J. Math. Kyoto Univ.
2-3 (1963) 335-364.

On increasing Markov process

By

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(Communicated by Prof. K. Ito, July 1, 1963)

1. Introduction

The aim of the present paper is to characterize increasing Markov processes on the line under certain conditions. A Markov process is called *increasing* if its sample functions are almost always non-decreasing. We shall consider a class $\mathfrak{M}$ of increasing Markov processes all of whose states are instantaneous, and whose Green's operator G_α maps bounded continuous functions vanishing near $+\infty$ into continuous functions, so that these Markov processes are strong Markov. Let us recall that the Green's operator is the Laplace transform of the semi-group H_t, determined by the transition probabilities of the process. We shall show (Theorem 5.1) that to each process in $\mathfrak{M}$ corresponds in a 1-1 way a family $n(a, db)$ of measures with the following properties:

1) $n(a, (-\infty, a))=0$, and $n(a, db)$ has no point masses;
2) $\int n(a, db)f(b)$ is continuous in a, whenever f is continuous and vanishes near $+\infty$ (i.e. in an interval of the form $[N, +\infty)$);
3) $n(a, db)$ has the maximum property; namely, if f is continuous and vanishes near $+\infty$, and $u(a)=\int n(a, db)f(b)$, has a maximum at $a=a_0$, then $f(a_0)\geqslant 0$.

We shall show that if the process is in addition, additive, then $n(a, db)$ has an explicit representation (Theorem 8.1). In section 9 we shall show that an increasing strong Markov process with continuous paths is deterministic.

It does not seem to be easy to obtain an adequate characterization of $\mathfrak{M}$ by a direct appeal to the Hille-Yosida theorem, since we know nothing more about the domain of the infinitesimal generator than the fact that it is dense. We shall, however, show by using Dynkin's formula [3, Section 2] that the infinitesimal generator exists and has a dense domain, a part of which is completely determined.

A crucial step in the whole proof is the solution of the integral equation (Lemms 5.1):

$$f+\alpha\int n(a, db)f(b) = g\,,$$

where n is the characteristic measure of the process (see § 3) which is concentrated in a half-line. The technique for solving this consists in breaking up $n(a, db)$ into smaller measures by using Dini's theorem on the uniform convergence of a monotone sequence of continuous functions to a continuous function [2, p. 121].

Finally it will be obvious from the proof that the corresponding results hold good in R^k. In this case, one can, for instance, define an increasing process by the property

$$P_a(x_t \in K_a) = 1$$

for every t, where $a=(a_1, \cdots, a_k)$, $K_a=(b : b_i \geqslant a_i)$.

The problem was suggested to the author by Professor K. Ito. His help cannot be overestimated. Thanks are due to Professor K. Chandrasekharan for constant encouragement and to Dr. K. Balagangadharan for valuable discussions and a critical reading of the manuscript.

2. Notations

For generalities on Markov processes see [3]. We recall a few notions.

M will denote a Markov process

$$M = (S,\ W,\ P_a,\ a \in S)\,,$$

where S is the state spase, W the sample space consisting of all right continuous functions on $[0, \infty) \rightarrow S$ and P_a probabilities on

W with the Markov property

$$P_a[w_t^- \in B_1,\ w_t^+ \in B_2] = E_a[P_{x_t}(B_2):\ w \in B_1] \tag{2.1}$$

where
$$\begin{aligned} x_t &= x_t(w) = w(t)\,, \\ w_t^+(s) &= w(t+s)\,, \quad s \geqslant 0\,, \\ w_t^-(s) &= w(t \wedge s)\,, \quad s \geqslant 0\,, \quad t \wedge s = \min(t, s)\,, \end{aligned}$$

and $B_1, B_2 \in B(W)$, the Borel algebra on W. $f \in (B(W))$ will mean that f is $B(W)$-measurable.

We shall write for $f \in (B(S))$,

$$H_t f(a) = E_a[f(x_t)] = \int_S P(t, a, db) f(b) \tag{2.2}$$

where $P(t, a, db) = P_a[x_t \in db]$. H_t defines a semi-group on the set of bounded Borel functions on S. The *Green's operator* $G_\alpha (\alpha > 0)$ is defined by

$$u(a) = G_\alpha f(a) = \int_0^\infty e^{-\alpha t} E_a[f(x_t)]\,dt\,. \tag{2.3}$$

G_α satisfies the *resolvent equation*

$$G_\alpha - G_\beta + G_\alpha G_\beta (\alpha - \beta) = 0\,. \tag{2.4}$$

In this paper we consider Markov processes on the real line R satisfying

(A. 1) *almost all sample functions are right continuous and increasing*;

and

(A. 2) $G_\alpha f(a)$ $(\alpha > 0)$ *is continuous for any bounded continuous function f vanishing near $+\infty$.*

Let $\tilde{C}$ be the class of all continuous functions that vanish near $+\infty$ (but might be unbounded near $-\infty$). (A. 1) and (A. 2) will imply

$$G_\alpha \tilde{C} \subset \tilde{C}\,. \tag{2.5}$$

Using the typical argument, we can easily see that (2. 5) *implies the strong Markov property of our process.*

It is easy to see that $G_\alpha : \tilde{C} \to G_\alpha \tilde{C}$ is one-to-one. *The infinitesimal generator* $\mathcal{G}$ is defined by

$$\mathcal{G}u = \alpha u - G_{\alpha}^{-1}u$$

where the domain $\mathfrak{D}(\mathcal{G})$ of $\mathcal{G}$ is $G_{\alpha}\tilde{C}$. This definition is independent of α because of the resolvent equation.

Let $\mathcal{G}^{i}$ be the generator of M_i for $i=1.2$. If then $\mathcal{G}^{1}=\mathcal{G}^{2}$, $M_1=M_2$.

Define for $b\in R$

$$\sigma_b(w) = \inf\{t : x_t(w) \geqslant b\} .$$

Then σ_b is a Markov time, i.e.

$$(\sigma_b \geqslant t) \in B_t = \{\dot{B} : (B = (w : w_t^{-} \in B')),\ B' \in B(W)\} ,$$

where B_t is the stopped Borel algebra at t, σ_b increases with b. If the paths are continuous it is the first arriving time at b if the starting point is to the left of b. We shall classify points of R in the following way.

1. a is a *trap* if $E_a[e^{-\sigma_b}]=0$, for every $b\geqslant a$;
2. a is an *exponential holding time point if*

$$0 < \lim_{b \downarrow a} E_a[e^{-\sigma_b}] < 1 ;$$

3. a is *instantaneous* if

$$\lim_{b \downarrow a} E_a[e^{-\sigma_b}] = 1 .$$

We shall call *a regular* if it is not a trap.

3. Characteristic measure of the process.

Proposition 3.1. *If a is not a trap, there exists a neighborhood $U(a)$ of a such that $E_c[\sigma_b]<\infty$ for $c, b\in U(a)$.*

Proof: If for every $u\in\mathfrak{D}(\mathcal{G})$, $\mathcal{G}u(a)=0$ then the fact that $\alpha G_{\alpha}f(a)=f(a)$ for every f with compact support implies that

$$H_t f(a) = E_a[f(x_t)] = f(a)$$

for every t, i.e. a is a trap. Hence there exists $u\in\mathfrak{D}(\mathcal{G})$, $\varepsilon>0$ and $U(a)$ such that $\mathcal{G}u(c)>\varepsilon$ for $c\in U(a)$. From Dynkin's formula, viz.

$$E_c\left[\int_0^{\sigma_b} \mathcal{G}u(x_t)dt\right] = E_c[u(x_{\sigma_b})] - u(c) ,$$

one gets

$$E_c[\sigma_b] \leqslant \frac{2||u||}{\varepsilon},$$

where $||u|| = \sup|u|$.

The set of regular poins is thus open. Let (λ, μ) be one of the component intervals.

Proposition 3. 2. *If* $\lambda < a < b < \mu$, *then*

$$E_a[\sigma_b] < \infty. \tag{3.1}$$

Proof: We have only to use Proposition 3. 1 and the fact that if a function is bounded in a neighbourhood of each point, then it is bounded in a compact set.

We shall assume hereafter that *there are no traps.* Then we see that the measure

$$n(a, db) = \int_0^\infty P(t, a, db)\,dt \tag{3.2}$$

exists in the sense that for every b,

$$\int_0^\infty P(t, a, (-\infty, b])\,dt < \infty.$$

Evidently this integral is equal to $E_a[\sigma_b]$. This is the probabilistic meaning of $n(a, db)$. Note that $E_a[\sigma_b]$ is bounded for a in a compact set.

Proposition 3. 3. *If f is continuous and has support in* $(-\infty, N]$ *for some* N, *then*

$$u(a) = \int_a^\infty n(a, db) f(b)$$

is continuous and vanishes in $(N, +\infty)$, *i.e.* $u \in \tilde{C}$, $G_\alpha f(a)$ *converges to* $u(a)$ *uniformly in* $a \geqslant -n$ *for evey* n.

Proof: Let M be such that

$$E_a[\sigma_N] < M, \quad -N \leqslant a, \tag{3.3}$$

Then if $-N \leqslant a$,

$$P_a[\sigma_N] < \frac{M}{\lambda}. \tag{3.4}$$

If $g(a)=E_a[\sigma_N]$, we see that

$$\int_0^\infty E_a[g(x_t)]\,dt = E_a\left[\int_0^N g(x_t)dt\right] \leqslant M^2, \quad -N \leqslant a. \tag{3.5}$$

It follows, using the Markov property, that

$$E_a[\sigma_N^2] \leqslant 2M^2, \quad -N \leqslant a. \tag{3.6}$$

Now

$$\begin{aligned}
|u(a)-G_\alpha f(a)| &= \left| E_a \int_0^\infty f(x_t)dt - G_\alpha f(a) \right| \\
&= E_a\left[\int_0^\infty (f(x_t)-e^{-\alpha t}f(x_t))dt\right] \\
&\leqslant E_a\left[\left|\int_0^N f(x_t)dt - \int_0^{\sigma_N} e^{-\alpha t} f(x_t)dt\right|\right] \\
&\leqslant E_a\left[\int_0^{\sigma_N} (1-e^{-\alpha t})dt\right] \|f\| \\
&= \|f\|\, E_a\left[\sigma_N - \frac{1-e^{-\alpha\sigma_N}}{\alpha}\right].
\end{aligned}$$

Also $x \geqslant 1-e^{-x}$ for $x \geqslant 0$ and $x-(1-e^{-x}) < x^2$ for $x<1$. We have therefore

$$\begin{aligned}
&E_a\left[\sigma_N - \frac{1-e^{-\alpha\sigma_N}}{\alpha}\right] \\
&\quad = E_a\left[\sigma_N - \frac{1-e^{-\alpha\sigma_N}}{\alpha} : \sigma_N > \lambda\right] + E_a\left[\sigma_N - \frac{1-e^{-\alpha\sigma_N}}{\alpha} : \sigma_N \leqslant \lambda\right] \\
&\quad \leqslant E_a[\sigma_N : \sigma_N > \lambda] + E_a\left[\sigma_N - \frac{1-e^{-\alpha\sigma_N}}{\alpha} : \sigma_N \leqslant \lambda\right] \\
&\quad \leqslant E_a[\sigma_N^2]\, P_a[\sigma_N > \lambda] + E_a\left[\sigma_N - \frac{1-e^{-\alpha\sigma_N}}{\alpha} : \sigma_N \leqslant \lambda\right].
\end{aligned}$$

Choose λ large so that $\frac{2M^3}{\lambda} < \varepsilon$ and then choose α such that $\alpha\lambda < 1$. We then have

$$E_a\left[\sigma_N - \frac{1-e^{-\alpha\sigma_N}}{\alpha}\right] \leqslant \varepsilon + E_a\left[\frac{\alpha^2\sigma^2 N}{\alpha} : \sigma_N \leqslant \lambda\right] \leqslant \varepsilon + \alpha\lambda^2.$$

Therefore $G_\alpha f(a)$ converges to $u(a)$ uniformly in $a \geqslant -N$ for every N and the continuity of $G_\alpha f$ implies that of u.

We shall call this measure *characteristic measure* of the process.

We defined G_α only for $\alpha>0$. Now we shall define G_0 by

$$G_0 f(a) = E_a\left(\int_0^\infty f(x_t)\,dt\right) = \int_a^\infty n(a,\ db)f(b)\,.$$

Then Proposition 3. 3 implies that, if $f\in\tilde{C}$, then $G_\alpha f(a)$ converges to $G_0 f(a)$ uniformly in $a\geqslant -n$ for every n and $G_0 f\in\tilde{C}$.

Proposition 3. 4.

$$\mathfrak{D}(\mathcal{G}) = G_0\tilde{C}\,;$$

$$\mathcal{G}u = -f \qquad \textit{for} \qquad u = G_0 f\,.$$

Proof: Letting $\beta\downarrow 0$ in the resolvent equation

$$G_\alpha f - G_\beta f + (\alpha-\beta)G_\alpha G_\beta f = 0 \qquad (f\in\tilde{C})\,.$$

We have

$$G_\alpha f - G_0 f + \alpha G_\alpha G_0 f = 0\,. \tag{3.7}$$

Letting $\beta\downarrow 0$ in $G_\alpha G_\beta = G_\beta G_\alpha$, we have

$$G_\alpha G_0 = G_0 G_\alpha$$

and so we have, by (3. 7),

$$G_\alpha f - G_0 f + \alpha G_0 G_\alpha f = 0\,. \tag{3.8}$$

Thus we have

$$G_0 f = G_\alpha(f+\alpha G_0 f) \tag{3.9}$$

and

$$G_\alpha f = G_0(f-\alpha G_\alpha f)\,. \tag{3.10}$$

$G_0\tilde{C}\subset G_\alpha C$ follows from (3. 9) and $G_\alpha\tilde{C}\subset G_0 C$ from (3. 10), and so we have

$$G_0\tilde{C} = G_\alpha\tilde{C} = \mathfrak{D}(\mathcal{G})\,.$$

Using (3. 9) we can see that, if $u=G_0 f$, then

$$\mathcal{G}u = \alpha u - (f+\alpha G_0 f) = -f\,.$$

4. Properties of $n(a, db)$

Proposition 4. 1. *If M_1 and M_2 are two Markov processes with same characteristic measures, they are identical.*

Proof: Let $\mathcal{G}^i$ and G_0^i correspond to M_i, $i=1,2$. We have

$$G_0^1 f = G_0^2 f\,, \qquad f \in \tilde{C}$$

by our assumption. If $u \in \mathfrak{D}(\mathcal{G}^1)$, then $u = G_0^1 f$ for some $f \in \tilde{C}$, and so we have

$$u = G_0^2 f \in \mathfrak{D}(\mathcal{G}^2)$$

and

$$\mathcal{G}^1 u = -f = \mathcal{G}^2 u\,.$$

Therefore $\mathcal{G}^2$ is an extension of $\mathcal{G}^1$. Similarly $\mathcal{G}^1$ is an extension of $\mathcal{G}^2$ and therefore $\mathcal{G}^1 = \mathcal{G}^2$. Hence the processes are identical.

Proposition 4. 2. *$n(a, db)$ has the maximum property, i.e. if*

$$u(a) = \int n(a,\ db) f(b)\,,$$

f vanishing in $[m, \infty)$ has a maximum in $[-n, m]$ at a_0, then $f(a_0) \geqslant 0$.

Proof: If

$$E_{a_0}\left[\int_0^\infty f(x_t)dt\right] \geqslant E_b\left[\int_0^\infty f(x_t)dt\right], \qquad b \geqslant a_0\,,$$

then

$$E_{a_0}\left[\int_0^\infty f(x_t)dt\right] \geqslant E_{a_0}\left[E_{x_s}\left[\int_0^\infty f(x_t)dt\right]\right],$$

so that

$$E_{a_0}\left[\int_0^\infty f(x_t)dt\right] \geqslant \int_s^\infty E_{a_0}[f(x_t)]\,dt\,,$$

i.e.

$$\int_0^s E_{a_0}[f(x_t)]\,dt \geqslant 0\,.$$

Divide by s and let $s \to 0$.

Corollary 1. $\int_{-\infty}^\infty f(b)n(a, db) \equiv 0$ *implies that* $f \equiv 0$.

Corollary 2. $\left\|\alpha \int_{-\infty}^\infty f(b)\,n(a, db) + f(a)\right\| \geqslant \left\|\alpha \int_{-\infty}^\infty f(b)\,n(a, db)\right\|$,

$\alpha > 0$, *where* $\|\ \|$ *denotes the supremum norm considered in fixed compact set.*

Proposition 4. 3. *The set of functions u of the form*

$$u = \int_{-\infty}^{\infty} f(b)\, n(\cdot, db) \tag{4.4}$$

is dense in the space of continuous functions vanishing at $+\infty$ *provided with the compact uniform topology.*

Proof: If f decreases and tends to zero at $+\infty$, then as $\alpha \to 0$ $\alpha G_\alpha f$ tends to f uniformly in compact sets, by Dini's theorem [2, p. 121]. It follows that this is true if f is continuous and tends to zero at $+\infty$. Let f vanish beyond some N. Then, as $\alpha \to \infty$,

$$\alpha^2 \int_0^{\infty} E_a[G_\alpha f(x_s)]\, ds - \alpha \int_0^{\infty} E_a[f(x_s)]\, ds \to f$$

uniformly on compact sets.

Proposition 4.4. *If* f *vanishes at* $+\infty$ *then* $E_a[f(x_t)]$ *is continuous.*

Proof: Let Λ be fixed and consider the process only $[\Lambda, \infty)$. Let E denote the Banach space of continuous functions in $[\Lambda, \infty)$ vanishing at $+\infty$. From Proposition 4.3 the resolvent G_α has its range in E. The Hille-Yosida theorem then gives a strongly continuous semi-group of operators $T_t : E \to E$ such that

$$\int_0^{\infty} e^{-\alpha t} T_t dt = G_\alpha .$$

But

$$G_\alpha f = \int_0^{\infty} e^{-\alpha t} E[f(x_t)]\, dt .$$

Since $E[f(x_t)]$ is right continuous in t we deduce

$$T_t f(a) = E_a[f(x_t)] ,$$

if f vanishes at $+\infty$ and is continuous in $[\Lambda, \infty)$. Since Λ was arbitrary the proposition is proved.

Proposition 4.5. $n(a, db)$ *is a continuous measure i.e., has no point mass, if and only if there are no exponential holding time points.*

Proof: If a is an exponential holding time point then

$$P_a[x_t = a] = e^{-\lambda_a t}, \quad 0 < \lambda_a < \infty .$$

It follows that

$$\int_0^\infty P_a[x_t = a]dt = \frac{1}{\lambda_a}.$$

Now suppose that

$$n(a, \{b\}) > 0$$

for some $b > a$. Then

$$\int_0^\infty P_a[x_t = b]dt > 0.$$

For an uncountable number of t we should have

$$P_a[x_t = b] > 0.$$

It follows that for some t, s, $t > s$,

$$P_a[x_t = b,\ x_s = b] > 0.$$

Using the Markov property

$$P_b[x_{t-s} = b] > 0,$$

i.e, b is an exponential holding time point.

5. The main theorem

We have seen that to a Markov process with increasing paths which go to $+\infty$ with probability one there corresponds a characteristic measure $n(a, db)$, which has the maximum property.

We shall now prove a partial converse to this. As we have proved above, all the following properties are true in the general case except perhaps (4), because $n(a, db)$ may have point masses; Proposition 4.5 shows that this can happen only when there are exponential holding time points.

Theorem 5.1. *Let $n(a, db)$ be mesure on R such that*

(1) $n(a, (-\infty, a]) = 0$; $n(a, (a, a+h)) > 0$, $h > 0$;

(2) $\int_a^\infty f(b)n(a, db)$ *is continuous if f is continuous and for $a \geqslant c$ for some c;*

(3) *if* $u(a) = \int_0^\infty f(b)n(a, db)$ *has a maximum in* $[\Lambda, c]$ *at* a_0, *then* $f(a_0) \geqslant 0$;

(4) *$n(a, db)$ is continuous, i.e. it has no point masses. Then there exists an increasing process for which $n(a, db)$ is the characteristic measure.*

For the proof of the theorem, the following lemma is fundamental.

Lemma 5.1. *Let $n(a, db)$ be measures on R^1 satisfying conditions* (1), (2) *and* (4) *of Theorem* 5.1. *Let Λ be fixed and consider a continuous function which vanishes beyond N. Let $\alpha>0$ be given. Then there exists a function g continuous in $[\Lambda, \infty)$ and vanishing outside $[\Lambda, N]$ such that*

$$g(a)+\alpha\int_{\Lambda}^{N} g(b)n(a, db) = f(a), \quad \Lambda\leqslant a\leqslant N. \tag{5.1}$$

Proof: Consider the function $n(a, b)=n(a, (\Lambda, b))$. Since $n(a, db)$ has no point masses, this is continuous non-decreasing in b, for fixed a. Since $\int_{-\infty}^{\infty} n(a, db)x(b)$ is continuous and since $n(a, db)$ has no point masses we see that $n(a, b)$ is continuous in a for fixed b. From Dini's theorem one deduces that $n(a, b)$ is continuous in (a, b).

From Dini's theorem again it follows now that there exists a $\delta>0$ such that

$$n(a, (b, b+h))<\frac{1}{\alpha+1} \quad \text{if} \quad h<\delta, \quad \Lambda\leqslant a, \ b\leqslant N. \tag{5.2}$$

(we use again the fact that $n(a, db)$ has no point masses).

If for $g\in E_N=\{$the set of functions continuous in $[\Lambda, \infty)$ with support in $[\Lambda, N]\}$, we define

$$Lg = \int_{b}^{b+h} g(b)n(a, db), \tag{5.3}$$

then the equation

$$g+\alpha Lg = h \tag{5.4}$$

has a solution for every $h\in E_N$, because $||\alpha L||<1$.

Consider a subdivision $(\Lambda+ih, 0\leqslant i\leqslant n)$ of $[\Lambda, n]$ into, say, n equal parts with $2h<\delta$.

Let $f_1\in E_N$ such that

$$f_1(a) = f(a), \quad \Lambda+(n-1)h\leqslant a\leqslant N;$$
$$f_1(a) = 0, \quad a\leqslant\Lambda+(n-2)h+h_1, \quad h_1<h.$$

Then there exists $g_1 \in E_N$ such that

$$g_1(a)+\alpha\int_{\Lambda+(n-2)h+h_1}^{\Lambda+nh} g_1(b)n(a,\ db) = f(a)\,,\quad \Lambda\leqslant a\leqslant N. \tag{5.5}$$

Let $f_2 \in E_{\Lambda+(n-1)h}$ be such that

$$f_2 = f-f-f_1-\alpha\int_{\Lambda+(n-3)h+h_1}^{\Lambda+(n-2)h+h_1} g_1(b)n(a,\ db)\,,\quad \Lambda+(n-2)h\leqslant a\leqslant N;$$
$$= 0\,,\qquad a\leqslant \Lambda+(n-3)h+h_1\,.$$

We can find $g_2 \in E_{\Lambda+(n-1)h}$ such that

$$g(a)+\alpha\int_{\Lambda+(n-3)h+h_1}^{\Lambda+(n-1)h} g_2(b)n(a,\ db) = f_2(a)\,,\quad \Lambda\leqslant a\leqslant N. \tag{5.6}$$

Adding (5.5) and (5.6) we see that

$$g_1(a)+g_2(a)+\alpha\int_{\Lambda+(n-3)h+h_1}^{N} [g_1(b)+g_2(b)]\,n(a,\ db)]\,n(a,\ db) = f_1+f_2\,,$$
$$\Lambda\leqslant a\leqslant N;$$

since $f_1+f_2=f$ for $\Lambda+(n-2)h\leqslant a\leqslant N$, we see that $g=g_1+g_2$ satisfies

$$g(a)+\alpha\int_{\Lambda+(n-3)h+h_1}^{N} g(b)n(a,\ db) = f\,,\quad \Lambda+(n-2)h\leqslant a\leqslant N. \tag{5.7}$$

It is clear how to complete the proof by proceeding backward in this fashion.

Now let us fix Λ, N and consider $[\Lambda, N]$. Proceeding exactly as in the Lemma 5.1, we can prove that given $f\in \boldsymbol{C}[\Lambda, N]$ (i.e. continuous functions on $[\Lambda, N]$) there exists $g\in \boldsymbol{C}[\Lambda, N]$ such that

$$f(\cdot) = \alpha\int_{\Lambda}^{N} n(\cdot,\ db)g(b)+g(\cdot)\,. \tag{5.8}$$

Proposition 5.1. *The g in the above equation is unique.*

The proof depends on this following lemma.

Lemma 5.2. *Let X be a compact Hausdorff space, f_n, $f\in \boldsymbol{C}(X)$ anb $f_n\to f$ uniformly. Let A be the set of maximum points of f and U be an open set containing A. Then there exists at least one n such that f_n has at least one maximum point in U.*

Proof: Let A_n be the set of maximum points of f_n, and K

the closure of $\bigcup_{n\geqq 1} A_n$. It is obviously enough to show that $K\cap A\neq\phi$. Suppose that $K\cap A=\phi$. Let

$$0<\beta=\sup_{x\in X}|f(x)|\,.$$

Since $\beta-f(x)>0$ on K we should have $\beta-f(x)>\varepsilon$ for some ε and for all $x\in K$. Choose n with $\|f_m-f\|<\frac{\varepsilon}{3}$ for $m\geqslant n$. Then if $x\in A_n$, $y\in A$,

$$f(x)>f_n(x)-\frac{\varepsilon}{3}>f_n(y)-\frac{\varepsilon}{3}\geqslant f(y)-\frac{2\varepsilon}{3}=\beta-\frac{2\varepsilon}{3}\,.$$

This is a contradiction.

Proof of Proposition 5.1. Suppose that

$$\alpha\int_\Lambda^N g(b)\,n(a,\ db)+g(a)\equiv 0\,.$$

Let $u(a)=\alpha\int_\Lambda^N g(b)n(a,\ db)$ and suppose that $\sup u>0$, and that the supremum is attained at a_0. Then since $u(N)=0$ we should have $a_0<N$ and then $g(a_0)<0$. Choose g_n such that $g_n=g$ for $a\leqslant N$ with support in $\left[\Lambda,\ N+\frac{1}{n}\right]$ and decreasing to g. Then $\alpha\int_\Lambda^{N+1/n} g_n(b)n(a,\ db)\downarrow\alpha\int_\Lambda^N g(b)n(a,\ db)$. The convergence is therefore uniform. Let A be the set of maximum points of u. A is compact and $N\notin A$. Further $g(a)<0$ for $a\in A$. According to the above lemma there is at least one g_n such that

$$u_o=\alpha\int_\Lambda^{N+1/n} g_n(b)\,n(a,\ db)$$

has at least one maximum point in U. It then follows the positive maximum property of $n(a,\ db)$ that $g_n(a)\geqslant 0$ at least at one point of U. Since $g_n=g$ in U this is a contradiction.

Replacing g by $-g$ and arguing in the same fashion we see that $u\equiv 0$. Hence $g\equiv 0$.

For every $f\in \boldsymbol{C}[\Lambda,\ N]$ define

$$G_\alpha f=\int_\Lambda^N n(a,\ db)g_\alpha(b)\,, \tag{5.9}$$

where g_α is given, by virtue of Lemma 5.1, by (5.1):

$$f(a) = \alpha \int_{\Lambda}^{N} n(a, db) g_{\alpha}(b) + g_{\alpha}(a).$$

Proposition 5. 2. *$G_{\alpha} f$ thus defined satisfies the resolvent equation*

$$G_{\alpha} - G_{\beta} + (\alpha - \beta) G_{\alpha} G_{\beta} = 0,$$

and $$\|\alpha G_{\alpha}\| \leqslant 1, \quad G_{\alpha} f \geqslant 0 \quad f \geqslant 0.$$

Proof: Integrating the equation defining g_{α}, we get

$$\int_{\Lambda}^{N} f(b) n(a, db) = \alpha \int_{\Lambda}^{N} n(a, db) \int_{\Lambda}^{N} n(b, dc) g_{\alpha}(c) + \int_{\Lambda}^{N} g_{\alpha}(b) n(a, db),$$

so that,

$$G_{\alpha}\left[\int_{\Lambda}^{N} f(b) n(\cdot, db)\right] = \int_{\Lambda}^{N} n(\cdot, db) \int_{\Lambda}^{N} n(b, dc) g_{\alpha}(c),$$

proving thereby that

$$G_{\alpha}\left[\int_{\Lambda}^{N} f(b) n(\cdot, db)\right] = \int_{\Lambda}^{N} n(\cdot, db) G_{\alpha} f(b).$$

Further, if

$$f = \beta \int_{\Lambda}^{N} n(a, db) g_{\beta}(b) + g_{\beta}(a),$$

then operating on both sides by G_{α}, we see that

$$G_{\alpha} f = \beta \int_{\Lambda}^{N} n(a, db) G_{\alpha} g_{\beta}(b) + G_{\alpha} g_{\beta}(a),$$

so that

$$G_{\beta} G_{\alpha} f = \int_{\Lambda}^{N} n(a, db) G_{\alpha} g_{\beta}(b) = G_{\alpha}\left[\int_{\Lambda}^{N} n(a, db) g_{\beta}(b)\right] = G_{\alpha} G_{\beta} f.$$

But

$$G_{\beta} G_{\alpha} f = G_{\beta}\left[\int_{\Lambda}^{N} n(\cdot, db) g_{\alpha}(b)\right] = \int_{\Lambda}^{N} n(a, db) G_{\beta} g_{\alpha}(b).$$

Hence $G_{\alpha} g_{\beta} = G_{\beta} g_{\alpha}$.

Finally,

$$\begin{aligned} G_{\alpha} f &= \beta G_{\alpha}\left[\int_{\Lambda}^{N} n(a, db) g_{\beta}(b)\right] + G_{\alpha} g_{\beta} \\ &= \beta G_{\alpha} G_{\beta} f + G_{\alpha} g_{\beta}, \end{aligned}$$

and

$$G_{\beta} f = \alpha G_{\alpha} G_{\beta} f + G_{\beta} g_{\alpha},$$

so that we have the resolvent equation

$$G_\alpha f - G_\beta f + (\alpha-\beta) G_\alpha G_\beta f \equiv 0 .$$

Let $u = G_\alpha f$. Suppose that $\sup u > 0$. Then it must be attained in $[\Lambda, N)$ and at least at one such point a, $g(a) \geqslant 0$. Thus $f > \alpha \sup u$ i.e., $\|f\| \geqslant \alpha \sup u$.

If $\inf G_\alpha f < 0$, at some such point, $g(a) \leqslant 0$ so that $f < 0$. The proposition is completely proved.

Proof of Theorem 5.1. Define for $f \in \boldsymbol{C}[\Lambda, N]$,

$$R_\alpha f(a) = G_\alpha f(a) + f(N)\left[\frac{1}{\alpha} - G_\alpha e(a)\right], \quad \Lambda \leqslant a \leqslant N, \tag{5.10}$$

where $e(a) \equiv 1$, $\Lambda \leqslant a \leqslant N$. Since $0 \leqslant G_\alpha e(a) \leqslant \frac{1}{\alpha}$ and $G_\alpha f \geqslant 0$ for $f \geqslant 0$, we see that $0 \leqslant \alpha R_\alpha f \leqslant 1$, if $0 \leqslant f \leqslant 1$ and $R_\alpha 1 = \frac{1}{\alpha}$. One easily verifies that

$$R_\alpha - R_\beta + (\alpha-\beta) R_\alpha R_\beta = 0 .$$

It is trivial to see that the set

$$\{u : u = R_1 f,\ f \geqslant 0\}$$

separate points of $[\Lambda, N]$. Now from a result of Ray [5, Theorem 1] we see that there exists a transition function Q_t:

$$Q_t f(x) = \int_{[\Lambda, N]} Q_t(x, dy) f(y), \quad t \geqslant 0, \tag{5.11}$$

where $Q_t f(x)$ is right continuous in t for $t \geqslant 0$ and

$$\int_0^\infty e^{-\alpha t} Q_t f(a)\, dt = R_\alpha f(a) .$$

Also $\lim_{\alpha\to\infty} \alpha R_\alpha f = g$ exists for every $f \in \boldsymbol{C}[\Lambda, N]$ and if $\mu_a = \mu$ is defined by

$$g(a) = \int_{[\Lambda, N]} \mu(a, db) f(b), \tag{5.12}$$

then

$$\int_{[\Lambda, N]} |g(b) - f(b)|\, \mu(a, db) = 0, \tag{5.13}$$

and

$$\int_{[\Lambda,N]} Q_t(a,\ db)f(b) = \lim_{\alpha\to\infty}\int_{[\Lambda,N]} Q_t(a,\ db)\alpha R_\alpha f(b)$$

$$= \int_{[\Lambda,N]} Q_t(a,\ db)g(b) = \int_{[\Lambda,N]} Q_t(a,\ db)\int_{[\Lambda,N]} f(c)\,\mu(b,\ dc)\,.$$

The last equation holding for every $f\in \boldsymbol{C}[\Lambda,\ N]$ implies that

$$\int_{[\Lambda,N]} Q_t(a,\ db)|f(b)-g(b)| = 0\,,\quad f\in \boldsymbol{C}[\Lambda,\ N]\,,$$

with $g(b)=\lim_{\alpha\to\infty}\alpha R_\alpha f(b)$.

Suppose that $f(N)=0$. If

$$f(a) = \alpha\int_\Lambda^N n(a,\ db)g_\alpha(b)+g_\alpha(a)\,,$$

and

$$\alpha[\sup_{a\in[\Lambda,N]} n(a,\ [\Lambda,\ N])]<1\,,$$

then evidently

$$g_\alpha = f-\alpha Lf+\alpha^2L^2f-\cdots\,,$$

where

$$Lf(a) = \int_\Lambda^N n(a,\ db)f(b)\,.$$

Hence $\lim_{\alpha\to 0} g_\alpha=f$ uniformly. This implies that

$$\int_\Lambda^N n(a,\ db)g_\alpha(b)\longrightarrow\int_\Lambda^N n(a,\ db)f(b)\,,$$

uniformly in $[\Lambda,\ N]$.

Since from (5.10), $R_\alpha\varphi=G_\alpha\varphi$ if $\varphi(N)=0$ we have if $f\geqslant 0$,

$$\lim_{\alpha\to\infty}\int_0^\infty e^{-\alpha t}Q_tfdt = \lim_{\alpha\to\infty} G_\alpha f = \lim_{\alpha\to\infty}\int_\Lambda^N n(a,\ db)g_\alpha(b)$$

$$= \int_\Lambda^N n(a,\ db)f(b)\,.$$

This proves that if $f(N)=0$ and $f\in \boldsymbol{C}[\Lambda,\ N]$,

$$\int_0^\infty Q_tfdt = \int_\Lambda^N n(\cdot,\ db)f(b) = \int_\Lambda^\infty n(\cdot,\ db)f(b)\,.$$

We shall now prove that $\lim_{\alpha\to\infty}\alpha G_\alpha f=f$ for every $f\in \boldsymbol{C}[\Lambda,\ N]$

with $f[N]=0$. Note that from the results of Ray [5, Theorem 1] quoted above, if

$$g=\lim_{\alpha\to\infty}\alpha G_\alpha\varphi\,,$$

then

$$\int_\Lambda^N n(a,\ db)g(b)=\int_\Lambda^N n(a,\ db)\varphi(b)\,.$$

Hence if
$$g(a)=\int_\Lambda^N n(a,\ db)f(b)\,,$$

then from (5.12),

$$\int \mu_a(db)g(b)=g(a)\,.$$

Fix $a_0\in[\Lambda,\ N]$. Choose f such that $f_h(a)=1$ for $a\leqslant a_0+\theta h$ where $\theta<1$, and $f_h(a)=0$ for $a\geqslant a_0+h$. We have

$$\int_{[\Lambda,N]}\mu_{a_0}(db)\int_\Lambda^N f_h(c)n(b,\ dc)=\int_\Lambda^N f_h(c)n(a_0,\ dc)\,,$$

so that

$$\frac{1}{n(a_0,(a_0,a_0+h))}\int_{[\Lambda,N]}\mu_{a_0}(db)\int_{a_0}^{a_0+h}f_h(c)n(b,\ dc)$$
$$=\frac{1}{n(a_0,(a_0,a_0+h))}\int_{a_0}^{a_0+h}f_h(c)n(a_0,\ dc)\,.$$

The right side exceeds $\dfrac{n(a_0,(a_0,a_0+\theta h))}{n(a_0,(a_0,a_0+h))}>\dfrac{1}{2}$, if θ is close to 1. It is clear that by choosing suitable f_h, θ etc., we can show that

$$\mu_{a_0}(a_0)\neq 0\,.$$

It follows that for every $a_0\in[\Lambda,\ N]$, $\mu_{a_0}(a_0)>0$. Hencs $\lim\limits_{\alpha\to\infty}\alpha G_\alpha f(a)=f(a)$ for every $a\in[\Lambda,\ N)$; since by (5.13)

$$\int \mu_a(db)|f(b)-g(b)|=0\,,\quad g=\lim_{\alpha\to\infty}\alpha G_\alpha f\,.$$

By routine patching methods one gets a system $P(t,\ a,\ db)$ such that

1. $0\leqslant P(t,\ a,\ db)\leqslant 1$;
2. $P(t+s,\ a,\ dc)=\int P(t,\ a,\ db)P(s,\ b,\ dc)$;

3. $\int_0^\infty P(t, a, (-\infty, b])dt = n(a, (-\infty, b])$ for every b;

4. $\lim_{t\to\infty}\int_{-\infty}^\infty P(t, a, db)f(b) = f(a)$;

5. $\int_0^\infty e^{-\alpha t}dt\int_{-\infty}^\infty P(t, a, db)f(b) = \int_{-\infty}^\infty n(a, db)g_\alpha(b)$;

6. $P(t, a, (-\infty, a)) = 0$ for every t.

In the next article we shall construct the process and this will complete the proof of Theorem 5.1.

6. Construction of the process

We shall prove the following

Theorem 6.1. *Let $P(t, a, db)\leqslant 1$ be measures on R such that*

(1) $P(t, a, (-\infty, a)) = 0$;

(2) $\int_{-\infty}^\infty P(t, a, db)P(s, b, dc) = P(t+s, a, dc)$;

(3) $\int_{-\infty}^\infty P(\delta, a, db)\left|\int_{-\infty}^\infty P(t, a, db)f(b)-\int_{-\infty}^\infty P(t, b, dc)f(c)\right| \to 0$

as $\delta\to 0$ for every t, if f is continuous and vanishes at $+\infty$.

Then there exists a Markov process with increasing paths having $P(t, a, db)$ for its transition measures.

Proof: Add $+\infty$ to R and say $\infty > a$ for every $a\in R$. Let $\Gamma=$ {the set of all functions on the set of non-negative rationals into $R\cup\infty$}. Using routine methods one can get probabilities on Γ such that if $\tilde{x}_r$ is the co-ordinate at r,

$$P_a[x_{r_i}\in E_i,\ 1\leqslant i\leqslant n] = \int_{E_1} P(r_1, a, da)\cdots\int_{E_n} P(r_n - r_{n-1}, a_{n-1}, da_n).$$

From (1) and the Markov property we see that

$$P_a[\tilde{x}_r \geqslant \tilde{x}_s, \text{ for every } r, s \text{ with } r\geqslant s] = 1.$$

Putting $t=0$ in (3), we get

$$\int_{-\infty}^\infty P(\delta, a, db)\,|f(b)-f(a)| \to 0 \tag{3'}$$

From (3′), we have

$$P_a[|\tilde{x}_\delta - a| > \varepsilon] \to 0 \quad \text{as} \quad \delta \to 0 .$$

Since

$$P_a[\,|\tilde{x}_{t+\delta} - x_t| > \varepsilon] = \int P(t,\ a,\ da_1) \int P(\delta,\ a_1,\ da_2) F(a_1,\ a_2),$$

with
$$F(a_1,\ a_2) = 0, \quad \text{if} \quad |a_1 - a_2| \leqslant \varepsilon,$$
$$= 1 \quad \text{if} \quad |a_1 - a_2| > \varepsilon,$$

we have

$$P_a[|\tilde{x}_{r+\delta} - \tilde{x}_r| > \varepsilon] \to 0 \quad \text{as} \quad \delta \to 0 . \tag{6.1}$$

One cannot conclude from (6.1) in general that $\tilde{x}.$ is right continuous at r with probability 1. (6.1) only shows that given a sequence $r_n \downarrow r$, $\tilde{x}_{r_n} \to \tilde{x}_r$ a.e. for some subsequence of r_n. Since in our case $\tilde{x}_r \geqslant \tilde{x}_s$ a.e., $r \geqslant s$, we should have right continuity at every rational r. Thus

$$P_a[\tilde{W} = \{\tilde{x}_n \text{ is increasing, right continuous at every } r\}] = 1 .$$

Given any right continuous increasing function $\tilde{x}_r$ on the rationals we get a right continuous function on $[0,\ \infty)$ into $R \cup \infty$ if we define

$$x_t = \inf_{r>t} \tilde{x}_r .$$

Let W be the set of all right continuous increasing functions on $[0,\ \infty)$ into $R \cup \infty$. The map

$$\tilde{x}. \to x.$$

gives a 1–1 map of $\tilde{W}$ onto W. This is cleary measurable and we get a probability P_a on W. We shall show that this satisfies the Markov property.

Let $f_1, \cdots, f_n$, f be bounded continuous functions. We have

$$E_a[f_1(x_{t_1}) \cdots f_n(x_{t_n}) f(x_t)] = \lim_{\substack{r_i \to t_i \\ r \to t}} E_a[f_1(x_{r_1}) \cdots f_n(x_{r_n}) f(x_r)],$$

where

$$t_i < r_i < t_{i+1}, \quad t_n < r_n < t < r .$$

And

$$E_a[f_1(x_{r_1}) \cdots f_n(x_{r_n}) f(x_r)] = E_a[f_1(x_{r_1}) \cdots f_n(x_{r_n}) E_{x_{r_n}}(f(x_{r-r_n}))] .$$

Letting $r_i \to t_i$, $1 \leqslant i \leqslant n-1$, $r \to t$, we get

$$E_a[f_1(x_{t_1}) \cdots f_n(x_{r_n})f(x_t)] = E_a[f_1(x_{t_1}) \cdots f_n(x_{r_n})E_{x_{r_n}}(f(x_{t-r_n}))].$$

Now the proof is completed by using (3).

Remarks. If $\int P(t, a, db)f(b)$ is continuous in a as in our case, then (3) follows from (3′).

(2) One can also use Doob's theorem on paths of a semi-martingale [1, Theorem 11.5], for constructing the process.

(3) The idea of the proof above can be combined with a modification of certain results of Nelson [4, §4] to give more general constructions.

It is very natural to expect that if

$$\int n(a, db)f(b) = \int n(db)f(b+a), \tag{6.2}$$

then the process is additive. We have

Theorem 6.2. *The process is additive if and only if*

$$\int n(a, db)f(b) = \int n(db)f(b+a).$$

Proof: We see from the hypothesis that

$$\int n(a+b, dc)f(c) = \int_a^\infty n(a, dc)f(b+c),$$

i.e., $\tau_a Lf = L\tau_a f$,

where $Lf(b) = \int n(b, dc)f(c)$ and $\tau_a f(b) = f(a+b)$. If $f = \alpha Lg_\alpha + g_\alpha$, then

$$\tau_a f = \alpha L\tau_a g_\alpha + \tau_a g_\alpha,$$

so that $G_\alpha \tau_a f = \tau_a G_\alpha f$, i.e.

$$\int_0^\infty e^{-\alpha t}dt \int P(t, b, dc)f(a+c) = \int_0^\infty e^{-\alpha t}dt \int P(t, a+b, dc)f(c).$$

we get $\int P(t, b, dc)f(a+c) = \int P(t, a+b, dc)f(c)$,

i.e., $\int P(t, a, dc)f(c) = \int P(t, o, dc)f(a+c)$.

This together with the Markov property implies that

$$P(t+s, o, dc) = P(t, o, dc) * P(s, o, dc). \tag{6.3}$$

Suppose that $t_1 < t_2 < \cdots < t_n$. We have only to prove that

$$P[x_{t_1} \in E_1, \ x_{t_2} - x_{t_1} \in E_2, \ \cdots, \ x_{t_n} - x_{t_{n-1}} \in E_n] = \Pi P_a[x_{t_i} - x_{t_{i-1}} \in E_i].$$

One easily gets this using the Markov property and (6.3).

7. Examples

Example 1. Let M be a strictly increasing function and

$$\int f(b)n(a, db) = \int_a^\infty f(b)dM(b).$$

If u is differentiable with respect to M then $u \in \mathfrak{D}(\mathcal{G})$ and

$$\mathcal{G}u = \frac{du}{dM}.$$

Example 2. Let M and N be strictly increasing and M bounded. Define

$$u(a) = \int n(a, db)f(b) = \int_{[a,\infty)} dM(y) \int_{[a,y)} dN(z)f(z).$$

If for every $b \geqslant a_0$,

$$u(a_0) \geqslant u(b),$$

then

$$\int_{[a_0,\infty)} dM(y) \int_{[a_0,y)} dN(z)f(z) - \int_{[a,\infty)} dM(y) \int_{[b,y)} dN(z)f(z) \geqslant 0,$$

i.e. $$\int_{[a_0,b)} dM(y) \int_{[a_0,y)} dN(z)f(z) + \left[\int_{[a_0,a)} dN(z)f(z)\right] dM(b, \infty) \geqslant 0.$$

If $f(a_0) < 0$, for b near a_0, $f(b) < 0$ so that the term on the left side The conditions of the main theorem are thus satisfied.

Example 3. For the Poisson process with mean $\lambda > 0$, it can be easily seen that the characteristic measure is concentrated on the non-negative integers, the mass at the point n being λ^{-n}, $n \geqslant 0$.

8. Additive increasing processes

The characterization of a Markov process given by a Lévy process is much simpler and in this case the characteristic measure has, in a sense, an explicit representation. In fact we have

Theorem 8.1. *An additive increasing Markov process is characterised by a measure m for which*

$$\int_{(0,\infty)} \frac{b}{b+1} m(db) < \infty, \tag{8.1}$$

in the sense that if $P(t, db) = P_0(x_t \in db)$, *then*

$$\int_0^\infty P(t, db)e^{-\alpha b} = \exp\left[-Kt\alpha - \int_0^\infty (1-e^{-\alpha b})m(db)\right], \tag{8.2}$$

where $K \geqslant 0$ *is a constant; and conversely, and* $K \geqslant 0$ *and* m *satisfying* (8.1) *give rise to a Markov increasing additive process. Further, if* n *is the corresponding characteristic measure* (§3), *we have, if* $K=0$

$$(m(u, \infty)du) * n(du) = du, \tag{8.3}$$

Proof: We prove the last statement. Consider equation (8.2) with $K=0$; then integrating both sides,

$$\int_0^\infty e^{-\alpha b} \int_0^\infty P(t, db)dt = \left[\int_0^\infty (1-e^{-\alpha u})m(du)\right]^{-1},$$

and by Fubini's theorem

$$\int_0^\infty e^{-\alpha b} n(db) = \left[\alpha \int_0^\infty e^{-\alpha u} m(u, \infty)du\right]^{-1},$$

i.e. $$\left[\int_0^\infty e^{-\alpha b} n(db)\right]\left[\int_0^\infty e^{-\alpha u} m(u, \infty)du\right] = \int_0^\infty e^{-\alpha u}du,$$

i.e. $$\int_0^\infty e^{-\alpha u}[m(u, \infty)du * n(du)] = \int_0^\infty e^{-\alpha u}du,$$

which is equivalent to (8.3).

Now we turn to the proof of the theorem. Suppose first $P(t, db)$ that corresponds to an additive increasing Markov process. Since

$$\int_0^\infty e^{-\alpha b}P(t+s, db) = \left[\int_0^\infty e^{-\alpha b}P(t, db)\int_0^\infty e^{-\alpha b}P(s, db)\right],$$

we see that

$$\int_0^\infty e^{-\alpha b}P(t, db) = e^{-tF(\alpha)},$$

where $F(\alpha)\geqslant 0$ and continuous. We have

$$\int_0^\infty \frac{1-e^{-\alpha b}}{b}\frac{bP(t, db)}{t} = \frac{1-e^{-tF(\alpha)}}{t}.$$

This shows that the family of measures $\frac{bP(t, db)}{t}$ is uniformly bounded on $[0, \infty)$. There exists then, by Helly's theorem, a measure M such that $\int_0^\infty M(db)<\infty$ and for every continuous function with compact support in $[0, \infty)$,

$$\int_{[0,\infty)} M(db)f(b) = \lim_{n\to\infty}\int_0^\infty \frac{bP(t_n, db)}{t_n}f(b),$$

for some subsequence t_n. Since $\frac{1-e^{-\alpha b}}{b}\to 0$ at $+\infty$, we see that

$$\int_{[0,\infty)} \frac{1-e^{-\alpha b}}{b} M(db) = \lim_{n\to\infty}\int_{[0,\infty)} b\frac{P(t_n, db)}{t_n}\frac{1-e^{-\alpha b}}{b} = F(\alpha),$$

i.e.
$$\alpha M(0)+\int_{(0,\infty)} \frac{1-e^{-\alpha b}}{b} M(db) = F(\alpha).$$

Put $\frac{M(db)}{b}=m(db)$, then

$$\alpha M(0)+\int_{(0,\infty)} (1-e^{-\alpha b})m(db) = F(\alpha).$$

$\int_{(0,\infty)} (1-e^{-\alpha b})m(db)<\infty$ is equivalent to $\int \frac{b}{b+1}m(db)<\infty$.

Now we shall prove the converse. This part of the proof is modelled on K. Ito's proof [3, Section 4] of the structure theorem for Lévy processes.

Let a measure $n(du)$ on $(0, \infty)$ be given and a constant $m\geqslant 0$ that such $\int_0^\infty \frac{u}{1+u}n(du)<\infty$. Then we shall determine a temporally homogeneous Lèvy process x_t such that

$$E(e^{-\alpha x_t}) = \exp\left[-\alpha mt-t\int_{(0,\infty)} (1-e^{-\alpha u})n(du)\right].$$

Let
$$S = \{(s, u):\ s\geqslant 0,\ u>0\},$$
$$S^N = \{(s, u):\ N\geqslant s\geqslant 0,\ u>0\},$$

and $\sigma(dsdu)$ the product measure on $B(S)$ of the Lebesgue measure and $n(du)$. Consider the space $\Omega=[0, \infty]^{B(S)}$ and let A be the algebra of all sets of the form $((x(E_1), \cdots, x(E_n))\in B^n)$ where $B^n\in B(R^n)$, for all n and all n-tuples of sets $E_1, \cdots, E_n$. We shall now define an elementary probability measure on A, which for fixed $E_1, \cdots, E_n$ gives a probability on $B(R^n)$. We then appeal to Kolmogoroff's existence theorem to get a probability on $[0, \infty]^{B(S)}$. We give the details below.

For any $E\in B(S)$, define

$$\begin{aligned} P[x(E) = n] &= e^{-\sigma(E)}\frac{[\sigma(E)]^n}{n!}, && \text{if } \sigma(E)<\infty; \\ &= 0, && \text{if } \sigma(E)=\infty; \\ P[x(E)=\infty] &= 1, && \text{if } \sigma(E)=\infty. \end{aligned}$$

Let $E=E_1\cup\cdots\cup E_r$ where $E_1, \cdots, E_r$ are disjoint. Then

$$\begin{aligned} P[x(E) = n] &= e^{-\sigma(E_1\cup\cdots\cup E_r)}\frac{[\sigma(E_1\cup\cdots\cup E_r)]^n}{n!} \\ &= \frac{e^{-[\sigma(E_1)+\cdots+\sigma(E_r)]}}{n!}[\sigma(E_1)+\cdots+\sigma(E_r)]^n \\ &= \frac{e^{-[\sigma(E_1)+\cdots+\sigma(E_r)]}}{n!}\sum_{i_1+\cdots+i_r=n}(n!)\frac{\sigma(E_1)^{i_1}\sigma(E_2)^{i_2}\cdots\sigma(E_r)^{i_r}}{i_1!\, i_2!\cdots i_r!} \\ &= \sum_{i_1+\cdots+i_r=n} P(x(E_1)=i_1)P(x(E_2)=i_2)\cdots P(x(E_r)=i_r). \end{aligned}$$

Let now $E_1, \cdots, E_n\in B(S)$. We have

$$\begin{aligned} E_1\cup\cdots\cup E_n = &\bigcup_i (E_i-\bigcup_{j\neq i}E_j)\bigcup_{i\neq j}[E_i\cap E_j-\bigcup_{k\neq i,j}E_k]\bigcup_{i\neq j\neq k}[E_i\cap E_j\cap E_k \\ &-\bigcup_{l\neq i,j,k}E_l\cdots\bigcup(E_1\cap E_2\cdots\cap E_n) \\ =&\hat{E}_1\cup\cdots\cup\hat{E}_{r(n)}, \quad \text{say.} \end{aligned}$$

In general $r(n)=2^n$. Then $\hat{E}_1, \cdots, \hat{E}_{r(n)}$, are disjoint and each set E_i is the disjoint union of some of the sets $\hat{E}_j$. Let

$$\begin{aligned} f^p(i) &= i, \quad \text{if } E_p\cap\hat{E}_i \text{ is non-empty;} \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{8.5}$$

Let $B\in B(R^n)$ and define

$$P[(x(E_1), \cdots, x(E_n))\in B] =$$
$$\sum_{k_1,\cdots,k_{r(n)}}\prod_{i=1}^{r(n)}P[x(\hat{E}_i)=k_i]\chi_B[(\sum_i f'(i)k_i, \cdots, \sum_i f^n(i)k_i)], \tag{8.6}$$

where χ_B is the characteristic function of B. From this definition of P it is clear that if τ is a permutation of $1, 2, \cdots, n$ then

$$P[(x(E_{\tau(1)}), \cdots, x(E_{\tau(n)})) \in \tau B] = P[(x(E_1), \cdots, x(E_n)) \in B],$$

where τB is defined in the obvious way. Let $F_1, \cdots, F_m$ be such that $F_i = E_i$, $1 \leqslant i \leqslant n$. Define the sets $\hat{F}_1, \cdots, \hat{F}_{r(m)}$ in the same way as in (8.4). We have

$$[(x(E_1), \cdots, x(E_n)) \in B] = [(x(F_1), \cdots, x(F_m)) \in B'],$$

where $\qquad B' = \{(\xi_1, \cdots, \xi_m) : (\xi_1, \cdots, \xi_n) \in B\}$,

and $\qquad \chi_{B'}[(\xi_1, \cdots, \xi_m)] = \chi_B[(\xi_1, \cdots, \xi_n)]$.

From formula (8.6) above, we have, if $g^q(j)$ is defined in a similar way as in (8.5), then

$$P[(x(F_1), \cdots, x(F_m)) \in B']$$
$$= \sum_{l_1, \cdots, l_{(r)m}} \prod_{j=1}^{r(m)} P[(x(\hat{F}_j) = l_j] \chi_{B'}[(\sum_j g'(j) l_i, \cdots, \sum_j g^m(j) l)]$$
$$= \sum_{l_1, \cdots, l_{(r)m}} \prod_{j=1}^{r(m)} P[(x(\hat{F}_j) = l_j] \chi_{B'}[(\sum_j g'(j) l_j, \cdots, \sum_j g^m(j) l_j)].$$

Also each of the $\hat{E}_j$'s can be expressed as a union of the $\hat{F}_k$'s and since the $\hat{E}_j$'s are disjoint each $\hat{F}_k$ can occur in at most one of the unions. Let $h^i(j) = 1$ if F_j occurs in the union for E_i and zero otherwise. Then since $\hat{E}_j$ = some union of sets $\hat{F}_k$,

$$P[x(\hat{E}_i) = k_i] = \sum_{k_i = \sum_j h(j) l_j} \prod_{j=1}^{r(m)} P[x(\hat{F}_j) = l_j].$$

Therefore noting that each $\hat{F}_k$ can occur in at most one expression or, equivalently, $h^i(j)$ for fixed j is not zero for at most one i

$$\sum_{k_i, \cdots, k_{r(n)}} \prod_{j=1}^{r(n)} P(x(\hat{E}_i) = k_i) \chi_B[(\sum_i f'(i) k_i, \cdots, \sum_i f^n(i) k_i)]$$
$$= \sum_{k_i, \cdots, k_{r(n)}} \sum_{k_1 = \sum_j h'(j) l_j, \cdots, k_{r(n)} = \sum_j h^{r(n)}(j) l_j}$$
$$\prod_{j=1}^{r(m)} P[x(\hat{F}_j) = l_j] \chi_B[(\sum_i f'(i) \sum_j h^i(j) l_j, \cdots, \sum_i f^n(i) \sum_j h^i(j) l_j])$$
$$= \sum_{k_1, \cdots, k_{r(n)}} \sum_{k_1 = \sum_j h'(j) l_j, \cdots, k_{r(n)} = \sum_j h^{r(n)}(j) l_j}$$
$$\prod_{j=1}^{r(m)} P[x(\hat{F}_j) = l_j] \chi_B[(\sum_j g'(j) l_j, \cdots, \sum_j g^n(j) l_j)],$$

since $\sum_{i=1}^{r(n)} f^p(i)h^i(j)=g^p(j)$,

$$=\sum_{l_1,\cdots,l_{r(m)}}\prod_{j=1}^{r(m)} P[x(\hat{F}_j)=l_j]\chi_B[(\sum_j g'(j)l_j,\ \cdots,\ \sum_j g^n(j)l_j],$$

i.e. $\quad P[(x(E_1),\ \cdots,\ x(E_n))\in B]=P[(x(F_1),\ \cdots,\ x(F_m))\in B']$.

Now suppose that $((x(E_1),\ \cdots,\ x(E_n))\in B_1)=((x(F_1),\ \cdots,\ x(F_m))\in B_2)$ and consider $G_1',\ G_2',\ \cdots,\ G_{m+n}'$, with $G_i'=E_i$, $1\leqslant i\leqslant n$ and $G_{n+j}=F_j$, $1\leqslant j\leqslant m$. Also consider $G_1^2,\ \cdots,\ G_{m+n}^2$ with $G_i^2=F_i$, $1\leqslant i\leqslant m$ and $G_{m+j}^2=E_j$, $1\leqslant j\leqslant n$. Define

$$B_1^1=((\xi_1,\ \cdots,\ \xi_{m+n}):\ (\xi_1,\ \cdots,\ \xi_n)\in B_1),$$
$$B_2^2=((\xi_1,\ \cdots,\ \xi_{m+n}):\ (\xi_1,\ \cdots,\ \xi_m)\in B_2).$$

From the above it then follows that

$$P[(x(E_1),\ \cdots,\ x(E_n))\in B_1]=P[(x(G_1^1),\ \cdots,\ x(G_{m+n}^1))\in B_1^1],$$
$$P[(x(F_1),\ \cdots,\ x(F_m))\in B_2]=P[(x(G_1^2),\ \cdots,\ x(G_{m+n}^2))\in B_2^2].$$

Since $\quad (x(G_1^1),\ \cdots,\ x(G_{m+n}^1))\in B_1^1)=((x(E_1),\ \cdots,\ x(E_n))\in B_1)$

$$=((x(F_1),\ \cdots,\ x(F_m))\in B_2)=((x(G_1^2),\ \cdots,\ x(G_{m+n}^2))\in B_2^2),$$

and $G_1^2=G_{\tau(i)}'$ where τ is the permutation $\tau(j)=n+j$, $1\leqslant j\leqslant m$; $\tau(m+j)=j$, it follows that $\tau B_1^1=B_2^2$ and hence

$$P[(x(E_1),\ \cdots,\ x(E_n))\in B_1]=P[(x(F_1),\ \cdots,\ x(F_m))\in B_2].$$

P is thus uniquently defined on A and defines a probability measure on $B(R^n)$ for fixed $E_1,\ \cdots,\ E_n$. We can then extend P to $B(A)$. From the formula (8.6), then, if $E_1,\ \cdots,\ E_n$ are disjoint, $x(E_1),\ \cdots,\ x(E_n)$ are independent. Further, if $E=E_1\cup\cdots\cup E_n$, $E_1,\ \cdots,\ E_n$ being disjoint, then $x(E)=x(E_1)+\cdots+x(E_n)$ with probability 1.

Let us understand by an elementary figure, a finite disjoint union of closed rectangles with rational vertices and contained in S. An elementary figure is always compact and is contained in S^N for some N. If $E\subset S^\infty$ and is at a positive distance from the t-axis,

$$\int_E \sigma(dsdu)=\int dsn(u:\ (s,\ u)\in E)<\infty,$$

since $\int_0^\infty \frac{u}{u+1}n(du)<\infty$. Therefore, $E[x(E)]=\int_E \sigma(dsdu)<\infty$ i.e.

$x(E) < \infty$ with probability 1. The set of all elementary figures is countable so that

$$P[x(E) < \infty \text{, for all elementary figures } E] = 1\,.$$

Also if $E, E_1, \cdots, E_n$ are elementary figures $E_1, \cdots, E_n$ disjoint and $E = \bigcup_{i=1}^{n} E_i$ then $x(E) = \sum_{i=1}^{n} x(E_i)$ with probability 1, the set of probability 0 depending on the tuple $(E, E_1, \cdots, E_n)$. The set of all such finite n-tuples being again countable we have

$$P(\Omega_0) = 1\,,$$

where

$$\Omega_0 = \{w : w \in \Omega = [0, \infty]^{B(S)}, \text{ such that } x(E) < \infty \text{ and } x(E) \text{ is additive on all elementary figures)}\}\,.$$

Define for U open $U \subset S$,

$$p(U, w) = \sup_{U \subset E} x(E, w)\,,$$

E running over all elementary figures; and for $B \in B(S)$

$$p(B, w) = \inf_{U \supset B, U \text{ open}} p(U, w)\,.$$

We can then show that for $w \in \Omega_0$, $p(B, w)$ is a measure on $B(S)$ which is finite on compact sets (since $x(E, w) < \infty$ for E an elementary figure). Since the class of all elementary figure is countable $p(U, w)$ is measurable in q, for every open set U. Then by the usual monotone-class argument and the fact that $p(\cdot, w)$ is a measure on $B(S)$, we can prove that $p(B, w)$ is measurable for every $B \in B(S)$.

Since $x(E)$ is a Poisson process, we can prove, using $E(x(E)) = \sigma(E)$, that if $E_n \in B(S)$, $E_n \uparrow E$, then

$$P[\lim_n x(E_n) = x(E)] = 1\,.$$

Let U be open. For every elementary figure $E \subset U$,

$$P[x(U) \geqslant x(E)] = 1\,,$$

so that $P[x(U) \geqslant x(E)$ for every elementary figure $E \subset U] = 1$. It follows that $P[x(U) \geqslant p(U)] = 1$. Let $E_n \uparrow U$ be elementary figures, Then

$$P[\lim_n x(E_n) = x(U)] = 1 .$$

But $\lim_n x(E_n) \leqslant p(U)$ for all w. Therefore

$$P[x(U) = p(U)] = 1 .$$

Again by using the monotone class argument, we can prove that

$$P[x(B) = p(B)] = 1 , \quad \text{for every} \quad B \in B(S) .$$

The finite dimensional distributions, therefore, of $\{p(B, w)\}$ are identical with those of $\{x(B, w)\}$. By considering simple functions, etc., we can show that

$$E\left[e^{-\alpha \int_{[0,N]\times(0,\infty)} up(dsdu)}\right] = \exp\left[-\int_{[0,N]\times(0,\infty)} (1-e^{-\alpha u})\sigma(dsdu)\right]$$
$$= \exp\left[-N\int_0^\infty (1-e^{-\alpha u})\, n(du)\right].$$

Since the right hand side is positive,

$$P\left[\int_{[0,N]\times(0,\infty)} up(dsdu) < \infty\right] > 0 .$$

We can see (by considering simple functions etc.) that $y_n = \int_{(n,n+1]\times[0,N]} up(dsdu)$ are independent random variables. From the above $\sum_{n=0}^{\infty} y_n = \int_{[0,N]\times(0,\infty)} up(dsdu) < \infty$, on a set of positive probability. Hence

$$P\left[\int_{[0,N]\times(0,\infty)} up(dsdu) < \infty\right] = 1 ,$$

so that $P\left[\int_{[0,t]\times(0,\infty)} up(dsdu) < \infty \text{ for every } t \geqslant 0\right] = 1$. Finally define,

$$x(w) = mt + \int_{[0,t]\times(0,\infty)} up(dsdu) .$$

It is not difficult to verify that $x_t(w)$ is a Lévy process and

$$E(e^{-\alpha x_t}) = \exp\left(-mt\alpha - t\int_0^\infty (1-e^{-\alpha u})\,n(du)\right).$$

9. Continuous increasing processes

In this case the problem is relatively simple. We have

Theorem 9.1. *If a process with increasing continuous paths is strongly Markovian then it is deterministic, i.e.*

$$P_a[\{w_a\}] = 1,$$

where the paths w_a are such that

$$w_{w_a(t)}(s) = w_a(t+s).$$

Proof: Let, as before, $\sigma_b = \inf\{t : x_t \geq b\}$. Then, by continuity $x(\sigma_b) = b$, if $\sigma_b < \infty$. We will prove that $P_a[\sigma_b < \infty] = 1$ or 0 Suppose that $P_a[\sigma_b < \infty] = 0$. Then for large t_0,

$$P_a[x_t \geqslant b] > 0, \quad \text{for} \quad t \geqslant t_0.$$

Since the paths increase, if $a \leqslant b$, then

$$\begin{aligned} P_a[x_t \geqslant b] &= P_a[x_t > c,\ x_t \geqslant b] = P_a[\sigma_c < t,\ x_t \geqslant b] \\ &\leqslant P_a[\sigma_c < \infty,\ x_t \geqslant b] \leqslant P_a[\sigma_c < \infty,\ x_{t+\sigma_c} \geqslant b] \\ &= P_a[\sigma_c < \infty]\, P_c[x_t \geqslant b]. \end{aligned}$$

Thus

$$P_a[x_t \geqslant b] \leqslant P_c[x_t \geqslant b], \quad a < c \leqslant b.$$

We have

$$\begin{aligned} P_a[x_{t+s} \geqslant b] &= P_a[x_t \geqslant b] + E_a[x_t < b : P_{x_t}(x_s \geqslant b)] \\ &\geqslant P_a[x_t \geqslant b] + P_a[x_t < b]\, P_a[x_t \geqslant b]. \end{aligned}$$

Letting $s \to \infty$ we see that

$$P_a[\sigma_b < \infty] \geqslant P_a[x_t \geqslant b] + P_a[x_t < b]\, P_a[\sigma_b < \infty],$$

i.e., $$P_a[\sigma_b < \infty] = 1 \quad \text{or} \quad 0.$$

We can prove that [3, Section 6] if $P_a[\sigma_b < \infty] = 1$, then

$$E_a[\sigma_b] < \infty.$$

From this we see that (see proposition 3, 4)

$$E_a[\sigma_b] < \infty .$$

Again, if $a < c_1 < c_2 < \cdots < c_n = b$,

$$\begin{aligned} P_a \sigma c_1 &< t_1,\ \sigma_{c_2} - \sigma_{c_1} < t_2,\ \cdots,\ \sigma_{c_n} - \sigma_{c_{n-1}} < t_n] \\ &= P_a[\sigma_{c_1} < t_1]\, P_{c_1}[\sigma_{c_2} < t_2] \cdots P_{c_{n-1}}[\sigma_{c_n} < t_n] \\ &= P_a[\sigma_{c_1} < t_1]\, P_a[\sigma_{c_2} - \sigma_{c_1} < t_2] \cdots P_a[\sigma_{c_n} - \sigma_{c_{n-1}} < t_n] . \end{aligned}$$

Thus σ_c, $a \leqslant c \leqslant b$, is an additive process. It is easily seen to be continuous. An appeal to Lévy's representation theorem or to Theorem 1, Section 4 in [3] shows tha σ_c is a constant. This is what we set out to prove.

Remark. In general in this case

$$G_\alpha \text{ does not map } \boldsymbol{C} \text{ into } \boldsymbol{C} .$$

If this is the case and $\lambda_a(t)$ is defined by

$$P_a[x_t = \lambda_a(t)] = 1 ,$$

then $n(a, db)$ is the measure induced on $[a, \infty)$ by the mapping of

$$[0, \infty) \rightarrow [a, \infty) ,$$

given by $t \rightarrow \lambda_a(t)$.

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