = (dU,\ dU)_{G_r} = -(dU,\ dU^{**})_{G_r} = -(dU,\ dV^*)_{G_r} = \int_{\partial G_r \cap G} UdV$$

We set $\theta_r = \partial G_r \cap G$ and denote components of θ_r by $\theta_r^{(i)}$ $(i=1, 2, \cdots, l(r))$. Then, by the same way as in the case of the proof of lemma 1, we can conclude that there exists a sequence $\{\theta_{r_n}\}$ such that

$$\sum_{i=1}^{l} \int_{\theta_{r_n}^{(i)}} |dU| \int_{\theta_{r_n}^{(i)}} |dV| \to 0 \quad (n \to \infty).$$

In such G_{r_n}, we have

$$||dU||^2_{G_{r_n}} = \int_{\theta_{r_n}} UdV = \sum_{i=1}^{l} \int_{\theta_{r_n}^{(i)}} UdV.$$

If $\theta_{r_n}^{(i)}$ is a closed curve, as $\int_{\theta_{r_n}^{(i)}} dV=0$, we have

$$\left|\int_{\theta_{r_n}^{(i)}} UdV\right| \leqslant \int_{\theta_{r_n}^{(i)}} |dU| \int_{\theta_{r_n}^{(i)}} |dV| .$$

If $\theta_{r_n}^{(i)}$ is a cross cut, let $p' \in c$ be a end point of $\theta_{r_n}^{(i)}$, then $U(p')=0$, hence

$$\left|\int_{\theta_{r_n}^{(i)}} UdV\right| = \left|\int_{\theta_{r_n}^{(i)}} (U(p)-U(p'))dV\right| \leqslant \int_{\theta_{r_n}^{(i)}} |dU| \int_{\theta_{r_n}^{(i)}} |dV| .$$

Consequently,

$$\left|\int_{\theta_{r_n}} UdV\right| \leqslant \sum_{i=1}^{l} \int_{\theta_{r_n}^{(i)}} |dU| \int_{\theta_{r_n}^{(i)}} |dV| \to 0 \quad (n \to \infty) .$$

Hence, G belongs to SO_{AD}. If we denote by O_{AD}^0 the class of Riemann surfaces each of which has no subregion not belonging to SO_{AD}, we have the following theorem proved by Kuroda [8]: *if the integral* $\int_0^R \frac{1}{\Lambda(r)} dr$ *is divergent, then* F *belongs to* O_{AD}^0.

Since $O_{AD}^0 \subsetneqq O_{AD}$ (Kuroda [8]), this is an improvement of the Sario's sufficient condition. Moreover, by the same way as above, we can generalize the above theorem in the following form.

THEOREM III. *If the integral* $\int_0^R \frac{dr}{\Lambda(r)}$ *is divergent for an exhaustion of* F *by regular regions, then*

$$(\omega, \sigma^*)_G = 0 ,$$

where $\sigma \in \Gamma_{he}(\bar{G})$ *and* ω *belongs to* $\Gamma_{he}(\bar{G})$, *that is,* $\omega = df$ *and the harmonic function* $f(p)$ *vanishes at every point of the relative boundary of* Γ.

8. The *special bilinear relation* is said to hold on F if the following is true (Accola [6]): if $\omega \in \Gamma_{h0}$, $\sigma \in \Gamma_{hse}$ and ω has a finite number of non-vanishing A and B-periods, then

$$(\omega, \sigma^*) = \sum_k \left(\int_{A_k} \omega \int_{B_k} \bar{\sigma} - \int_{A_k} \bar{\sigma} \int_{B_k} \omega\right), \quad \text{(a finite sum)} .$$

Let Γ_{hm} be the orthogonal complement in Γ_h of Γ_{hse}. In [6] the following theorem is proved: *validity of the special bilinear rela-*

tion on F is equivalent to $\Gamma_{hm}=\Gamma_{h0}\cap\Gamma_{he}$. Also a surface on which $\Gamma_{he}\cap\Gamma_{h0}\supsetneqq\Gamma_{hm}=\phi$ holds is constructed. The surface evidently does not belong to O_{HD}. Since $(\omega,\sigma^*)=\overline{(\sigma^*,\omega)}=\overline{(\sigma^{**},\omega^*)}=-\overline{(\sigma,\omega^*)}$, if $\int_0^R\frac{dr}{\Lambda_0(r)}=\infty$ and $\omega\in\Gamma_{h0}$ has a finite number of non-vanishing A- and B-periods, then by theorem I we have

$$(\omega,\sigma^*)=\sum_k\left(\int_{A_k}\omega\int_{B_k}\bar{\sigma}-\int_{A_k}\bar{\sigma}\int_{B_k}\omega\right),\qquad\text{(a finite sum)}.$$

Therefore we know that *if* $\int_0^R\frac{dr}{\Lambda_0(r)}=\infty$, *then* $\Gamma_{hm}=\Gamma_{he}\cap\Gamma_{h0}$.

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REFERENCES

[1] Ahlfors, L.; Normalintegral auf offenen Riemannschen Fläche. Ann. Acad. Sci. Fenn. Ser. A. 1. 35 (1947).

[2] Ahlfors, L. and Sario, L; Riemann surfaces. Princeton. (1960).

[3] Pfruger, A.; Theorie der Riemannschen Flächen. Berlin. (1957).

[4] Pfruger, A.; Uber die Riemannsche Periodenrelation auf transzendenten hyperelliptischen Flächen. Comm. Math. Helv. vol. 30 (1956) 998-106.

[5] Kusunoki, Y.; On Riemann's period relations on open Riemann surfaces. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. Vol. 30 (1956) 1-22.

[6] Accola, R.; The bilinear relation on open Riemann surfaces. Trans. Amer. Math. Soc. vol. 96 (1960) 143-161.

[7] Noshiro, K.; Cluster sets. Berlin (1960).

[8] Kuroda, T.; On some theorems of Sario. Bull. Math. de la Soc. Math. Phys. de la R.p.R. Tom 2 (50) n 4, (1958) 411-417.

[9] Sario, L.; Uber Riemannschen Flächen mit hebbarem Rand. Ann. Acad. Sci. Fenn. Ser. A. 1 50 (1948).