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Branching Markov processes III

By

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Contents

IV. Branching semi-groups

The definition of a branching Markov process was introduced in Chapter I: it is a Markov process on $\widehat{S}$ whose semi-group satisfies (1.2). We shall say that non-negative contraction semi-group $\boldsymbol{T}_t$ on $\boldsymbol{B}(\widehat{S})$ with the property (1.2) *has the branching property* or, simply, that it is *a branching semi-group*. Therefore, the study of branching processes is, as a problem in analysis, the study of branching semi-groups. In §1.3 we have introduced two fundamental equations for a branching semi-group; *M-equation* and *S-equation*. The *M*-equation is a usual renewal type integral equation for a semi-group (the so called Desiré-André's equation or the first passage time relation applied to the first splitting time τ). When we look at the *M*-equation on S only, then, by virtue of the branching property, we have non-linear integral equation, which we have called the S-equation.

In this chapter we shall give these equations independent of the branching Markov processes only in terms of the *fundamental system* (T_0^t, K, π): T_t^0 and K are defined through (4.2) and (4.3) from a Markov process X^0 on S, and π is a substochastic kernel on $S\times S$ such that $\pi(x, S)=0$ for every $x\in S$. Given an M-equation, we shall construct its solutions according to Moyal [33] and show that the minimal solution of the M-equation defines a branching semi-group. This will give another analytical method of constructing an (X^0, π)-branching Markov process from a given X^0 and π. Also, one can construct an (X^0, π)-branching Markov process through the solutions of the S-equation: we shall first construct the solutions of the S-equation by the usual method of successive approximation and then define a branching semi-group from these solutions. In §4.5, we shall discuss the theory of infinitesimal generators of a branching semi-group under certain regularity assumptions on the fundamental system. As a consequence, we shall have two types of differential equations, the *backward equation*, which is a semi-linear evolution equation, and the *forward equation*, which is a system of linear evolution equations involving functional derivatives. In §4.6, the equations related to the number of particles will be discussed.

§4.1. Fundamental system, *M*-equation and *S*-equation

Let $X^0=\{x_t^0, P_x^0, \mathscr{B}_t^0, \zeta^0\}$ be a right continuous strong Markov process on $S\cup\{\Delta\}$, with Δ as the terminal point such that $\overline{\mathscr{B}}_{t+0}^0=\mathscr{B}_t^0$. Throughout this chapter we assume that (i)

(4.1) $\qquad P_x^0[x_{\zeta^0-}^0 \text{ exists}, \ \zeta^0<\infty]=P_x^0[\zeta^0<\infty]$

for every x and

(ii) $\qquad P_x^0[\zeta^0=s]=0 \quad \text{for every } x\in S \text{ and } s\geq 0.$

Define a semi-group T_t^0 on $\boldsymbol{B}(S)$ and a kernel $K(x;\, dt\, dy)$ on $S\times([0, \infty)\times S)$ by

(4.2) $\qquad T_t^0 f(x)=E_x^0[f(x_t^0);\ t<\zeta^0]$

(4. 3) $$K(x;\, dt\, dy) = P_x^0[\zeta^0 \in dt,\, x_{\zeta^0-}^0 \in dy].$$

Then we have clearly

(4. 4) $$\int_0^t \int_S K(x;\, dr\, dy) f(y) + T_t^0\left[\int_0^s \int_S K(\cdot\,;\, dr\, dy) f(y)\right] = \int_0^{t+s} \int_S K(x;\, dr\, dy) f(y)$$

and

(4. 5) $$T_t^0 1(x) + \int_0^t \int_S K(x;\, dr\, dy) = 1.$$

Let $\pi(x, dy)$ be a substochastic kernel[1] on $S \times S$ such that $\pi(x, S) = 0$ for every x.

Definition 4. 1. We shall call (T_t^0, K, π) a *fundamental system (defined by* X^0 *and* π*)*. When this system is defined by a branching Markov process $\boldsymbol{X}$, i.e., when X^0 is the non-branching part[2] of $\boldsymbol{X}$ and π is the branching law[3] of $\boldsymbol{X}$, we shall call (T_t^0, K, π) the *fundamental system of the branching Markov process* $\boldsymbol{X}$.

A class of fundamental systems we shall consider quite often in the future is the following: let $X = \{x_t, P_x, \mathcal{B}_t\}$ be a conservative right continuous strong Markov process on S such that $\overline{\mathcal{B}}_{t+0} = \mathcal{B}_t$ and T_t be its semi-group; $T_t f(x) = E_x[f(x_t)]$, $f \in \boldsymbol{B}(S)$. Let k be a non-negative measurable function and $X^0 = \{x_t^0, P_x^0, \zeta^0\}$ be $e^{-\int_0^t k(x_s)ds}$-subprocess of X, (cf. Definition 0. 8).

Definition 4. 2. When the process X^0 which defines (T_t^0, K) is given as above we shall call (T_t^0, K, π) the *fundamental system determined by* $[X, k, \pi]$.

When (T_t^0, K, π) is determined by $[X, k, \pi]$, then T_t^0 and K are given by

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1) As we remarked in §3. 3 it is equivalent to give a stochastic kernel π on $S \times \widehat{\boldsymbol{S}}$ such that $\pi(x, S) \equiv 0$.

2) Definition 1. 2. In this chapter, we shall assume that every branching Markov process satisfies (C. 2).

3) Definition 1. 3.

(4.6) $$T_t^0 f(x) = E_x[e^{-\int_0^t k(x_s)ds} f(x_t)]$$

(4.7) $$\int_0^t \int K(x;\, ds\,dy) f(y) = E_x\left[\int_0^t e^{-\int_0^s k(x_u)du} k(x_s) f(x_s) ds\right]$$
$$= \int_0^t T_s^0(kf)(x) ds.$$

(cf. [37]).

Given a fundamental system, we shall define kernels $\boldsymbol{T}_t^0(\boldsymbol{x}, d\boldsymbol{y})$ and $\psi(\boldsymbol{x};\, dt\,d\boldsymbol{y})$, $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S}$, $t \in [0, \infty)$, by

(4.8) $$\boldsymbol{T}_t^0 \widehat{f}(\boldsymbol{x}) \left(\equiv \int_{\hat{S}} \boldsymbol{T}_t^0(\boldsymbol{x}, d\boldsymbol{y}) \widehat{f}(\boldsymbol{y})\right) = \widehat{T_t^0 f}(\boldsymbol{x}), \quad f \in \boldsymbol{C}^*(S)^+$$

and

(4.9) $$\int_0^t \int_{\hat{S}} \psi(\boldsymbol{x};\, ds\,d\boldsymbol{y}) \widehat{f}(s, \boldsymbol{y})$$
$$= \int_0^t \langle T_s^0 f(s, \cdot) \,|\, \int_S K(\cdot\,;\, ds\,dz) F(z; f(s, \cdot)) \rangle(\boldsymbol{x}),^{4)}$$
$$f \in \boldsymbol{C}^*([0, \infty) \times S)^+,$$

where we put

(4.10) $$F(x;\, g) = \int_{\hat{S}} \pi(x, d\boldsymbol{y}) \widehat{g}(\boldsymbol{y}), \qquad g \in \overline{\boldsymbol{B}^*(S)}.$$

$\boldsymbol{T}_t^0$ and ψ are well defined by virtue of Lemma 0.3. It is clear that $\boldsymbol{T}_t^0$ defines, for each $n = 1, 2, \cdots$, a semi-group on $\boldsymbol{B}(S^n)$.

Theoren 4.1. *When (T_t^0, K, π) is the fundamental system of a branching Markov process $\boldsymbol{X}$,*[5] *$\boldsymbol{T}_t^0$ and ψ coincide with $\boldsymbol{T}_t^0$ and ψ defined by*

$$\boldsymbol{T}_t^0 f(\boldsymbol{x}) = \boldsymbol{E}_{\boldsymbol{x}}[f(\boldsymbol{X}_t);\, t < \tau] \qquad \textit{and}$$
$$\psi(\boldsymbol{x};\, ds\,d\boldsymbol{y}) = \boldsymbol{P}_{\boldsymbol{x}}[\tau \in ds,\, \boldsymbol{X}_\tau \in d\boldsymbol{y}].$$

Proof. Looking at the relation

$$\boldsymbol{P}_x[\tau \le t,\, \boldsymbol{X}_\tau \in d\boldsymbol{y}] = \int_0^t \int_S K(x;\, ds\,dz) \pi(z, d\boldsymbol{y})$$

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4) The right hand side of (4.9) is, if $\boldsymbol{x} = [x_1, x_2, \cdots, x_n] \in S^n$,
$$\sum_{i=1}^n \int_0^t \int_S K(x_i;\, ds\,dz) \{F(z; f(s, \cdot)) \prod_{j \neq i} T_s^0 f(s, \cdot)(x_j)\}.$$
We remark also that $T_s^0 f(s, \cdot)(x) = \int_S f(s, y) T_s^0(x, dy)$.

5) We assume that $\boldsymbol{X}$ possesses the branching law.

which is a direct consequence of the definition of the branching law, the assertion follows at once from the fact that $\boldsymbol{X}$ has the property B. III by Theorem 1. 3.

Lemma 4. 1. *For a given fundamental system* (T_t^0, K, π) *the above* $\boldsymbol{T}_t^0$ *and* ψ *satisfy*

$$\boldsymbol{T}_t^0 1(\boldsymbol{x}) + \psi(\boldsymbol{x};\ [0, t] \times \boldsymbol{S}) \leq 1 \tag{4.11}$$

and

$$\int_0^t \int_{\boldsymbol{S}} \psi(\boldsymbol{x};\ dr d\boldsymbol{y}) f(\boldsymbol{y}) + \boldsymbol{T}_t^0 \Big[\int_0^s \int_{\boldsymbol{S}} \psi(\cdot\ ;\ dr d\boldsymbol{y}) f(\boldsymbol{y})\Big](\boldsymbol{x}) \tag{4.12}$$
$$= \int_0^{t+s} \int_{\boldsymbol{S}} \psi(\boldsymbol{x};\ dr d\boldsymbol{y}) f(\boldsymbol{y}), \qquad f \in \boldsymbol{B}(\boldsymbol{S}).$$

Proof. Since $F(x;\ 1) \leq 1$ for every $x \in S$,

$$\psi(\boldsymbol{x};\ [0, t] \times \boldsymbol{S}) \leq \int_0^t \langle T_s^0 1 | \int_S K(\cdot\ ;\ drdz) \rangle (\boldsymbol{x}).$$

But
$$T_t^0 1(x) + \int_0^t \int_S K(x;\ drdz) \equiv 1,$$

and hence
$$\int_S K(x;\ drdz) = -d_r (T_r^0 1(x)).$$

Therefore

$$\int_0^t \langle T_t^0 1 | \int_S K(\cdot\ ;\ drdz) \rangle = \int_0^t \langle T_r^0 1 | -d_r (T_r^0 1) \rangle$$
$$= \int_0^t -d_r (\widehat{T_r^0 1}) = 1 - \widehat{T_t^0 1} = 1 - \boldsymbol{T}_t^0 1,$$

which proves (4. 11). Next we have

$$\int_0^{t+s} \int_{\boldsymbol{S}} \psi(\boldsymbol{x};\ dr d\boldsymbol{y}) \widehat{g}(\boldsymbol{y}) = \int_0^t \int_{\boldsymbol{S}} \psi(\boldsymbol{x};\ dr d\boldsymbol{y}) \widehat{g}(\boldsymbol{y})$$
$$+ \int_t^{t+s} \langle T_r^0 g | \int_S K(\cdot\ ;\ drdz) F(z;\ g) \rangle (\boldsymbol{x}),$$

and by (4. 4) the second term of the right hand side is equal to

$$\int_0^s \langle T_{r+t}^0 g | \int_S K(\cdot\ ;\ dr+t, dz) F(z;\ g) \rangle (\boldsymbol{x})$$
$$= \int_0^s \langle T_t^0 T_r^0 g | \int_S T_t^0 K(\cdot\ ;\ drdz) F(z;\ g) \rangle (\boldsymbol{x})$$

$$= \boldsymbol{T}_t^0 \int_0^s \langle T_r^0 g | \int_S K(\cdot\,; drdz) F(z;\, g) \rangle (\boldsymbol{x})^{6)}$$

$$= \boldsymbol{T}_t^0 \left[\int_0^s \int_S \psi(\cdot\,; drdy) \hat{g}(y) \right] (\boldsymbol{x}).$$

This proves (4.12) if f is of the form $\hat{g}$, $g \in \boldsymbol{C}^*(S)$. By virtue of Lemma 0.2, (4.12) holds for every $f \in \boldsymbol{B}(S)$.

Example 4.1. When $S = \{a\}$, (cf. Examples 0.1 and 0.3), $f \in \boldsymbol{B}^*(S)^+$ is given by a number f such that $0 \leq f < 1$. Then $T_t^0 f = e^{-ct} f$, where $0 \leqq c < \infty$, and $K(dt)f = c e^{-ct} f dt$. Now $\boldsymbol{S} \simeq \boldsymbol{Z}^+ = \{0, 1, 2, \cdots\}$. Let $\pi(1, \{n\}) = \pi_n$, $n = 0, 2, 3, \cdots$, $(0 \leq \pi_n, \sum_{n=0}^{\infty} \pi_n \leq 1)$. Then $\boldsymbol{T}_t^0(n, dy) = e^{-cnt} \delta_{\{n\}}(dy)$, $y \in \boldsymbol{S}$, and

$$\psi(n;\, dsdy) = cne^{-cns} ds \sum_{j \geqq n-1}^{\infty} \pi_{j-n+1} \delta_{\{j\}}(dy).$$

Definition 4.3. Given a fundamental system (T_t^0, K, π), we construct $\boldsymbol{T}_t^0$ and ψ by (4.8) and (4.9). For a given $f \in \boldsymbol{B}(S)$, consider the following integral equation

$$u(t, \boldsymbol{x}) = \boldsymbol{T}_t^0 f(\boldsymbol{x}) + \int_0^t \int_{\boldsymbol{S}} \psi(\boldsymbol{x};\, dsdy) u(t-s, y), \tag{4.13}$$
$$\boldsymbol{x} \in \boldsymbol{S},\ t \in [0, \infty)$$

call it the *M-equation (corresponding to the system* (T_t^0, K, π)*).* A solution $u(t, \boldsymbol{x})$ of (4.13) is called a *solution of the M-equation with the initial value f.*

Theorem 4.2. *Let* $\boldsymbol{X}$ *be a branching Markov process and set* $u(t, \boldsymbol{x}) = \boldsymbol{T}_t f(\boldsymbol{x}) = \boldsymbol{E}_{\boldsymbol{x}}[f(\boldsymbol{X}_t)]$, $f \in \boldsymbol{B}(S)$. *Then* $u(t, \boldsymbol{x})$ *is a solution of the M-equation corresponding to the system* (T_t^0, K, π) *of the process* $\boldsymbol{X}$ *with the initial value f.*

Proof. By the strong Markov property[7] applied to the first

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6) It is easy to see that $\boldsymbol{T}_t^0 \langle f | g \rangle = \langle T_t^0 f | T_t^0 g \rangle$; in fact

$$\boldsymbol{T}_t^0 \langle f | g \rangle = \lim_{\epsilon \to 0} \boldsymbol{T}_t^0 \{ (\widehat{f + \epsilon g} - \hat{f}) / \epsilon \} = \lim_{\epsilon \to 0} [\widehat{T_t^0 (f + \epsilon g)} - \widehat{T_t^0 f}] / \epsilon$$
$$= \lim_{\epsilon \to 0} (\widehat{T_t^0 f + \epsilon T_t^0 g} - \widehat{T_t^0 f}) / \epsilon = \langle T_t^0 f | T_t^0 g \rangle \text{ by } (0.36).$$

7) It should be remembered that we are always assuming $\boldsymbol{X}$ is strong Markov such that $\overline{\mathscr{B}}_{t+0} = \mathscr{B}_t$.

spilitting time τ, we have

$$\begin{aligned} u(t, \boldsymbol{x}) &= \boldsymbol{E}_{\boldsymbol{x}}[f(\boldsymbol{X}_t)] = \boldsymbol{E}_{\boldsymbol{x}}[f(\boldsymbol{X}_t);\ t<\tau] + \boldsymbol{E}_{\boldsymbol{x}}[f(\boldsymbol{X}_t);\ \tau\leqq t] \\ &= \boldsymbol{T}_t^0 f(\boldsymbol{x}) + \boldsymbol{E}_{\boldsymbol{x}}[\boldsymbol{E}_{\boldsymbol{X}_\tau}[f(\boldsymbol{X}_{t-s})]\,|_{s=\tau};\ \tau\leq t] \\ &= \boldsymbol{T}_t^0 f(\boldsymbol{x}) + \int_0^t \int_{\boldsymbol{S}} \psi(\boldsymbol{x};\ dsd\boldsymbol{y}) u(t-s, \boldsymbol{y}) \end{aligned}$$

by Theorem 4.1.

Definition 4.4. Given a fundamental system (T_t^0, K, π) and given $f \in B^*(S)$, consider the following integral equation

$$u(t, x) = T_t^0 f(x) + \int_0^t \int_S K(x;\ dsdy) F(y;\ u(t-s, \cdot)), \qquad (4.14)$$
$$x \in S,\ t \in [0, \infty)$$

where $F(x; u)$ is defined by (4.10). We shall call it *S-equation* (*corresponding to the system* (T_t^0, K, π)). A solution $u(t, x)$ of (4.14) such that $|u(t, x)| \leq 1$ is called a *solution of the S-equation with the initial value* f.

Theorem 4.3. *Let* $\boldsymbol{X}$ *be a branching Markov process and set* $u(t, x) = \boldsymbol{T}_t \hat{f}(x) = \boldsymbol{E}_x[\hat{f}(\boldsymbol{X}_t)]$, $f \in \boldsymbol{C}^*(S)$, $x \in S$ *then* $u(t, x)$ *is a solution of the S-equation corresponding to the system* (T_t^0, K, π) *of* $\boldsymbol{X}$ *with the initial value* f.

Proof. Since $\boldsymbol{T}_t \hat{f}(\boldsymbol{x}) = \widehat{\boldsymbol{T}_t \hat{f}|_S}(\boldsymbol{x}) = \widehat{u(t, \cdot)}(\boldsymbol{x})$ we obtain (4.14) from (4.13) by restricting it on S.

§4.2. Construction of a branching semi-group through the M-equation

First of all we shall give the following

Definition 4.5. A semi-group $\boldsymbol{U}_t$ on $\boldsymbol{B}(\boldsymbol{S})$ is called a *branching semi-group* if it is a non-negative contraction semi-group (i.e. the kernel $\boldsymbol{U}_t(\boldsymbol{x}, d\boldsymbol{y})$ of $\boldsymbol{U}_t$ is substochastic for every t) with the following property (called the *branching property*);

$$\boldsymbol{U}_t \hat{f}(\boldsymbol{x}) = \widehat{\boldsymbol{U}_t \hat{f}|_S}(\boldsymbol{x}).$$

Let (T_t^0, K, π) be a given fundamental system and $\boldsymbol{T}_t^0$ and ψ be defined through (4.8) and (4.9). Define kernels $\psi^{(n)}(x;\, dtdy)$ $(n=0,1,2,\cdots)$ on $S\times([0,\infty)\times S)$ by[8)]

$$(4.15)\qquad \Phi^{(0)}(x;\, t, dy)=\delta_{\{x\}}(dy),$$

$$\Phi^{(1)}(x;\, t, dy)=\int_0^t \psi(x;\, dsdy),$$

and $$\Phi^{(n)}(x;\, t, dy)=\int_0^t\int_S \psi(x;\, dvdz)\Phi^{(n-1)}(z;\, t-v, dy).$$

Then $$\psi^{(n)}(x;\, dtdy)=d_t\Phi^{(n)}(x;\, t, dy).$$

Set for each $n=0,1,2,\cdots$,

$$(4.16)\qquad \boldsymbol{T}_t^{(n)}(x, dy)=\int_0^t\int_S \psi^{(n)}(x;\, dsdz)\,\boldsymbol{T}_{t-s}^0(z, dy).^{9)}$$

Lemma 4.2. $\boldsymbol{T}_t^{(n)}$ and $\psi^{(n)}$ *satisfy the following relations for* $f\in\boldsymbol{B}(S)$[10)] *and* $0\leq k\leq n$;

$$(4.17)\qquad \Phi^{(n)}(t)f(x)=\int_0^t \psi^{(n-k)}(dr)\Phi^{(k)}(t-r)f(x),$$

$$(4.18)\qquad \boldsymbol{T}_t^{(n)}f(x)=\int_0^t \psi^{(n-k)}(dr)\,\boldsymbol{T}_{t-r}^{(k)}f(x),$$

$$(4.19)\qquad \boldsymbol{T}_v^{(0)}\boldsymbol{T}_{t-v}^{(n)}f(x)=\int_v^t \psi(dr)\,\boldsymbol{T}_{t-r}^{(n-1)}f(x),$$

$$(4.20)\qquad \Phi^{(n)}(t)f(x)=\Phi^{(n)}(s)f(x)+\sum_{j=1}^{n}\boldsymbol{T}_s^{(n-j)}\Phi^{(j)}(t-s)f(x),$$
for $0\leq s\leq t$.

Proof. (4.17) is the usual formula for iteration of convolutions and can be proved easily. (4.18) follows from (4.16) and (4.17). Now

8) Let $\Phi^{(n)}(x;\, t, dy)=\int_0^t \psi^{(n)}(x;\, dsdy)$. Clearly it is equivalent to give $\psi^{(n)}$ and $\Phi^{(n)}$.

9) Hence it is clear that $\boldsymbol{T}_t^{(0)}=\boldsymbol{T}_t^0$ and $\boldsymbol{T}_t^{(n)}$, $n=0,1,2,\cdots$ are non-negative kernels.

10) We write $\boldsymbol{T}_t^{(n)}f(x)=\int_S \boldsymbol{T}_t^{(n)}(x, dy)f(y)$, $\Phi^{(n)}(t)\cdot f(x)=\int_S \Phi^{(n)}(x;\, t, dy)f(y)$ and $\psi^{(n)}(dt)f(x)=\int_S \psi^{(n)}(x;\, dtdy)f(y)$.

$$\begin{aligned} T_v^{(0)} T_{t-v}^{(n)} f(x) &= T_v^{(0)} \left\{ \int_0^{t-v} \psi(dr)\, T_{t-v-r}^{(n-1)} f \right\}(x) \\ &= \int_0^{t-v} T_v^{(0)} \psi(dr)\, T_{t-v-r}^{(n-1)} f(x), \end{aligned}$$

and by (4.12) this is equal to

$$\int_0^{t-v} d_r \Phi(r+v)\, T_{t-r-v}^{(n-1)} f(x) = \int_v^t \psi(dr)\, T_{t-r}^{(n-1)} f(x).$$

This proves (4.19). For the proof of (4.20), first we note that if $n=1$, (4.20) is just (4.12). Assume that it holds for $n=1, 2, \cdots, n$; then

$$\begin{aligned} \Phi^{(n+1)}(t) f(x) &= \int_0^t \psi(dr) \Phi^{(n)}(t-r) f(x) \\ &= \int_0^s \psi(dr) \Phi^{(n)}(t-r) f(x) + \int_s^t \psi(dr) \Phi^{(n)}(t-r) f(x) \\ &= \int_0^s \psi(dr) \{ \Phi^{(n)}(s-r) f + \sum_{j=1}^{n} T_{s-r}^{(n-j)} \Phi^{(j)}(t-s) f \}(x) \\ &\qquad + \int_s^t \psi(dr) \Phi^{(n)}(t-r) f(x) \\ &= \Phi^{(n+1)}(s) f(x) + \sum_{j=1}^{n} T_s^{(n-j+1)} \Phi^{(j)}(t-s) f(x) \\ &\qquad + \int_0^{t-s} T_s^{(0)} \psi(dr) \Phi^{(n)}(t-r) f(x) \\ &= \Phi^{(n+1)}(s) f(x) + \sum_{j=1}^{n+1} T_s^{(n+1-j)} \Phi^{(j)}(t-s) f(x) \end{aligned}$$

by (4.12) and (4.18). This proves (4.20) for every n.

Lemma 4.3. $\sum_{n=0}^{\infty} T_t^{(n)}(x, S) \leq 1$ *for every* $x \in S$.

Proof. By (4.11) we have

$$\begin{aligned} T_t^{(1)}(x, S) = T_t^{(1)} 1(x) &= \int_0^t \psi(dv)\, T_{t-v}^{(0)} 1(x) \\ &\leq \int_0^t \psi(dv) (1 - \psi(\cdot\,;\ [0, t-v] \times S)) \\ &= \Phi^{(1)}(t) 1(x) - \Phi^{(2)}(t) 1(x) \end{aligned}$$

and

$$\begin{aligned}\boldsymbol{T}_t^{(2)}(\boldsymbol{x}, \boldsymbol{S}) &= \int_0^t \psi(dv)\, T_{t-v}^{(1)} 1(\boldsymbol{x}) \\ &\leq \int_0^t \psi(dv)\, [\boldsymbol{\Phi}(t-v) 1(\boldsymbol{x}) - \boldsymbol{\Phi}^{(2)}(t-v) 1(\boldsymbol{x})] \\ &= \boldsymbol{\Phi}^{(2)}(t) 1(\boldsymbol{x}) - \boldsymbol{\Phi}^{(3)}(t) 1(\boldsymbol{x}).\end{aligned}$$

Repeating this we have for every $n=1, 2, \cdots$,

$$\boldsymbol{T}_t^{(n)}(\boldsymbol{x}, \boldsymbol{S}) \leq \boldsymbol{\Phi}^{(n)}(t) 1(\boldsymbol{x}) - \boldsymbol{\Phi}^{(n+1)}(t) 1(\boldsymbol{x})$$

and therefore we have

$$\sum_{n=0}^{\infty} \boldsymbol{T}_t^{(n)}(\boldsymbol{x}, \boldsymbol{S}) \leq \boldsymbol{T}_t^0 1(\boldsymbol{x}) + \boldsymbol{\Phi}^{(1)}(t) 1(\boldsymbol{x}) \boldsymbol{x} \leq 1$$

by (4.11).

Thus for each $t \in [0, \infty)$,

$$\boldsymbol{T}_t(\boldsymbol{x}, d\boldsymbol{y}) = \sum_{n=0}^{\infty} \boldsymbol{T}_t^{(n)}(\boldsymbol{x}, d\boldsymbol{y}) \tag{4.21}$$

defines a substochastic kernel on $\boldsymbol{S} \times \boldsymbol{S}$. Let

$$\boldsymbol{T}_t f(\boldsymbol{x}) = \int_{\boldsymbol{S}} \boldsymbol{T}_t(\boldsymbol{x}, d\boldsymbol{y}) f(\boldsymbol{y}), \qquad f \in \boldsymbol{B}(\boldsymbol{S}). \tag{4.22}$$

Now we shall show that $\boldsymbol{T}_t$ is a semi-group on $\boldsymbol{B}(\boldsymbol{S})$. In fact

$$\begin{aligned}\boldsymbol{T}_t^{(n)} f(\boldsymbol{x}) &= \int_0^t \psi^{(n)}(dr)\, \boldsymbol{T}_{t-r}^{(0)} f(\boldsymbol{x}) \\ &= \int_0^s \psi^{(n)}(dr)\, \boldsymbol{T}_{t-r}^{(0)} f(\boldsymbol{x}) + \int_s^t \psi^{(n)}(dr)\, \boldsymbol{T}_{t-r}^{(0)} f(\boldsymbol{x}).\end{aligned}$$

Then by (4.20) the second term of the last expression is equal to

$$\begin{aligned}&\sum_{j=1}^{n} \boldsymbol{T}_s^{(n-j)} \int_0^{t-s} \psi^{(j)}(dr)\, \boldsymbol{T}_{t-s-r}^{(0)} f(\boldsymbol{x})^{11)} \\ &= \sum_{j=1}^{n} \boldsymbol{T}_s^{(n-j)} \boldsymbol{T}_{t-s}^{(j)} f(\boldsymbol{x}).\end{aligned}$$

Also the first term is equal to

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11) By (4.20), one can easily prove for $f(r, \boldsymbol{x}) \in \boldsymbol{B}([0, \infty] \times \boldsymbol{S})$

$$\int_s^t \psi(dr) f(r, \cdot)(\boldsymbol{x}) = \sum_{j=1}^{n} \boldsymbol{T}_s^{(n-j)} \int_0^{t-s} \psi^{(j)}(dr) f(r+s, \cdot)(\boldsymbol{x}).$$

$$\int_0^s \psi^{(n)}(dr)\, \boldsymbol{T}_{s-r}^{(0)} \boldsymbol{T}_{t-s}^{(0)} f(\boldsymbol{x}) = \boldsymbol{T}_s^{(n)} \boldsymbol{T}_{t-s}^{(0)} f(\boldsymbol{x}),$$

and hence we have

$$\boldsymbol{T}_t^{(n)} f(\boldsymbol{x}) = \sum_{j=0}^{n} \boldsymbol{T}_s^{(n-j)} \boldsymbol{T}_{t-s}^{(j)} f(\boldsymbol{x}).$$

Therefore

$$\begin{aligned} \boldsymbol{T}_t f(\boldsymbol{x}) &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \boldsymbol{T}_s^{(n-j)} \boldsymbol{T}_{t-s}^{(j)} f(\boldsymbol{x}) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \boldsymbol{T}_s^{(n)} \boldsymbol{T}_{t-s}^{(m)} f(\boldsymbol{x}) \\ &= \boldsymbol{T}_s(\boldsymbol{T}_{t-s} f)(\boldsymbol{x}), \end{aligned}$$

which proves $\boldsymbol{T}_t$ is a semi-group on $\boldsymbol{B}(\boldsymbol{S})$.

Next we shall show that $u(t, \boldsymbol{x}) = \boldsymbol{T}_t f(\boldsymbol{x})$ is a solution of the M-equation (4.13). Moreover, it is the minimal solution in the sence that if $f \geqq 0$, then $u(t, \boldsymbol{x})$ is the smallest of all non-negative solutions of (4.13). In fact,

$$\begin{aligned} u(t, \boldsymbol{x}) &= \boldsymbol{T}_t f(\boldsymbol{x}) \\ &= \boldsymbol{T}_t^{(0)} f(\boldsymbol{x}) + \sum_{j=1}^{\infty} \boldsymbol{T}_t^{(j)} f(\boldsymbol{x}) \\ &= \boldsymbol{T}_t^{(0)} f(\boldsymbol{x}) + \int_0^t \psi(ds) \sum_{i=0}^{\infty} \boldsymbol{T}_{t-s}^{(i)} f(\boldsymbol{x}) \\ &= \boldsymbol{T}_t^{(0)} f(\boldsymbol{x}) + \int_0^t \psi(ds)\, \boldsymbol{T}_{t-s} f(\boldsymbol{x}), \end{aligned}$$

which proves $u(t, \boldsymbol{x})$ is a solution of the M-equation (4.13). Now let $0 \leqq v$ be a solution of (4.13); then

$$v(t, \boldsymbol{x}) = \boldsymbol{T}_t^0 f(\boldsymbol{x}) + \int_0^t \psi(dr) v(t-r, \cdot)(\boldsymbol{x}) \geqq \boldsymbol{T}_t^0 f(\boldsymbol{x})$$

and if we suppose $v(t, \boldsymbol{x}) \geqq \sum_{i=0}^{n} \boldsymbol{T}_t^{(i)} f(\boldsymbol{x})$, then

$$\begin{aligned} v(t, \boldsymbol{x}) &\geqq \boldsymbol{T}_t^0 f(\boldsymbol{x}) + \int_0^t \psi(dr) \Big(\sum_{i=0}^{n} \boldsymbol{T}_{t-r}^{(i)} f\Big)(\boldsymbol{x}) \\ &= \sum_{i=0}^{n+1} \boldsymbol{T}_t^{(i)} f(\boldsymbol{x}). \end{aligned}$$

This proves $v(t, \boldsymbol{x}) \geq \sum_{i=0}^{n} \boldsymbol{T}_t^{(i)} f(\boldsymbol{x})$ for all n, and hence letting $n \to \infty$, we have $v(t, \boldsymbol{x}) \geq \boldsymbol{T}_t f(\boldsymbol{x})$.

Finally we must show that $\boldsymbol{T}_t$ is a branching semi-group, but this was proved already in Proposition 1.3.[12]

Summarizing, we have the following

Theorem 4.4. *For a given fundamental system* (T_t^0, K, π), *we construct a kernel* $\boldsymbol{T}_t(\boldsymbol{x}, d\boldsymbol{y})$ *on* $\boldsymbol{S} \times \boldsymbol{S}$ *by* (4.15), (4.16) *and* (4.21). *Then* $\boldsymbol{T}_t f(\boldsymbol{x}) \equiv \int_{\boldsymbol{S}} \boldsymbol{T}_t(\boldsymbol{x}, d\boldsymbol{y}) f(\boldsymbol{y})$, $f \in \boldsymbol{B}(\boldsymbol{S})$, *defines a branching semi-group.* $u(t, \boldsymbol{x}) = \boldsymbol{T}_t f(\boldsymbol{x})$, $f \in \boldsymbol{B}(\boldsymbol{S})$, *is a solution of the M-equation corresponding to the given system with the initial value* f, *and if* $f \geq 0$, *then* $u(t, \boldsymbol{x})$ *is the minimal solution among all non-negative solutions with the initial value* f.

Corollary. $u(t, x) = \boldsymbol{T}_t \widehat{f}(\boldsymbol{x})$, $f \in \boldsymbol{B}^*(S)$, *is a solution of the S-equation corresponding to the given system with the initial value* f.

Proof is the same as that of Theorem 4.3.

To this semi-group there corresponds a unique (up to equivalence) branching Markov process $\boldsymbol{X}$. If we compare the above construction with the probabilistic construction given in Chapter III we see at once that $\boldsymbol{X}$ is the (X_t^0, π)-branching Markov process, and hence it is a right continuous strong Markov process.

Example 4.2. Consider Example 4.1. Then the construction of $\boldsymbol{T}_t$ is just the usual analytical construction of the semi-group of the minimal Markov chain $(X_t, \boldsymbol{P}_i)$ on $i \in \boldsymbol{Z}^+ = \{0, 1, 2, \cdots\}$ such that $\boldsymbol{E}_i(\tau) = \frac{1}{ci}$ and $\boldsymbol{P}_i(X_\tau = j) = \pi_{j-i+1}$, where τ is the first jumping time. Hence by the above theorem, we see in particular that such a Markov chain is a branching process, i.e. the transition matrix satisfies (1.3).

12) When the fundamental system satisfies the condition (U) of Definition (4.2) given below, we can give a simpler proof of the branching property by Theorem 4.7 and Theorem 4.5, Cor. Cf. §4.4.

This fundamental fact is, of course, well known in the theory of branching processes, (cf. Harris [8], Chapter V).

Finally we shall discuss the uniqueness of the solution of the M-equation. The following class of fundamental systems plays an important rôle in the future discussions.

Definition 4.6. A fundamental system (T_t^0, K, π) is said to *satisfy the condition* (U) if T_t^0 satisfies

(U) $$\inf_{x\in S}\inf_{0\leq t\leq\sigma} T_t^0 1(x)>0, \qquad \text{for every } \sigma>0.$$

It is clear that a fundamental system (T_t^0, K, π) satisfies the condition (U) if it is determined by $[X, k, \pi]$ (cf. Definition 4.2) and k is bounded (i.e., $k\in \boldsymbol{B}(S)^+$); in fact,

$$T_t^0 1(x)=E_x[e^{-\int_0^t k(x_s)ds}]\geqq e^{-t\|k\|}$$

and hence for every $\sigma>0$

$$\inf_{x\in S}\inf_{0\leq t\leq\sigma} T_t^0 1(x)\geqq e^{-\sigma\|k\|}>0.$$

Theorem 4.5. *Suppose* (T_t^0, K, π) *satisfies the condition* (U). *Then the solution* $u(t, \boldsymbol{x})$ *of the M-equation with the initial value* $f(\boldsymbol{x})$ *such that* $\lim_{\boldsymbol{x}\to\Delta}\sup_{0\leq t\leq\sigma}|u(t, \boldsymbol{x})|=0$ *is unique.*

Proof. First we remark that for each $n=1, 2, \cdots$, and $\sigma>0$, we have

(4.23) $$\sup_{\boldsymbol{x}\in S^n}\psi(\boldsymbol{x}; [0, \sigma]\times \boldsymbol{S})<1.$$

For, by (4.11) and (U),

$$\sup_{\boldsymbol{x}\in S^n}\psi(\boldsymbol{x}; [0, \sigma]\times \boldsymbol{S})\leq 1-\inf_{\boldsymbol{x}\in S^n}\boldsymbol{T}_\sigma^0 1(\boldsymbol{x})=1-\inf_{\boldsymbol{x}\in S^n}\widehat{T_\sigma^0 1}(\boldsymbol{x})<1,$$

Now suppose that there exist two solutions u_1 and u_2 of (4.13) satisfying the condition of the theorem, then $\varphi_t(\boldsymbol{x})=u_1(t, \boldsymbol{x})-u_2(t, \boldsymbol{x})$ is a solution of

$$\varphi_t(\boldsymbol{x})=\int_0^t\int_{\boldsymbol{S}}\psi(\boldsymbol{x};\, dr d\boldsymbol{y})\varphi_{t-r}(\boldsymbol{y})$$

such that

(4.24) $$\lim_{x\to\Delta}\sup_{0\le t\le\sigma}|\varphi_t(\boldsymbol{x})|=0.$$

Assume

$$a=\sup_{y\in S}\sup_{0\le s\le\sigma}|\varphi_s(\boldsymbol{y})|>0.$$

Then by (4.24) there exists m such that

(4.25) $$a=\sup_{y\in S^m}\sup_{0\le s\le\sigma}|\varphi_s(\boldsymbol{y})|.$$

On the other hand

$$0<\sup_{x\in S^m}\sup_{0\le t\le\sigma}|\varphi_t(\boldsymbol{x})|\le\sup_{x\in S^m}\sup_{0\le t\le\sigma}\int_0^t\int_S\psi(\boldsymbol{x};\,drd\boldsymbol{y})\{\sup_{y\in S}\sup_{0\le s\le\sigma}|\varphi_s(\boldsymbol{y})|\}$$
$$\le\sup_{x\in S^m}\psi(\boldsymbol{x},\,[0,\sigma]\times\boldsymbol{S})\{\sup_{y\in S}\sup_{0\le s\le\sigma}|\varphi_s(\boldsymbol{y})|\}$$

and hence by (4.23), we have

$$0<\sup_{x\in S^m}\sup_{0\le t\le\sigma}|\varphi_t(\boldsymbol{x})|<\sup_{y\in S}\sup_{0\le s\le\sigma}|\varphi_s(\boldsymbol{y})|=a,$$

which contradicts (4.25). Therefore $\varphi_t(\boldsymbol{x})=0$ for all $t\in[0,\sigma]$ and $\boldsymbol{x}\in\boldsymbol{S}$. Since σ is arbitrary, $u_1=u_2$, which proves the theorem.

Corollary. *Suppose* (T_t^0, K, π) *satisfies the condition* (U), *and let* $\boldsymbol{U}_t$ *be a branching semi-group on* $\boldsymbol{B}(\boldsymbol{S})$ *such that, for every* $f\in\boldsymbol{B}(\boldsymbol{S})$, $u(t,\boldsymbol{x})=\boldsymbol{U}_tf(\boldsymbol{x})$ *defines a solution of the M-equation* (4.13). *Then* $\boldsymbol{U}_t$ *coincides with the semi-group* $\boldsymbol{T}_t$ *constructed in Theorem* 4.4.

Proof. Let $f\in\boldsymbol{B}^*(S)^+$; then $u(t,\boldsymbol{x})=\boldsymbol{U}_t\widehat{f}(\boldsymbol{x})=\widehat{u(t,\cdot)}(\boldsymbol{x})$ is a solution of the M-equation with the initial value $\widehat{f}$, where $u(t,x)=\boldsymbol{U}_t\widehat{f}|_S(x)$. We shall show that

(4.26) $$\lim_{x\to\Delta}\sup_{0\le t\le\sigma}|u(t,\boldsymbol{x})|=0,\qquad\text{for every }\sigma>0.$$

For, since $u(t,x)$ is a solution of the S-equation (4.14), we have

$$0\le u(t,x)=T_t^0f(x)+\int_0^t\int_S K(x;\,dsdy)F(y;\,u(t-s,\cdot))$$
$$\le T_t^0f(x)+\int_0^t\int_S K(x;\,dsdy)=T_t^0f(x)+1-T_t^01(x)$$
$$=1-T_t^0(1-f)(x)\le 1-(1-\|f\|)\inf_{x\in S,0\le t\le\sigma}T_t^01(x)<1$$

for every $t \in [0, \sigma]$ and $x \in S$; therefore, (4.26) is satisfied. In the same way we see that $v(t, \boldsymbol{x}) = \boldsymbol{T}_t \widehat{f}(\boldsymbol{x})$, $f \in B^*(S)^+$ satisfies the same equation and (4.26). Hence by Theorem 4.5, we have $u(t, \boldsymbol{x}) \equiv v(t, \boldsymbol{x})$, i.e., $\boldsymbol{T}_t \widehat{f}(\boldsymbol{x}) = \boldsymbol{U}_t \widehat{f}(x)$ for every $f \in \boldsymbol{B}^*(S)^+$. By Lemma 0.2 we have $\boldsymbol{T}_t \equiv \boldsymbol{U}_t$ on $\boldsymbol{B}(\boldsymbol{S})$.

§4.3. S-equation

Let (T_t^0, K, π) be a given fundamental system. In Definition 4.4 of §4.1 the S-equation was defined as

$$u(t, x) = T_t^0 f(x) + \int_0^t \int_S K(x;\, dsdy) F(y;\, u_{t-s}), \tag{4.14}$$

where $u_t(x) = u(t, x)$. A solution of (4.14) can be constructed by the usual method of successive approximation.

Theorem 4.6. *For a given* $f \in \overline{\boldsymbol{B}^*(S)^+}$, define $\{u_n(t, x)\}$ *inductively by*

$$\begin{aligned} &u_0(t, x) \equiv 0, \\ &u_n(t, x) = T_t^0 f(x) + \int_0^t \int_S K(x;\, dsdy) F(y;\, u_{n-1}(t-s, \cdot)). \end{aligned} \tag{4.27}$$

Then

(i) $$0 \leq u_n \leq u_{n+1} \leq 1 - T_t^0 (1-f),$$

and hence

$$u_\infty(t, x) \equiv \lim_{n\to\infty} u_n(t, x) \tag{4.28}$$

exists for every $t \in [0, \infty)$ *and* $x \in S$.

(ii) u_∞ *is a solution of the* S*-equation* (4.14), *and it is the minimal solution of* (4.14) *in the sense that if* $v (0 \leq v \leq 1)$ *is any solution of* (4.14), *then* $u_\infty \leq v$.

(iii) u_∞ *has the following representation by a (uniquely determined) substochastic kernel* $\mu_t(x, dy)$ *on* $S \times \boldsymbol{S}$;

$$u_\infty(t, x) = \int_S \mu_t(x, dy) \widehat{f}(y). \tag{4.29}$$

Proof. First of all we remark that, since

$$F(x; f) = \int_S \pi(x, dy) \hat{f}(y)$$

and π is a substochastic kernel, if $0 \leq g_1 \leq g_2 \leq 1$, then $0 \leq F(x; g_1) \leq F(x; g_2) \leq 1$. Then

$$\begin{aligned} u_0 \equiv 0 \leq u_1 &= T_t^0 f(x) + \int_0^t \int_S K(x; dsdy) \pi(y; \{\partial\}) \\ &\leq T_t^0 f(x) + \int_0^t \int_S K(x; dsdy) \\ &= T_t^0 f(x) + 1 - T_t^0 1(x) \\ &= 1 - T_t^0 (1-f)(x), \end{aligned}$$

and if we suppose $0 \leq u_{k-1} \leq u_k \leq 1 - T_t^0(1-f)$, then

$$\begin{aligned} 0 \leq u_k(t, x) &= T_t^0 f(x) + \int_0^t \int_S K(x; dsdy) F(y; u_{k-1}(t-s, \cdot)) \\ &\leq T_t^0 f(x) + \int_0^t \int_S K(x; dsdy) F(y; u_k(t-s, \cdot)) \\ &= u_{k+1}(t, x) \leq T_t^0 f(x) + \int_0^t \int_S K(x; dsdy) = 1 - T_t^0 (1-f)(x), \end{aligned}$$

which proves (i). Now it is clear that $u_\infty(t, x) \equiv \lim_{n\to\infty} u_n(t, x)$ is a solution of (4. 14). Suppose that $0 \leq v \leq 1$ is a solution of (4. 14); then $u_0 \equiv 0 \leq v$, and if we suppose $u_k \leq v$, then

$$\begin{aligned} u_{k+1}(t, x) &= T_t^0 f(x) + \int_0^t \int_S K(x; dsdy) F(y; u_k(t-s, \cdot)) \\ &\leq T_t^0 f(x) + \int_0^t \int_S K(x; dsdy) F(y; v(t-s, \cdot)) \\ &= v(t, x). \end{aligned}$$

This proves $u_k \leq v$ for every k, and hence $u_\infty \leq v$. Therefore (ii) is proved. Finally we shall prove (iii). By Lemma 0. 3 it is easy to see that each $u_k(t, x)$ has the expression

$$u_k(t, x) = \int_S \hat{f}(y) \mu_t^{(k)}(x, dy),$$

where $\mu_t^{(k)}(x, dy)$ is (for each fixed t) a substochastic kernel on $S \times S$. Thus (4. 29) holds with $\mu_t(x, dy)$ which is a weak limit (on $\widehat{S}$) of $\mu_t^{(k)}$ when $k\to\infty$.

As already stated in the Corollary of Theory 4.4, the minimal solution of the M-equation supplies a solution of the S-equation. Conversely, we can construct a solution of the M-equation from a solution of S-equation as we shall see in the following

Theorem 4.7. *Let $f\in\overline{B^*(S)^+}$ and $u(t,x)$ be a solution of the S-equation* (4.14); *then $\boldsymbol{u}(t,\boldsymbol{x})$ defined by*

$$\boldsymbol{u}(t,\boldsymbol{x})=\widehat{u(t,\cdot)}(\boldsymbol{x}),\qquad \boldsymbol{x}\in\boldsymbol{S}, \tag{4.30}$$

is a solution of the M-equation (4.13).

The theorem follows at once from the following Lemma by setting $s=0$ in (4.31).

Lemma 4.4. *Let $u(t,x)=u_t(x)$ be a solution of the S-equation* (4.14); *then*

$$\widehat{T_s^0u_{t-s}}(\boldsymbol{x})=\widehat{T_t^0f}(\boldsymbol{x})+\int_s^t\langle T_r^0u_{t-r}|\int_S K(\cdot\,;\,drdy)F(y;\,u_{t-r})\rangle(\boldsymbol{x}) \tag{4.31}$$

where $s<t$.

Proof. When $\boldsymbol{x}=\partial$ or Δ, it is obvious. Suppose $\boldsymbol{x}\in S^n$. We shall prove (4.31) by induction on n. When $n=1$ we have by (4.14)

$$u_{t-s}=T_{t-s}^0f+\int_0^{t-s}\int_S K(\cdot\,;\,drdy)F(y;\,u_{t-s-r})$$

and by (4.4)

$$\begin{aligned}T_s^0u_{t-s}&=T_s^0T_{t-s}^0f+T_s^0\int_0^{t-s}\int_S K(\cdot\,;\,drdy)F(y;\,u_{t-s-r})\\&=T_t^0f+\int_s^t\int_S K(\cdot\,;\,drdy)F(y;\,u_{t-r}).\end{aligned}$$

Thus (4.31) holds for $n=1$. Suppose it is true for $\boldsymbol{x}\in S^{n-1}$ $(n\geqq 2)$. Then for $\boldsymbol{x}=[x_1,x_2,\cdots,x_n]\in S^n$, we have by setting $\boldsymbol{x}'=[x_2,x_3,\cdots,x_n]$,

$$\begin{aligned}&\widehat{T_s^0u_{t-s}}(\boldsymbol{x})=T_s^0u_{t-s}(x_1)\prod_{j=2}^n(T_s^0u_{t-s})(x_j)\\&=\left\{T_t^0f(x_1)+\int_s^t\int_S K(x_1;\,drdy)F(y;\,u_{t-r})\right\}\left\{\widehat{T_t^0f}(\boldsymbol{x}')\right.\end{aligned}$$

$$+\int_s^t \langle T_v^0 u_{t-v} | \int_S K(\cdot\,;\, dvdz) F(z;\, u_{t-v}) \rangle (\boldsymbol{x}') \Big\}$$

$$= \widehat{T_t^0 f}(\boldsymbol{x}) + \int_s^t \int_S K(x_1;\, drdy) F(y;\, u_{t-r}) \widehat{T_t^0 f}(\boldsymbol{x}')$$

$$+ T_t^0 f(x_1) \int_s^t \langle T_v^0 u_{t-v} | \int_S K(\cdot\,;\, dvdz) F(z;\, u_{t-v}) \rangle (\boldsymbol{x}')$$

$$+ \int_s^t \int_S K(x_1;\, drdy) F(y;\, u_{t-r}) \int_s^t \langle T_v^0 u_{t-v} | \int_S K(\cdot\,;\, dvdz) F(z;\, u_{t-v}) \rangle (\boldsymbol{x}')$$

$= I$, say.

Now consider the last term:

$$\int_s^t \int_S K(x_1;\, drdy) F(y;\, u_{t-r}) \int_s^t \langle T_v^0 u_{t-v} | \int_S K(\cdot\,;\, dvdz) F(z;\, u_{t-v}) \rangle (\boldsymbol{x}')$$

$$= \int_s^t \int_S K(x_1;\, drdy) F(y;\, u_{t-r}) \Big\{ \int_r^t \langle T_v^0 u_{t-v} | \int_S K(\cdot\,;\, dvdz) F(z;\, u_{t-v}) \rangle (\boldsymbol{x}')$$

$$+ \int_s^r \langle T_v^0 u_{t-v} | \int_S K(\cdot\,;\, dvdz) F(z;\, u_{t-v}) \rangle (\boldsymbol{x}') \Big\}$$

$$= \int_s^t \int_S K(x_1;\, drdy) F(y;\, u_{t-r}) \int_r^t \langle T_v^0 u_{t-v} | \int_S K(\cdot\,;\, dvdz) F(z;\, u_{t-v}) \rangle (\boldsymbol{x}')$$

$$+ \sum_{j=2}^n \int_s^t \int_S K(x_j;\, dvdy) F(y;\, u_{t-v}) \int_v^t \int_S K(x_1;\, drdy) F(y;\, u_{t-r})$$

$$\times \prod_{k=2,\, k\neq j}^n T_v^0 u_{t-v}(x_k).$$

Hence

$$I = \widehat{T_t^0 f}(\boldsymbol{x}) + \int_s^t \int_S K(x_1;\, drdy) F(y;\, u_{t-r}) \Big\{ \widehat{T_t^0 f}(\boldsymbol{x}')$$

$$+ \int_r^t \langle T_v^0 u_{t-v} | \int_S K(\cdot\,;\, dvdz) F(z;\, u_{t-v}) \rangle (\boldsymbol{x}') \Big\}$$

$$+ \sum_{j=2}^n \int_s^t \int_S K(x_j;\, drdy) F(y;\, u_{t-r}) \Big\{ T_t^0 f(x_1)$$

$$+ \int_r^t \int_S K(x_1;\, dvdz) F(z;\, u_{t-v})) \times \prod_{k=2,\, k\neq j}^n T_r^0 u_{t-r}(x_k) \Big\}$$

and, by induction hypothesis, this is equal to

$$\widehat{T_t^0 f}(\boldsymbol{x}) + \int_s^t \int_S K(x_1;\, drdy) F(y;\, u_{t-r}) \widehat{T_r^0 u_{t-r}}(\boldsymbol{x}')$$

$$+ \sum_{j=2}^n \int_s^t \int_S K(x_j;\, drdy) F(y; u_{t-r}) \prod_{k=1,\, k\neq j}^n T_r^0 u_{t-r}(x_k)$$

$$= \widehat{T_t^0 f}(\boldsymbol{x}) + \sum_{j=1}^{n} \int_s^t \int_S K(x_j;\ drdy) F(y; u_{t-r}) \prod_{k=1, k\neq j}^{n} T_r^0 u_{t-r}(x_k)$$

$$= \widehat{T_t^0 f}(\boldsymbol{x}) + \int_s^t \int_S \langle T_r^0 u_{t-r} | \int_S K(\cdot\,;\ drdy) F(y;\ u_{t-r})\rangle(\boldsymbol{x}).$$

Thus (4.31) is proved.

Corollary 1. *Suppose* (T_t^0, K, π) *satisfies the condition* (U); *then the solution* $u(t, x)$ $(0\leq u\leq 1)$ *of the S-equation* (4.14) *with the initial value* $f\in \boldsymbol{B}^*(S)^+$ *is unique, and hence it coincides with* $u_\infty(t, x)$ *of Theorem* 4.6.

Proof. Let $u(t, x)$ be a solution of the S-equation (4.14) then just as in the proof of Corollary of Theorem 4.5, we have

$$\sup_{x\in S}\sup_{0\leq t\leq\sigma} |u(t, x)| \leq 1-(1-\|f\|) \inf_{x\in S, 0\leq t\leq\sigma} T_t^0 1(x) < 1.$$

Then $\hat{u}(t, \cdot)(\boldsymbol{x})$ is a solution of the M-equation with the initial value $\hat{f}(\boldsymbol{x})$ satisfying $\lim_{\boldsymbol{x}\to\Delta} \sup_{0\leq t\leq\sigma} |\hat{u}(t, \cdot)(\boldsymbol{x})| = 0$. By Theorem 4.5 $\hat{u}(t, \cdot)(\boldsymbol{x})$ is the unique solution and therefore $u(t, x)$ must be unique.

Corollary 2. *Let* $\boldsymbol{T}_t$ *be the branching semi-group constructed in Theorem* 4.4 (i.e., *the semi-group of the* (X^0, π)*-branching Markov process*). *Then for* $f\in \overline{B^*(S)^+}$, $u(t, x) = \boldsymbol{T}_t \hat{f}|_S(x)$ *is the minimal solution of the S-equation with the initial value f, that is, we have*

$$\boldsymbol{T}_t \hat{f}|_S(x) = u_\infty(t, x),$$

where u_∞ *is defined in Theorem* 4.6.

Proof. Let $v(t, x)$ $(0\leq v\leq 1)$ be a solution of the S-equation with the initial value f; then by Theorem 4.7 $\boldsymbol{v}(t, \boldsymbol{x}) = \hat{v}(t, \cdot)(\boldsymbol{x})$ is a solution of the M-equation with the initial value $\hat{f}(\boldsymbol{x})$. By Theorem 4.4 we have $\boldsymbol{T}_t \hat{f}(\boldsymbol{x}) \leq \boldsymbol{v}(t, \boldsymbol{x})$; in particular, we have $u(t, x) \leq v(t, x)$.

One of the consequences of Corollary 2 is the following. Let $f\equiv 1$; then $\boldsymbol{T}_t \hat{1}|_S(x) = \boldsymbol{E}_x[\hat{1}(\boldsymbol{X}_t)] = \boldsymbol{P}_x[e_\Delta > t]$. Thus $\boldsymbol{P}_x[e_\Delta > t]$ is the minimal solution of S-equation with the initial value 1. In particular we have the following

Corollary 3. *For an* (X^0, π)*-branching Markov process* $\boldsymbol{X}$, $\boldsymbol{P}_x[e_\Delta = +\infty] = 1$ *for every* x *if and only if* $u(t, x) \equiv 1$ *is the unique solution of the S-equation corresponding to the system* (T_t^0, K, π) *of* $\boldsymbol{X}$ *with the initial value* 1.

Now we shall discuss the regularity of a solution of the S-equation assuming some regularity conditions on the fundamental system (T_t^0, K, π). Let $H \subset \boldsymbol{B}(S)$ be a closed linear subspace of $\boldsymbol{B}(S)$ satisfying:

(H. 1) $H \cap \boldsymbol{C}(S)$ is dense in $\boldsymbol{C}(S)$ in the sense of w-convergence.[13]

(H. 2) The function $f(x) = \int_a^b u_t(x)dt$ belongs to H if $u_t \in H$ for each $t \in [a, b]$, u_t is right-continuous in t for each $x \in S$ and $\sup_{t \in [a, b]} \|u_t\| < \infty$.

Given a stochastically continuous[14] non-negative contraction semi-group U_t on $\boldsymbol{B}(S)$ such that $U_t(H) \subset H$, we set according to Dynkin [6]

$$H_0 \equiv H_0^{(U)} = \{f \in H;\ s\text{-}\lim U_t f = f\},^{15)} \tag{4.32}$$

$$\widetilde{H}_0 \equiv \widetilde{H}_0^{(U)} = \{f \in H;\ w\text{-}\lim U_t f = f\}. \tag{4.33}$$

The *H-infinitesimal generator* A_H and the weak H-infinitesimal generator $\widetilde{A}_H$ of U_t are defined as in [6]; in particular A_H is the infinitesimal generator in the Hille-Yosida sense of U_t restricted on H_0.

Definition 4. 7. A fundamental system (T_t^0, K, π) is called *H-regular* if it is determined by $[X, k, \pi]$ (cf. Definition 4. 2) such that, if T_t is the semi-group of X,

(i) $T_t(H) \subset H$,

(ii) $k \cdot f \in H_0 (\equiv H_0^{(T)})$, if $f \in H_0$, and

13) Let $\{f_s\} \subset \boldsymbol{B}(S)$ then $w\text{-}\lim_{s \to s_0} f_s = f_{s_0}$ if and only if $\sup_s \|f_s\| < \infty$ and $\lim_{s \to s_0} f_s(x) = f_{s_0}(x)$ for every $x \in S$.

14) i.e. $\lim_{t \downarrow 0} U_t f(x) = f(x)$ for every $f \in \boldsymbol{C}(S)$. Every semi-group corresponding to a right continuous Markov process on S is stochastically continuous.

15) $s\text{-}\lim_{s \to s_0} f_s = f_{s_0}$ if and only if $\|f_s - f_{s_0}\| \to 0$, $(s \to s_0)$.

(iii) $F(\cdot\,;f)\in H_0$, if $f\in H_0\cap \boldsymbol{B}^*(S)^+$.

When $H=H_0=\boldsymbol{C}(S)$ we shall call the H-regular fundamental system simply as *regular*.

Definition 4.8. A fundamental system (T_t^0, K, π) is called *weakly H-regular* if it is determined by $[X, k, \pi]$ such that, if T is the semi-group of X,

(i) $T_t(H)\subset H$,

(ii) $k\cdot f\in \widetilde{H}_0(\equiv \widetilde{H}_0^{(T)})$ if $f\in\widetilde{H}_0$,

(iii) $F(\cdot\,;f)\in\widetilde{H}_0$ if $f\in\widetilde{H}_0\cap\boldsymbol{B}^*(S)$, and

(iv) the function $f(x)=\int_0^t T_s^0(v_{t-s})ds$ belongs to $\widetilde{H}_0$, if $v_s\in\widetilde{H}_0$ for every $s\in[0,t]$, $v_s(x)$ is right continuous in s and $\sup_{s\in[0,t]}\|v_s\|<\infty$.

Remark 4.1. (i) The weak H-regularity does not necessarily imply the H-regularity.

(ii) If a system (T_t^0, K, π) is H-regular or weakly H-regular, then it satisfies the condition (U) since $k\in\boldsymbol{B}(S)^+$; hence the solution of the S-equation with the initial value $f\in\boldsymbol{B}^*(S)^+$ is unique. (Therefore it must coincide with u_∞ of Theorem 4.6 (4.28)).

(iii) If (T_t^0, K, π) is H-regular (weakly H-regular), then $T_t^0(H)\subset H$ and $H_0^{(T^0)}=H_0$ (resp. $\widetilde{H}_0^{T^0}=\widetilde{H}_0$). Let $A_H(\widetilde{A}_H)$ and $A_H^0(\widetilde{A}_H^0)$ be the H-infinitesimal generator (resp. weak H-infinitesimal generator) of T_t and T_t^0 respectively. Then $D(\widetilde{A}_H)=D(\widetilde{A}_H^0)$ (resp. $D(A_H)=D(A_H^0)$) and $A_H^0=A_H-k$, (resp. $\widetilde{A}_H^0=\widetilde{A}_H-k$).

(iv) (T_t^0, K, π) is regular if and only if it is determined by $[X, k, \pi]$ where the semi-group T_t of X is a strongly continuous semi-group on $\boldsymbol{C}(S)$, $k\in\boldsymbol{C}(S)^+$ and $F(\cdot\,;f)\in\boldsymbol{C}(S)$ if $f\in\boldsymbol{C}^*(S)^+$.

Theorem 4.8. *Suppose we are given an H-regular (weakly H-regular) fundamental system* (T_t^0, K, π). *If* $f\in H_0\cap\boldsymbol{B}^*(S)^+$ (resp. $f\in\widetilde{H}_0\cap\boldsymbol{B}^*(S)^+$), *then the solution of the S-equation* $u(t,x)\equiv u_t(x;f)$ *with the initial value f (which is unique*[16] *by Remark* 4.1

16) We shall give another direct proof of the uniqueness of the solution in §4.4.

(ii)) *belongs to* H_0 (resp. $\widetilde{H}_0$), *and* $u(t, \cdot)$ *is strongly continuous* (resp. *weakly right continuous*) *in* t.

Proof. Assume (T_t^0, K, π) is H-regular. By (4.7) $K(x; dsdy) = T_s^0(x, dy)k(y)ds$, where $T_s^0(x, dy)$ is the kernel of the semi-group T_s^0. Thus the S-equation has the form $u_t = T_t^0 f + \int_0^t T_s^0(k \cdot F(\cdot\,; u_{t-s}))ds$. Let $\{u_n(t, \cdot)\}$ $(n=0, 1, 2, \cdots)$ be defined by (4.27); then $u_n \leq u_{n+1}$ and $\lim_{n\to\infty} u_n = u$. Also by Theorem 4.6 (i) $\sup_{0\leq t\leq\sigma} \|u(t, \cdot)\| \leq 1-(1-\|f\|)e^{-\sigma\|k\|} \equiv A_\sigma < 1$ for every $\sigma > 0$. Next, we remark that if $g, h \in \boldsymbol{B}_r^*(S)^+$ where $r<1$, then by Lemma 0.1 (0.33) $\|\hat{g}-\hat{h}\|_S \leq a_r \|g-h\|$, and hence

$$(4.34) \qquad \|F(\cdot\,; g)-F(\cdot\,; h)\| = \sup_{x\in S} \Big| \int_S \pi(x, dy)(\widehat{f}(y) - \hat{g}(y)) \Big| \leq a_r \|g-h\|.$$

Now suppose $u_n(t, \cdot) \in H_0$ for every t and is strongly continuous in t. (For $n=0$, $u_n \equiv 0$, and hence it is trivially true). Then by the H-regularity of (T_t^0, K, π), $kF(\cdot\,; u_n(s, \cdot)) \in H_0$, and hence $v_s \equiv T_{t-s}^0(k \cdot F(\cdot\,; u_n(s, \cdot))) \in H_0$ every $0 \leq s \leq t$. We shall prove that v_s is strongly continuous in s on $[0, t]$. For,

$$\begin{aligned}
\|v_{s+h} - v_s\| &= \| T_{t-s-h}^0(k \cdot F(\cdot\,; u_n(s+h, \cdot))) - T_{t-s}^0(k \cdot F(\cdot\,; u_n(s, \cdot)))\|, \\
&\leq \| T_{t-s-h}^0(k \cdot \{F(\cdot\,; u_n(s+h, \cdot)) - F(\cdot\,; u_n(s, \cdot))\})\| \\
&\qquad + \|(T_{t-s-h}^0 - T_{t-s}^0)(k \cdot F(\cdot\,; u_n(s, \cdot)))\| \\
&\leq \|k\| \|F(\cdot\,; u_n(s+h, \cdot)) - F(\cdot\,; u_n(s, \cdot))\| \\
&\qquad + \| T_{t-s-h}^0 - T_{t-s}^0)(k \cdot F(\cdot\,; u_n(s, \cdot)))\| \\
&\leq a' \|k\| \cdot \|u_n(s+h, \cdot) - u_n(s, \cdot)\| \\
&\qquad + \|(T_{t-s-h}^0 - T_{t-s}^0)(k \cdot F(\cdot\,; u_n(s, \cdot)))\| \\
&\to 0
\end{aligned}$$

when $h \to 0$, where we set $a' = A_t$. Therefore,

$$w_t = \int_0^t v_s ds = \int_0^t T_s^0(k \cdot F(\cdot\,; u_n(t-s, \cdot)))ds \in H_0$$

and

$$\begin{aligned}\|w_{t+h}-w_t\| &\leq \int_t^{t+h}\|T^0_{t+h-s}(k\cdot F(\cdot\,;u_n(s,\cdot))\|ds\\ &+\int_0^t\|(T^0_{t+h-s}-T^0_{t-s})(k\cdot F(\cdot\,;u_n(s,\cdot)))\|ds\\ &\to 0\end{aligned}$$

when $h\to 0$. Thus w_t is strongly continuous and therefore $u_{n+1}(t,\cdot)$ $=T_t^0 f+w_t\in H_0$ and is strongly continuous in t. Hence, for every $n=0,1,2,\cdots$, $u_n(t,\cdot)\in H_0$ and is strongly continuous in t. Now if $t\leqq\sigma$, then, setting $a'=a_{A_\sigma}$, we have

$$\begin{aligned}\|u_n(t,\cdot)-u_{n-1}(t,\cdot)\| &\leq \Big\|\int_0^t T_s^0(k\cdot\{F(\cdot\,;u_{n-1}(t-s,\cdot))\\ &\qquad -F(\cdot\,;u_{n-2}(t-s,\cdot))\})ds\Big\|\\ &\leq \|k\|\int_0^t\|F(\cdot\,;u_{n-1}(t-s,\cdot))-F(\cdot\,;u_{n-2}(t-s,\cdot))\|ds\\ &\leq a'\|k\|\int_0^t\|u_{n-1}(s,\cdot)-u_{n-2}(s,\cdot)\|ds\\ &\leq (a'\|k\|)^2\int_0^t\int_0^{t_1}\|u_{n-2}(s,\cdot)-u_{n-3}(s,\cdot)\|dsdt_1\\ &\cdots\cdots\cdots\\ &\leqq (a'\|k\|)^n\int_0^t\cdots\int_0^{t_{n-1}}\|u_1(s,\cdot)\|dsdt_{n-1}dt_{n-2}\cdots dt_1\\ &\leqq \frac{\{a'\|k\|\}^n}{n!}\sigma^n.\end{aligned}$$

Hence for every $\sigma>0$,

$$\sup_{0\leq t\leq\sigma}\|u_t(\cdot\,;f)-u_n(t,\cdot)\|\leq\sum_{m\geq n}\frac{\{a'\|k\|\}^m}{m!}\sigma^m\to 0$$

when $n\to\infty$, which proves $u_t(\cdot\,;f)\in H_0$ and is strongly continuous in t.

The proof for the case of weak H-regular is similar. We only remark that we use the condition (iv) of Definition 4.8 to show that $\int_0^t T_s^0(k\cdot F(\cdot\,;u_n(t-s,\cdot))ds\in\widetilde{H}_0$ by assuming $u_n(s,\cdot)\in\widetilde{H}_0$ and is weakly right continuous in s.

Further regularity of the solution $u_t(\cdot\,;f)$, when $f\in D(A_H)\cap$ $\boldsymbol{B}^*(S)^+$ (resp. $f\in D(\widetilde{A}_H)\cap\boldsymbol{B}^*(S)^+$), will be disscussed in §4.5.

§4. 4. Construction of a branching semi-group through the S-equation

Given a fundamental system (T_t^0, K, π), we constructed in §4. 2 a branching semi-group as the minimal solution of the M-equation. We shall now give another construction of a branching semi-group using the solution u_∞ of the S-equation obtained in Theorem 4. 6. For this *we shall assume in this section that* (T_t^0, K, π) *is determined by* $[X, k, \pi]$, *where* $k \in \boldsymbol{B}(S)^+$. Then this fundamental system satisfies the condition (U) and hence u_∞ is the unique solution of the S-equation if the initial value f is in $\boldsymbol{B}^*(S)^+$. But the proof of Corollary 1 of Theorem 4. 7 involves arguments on the M-equation; therefore we shall give first of all a direct proof of the uniqueness of the solution so that future discussions will be self-contained and independent of the discussion given in §4. 2.

Let $u_t = u(t, x)$ $(0 \leq u \leq 1)$ be a solution of the S-equation (4. 14) with the initial value $f \in \boldsymbol{B}^*(S)^+$. Then

$$\begin{aligned} 0 \leq u_t = T_t^0 f + \int_0^t T_s^0 (k \cdot F(\cdot\,; f)) ds &\leq T_t^0 f + (1 - T_t^0 1) \\ &\leqq 1 - (1 - \|f\|) e^{-\|k\| \cdot t} \equiv A_t < 1. \end{aligned}$$

If $v_t = v(t, x)$ $(0 \leq v \leq 1)$ is another solution, then we have from (4. 34) that if $t \leqq \sigma$

$$\begin{aligned} \|u_t - v_t\| &= \left\| \int_0^t T_s^0 \{ k(F(\cdot, u_{t-s}) - F(\cdot, v_{t-s})) \} ds \right\| \\ &\leq a' \|k\| \int_0^t \|u_s - v_s\| ds \\ &\leqq (a' \|k\|)^2 \int_0^t \int_0^{t_1} \|u_s - v_s\| ds\, dt_1 \\ &\cdots\cdots\cdots \\ &\leqq (a' \|k\|)^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \|u_s - v_s\| ds\, dt_{n-1} \cdots dt_1 \\ &\leqq \frac{(a' \|k\|)^n}{n!} \sigma^n, \end{aligned}$$

where $a' = a_{A_\sigma}$. Hence

$$\sup_{0\leq t\leq\sigma}\|u_t-v_t\|\leq\frac{(a'\|k\|)^n}{n!}\sigma^n\to 0 \qquad (n\to\infty)$$

which proves $u_t\equiv v_t$; i.e., the solution of the S-equation with the initial value $f\in\boldsymbol{B}^*(S)^+$ is unique, and hence it must coincide with u_∞ of Theorem 4.6. We set $u_t(x;f)\equiv u_\infty(t,x)$. Then by Theorem 4.6

$$\sup_{0\leq t\leq\sigma}\|u_t(\cdot\,;f)\|\leq 1-(1-\|f\|)e^{-\|k\|\sigma}<1, \qquad \text{for all } \sigma>0, \tag{4.35}$$

and u_t has the following expression

$$u_t(x;f)=\int_S \mu_t(x,dy)\hat{f}(y) \tag{4.36}$$

where $\mu_t(x,dy)$ is a (uniquely determined) substochastic kernel on $S\times S$. By Lemma 0.3 there exists a (uniquely determined) substochastic kernel $\widetilde{\boldsymbol{T}}_t(\boldsymbol{x},d\boldsymbol{y})$ on $\boldsymbol{S}\times\boldsymbol{S}$ such that for every $f\in\boldsymbol{B}^*(S)^+$,

$$\widehat{u_t(\cdot\,;f)}(\boldsymbol{x})=\int_{\boldsymbol{S}}\widetilde{\boldsymbol{T}}_t(\boldsymbol{x},d\boldsymbol{y})\hat{f}(\boldsymbol{y}), \quad t\in[0,\infty),\ \boldsymbol{x}\in\boldsymbol{S}. \tag{4.37}$$

We shall show that $\widetilde{\boldsymbol{T}}_t g(\boldsymbol{x})=\int\widetilde{\boldsymbol{T}}_t(\boldsymbol{x},d\boldsymbol{y})g(\boldsymbol{y})$, $g\in\boldsymbol{B}(\boldsymbol{S})$, defines a semi-group on $\boldsymbol{B}(\boldsymbol{S})$. For this we shall prove

$$u_{t+s}(\cdot\,;f)=u_t(\cdot\,;u_s(\cdot\,;f)), \qquad f\in\boldsymbol{B}^*(T)^+. \tag{4.38}$$

In fact,

$$\begin{aligned}
u_{t+s}(\cdot\,;f)&=T^0_{t+s}f+\int_0^{t+s}\int_S K(\cdot\,;drdy)F(y;u_{t+s-r}(\cdot\,;f))\\
&=T^0_tT^0_sf+\int_0^t\int_S K(\cdot\,;drdy)F(y;u_{t+s-r}(\cdot\,;f))\\
&\qquad+\int_t^{t+s}\int_S K(\cdot\,;drdy)F(y;u_{t+s-r}(\cdot\,;f))\\
&=I,\ \text{say};
\end{aligned}$$

applying (4.4) to the last term of the above we have

$$\begin{aligned}
I=T^0_tT^0_sf&+\int_0^t\int_S K(\cdot\,;drdy)F(y;u_{t+s-r}(\cdot\,;f))\\
&+T^0_t\int_0^s\int_S K(\cdot\,;drdy)F(y;u_{s-r}(\cdot\,;f))
\end{aligned}$$

$$= T_t^0(T_s^0 f + \int_0^s \int_S K(\cdot\,; drdy) F(y; u_{s-r}(\cdot\,; f)))$$
$$+ \int_0^t \int_S K(\cdot\,; drdy) F(y; u_{t+s-r}(\cdot\,; f))$$
$$= T_t^0 u_s(\cdot\,; f) + \int_0^t \int_S K(\cdot\,; drdy) F(y; u_{t+s-r}(\cdot\,; f)).$$

This proves that $v_t = u_{t+s}(\cdot\,; f)$ is a solution of the S-equation with the initial value $u_s(\cdot\,; f) \in \boldsymbol{B}^*(S)^+$, and by the uniqueness of the solution we have (4.38). Then for $f \in \boldsymbol{C}^*(S)^+$ we have

$$\widetilde{\boldsymbol{T}}_{t+s}\widehat{f}(\boldsymbol{x}) = \widehat{u_{t+s}(\cdot\,; f)}(\boldsymbol{x}) = \widehat{u_t(\cdot\,; u_s(\cdot\,; f))}(\boldsymbol{x})$$
$$= \widetilde{\boldsymbol{T}}_t(\widehat{u_s(\cdot\,; f)})(\boldsymbol{x}) = \widetilde{\boldsymbol{T}}_t(\widetilde{\boldsymbol{T}}_s\widehat{f})(\boldsymbol{x}).$$

By Lemma 0.2, $\widetilde{\boldsymbol{T}}_{t+s} g(\boldsymbol{x}) = \widetilde{\boldsymbol{T}}_t(\widetilde{\boldsymbol{T}}_s g)(\boldsymbol{x})$ holds for all $\boldsymbol{C}_0(\boldsymbol{S})$ and hence for all $g \in \boldsymbol{B}(\boldsymbol{S})$. Thus $\widetilde{\boldsymbol{T}}_t$ is a semi-group on $\boldsymbol{B}(\boldsymbol{S})$, and by its definition it is a branching semi-group. In this way we have constructed a branching semi-group $\widetilde{\boldsymbol{T}}_t$ from a given fundamental system. We shall assume further that (T_t^0, K, π) is H-regular or weakly H-regular; then we have the following

Theorem 4.9. (i) *Suppose (T_t^0, K, π) is H-regular. Then $\widetilde{\boldsymbol{T}}_t$ is a strongly continuous semi-group on the smallest closed linear subspace $\boldsymbol{H}_0$ in $\boldsymbol{B}(\boldsymbol{S})$ containing $\{\widehat{f};\, f \in H_0 \cap \boldsymbol{B}^*(S)^+\}$. In particular if (T_t^0, K, π) is regular, then $\widetilde{\boldsymbol{T}}_t$ is a strongly continuous semi-group on $\boldsymbol{C}_0(\boldsymbol{S})$, and hence the corresponding branching Markov process is a Hunt process.*
(ii) *Suppose (T_t^0, K, π) is weakly H-regular. Then $\widetilde{\boldsymbol{T}}_t$ is weakly continuous on the smallest closed linear subspace $\widetilde{\boldsymbol{H}}_0$ in $\boldsymbol{B}(\boldsymbol{S})$ containing $\{\widehat{f};\, f \in \widetilde{H}_0 \cap B^*(\boldsymbol{S})\}$. Also, $\widetilde{\boldsymbol{T}}_t$ is strongly continuous on the smallest closed linear subspace containing $\{\widehat{f}: f \in H_0^{(T^0)} \cap \boldsymbol{B}^*(S)\}$.*[17]

Proof. Proof of (i) is almost immediate from Theorem 4.8: in fact if $f \in H_0 \cap \boldsymbol{B}^*(S)^+$, then

17) In the case of H-regular we have $H_0^{(T^0)} = H_0 (\equiv H_0^{(T)})$ but in the case of weakly H-regular they do not coincide in general.

$$u_t(\cdot\,;f)=\widetilde{\boldsymbol{T}}_t\widehat{f}\,|_S\in H_0\cap\boldsymbol{B}^*(S)^+ \quad\text{and}\quad \|u_t(\cdot\,;f)-f\|\to 0$$

when $t\to 0$. Then $\widetilde{\boldsymbol{T}}_t\widehat{f}\in H_0$ and

$$\|\widetilde{\boldsymbol{T}}_t\widehat{f}-\widehat{f}\|_S\le a_{A_0}\|u_t(\cdot\,;f)-f\|\to 0$$

when $t\to 0$. The first assertion of (ii) is proved similarly. As for the second assertion, we see from the Corollary of Theorem 4. 10 given below that if (T_t^0, K, π) is weakly H-regular, then $f\in D(\widetilde{A}_H)\cap\boldsymbol{B}^*(S)^+$ implies $u_t(\cdot\,;f)\in D(\widetilde{A}_H)\cap\boldsymbol{B}^*(S)^+\subset H_0^{(T^0)}\cap\boldsymbol{B}^*(S)$; therefore $\|u_t(\cdot\,;f)-f\|\to 0$. Then the proof is the same as in (i).

In §4. 2 we have constructed a branching semi-group $\boldsymbol{T}_t$ as the minimal solution of the M-equation and, it is the semi-group corresponding to the (X^0,π)-branching Markov process. We now claim that $\widetilde{\boldsymbol{T}}_t=\boldsymbol{T}_t$; i.e., the semi-group $\widetilde{\boldsymbol{T}}_t$ is the semi-group corresponding to the (X^0,π)-branching Markov process. This follows from Theorem 4. 4, Corollary or Theorem 4. 7 and Theorem 4. 5, Corollary. But in the case when (T_t^0, K, π) is regular, we can give the following direct proof independent of the arguments involving the M-equation. Thus we shall see that, at least in the case of a regular fundamental system, the construction of the (X^0,π)-branching Markov process given in this section is completely self-contained.

Suppose, therefore, (T_t^0, K, π) is regular; then branching Markov process $\boldsymbol{X}$ corresponding to the semi-group $\widetilde{\boldsymbol{T}}_t$ is a Hunt process,[18] and we shall show that $\boldsymbol{X}$ is the (X^0,π)-branching Markov process. By Theorem 4. 10 given below, if $f\in D(A^0)\cap\boldsymbol{B}^*(S)^+$, then

$$\left\|\frac{1}{t}(\widetilde{\boldsymbol{T}}_t\widehat{f}-\widehat{f})-\langle f\,|\,A^0f+kF(f)\rangle\right\|_S\to 0 \quad\text{when } t\to 0.$$

In particular we have

$$\left\|\frac{1}{t}(\widetilde{\boldsymbol{T}}_t\widehat{f}\,|_S-f)-A^0f-k\int_S\pi(\cdot\,,dy)\widehat{f}(y)\right\|\to 0 \quad\text{when } t\to 0.$$

18) Hence it satisfies the conditions (C. 1) and (C. 2), cf. §1. 2.

If we consider λf, $|\lambda|\leq 1$, then we see easily that[19)]

$$\sup_{x\in S}\left|\frac{1}{t}\int_{S^n}\widetilde{\boldsymbol{T}}_t(x, dy)\widehat{f}(y)-k(x)\int_{S^n}\pi(x, dy)\widehat{f}(y)\right|\to 0,$$
$$(n=0, 2, 3, \cdots)$$

and

$$\sup_{x\in S}\left|\frac{1}{t}\left\{\int_S\widetilde{\boldsymbol{T}}_t(x, dy)f(y)-f(x)\right\}-A^0f(x)\right|\to 0, \quad \text{when } t\to 0.$$

From the first formula we can conclude, as in Ikeda-Watanabe [18], that $\pi(x, dy)$ is the branching law of $\boldsymbol{X}$ and further

$$\boldsymbol{P}_x[\tau\leq t,\ \boldsymbol{X}_\tau\in E]=\int_0^t\int_S T_s^*(x, dy)k(y)\pi(y, E)ds,$$

where $T_s^*(x, dy)$ is the kernel of the semi-group of the non-branching part X^* of $\boldsymbol{X}$. From this we have $\sup_{x\in S}\boldsymbol{P}_x[\tau\leq t]=0(t)$. We shall now prove that X^* is epuivalent to X^0, i.e., $T_s^*\equiv T_s^0$. It is sufficient to show that

$$(*)\qquad \sup_{x\in S}\boldsymbol{E}_x[f(\boldsymbol{X}_t);\ t\geq\tau,\ \boldsymbol{X}_t\in S]=\mathrm{o}(t)\qquad (t\downarrow 0),$$

since then we have, for $f\in D(A^0)\cap\boldsymbol{B}^*(S)^+$,

$$\begin{aligned}&\sup_{x\in S}\left|\frac{1}{t}\left\{\int_S T_t^*(x, dy)f(y)-f(x)\right\}-A^0f(x)\right|\\ &\leq\sup_{x\in S}\left|\frac{1}{t}\left\{\int_S\widetilde{\boldsymbol{T}}_t(x, dy)f(y)-f(x)\right\}-A^0f(x)\right|\\ &\qquad+\frac{1}{t}\sup_{x\in S}\boldsymbol{E}_x[f(\boldsymbol{x}_t);\ t\geq\tau,\ \boldsymbol{X}_t\in S]\to 0.\end{aligned}$$

This proves that $D(A^0)\subset D(A^*)$ and $A^*f=A^0f$ on $D(A^0)$, and hence $T_t^0\equiv T_t^*$. But we have

$$\boldsymbol{E}_x[f(\boldsymbol{X}_t);\ t\geq\tau,\ \boldsymbol{X}_t\in S]=\boldsymbol{E}_x[\boldsymbol{E}_{\boldsymbol{X}_\tau}[f(\boldsymbol{X}_{t-u});\ \boldsymbol{X}_{t-u}\in S]\,|_{u=\tau};\ \tau\leq t]$$

and

$$\begin{aligned}\boldsymbol{E}_x[f(\boldsymbol{X}_r);\ \boldsymbol{X}_r\in S]&=\boldsymbol{T}_r\langle 0|f\rangle(\boldsymbol{x})=\lim_{\epsilon\to 0}\frac{1}{\epsilon}(\widehat{T_r\widehat{\epsilon f}-T_r0})(\boldsymbol{x})\\ &=\langle\boldsymbol{T}_r\widehat{0}|_S|\,\boldsymbol{T}_r\langle 0|f\rangle|_S\rangle(\boldsymbol{x}).\end{aligned}$$

---

19) Generally, if a sequence of a Banach space valued analytic functions $\{f_n(\lambda)\}$ is such that $\|f_n(\lambda)\|\to 0(|\lambda|\leq 1)$ when $n\to\infty$, then $\|f_n^{(\nu)}(0)\|\to 0$ $(n\to\infty)$ where $f_n^{(\nu)}$ is ν-th derivative.

Since $\sup_{x\in S} T_r\widehat{0}(x) = 0(r)$ we have

$$\sup_{x\in S-S_\cdot} \langle T_r\widehat{0}|_S | \boldsymbol{T}_r\langle 0|f\rangle|_S\rangle(\boldsymbol{x}) = 0(r).$$

Combining this with $\sup_{x\in S}\boldsymbol{P}_x[\tau\leq t] = 0(t)$ we have (*). Thus $\boldsymbol{X}$ is the (X^0, π)-branching Markov process.

§4. 5. Backward and forward equations

We shall discuss in this section the theory of the infinitesimal generator of a branching semi-group $\boldsymbol{T}_t$ corresponding to the (X^0, π)-branching Markov process. As in [6], the strong and the weak infinitesimal generators $\boldsymbol{A}$ and $\widetilde{\boldsymbol{A}}$ of $\boldsymbol{T}_t$ are defined by

$$\boldsymbol{A}f = s\text{-}\lim_{t\to 0}\frac{\boldsymbol{T}_t f - f}{t} \quad \text{and} \quad \widetilde{\boldsymbol{A}}f = w\text{-}\lim_{t\to 0}\frac{\boldsymbol{T}_t f - f}{t}$$

with domain of definitions

$$D(\boldsymbol{A}) = \left\{f : f\in \boldsymbol{B}(\boldsymbol{S}) \text{ such that } s\text{-}\lim_{t\to 0}\frac{\boldsymbol{T}_t f - f}{t} = \boldsymbol{A}f \text{ exists}\right\}$$

and

$$D(\widetilde{\boldsymbol{A}}) = \left\{f; f\in \boldsymbol{B}(\boldsymbol{S});\ w\text{-}\lim_{t\to 0}\frac{\boldsymbol{T}_t f - f}{t} = \widetilde{\boldsymbol{A}}f \text{ exists such that } w\text{-}\lim_{t\downarrow 0}\boldsymbol{T}_t(\widetilde{\boldsymbol{A}}f) = \widetilde{\boldsymbol{A}}f\right\}.$$

It seems difficult to discuss $\boldsymbol{A}$ or $\widetilde{\boldsymbol{A}}$ without some additional condition on the system (T_t^0, K, π) and so we shall assume it is H-regular or weakly H-regular for some closed linear subspace H satisfying the conditions (H. 1) and (H. 2) of §4. 3.

Lemma 4. 5. *Suppose (T_t^0, K, π) is H-regular (weakly H-regular) and let $v_t\in H$, $t\in[0,\infty)$ and $f\in \boldsymbol{B}^*(S)^+\cap H_0$* (resp. *$f\in \boldsymbol{B}^*(S)^+\cap \widetilde{H}_0$) such that $\|v_t - f\|\to 0$ when $t\to 0$. Then*

$$s\text{-}\lim_{t\downarrow 0}\frac{\int_0^t T_s^0(k\cdot F(\cdot\,;\, v_{t-s}))ds}{t} = kF(f) \tag{4. 39}$$

$$\left(\text{resp. } w\text{-}\lim_{t\downarrow 0}\frac{\int_0^t T_s^0(k\cdot F(\cdot\,;\, v_{t-s}))ds}{t} = kF(f)\right).$$

Proof. From the condition $\|v_t - f\| \to 0$ $(t \to 0)$ and $f \in \boldsymbol{B}^*(S)^+$ we may assume $\sup_{0 \le t \le t_0} \|v_t\| \leqq r < 1$ for some $t_0 > 0$. We shall put for $t \leqq t_0$

$$\frac{1}{t}\int_0^t T_s^0(k \cdot F(\cdot\,;\, v_{t-s}))ds - kF(\cdot\,;f) = I_1 + I_2$$

where

$$I_1 = \frac{1}{t}\int_0^t T_s^0\{kF(\cdot\,;\, v_{t-s}) - kF(\cdot\,;f)\}ds$$

and

$$I_2 = \frac{1}{t}\int_0^t T_s^0(kF(\cdot\,;f))ds - kF(\cdot\,;f).$$

By (4.34) we have

$$\begin{aligned}\|I_1\| &\leq \frac{1}{t}\int_0^t \|T_s^0\{k(F(\cdot\,;\, v_{t-s}) - F(\cdot\,;f))\}\|ds \\ &\leq \frac{1}{t}\|k\| a_r \int_0^t \|v_s - f\| ds \to 0 \quad (t \to 0).\end{aligned}$$

If (T_t^0, K, π) is H-regular (weakly H-regular) and $f \in H_0$ (resp. $f \in \widetilde{H}_0$), then $kF(\cdot\,;f) \in H_0$ (resp. $kF(\cdot\,;f) \in \widetilde{H}_0$) and hence

$$s\text{-}\lim_{t\to 0} T_t^0(kF(\cdot\,;f)) = kF(\cdot\,;f)$$
$$(\text{resp. } w\text{-}\lim_{t\to 0} T_t^0(kF(\cdot\,;f)) = kF(\cdot\,;f)).$$

Then we have clearly that

$$s\text{-}\lim_{t\to 0} I_2 = 0 \quad (\text{resp. } w\text{-}\lim_{t\to 0} I_2 = 0),$$

and the proof of the lemma is now complete.

Theorem 4.10. (i) *Suppose* (T_t^0, K, π) *is H-regular. If* $f \in D(A_H^0) \cap \boldsymbol{B}^*(S)^+ (= D(A_H) \cap \boldsymbol{B}^*(S)^+)$, *then* $\hat{f} \in D(\widetilde{\boldsymbol{A}})$ *and* $\widetilde{\boldsymbol{A}}\hat{f}$ *is given by*

$$\boldsymbol{A}\hat{f} = \langle f \,|\, c(f) \rangle, \tag{4.40}$$

where

$$\begin{aligned} c(f) &= A_H^0 f + kF(\cdot\,;f) \\ &= A_H f + k(F(\cdot\,;f) - f). \end{aligned} \tag{4.41}$$

Conversely, if $f\in H\cap \boldsymbol{B}^*(S)^+$ *is such that* $\widehat{f}\in D(\boldsymbol{A})$, *then* $f\in D(A_H^0)$ $(=D(A_H))$ *and hence* $\boldsymbol{A}\widehat{f}$ *is given by* (4.40).

(ii) *Suppose* (T_t^0, K, π) *is weakly H-regular. If* $f\in D(\widetilde{A}_H^0)\cap \boldsymbol{B}^*(S)^+$ $(=D(\widetilde{A}_H)\cap \boldsymbol{B}^*(S)^+)$, *then* $\widehat{f}\in D(\widetilde{\boldsymbol{A}})$ *and* $\widetilde{\boldsymbol{A}}\widehat{f}$ *is given by*

$$\widetilde{\boldsymbol{A}}\widehat{f}=\langle f\,|\,\tilde{c}(f)\rangle, \tag{4.42}$$

where

$$\tilde{c}(f)=\widetilde{A}_H^0 f+kF(\cdot\,;f)=\widetilde{A}_H f+k(F(\cdot\,;f)-f). \tag{4.43}$$

Conversely, if $f\in H\cap \boldsymbol{B}^*(S)^+$ *is such that* $\widehat{f}\in D(\widetilde{\boldsymbol{A}})$, *then* $f\in D(\widetilde{A}_H^0)$ $=D(\widetilde{A}_H)$ *and hence* $\widetilde{\boldsymbol{A}}\widehat{f}$ *is given by* (4.42).

Proof. We shall first prove (i). Suppose $f\in D(A_H^0)\cap \boldsymbol{B}^*(S)^+$ then by Theorem 4.8, $u_t(\cdot\,;f)=\boldsymbol{T}_t\widehat{f}\,|_S\in H_0$ and $\|u_t(\cdot\,;f)-f\|\to 0$ when $t\downarrow 0$. Now if $c(f)$ is defined by (4.41), we have

$$\Big(\frac{u_t(\cdot\,;f)-f}{t}-c(f)\Big)=\Big(\frac{T_t^0 f-f}{t}-A_H^0 f\Big)$$
$$+\Big(\frac{\int_0^t T_s^0(k\cdot F(\cdot\,;u_{t-s}))ds}{t}-kF(\cdot\,;f)\Big).$$

Clearly the first term converges strongly (i.e., in the norm) to zero when $t\downarrow 0$ and so does also the second term by Lemma 4.5. Thus $\Big\|\frac{1}{t}(u_t(\cdot\,;f)-f)-c(f)\Big\|\to 0$ when $t\to 0$. Then, if $t\leq\sigma$, we have by Lemma 0.1 (0.35)

$$\Big\|\frac{\boldsymbol{T}_t\widehat{f}-\widehat{f}}{t}-\langle f\,|\,c(f)\rangle\Big\|_S=\Big\|\frac{\widehat{u_t(\cdot\,;f)}-\widehat{f}}{t}-\langle f\,|\,c(f)\rangle\Big\|_S$$
$$\leq d_{A_\sigma}\Big\|\frac{1}{t}(u_t(\cdot\,;f)-f)-c(f)\Big\|+e_{A_\sigma}\|c(f)\|\,\|u_t(\cdot\,;f)-f\|^{[20]}$$
$$\to 0$$

when $t\to 0$ proving that $\widehat{f}\in D(\boldsymbol{A})$ and $\boldsymbol{A}\widehat{f}=\langle f\,|\,c(f)\rangle$.

Conversely let $f\in H\cap \boldsymbol{B}^*(S)^+$ be such that $\widehat{f}\in D(\boldsymbol{A})$. Then

$$\Big\|\frac{1}{t}(\boldsymbol{T}_t\widehat{f}-\widehat{f})-\boldsymbol{A}\widehat{f}\Big\|_S\to 0 \qquad (t\to 0)$$

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20) $A_\sigma=1-(1-\|f\|)e^{-\sigma\|k\|}<1$.

and *a fortiori*

$$\left\|\frac{1}{t}(u_t(\cdot\,;f)-f)-A\hat{f}|_S\right\|\to 0 \qquad (t\to 0);$$

that is,

$$\left\|\frac{T_t^0 f-f}{t}+\frac{1}{t}\int_0^t T_s^0(k\cdot F(\cdot\,;u_{t-s}))ds-A\hat{f}|_S\right\|\to 0 \quad (t\to 0). \tag{4.44}$$

From (4.44) we see in particular that $\left\|\frac{T_t^0 f-f}{t}\right\|$ is bounded in t and hence $\|T_t^0 f-f\|\to 0$. Therefore $f\in H_0$ and this implies, by Theorem 4.8, that $u_t(\cdot\,;f)\in H_0$ and $\|u_t(\cdot\,;f)-f\|\to 0$. Then by Lemma 4.5 $s\text{-}\lim_{t\downarrow 0}\frac{1}{t}\int_0^t T_s^0(k\cdot F(\cdot\,;u_{t-s}))ds=k\cdot F(\cdot\,;f)$. Combining this with (4.44) we see that $s\text{-}\lim\frac{T_t^0 f-f}{t}$ exists and is equal to $A\hat{f}|_S-k\cdot F(\cdot\,;f)$ which proves $f\in D(A_H^0)$.

The proof of (ii) is quite similar, and therefore it is omitted.

Corollary. *Suppose the fundamental system* (T_t^0, K, π) *is H-regular (weakly H-regular). If* $f\in D(A_H)\cap \boldsymbol{B}^*(S)^+$ (resp. $f\in D(\tilde{A}_H)\cap \boldsymbol{B}^*(S)^+$), *then* $u_t=u_t(\cdot\,;f)=\boldsymbol{T}_t\hat{f}|_S\in D(A_H)$ (resp. $u_t\in D(\tilde{A}_H)$) *for every* $t\in[0,\infty)$ *and* $\frac{du_t}{dt}$ *exists strongly* (resp. $\frac{d^+u_t}{dt}$ *exists weakly*);[21] *further, we have*

$$\frac{du_t}{dt}=A_H u_t+k(F(\cdot\,;u_t)-u_t) \tag{4.45}$$

$$\left(\text{resp. } \frac{d^+u_t}{dt}=\tilde{A}_H u_t+k(F(\cdot\,;u_t)-u_t)\right)$$

and

$$\|u_t-f\|\to 0, \qquad (t\to 0). \tag{4.46}$$

Proof. If $f\in D(A_H)\cap \boldsymbol{B}^*(S)$, then $\hat{f}\in D(\boldsymbol{A})$. Therefore, by the general theory of semi-groups we see that $\boldsymbol{T}_t\hat{f}(\boldsymbol{x})=\widehat{u_t(\cdot\,;f)}(\boldsymbol{x})\in D(\boldsymbol{A})$ and is strongly differentiable[22] in t and $\frac{d\boldsymbol{T}_t\hat{f}}{dt}=\boldsymbol{A}\boldsymbol{T}_t\hat{f}=\boldsymbol{T}_t\boldsymbol{A}\hat{f}$. Then

21) $\frac{d^+u_t}{dt}$ denotes the right hand derivative.

22) With respect to the Banach space $\boldsymbol{B}(\boldsymbol{S})$.

$u_t(\cdot\,; f)$ is strongly differentiable in t and $u_t \in D(A_H)$ by the second part of (i) of the previous theorem. By the same theorem we have (4. 45). The proof of the case of weakly H-regular is quite similar and hence it is omitted.

Definition 4. 9. The equation (4. 45) with the initial condition (4. 46) is called the *backward equation* corresponding to the system (T_t^0, K, π).

Thus the backward equation is a semi-linear evolution equation and the semi-group of the (X^0, π)-branching Markov process defines its solution.

Now we shall consider the equation

$$(4\cdot 47) \qquad \frac{\partial \boldsymbol{T}_t \widehat{f}}{\partial t} = \boldsymbol{T}_t \boldsymbol{A} \widehat{f} = \boldsymbol{T}_t \langle f \,|\, c(f) \rangle.$$

For simplicity, we shall assume that the fundamental system (T_t^0, K, π) is regular, though a similar argument can be carried over for H-regular or weakly H-regular fundamental systems. Then the branching semi-group $\boldsymbol{T}_t$ is a strongly continuous semi-group on $\boldsymbol{C}_0(\boldsymbol{S})$ such that if $f \in D(A) \cap \boldsymbol{C}^*(S)^+$,[23] then $\widehat{f} \in D(\boldsymbol{A})$ and

$$(4.\,48) \qquad \boldsymbol{A} \widehat{f} = \langle f \,|\, c(f) \rangle,$$

where $c(f)$ is given by $c(f) = Af + k(F(\cdot\,; f) - f)$. (4. 48) determines the semi-group uniquely: in fact we have the following

Theorem 4. 11. *Let (T_t^0, K, π) be a regular fundamental system. Let $\boldsymbol{U}_t$ be a non-negative contraction semi-group on $\boldsymbol{B}(\boldsymbol{S})$ such that if $f \in D(A) \cap \boldsymbol{C}^*(S)^+$, then $\widehat{f} \in D(\boldsymbol{A}_U)$[24] and*

$$(4.\,49) \qquad \boldsymbol{A}_U \widehat{f} = \langle f \,|\, c(f) \rangle,$$

where

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23) In the case of $H = \boldsymbol{C}(S)$ we write A_H simply as A.

24) $D(\boldsymbol{A}_U)$ is the domain of the strong infinitesimal generator $\boldsymbol{A}_U$ of $\boldsymbol{U}_t$;

$$D(\boldsymbol{A}_U) = \left\{ f \in \boldsymbol{B}(\boldsymbol{S});\ s\text{-}\lim \frac{\boldsymbol{U}_t f - f}{t} \equiv \boldsymbol{A}_U f \text{ exists} \right\}.$$

(4.50) $$c(f)=A^0f+k\cdot F(\cdot\,;f)=Af+k(F(\cdot\,;f)-f).$$

Then $U_t=T_t$, that is, U_t is the semi-group of (X^0,π)-branching Markov process.

Before proving the theorem we shall give the following remark. Let B be a Banach space and $\mathscr{D}$ be an open subset of B. A real valued function $\Phi(f)$ defined on $\mathscr{D}$ is said to be *G-differentiable*[25] *in* $\mathscr{D}$ if for every $f\in\mathscr{D}$ and $g\in B$

$$\lim_{\epsilon\downarrow 0}\frac{\Phi(f+\epsilon g)-\Phi(f)}{\epsilon}\equiv\delta\Phi(f;\,g)$$

exists. $\delta\Phi(f;\,g)$ is called the first variation with increment g of f. Now we take $C(S)$ as B and

(4.51) $$\mathscr{D}(S)=\{f\in C(S);\ 0<f<1\}$$

as $\mathscr{D}$. Given a bounded measure μ on S define $\Phi(f)$, $f\in\mathscr{D}$ by

$$\Phi(f)=\int_S\hat{f}(\boldsymbol{x})\mu(d\boldsymbol{x}).$$

Then by (1.49), $\Phi(f)$ is G-differentiable in $\mathscr{D}$ and

(4.52) $$\delta\Phi(f;\,g)=\int_S\langle f\,|\,g\rangle(\boldsymbol{x})\mu(d\boldsymbol{x}),\quad f\in\mathscr{D}(S),\ g\in C(S).$$

Remark 4.2. Such $\Phi(f)$ has all higher order derivatives and in fact it is an analytic function of $f\in\mathscr{D}(S)$ in the sense of [9]. One can develop the theory of branching semi-groups on the basis of analytic functions defined on $\mathscr{D}(S)$ instead of using the symmetric direct product spaces: for such an approach see Mullikin [36].

Now let U_t be a semi-group satisfying the condition of the theorem. If we set

(4.53) $$\Phi_{\boldsymbol{x},t}(f)=U_t\hat{f}(\boldsymbol{x}),\qquad f\in\mathscr{D}(S),$$

then for each $\boldsymbol{x}\in S$ we have that

(i) for fixed $f\in\mathscr{D}(S)$, it is continuous in t,[26]

25) Cf. Hille-Phillips [9] p. 71.

26) (i) is a consequence of (ii). Note that the linear hull of $\{f;\,\hat{f}\in D(A)\cap\mathscr{D}(S)\}$ is dense in $C_0(S)$.

(ii) for fixed $f \in D(A) \cap \mathcal{D}(S)$, it is continuously differentible in t, and

(iii) for fixed t, it is G-differentiable in $f \in \mathcal{D}(S)$.

By (4.47) and (4.52) we have for $f \in D(A) \cap \mathcal{D}(S)$

$$\frac{\partial \Phi_{x,t}}{\partial t}(f) = \delta\Phi_{x,t}(f;\, c(f)), \quad \Phi_{x,0+}(f) = \hat{f}(x).$$

Definition 4.10. For a given regular fundamental system (T_t^0, K, π) and a function $\Phi(f)$ defined on $\mathcal{D}(S)$,

$$(4.54) \qquad \begin{cases} \dfrac{\partial \Phi_t(f)}{\partial t} = \delta\Phi_t(f;\, c(f)), & f \in D(A) \cap \mathcal{D}(S) \\ \Phi_{0+}(f) = \Phi(f) \end{cases}$$

is called the *forward equation corresponding to the system* (T_t^0, K, π). A function $\Phi_t(f)$ of (t, f) defined on $[0, \infty) \times \mathcal{D}(S)$ is called a *solution of* (4.54) *with the initial value* $\Phi(f)$ if it satisfies the conditions (i), (ii), (iii) above and (4.54).

Example 4.3. In the simplest case when $S = \{a\}$ and if the fundamental system is given by c and $\{\pi_i\}_{i=0}^{\infty}$ (cf. Example 4.1),[27] then the forward equation (4.54) is given as

$$\frac{\partial \Phi_t(f)}{\partial t} = c(f) \frac{\partial \Phi_t(f)}{\partial f},$$

where $$c(f) = c \cdot \left(\sum_{j=0}^{\infty} \pi_j f^j - f \right).$$

If $\Phi_{i,t}(f) = \sum_{j=0}^{\infty} P_{ij}(t) f^j$, then the above equation is equivalent to

$$\frac{\partial P_{ij}(t)}{\partial t} = -jcP_{ij}(t) + c \sum_{k=1}^{j+1} P_{ik} k \cdot \pi_{j-k+1}.$$

This is just the classical Kolmogorov's forward differential equation for a Markov chain (X_t, P_i) such that $E_i(\tau) = \dfrac{1}{ic}$ and $P_i[x_\tau = j] = \pi_{j-i+1}$, where τ is the first jumping time.

Thus $\Phi_{x,t}(f) = U_t \hat{f}(x), f \in \mathcal{D}(S)$, defines a solution of the forward

27) Clearly it is a regular fundamental system.

equation (4.54) with the initial value $\Phi(f)\equiv\widehat{f}(x)$ for each fixed $x\in S$. Hence the theorem will be proved if we can prove the following

Theorem 4.11′. *Let* (T_t^0, K, π) *be a given regular fundamental system and* U_t *be a non-negative contraction semi-group on* $B(S)$ *such that for each* $x\in S$, $\Phi_{x,t}(f)\equiv U_t\widehat{f}(x)$, $f\in\mathcal{D}(S)$, *defines a solution of the forward equation* (4.54) *with the initial value* $\Phi(f)\equiv\widehat{f}(x)$. *Then* $U_t=T_t$.

Proof. Set $\Phi'_{x,t}(f)=T_t\widehat{f}(x)$, $f\in\mathcal{D}(S)$; then we know that for each fixed x, $\Phi'_{x,t}(f)$ is also a solution of (4.54) with the initial value $\Phi(f)=\widehat{f}(x)$. Since U_t is a contraction semi-group, we have by Lemma 0.1,

$$|\Phi_{x,t}(f)-\Phi_{x,t}(g)|=|U_t(\widehat{f}-\widehat{g})(x)|\leq\|\widehat{f}-\widehat{g}\|_S\leq a_r\|f-g\|,$$
$$f,g\in\mathcal{D}(S)\cap C_r^*(S),$$

and noting (4.52) we have, provided $f,g\in\mathcal{D}(S)\cap C_r^*(S)$,

$$\begin{aligned}&|\delta\Phi_{x,t}(f;\,c(f))-\delta\Phi_{x,t}(g;\,c(g))|\\&=|U_t(\langle f|c(f)\rangle-\langle g|c(g)\rangle)|\\&\leq\|\langle f|c(f)\rangle-\langle g|c(g)\rangle\|_S\\&\leq b_r\|c(f)\|\|f-g\|+c_r\|c(f)-c(g)\|.\end{aligned}$$

Clearly we have similar results for $\Phi'_{x,t}$. Hence if we set $\Phi_t(f)=\Phi_{x,t}(f)-\Phi'_{x,t}(f)$, $f\in\mathcal{D}(S)$, then $\Phi_t(f)$ is a solution of (4.54) with the initial value $\Phi(f)\equiv 0$ such that for every $r<1$

$$|\Phi_t(f)-\Phi_t(g)|\leq\alpha_r\|f-g\| \tag{4.55}$$

and

$$\begin{aligned}&|\delta\Phi_t(f;\,c(f))-\delta\Phi_t(g;\,c(g))|\\&\leq\beta_r\|c(f)\|\|f-g\|+\gamma_r\|c(f)-c(g)\|\end{aligned} \tag{4.56}$$

for every f and g in $\mathcal{D}(S)\cap C_r^*(S)$, where α_r, β_r and γ_r are constants depending on r. By the following lemma we have $\Phi_t(f)\equiv 0$ and hence $U_t\widehat{f}(x)=\Phi_{x,t}(f)=\Phi'_{x,t}(f)=T_t\widehat{f}(x)$ for every $f\in\mathcal{D}(S)$.

Since the linear hull of $\{\hat{f}; f\in\mathcal{D}(S)\}$ is dense in $\boldsymbol{C}_0(S)$ we have $\boldsymbol{U}_t=\boldsymbol{T}_t$ on $\boldsymbol{C}_0(S)$ and hence on $\boldsymbol{B}(S)$.

Lemma 4.6. *Let $\Phi_t(f)$ be a solution of the forward equation* (4.54) *with the initial value $\Phi(f)\equiv 0$ satisfying* (4.55) *and* (4.56) *for every $r<1$. Then $\Phi_t(f)=0$ for every $t\geq 0$ and $f\in\mathcal{D}(S)$.*

Proof. Since $D(A)\cap\mathcal{D}(S)$ is dense in $\mathcal{D}(S)$ and $\Phi_t(f)$ is continuous in $f\in\mathcal{D}(S)$ by (4.55), it is sufficient to show $\Phi_t(f)\equiv 0$ for every $f\in D(A)\cap\mathcal{D}(S)$. So assume $f\in D(A)\cap\mathcal{D}(S)$ and let $u_t\equiv u_t(\cdot\,; f)$ be the solution of S-equation with the initial value f; then we know that $u_t\in D(A)\cap\mathcal{D}(S)$ by Cor. of Theorem 4.10 and $\sup_{0\leq t\leq\sigma}\|u_t\|\leq A_\sigma<1$ for every $\sigma<0$. We shall now prove that $\dfrac{d\psi_\sigma(t)}{dt}\equiv 0$ in $t\in(0,\sigma)$, where we set $\psi_\sigma(t)=\Phi_t(u_{\sigma-t})$, $t\in[0,\sigma]$, for each fixed $\sigma>0$. If this is proved, then $\psi_\sigma(t)$ is constant in t, and hence $\psi_\sigma(\sigma)=\Phi_\sigma(f)=\psi_\sigma(0)=\Phi_0(u_\sigma)=0$ for every $\sigma>0$; therefore, the lemma will be proved.

Now

$$\frac{1}{h}[\psi_\sigma(t+h)-\psi_\sigma]=\frac{1}{h}[\Phi_{t+h}(u_{\sigma-t-h})-\Phi_t(u_{\sigma-t})]$$
$$=\frac{1}{h}[\Phi_{t+h}(u_{\sigma-t-h})-\Phi_t(u_{\sigma-t-h})]+\frac{1}{h}[\Phi_t(u_{\sigma-t-h})-\Phi_t(u_{\sigma-t})]$$
$$=I_1+I_2\,,$$

where we set

$$I_1=\frac{1}{h}[\Phi_{t+h}(u_{\sigma-t-h})-\Phi_t(u_{\sigma-t-h})]$$

and

$$I_2=\frac{1}{h}[\Phi_t(u_{\sigma-t-h})-\Phi_t(u_{\sigma-t})].$$

Set $\Theta_t(f)=\dfrac{\partial\Phi_t}{\partial t}(f)\Big(=\delta\Phi_t(f;\,c(f))\Big)$; then $I_1=\Theta_{t+\theta h}(u_{\sigma-t-h})$ for some $\theta=\theta(h)$ such that $0<\theta<1$, and hence

$$|I_1-\Theta_t(u_{\sigma-t})|=|\Theta_{t+\theta h}(u_{\sigma-t-h})-\Theta_t(u_{\sigma-t})| \tag{4.57}$$
$$\leq|\Theta_{t+\theta h}(u_{\sigma-t-h})-\Theta_{t+\theta h}(u_{\sigma-t})|+|\Theta_{t+\theta h}(u_{\sigma-t})-\Theta_t(u_{\sigma-t})|.$$

Since Θ_s is continuous in s by the condition (ii) of a solution, the second term tends to zero when $h\to 0$. The first term is equal to

$$|\delta\Phi_{t+\theta h}(u_{\sigma-t-h};\ c(u_{\sigma-t-h}))-\delta\Phi_{t+\theta h}(u_{\sigma-t};\ c(u_{\sigma-t}))|,$$

and by (4. 56) this is majorized by

$$K_1\|u_{\sigma-t-h}-u_{\sigma-t}\|+K_2\|c(u_{\sigma-t-h})-c(u_{\sigma-t})\|.$$

The first term tends to zero when $h\to 0$. By Theorem 4. 11 and its corollary, $c(u_{\sigma-t-h})=(A\hat{u}_{\sigma-t+h})|_S=(T_{\sigma-t-h}A\hat{f})|_S$ and similarly $c(u_{\sigma-t})=(T_{\sigma-t}A\hat{f})|_S$; therefore, the second term is majorized by $\|(T_{\sigma-t-h}-T_{\sigma-t})A\hat{f}\|$ which tends to zero when $h\to 0$. This proves $|I_1-\Theta(u_{\sigma-t})|\to 0$ when $h\to 0$.

Next consider I_2; setting $g=c(u_{\sigma-t})$,

$$\begin{aligned}I_2&=\frac{1}{h}\{\Phi_t(u_{\sigma-t-h})-\Phi(u_{\sigma-t})\}\\&=\frac{1}{h}\{\Phi_t(u_{\sigma-t-h})-\Phi_t(u_{\sigma-t}-h\cdot g)\}\\&\quad+\frac{1}{h}\{\Phi_t(u_{\sigma-t}-h\cdot g)-\Phi_t(u_{\sigma-t})\}\end{aligned}$$

and the second term tends to $-\delta\Phi_t(u_{\sigma-t};\ g)$ when $h\to 0$ by the definition of the functional derivative δ. By (4. 55) the first term is majorized by

$$\frac{K}{h}\|u_{\sigma-t-h}-u_{\sigma-t}+h\cdot g\|,$$

and this tends to zero since

$$\left\|\frac{u_{\sigma-t-h}-u_{\sigma-t}}{h}+g\right\|=\left\|\frac{1}{h}\{T_{\sigma-t}\hat{f}|_S-T_{\sigma-t-h}\hat{f}|_S\}-(AT_{\sigma-t}\hat{f})|_S\right\|$$
$$\to 0\quad(h\to 0).$$

Thus $I_2\to-\delta\Phi(u_{\sigma-t};\ g)$ and hence

$$I_1+I_2\to\Theta_t(u_{\sigma-t})-\delta\Phi_t(u_{\sigma-t};\ g)=\left(\frac{\partial}{\partial t}\Phi-\delta\Phi_t\right)=0.$$

This proves $\dfrac{d\psi_\sigma(t)}{dt}=0.$

Finally we shall give a direct proof that the semi-group $\boldsymbol{T}_t$ constructed in Theorem 4. 4 as the minimal solution of the M-equation satisfies the forward equation. This will give us a new proof of the branching property of $\boldsymbol{T}_t$ at least in the case when (T_t^0, K, π) is regular. This point can be seen more clearly in the following way: if $\Phi_{x,t}(f)=\boldsymbol{T}_t\widehat{f}(\boldsymbol{x})$, $f\in\mathcal{D}(S)$, defines a solution of the forward equation with the initial value $\Phi(f)=\widehat{f}(\boldsymbol{x})$ for each $\boldsymbol{x}\in\boldsymbol{S}$, then $\Phi'_{x,t}(f)=\widehat{\boldsymbol{T}_t\widehat{f})|_S}(\boldsymbol{x})$ defines also a solution of the same equation with the same initial value. Hence by Lemma 4. 5 we have

$$\Phi'_{x,t}-\Phi_{x,t}\equiv 0,\quad \text{i.e.,}\quad \boldsymbol{T}_t\widehat{f}(\boldsymbol{x})=(\widehat{\boldsymbol{T}_t\widehat{f})|_S}(\boldsymbol{x}).$$

This proves $\boldsymbol{T}_t$ has the branching property.[28]

Now, $\boldsymbol{T}_t$ was constructed as

$$\boldsymbol{T}_t f=\sum_{n=0}^{\infty}\boldsymbol{T}_t^{(n)}f,\qquad f\in\boldsymbol{B}(\boldsymbol{S}),$$

where $\boldsymbol{T}_t^{(n)}$, $n=0,1,2,\cdots$ were defined by (4. 16). Let $\mu(\boldsymbol{x}, d\boldsymbol{y})$ be a kernel on $\boldsymbol{S}\times\boldsymbol{S}$ defined by

$$\int_{\boldsymbol{S}}\mu(\boldsymbol{x}, d\boldsymbol{y})\widehat{f}(\boldsymbol{y})=\langle f\,|\,k\cdot F(\cdot\,;f)\rangle(\boldsymbol{x}). \tag{4.58}$$

Such a kernel exists and uniquely determined by Lemma 0. 3. Set

$$\phi(t, \boldsymbol{x}, d\boldsymbol{y})=\int_{\boldsymbol{S}}\boldsymbol{T}_t^0(\boldsymbol{x}, d\boldsymbol{z})\mu(\boldsymbol{z}, d\boldsymbol{y}); \tag{4.59}$$

then the kernel $\psi(\boldsymbol{x}; ds\,d\boldsymbol{y})$ defined by (4. 9) is given by

$$\psi(\boldsymbol{x}; ds\,d\boldsymbol{y})=\phi(s, \boldsymbol{x}, d\boldsymbol{y})ds.$$

Now set

$$\phi^*(t, \boldsymbol{x}, d\boldsymbol{y})=\int_{\boldsymbol{S}}\mu(\boldsymbol{x}, d\boldsymbol{z})\,T_t^0(\boldsymbol{z}, d\boldsymbol{y}); \tag{4.60}$$

then clearly

28) This argument is similar to that given in Harris [8] to prove that a minimal Markov chain such that $P_i(X_T=j)=\pi_{j-i+1}$ and $E_i(\tau)=\frac{1}{c_i}$ (cf. Example 2) is a branching process.

$$(4.61)\qquad \int_S \phi(s, x, dz)\, T^0_{t-s}(z, dy) = \int_S T^0_s(x, dz)\phi^*(t-s, z, dy).$$

Rewriting (4.16) by ϕ and ϕ^*, we have

$$T_t^{(1)}(x, dy) = \int_0^t \int_S \phi(s, x, dz)\, T^0_{t-s}(z, dy) ds = \int_0^t \int_S T^0_s(x, dz)\phi^*(t-s, z, dy) ds,$$

$$T_t^{(2)}(x, dy) = \int_0^t \int_S \phi^{(2)}(s, x, dz)\, T^{(0)}_{t-s}(z, dy) ds^{29)}$$
$$= \int_0^t \int_S T^0_s(x, dz)\phi^{*(2)}(t-s, z, dy) ds$$
$$= \int_0^t \int_S T^{(1)}_s(x, dz)\phi^*(t-s, z, dy) ds,$$
$$\vdots$$
$$T_t^{(n)}(x, dy) = \int_0^t \int_S T^{(n-1)}_s(x, dz)\phi^*(t-s, z, dy) ds$$

and hence

$$T_t f(x) = \sum_{n=0}^{\infty} T_t^{(n)} f(x) = T^0_t f(x) + \int_0^t ds\, T_s\Big(\int_S \phi^*(t-s, \cdot, dy) f(y)\Big)(x)$$

for every $f \in B(S)$. In particular for $\widehat{f}(x)$, $f \in B^*(S)^+$, we have by (4.60) and (4.58)

$$(4.62)\qquad T_t \widehat{f}(x) = \widehat{T^0_t f}(x) + \int_0^t ds\, T_s(\langle T^0_{t-s} f \mid kF(\cdot\,;\ T^0_{t-s} f)\rangle)(x).$$

Therefore, if $f \in C^*(S)^+ \cap D(A)$,

$$\left\| \frac{T_t \widehat{f} - \widehat{f}}{t} - \langle f \mid A^0 f + k\cdot F(\cdot\,; f)\rangle \right\|$$
$$\leq \left\| \frac{\widehat{T^0_t f} - \widehat{f}}{t} - \langle f \mid A^0 f\rangle \right\|$$
$$+ \left\| \frac{1}{t}\int_0^t ds \{T_s(\langle T^0_{t-s} f \mid k\cdot F(\cdot\,;\ T^0_{t-s} f)\rangle) - \langle f \mid kF(\cdot\,; f)\rangle\} \right\|$$
$$= \|I_1\| + \|I_2\|$$

29) $\phi^{(2)}(t, x, dy) = \int_S \int_0^t \phi(t-s, x, dz)\phi(s, z, dy) ds$. $\phi^{*(2)}$ is defined similarly.

and by Lemma 0.1 (0.35),

$$\|I_1\|\leq d\left\|\frac{1}{t}(T_t^0 f-f)-A^0 f\right\|+e\|A^0 f\|\cdot\|T_t^0 f-f\|\to 0$$

when $t\to 0$. Also by (0.34) and (4.34)

$$\begin{aligned}\|I_2\|&\leq\frac{1}{t}\int_0^t ds\|\langle T_{t-s}^0 f\,|\,kF(\cdot\,;\,T_{t-s}^0 f)\rangle-\langle f\,|\,kF(\cdot\,;f)\rangle\|\\&\leq\frac{1}{t}\int_0^t ds\|\langle T_{t-s}^0 f\,|\,k\cdot F(\cdot\,;\,T_{t-s}^0 f)\rangle-\langle T_{t-s}^0 f\,|\,k\cdot F(\cdot\,;f)\rangle\|\\&\quad+\frac{1}{t}\int_0^t ds\|\langle T_{t-s}^0 f\,|\,k\cdot F(\cdot\,;f)\rangle-\langle f\,|\,kK(\cdot\,;f)\rangle\|\\&\leq\frac{K}{t}\int_0^t ds\|k\cdot F(\cdot\,;\,T_{t-s}^0 f)-k\cdot F(\cdot\,;f)\|\\&\quad+\frac{K'}{t}\int_0^t\|T_{t-s}^0 f-f\|ds\\&\leq\frac{K''}{t}\int_0^t\|T_{t-s}^0 f-f\|ds\end{aligned}$$

for some constants K, K' and K'' and $t\in[0,\sigma]$ if σ is sufficiently small. Hence $\|I_2\|\to 0$ where $t\to 0$. Hence $\widehat{f}\in D(\boldsymbol{A})$ and $\boldsymbol{A}\widehat{f}=\langle f\,|\,A^0 f+k\cdot F(\cdot\,;f)\rangle$. This implies, as we have seen above, that $\Phi_{t,\boldsymbol{x}}(f)=\boldsymbol{T}_t\widehat{f}(\boldsymbol{x})$, $f\in\mathcal{D}(S)$, satisfies the forward equation.

§4.6. Number of particles and related equations

Let $\boldsymbol{X}=(\boldsymbol{X}_t, \boldsymbol{P}_{\boldsymbol{x}})$ be a branching Markov process; we assume

(4.63) $\qquad \boldsymbol{P}_{\boldsymbol{x}}[e_\Delta=+\infty]=1 \qquad$ for every $\boldsymbol{x}\in\boldsymbol{S}$.

This is equivalent to the following weaker condition:

(4.64) $\qquad \boldsymbol{P}_x[e_\Delta=+\infty]=1 \qquad$ for every $x\in S$

since, if $\boldsymbol{x}=[x_1, x_2, \cdots, x_n]$,

$$\boldsymbol{P}_x[e_\Delta=+\infty]=\lim_{t\to\infty}\boldsymbol{T}_t\widehat{1}(\boldsymbol{x})=\lim_{t\to\infty}\prod_{i=1}^n\boldsymbol{T}_t\widehat{1}(x_i)=\prod_{i=1}^n\boldsymbol{P}_{x_i}[e_\Delta=+\infty].$$

The mapping $f\in\mathfrak{B}(S)\to\check{f}\in\mathfrak{B}(\boldsymbol{S})$ is defined by (0.32);

$$\check{f}(\boldsymbol{x})=\begin{cases}0, & \text{if } \boldsymbol{x}=\partial \\ f(x_1)+f(x_2)+\cdots+f(x_n), & \text{if } \boldsymbol{x}=[x_1, x_2, \cdots, x_n]\in S^n, \\ & \quad n=1, 2, \cdots. \end{cases}$$

We shall sometimes write $(f)^{\vee}$ instead of $\check{f}$. It is clear for $f\in \boldsymbol{B}(S)^+$ and $0\leq\lambda\leq 1$ that if g is defined by $g(x)=\lambda^{f(x)}$, then

(4.65) $$\hat{g}(\boldsymbol{x})=\lambda^{\check{f}(\boldsymbol{x})}, \qquad \boldsymbol{x}\in \boldsymbol{S}.$$

The operation "$\vee$" is linear:

(4.66) $$(f_1+f_2)^{\vee}=\check{f}_1+\check{f}_2 .$$

In this section we shall discuss $\xi_t^f(\omega)$ defined by

(4.67) $$\xi_t^f(\omega)=\check{f}(\boldsymbol{X}_t).$$

If I_D is the indicator function of a set $D\in\mathscr{B}(S)$

(4.68) $$\xi_t^D(\omega)\equiv\xi_t^{I_D}(\omega)=\check{I}_D(\boldsymbol{X}_t)$$

stands for the number of particles in the set D.

Lemma 4.7. *For $f\in \boldsymbol{B}(S)^+$ and $h\in\overline{\boldsymbol{B}^*(S)^+}$ we have, for $\boldsymbol{x}=[x_1, x_2, \cdots, x_n]\in S^n$,*

(4.69) $$\boldsymbol{T}_t(\widehat{h}(\check{f})^k)(\boldsymbol{x})=\sum_{(k_1,k_2,\cdots,k_n)}^{(k)}\frac{k!}{k_1!\,k_2!\cdots k_n!}\prod_{j=1}^{n}\boldsymbol{T}_t(\widehat{h}(\check{f})^{k_j})(x_j).$$ [30]

Proof. We assume first $h\in \boldsymbol{B}^*(S)^+$; then there exists some $\lambda_0>0$ such that, if $|\lambda|\leq\lambda_0$, $\|e^{\lambda f}\|\ \|h\|<1$. Therefore, setting $g=e^{\lambda f}$, we have for $\boldsymbol{x}=[x_1, x_2, \cdots, x_n]$ and $|\lambda|\leq\lambda_0$

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\boldsymbol{T}_t(\widehat{h}(\check{f})^k)(\boldsymbol{x})\\ &=\boldsymbol{T}_t(\widehat{h\cdot g})(\boldsymbol{x})\\ &=\prod_{j=1}^{n}\boldsymbol{T}_t(\widehat{h\cdot g})(x_j)\\ &=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\frac{\lambda^{k_1+k_2+\cdots+k_n}}{k_1!\,k_2!\cdots k_n!}\prod_{j=1}^{n}\boldsymbol{T}_t(\widehat{h}(\check{f})^{k_j})(x_j).\end{aligned}$$

30) This equality is true including the case $+\infty=+\infty$. $\sum\limits_{(k_1,k_2,\cdots,k_n)}^{(k)}$ denotes the sum over all $(k_1, k_2, \cdots, k_n)$ such that $k_i\geq 0$ and $k_1+k_2+\cdots+k_n=k$.

Comparing the coefficients of λ^k we have (4.69). When $h\in\overline{\boldsymbol{B}^*(S)^+}$, we have (4.96) by the monotone convergence theorem, taking $h_n\in\boldsymbol{B}^*(S)^+$ such that $h_n\uparrow h$.

Corollary. *For* $f\in\boldsymbol{B}(S)^+$,

(4.70) $$\boldsymbol{T}_t\check{f}(\boldsymbol{x})=(\boldsymbol{T}_t\check{f}|_S)^{\vee}(\boldsymbol{x}).$$

Now set for $f\in\boldsymbol{B}(S)^+$

(4.71) $$M_tf(x)=\boldsymbol{T}_t\check{f}|_S(x)=\boldsymbol{E}_x[\check{f}(\boldsymbol{X}_t)].$$

Let

$$\boldsymbol{B}^1=\{f\in\boldsymbol{B}(S),\ M_t|f|\in\boldsymbol{B}(S)\ \text{for every}\ t>0\}.$$

It is clear that if f belongs to $\boldsymbol{B}^1$ then both $f^+=f\vee 0$ and $f^-=(-f)\vee 0$ belong to $\boldsymbol{B}^1$. We define $M_tf(x)$ for $f\in\boldsymbol{B}^1$ by

(4.72) $$M_tf(x)=M_tf^+(x)-M_tf^-(x).$$

If we define a kernel $M_t(x,dy)$ on $S\times S$ by

(4.73) $$M_t(x,E)=M_tI_E(x),\ x\in S,\ E\in\mathcal{B}(S),$$

then we have clearly

(4.74) $$M_tf(x)=\int_S f(y)M_t(x,dy),\ f\in\boldsymbol{B}^1.$$

By (4.70) we have

$$\begin{aligned}M_{t+s}f(x)&=\boldsymbol{T}_{t+s}\check{f}(x)=\boldsymbol{T}_t(\boldsymbol{T}_s\check{f})(x)=\boldsymbol{T}_t(\boldsymbol{T}_s\check{f}|_S)^{\vee}(x)\\&=\boldsymbol{T}_t(M_sf)^{\vee}(x)=M_t(M_sf)(x).\end{aligned}$$

Thus we have the following

Theorem 4.12. $\int M_tf(x)=\int M_t(x,dy)f(y)=\boldsymbol{E}_x[\check{f}(\boldsymbol{X}_t)]$, $x\in S$, $f\in\boldsymbol{B}^1$, *defines a non-negative semi-group on* $\boldsymbol{B}^1$.

Definition 4.10. The non-negative semi-group M_t is called the *expectation semi-group of the process* $\boldsymbol{X}_t$.

From now on we assume $\boldsymbol{X}$ is an (X^0,π)-branching Markov process and let (T_t^0,K,π) be the fundamental system of $\boldsymbol{X}$.

Lemma 4.8. *Let* $h \in \overline{\boldsymbol{B}^*(S)^+}$ *and* $f \in \boldsymbol{B}(S)^+$; *then for each* $k=0, 1, 2, \cdots$, *we have*

$$(4.75)\qquad \boldsymbol{T}_t(\widehat{h}(\check{f})^k)|_S(x) = T_t^0(h \cdot f^k)(x) + \int_0^t\int_S K(x;\, dsdy)$$
$$\cdot \sum_{n=0}^{\infty} \sum_{(k_1, k_2, \cdots, k_n)}^{(k)} \frac{k!}{k_1!\, k_2! \cdots k_n!} \int_{S^n} \pi(y,\, d\boldsymbol{z}) \prod_{j=1}^{n} \boldsymbol{T}_{t-s}(\widehat{h}(\check{f})^{k_j})|_S(z_j).^{31)}$$

Proof. We assume first that $h \in \boldsymbol{B}^*(S)^+$. Then there exists some $\lambda_0 > 0$ such that, if $|\lambda| \leq \lambda_0$, $\|e^{\lambda f}\| \|h\| < 1$. We know that $v(t, x) = \boldsymbol{T}_t(\widehat{h} \cdot \widehat{e^{\lambda f}})|_S(x)$ satisfies the S-equation:

$$(4.76)\qquad v(t, x) = T_t^0(h \cdot e^{\lambda f}) + \int_0^t\int_S K(x;\, dsdy) \int_S \pi(y,\, d\boldsymbol{z}) \widehat{v}(t-s, \cdot)(\boldsymbol{z}).$$

Since $v(t, x) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \boldsymbol{T}_t(\widehat{h}(\check{f})^k)|_S(x)$, $|\lambda| < \lambda_0$, we have

$$(4.77)\qquad \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \boldsymbol{T}_t(\widehat{h}(\check{f})^k)|_S(x)$$
$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \{T_t^0(h \cdot f^k)(x) + \int_0^t\int_S K(x;\, dsdy) \sum_{n=0}^{\infty} \int_{S^n} \pi(y,\, d\boldsymbol{z})$$
$$\cdot \sum_{(k_1, k_2, \cdots k_n)}^{(k)} \frac{k!}{k_1!\, k_2! \cdots k_n!} \prod_{j=1}^{n} \boldsymbol{T}_{t-s}(\widehat{h}(\check{f})^{k_j}|_S(z_j),$$
$$(\boldsymbol{z} = [z_1, z_2, \cdots, z_n]).$$

Comparing the coefficients of λ^k we have (4.75). When $h \in \overline{\boldsymbol{B}^*(S)^+}$, taking $h_n \in \boldsymbol{B}(S)^+$ such that $h_n \uparrow h$, we have (4.75) by the monotone convergence theorem.

If $h \equiv 1$ and $k=1$, we have from (4.75)

$$(4.78)\qquad \boldsymbol{T}_t(\check{f})|_S(x) = T_t^0 f(x) + \int_0^t\int_S K(x;\, dsdy) \sum_{n=0}^{\infty} \int_{S^n} \pi(y,\, d\boldsymbol{z})$$
$$\cdot \sum_{i=1}^{n} \boldsymbol{T}_{t-s}(\check{f})|_S(z_j)$$
$$= T_t^0 f(x) + \int_0^t\int_S K(x;\, dsdy) \int_S \pi(y,\, d\boldsymbol{z}) [(T_{t-s}\check{f}|_S)^{\vee}(\boldsymbol{z}).$$

Theorem 4.13. $u(t, x) = M_t f(x)$, $f \in \boldsymbol{B}^+$ *satisfies the following* (linear) *integral equation*

31) $\boldsymbol{z} = [z_1, z_2, \cdots, z_n]$.

$$(4.79) \qquad u(t,x)=T_t^0f(x)+\int_0^t\int_S K(x;\,dsdy)G(y;\,u(t-s,;\,\cdot)),$$

where

$$(4.80) \qquad G(y;\,g)=\int_S \pi(x,dy)\check{g}(y).$$

Further, $u(t,x)=M_tf(x)$ *defines the smallest solution among all non-negative solutions of* (4.79).

Proof. (4.79) follows from (4.78). To prove the second assertion, we need the following

Lemma 4.9. *If* $\boldsymbol{x}=[x_1,x_2,\cdots,x_n]\in S^n$,

$$(4.81) \quad \boldsymbol{T}_t^{(0)}\check{f}(\boldsymbol{x})=\langle T_t^0 1\,|\,T_t^0 f\rangle(\boldsymbol{x})\equiv\sum_{i=1}^n\{\prod_{j\neq i}T_t^0 1(x_j)\}\,T_t^0 f(x_i),$$

$$(4.82) \quad \int_0^s\int_S \psi(\boldsymbol{x};\,dsdy)\hat{g}(s,\cdot)(y)$$

$$=\sum_{i=1}^n\int_0^t T_s^0 g(s,\cdot)(x_i)\,[-d_s(\prod_{j\neq i}T_s^0 1(x_j))]$$

$$+\sum_{i=1}^n\int_0^t\{\prod_{j\neq i}T_s^0 1(x_j)\}\int_S K(x_i;\,dsds)G(y;\,g(s,\cdot)),$$

for every $f\in\mathfrak{B}(S)^+$ *and* $g\in\mathfrak{B}([0,\infty)\times S)^+$.[32]

Proof. Let $h=e^{-\lambda f}$; then (4.81) is obtained from $\boldsymbol{T}_t^{(0)}\hat{h}(\boldsymbol{x})=\widehat{T_t^0 h}(\boldsymbol{x})$ by differentiating with respect to λ and then putting $\lambda=0$. (4.82) can be proved in a similar way.

Now let $v_t\equiv v(t,x)\,(0\leq v\leq+\infty)$ be a solution of (4.79). Then, for $\boldsymbol{x}=[x_1,x_2,\cdots,x_n]\in S^n$,

$$\begin{aligned}\check{v}_t(\boldsymbol{x})&=\sum_{i=1}^n v(t,x_i)\\ &=\sum_{i=1}^n T_t^0 f(x_i)+\sum_{i=1}^n\int\int_0^t K(x_i;\,dsdy)G(y;\,v_{t-s})\\ &=\sum_{i=1}^n\{\prod_{j\neq i}T_t^0 1(x_j)\}\,T_t^0 f(x_i)+\sum_{i=1}^n(1-\prod_{j\neq i}T_t^0 1(x_j))\,T_t^0 f(x_i)\\ &\quad+\sum_{i=1}^n\int_0^t\prod_{j\neq i}T_s^0 1(x_j)\int_S K(x_i;\,dsdy)G(y;\,v_{t-s})\\ &\quad+\sum_{i=1}^n\int_0^t(1-\prod_{j\neq i}T_s^0 1(x_j))\int_S K(x_i;\,dsdy)G(y;\,v_{t-s})\\ &\equiv I_1+I_2+I_3+I_4,\ \text{say};\end{aligned}$$

32) As for the definitions of $\boldsymbol{T}_t$ and ψ, see §4.1.

then

$$I_4=\sum_{i=1}^{n}\int_0^t\int_0^s d_r\Big(-\prod_{j\neq i}T_r^0 1(x_j)\Big)\int_S K(x_i;\ dsdy)G(y;\ v_{t-s})$$
$$=\sum_{i=1}^{n}\int_0^s\Big[\int_r^t\int_S K(x_i;\ dsdy)G(y;\ v_{t-s})\Big]d_r\Big(-\prod_{j\neq i}T_r^0 1(x_j)\Big),$$

and hence

$$I_2+I_4=\sum_{i=1}^{n}\int_0^t\Big[T_t^0 f(x_i)+\int_r^t\int_S K(x_i;\ dsdy)G(y;\ v_{t-s})\Big]d_r\Big(-\prod_{j\neq i}T_r^0 1(x_j)\Big)$$
$$=\sum_{i=1}^{n}\int_0^t[T_r^0 v_{t-r}(x_i)]\,d_r\Big(-\prod_{j\neq i}T_r^0 1(x_j)\Big).$$

where we used (4. 4) to single out T_0^r.

Therefore by (4. 82)

$$I_2+I_4+I_3=\int_0^t\int_S \psi(\boldsymbol{x};\ ds d\boldsymbol{y})\check{v}_{t-s}(\boldsymbol{y}).$$

By (4. 81)

$$I_1=\boldsymbol{T}_t^{(0)}\check{f}(\boldsymbol{x}).$$

Hence we have

$$\check{v}_t(\boldsymbol{x})=\boldsymbol{T}_t^{(0)}\check{f}(\boldsymbol{x})+\int_0^t\int_S \psi(\boldsymbol{x};\ ds d\boldsymbol{y})\check{v}_{t-s}(\boldsymbol{y});$$

i.e., $\check{v}_t(\boldsymbol{x})$ is a a solution of the M-equation with the initial value $\check{f}(\boldsymbol{x})$. In §4. 2 we have shown that $\boldsymbol{T}_t\check{f}(\boldsymbol{x})$ is the smallest of all such solutions, and therefore

$$\boldsymbol{T}_t\check{f}(\boldsymbol{x})\leq\check{v}_t(\boldsymbol{x})$$

which implies, in particular, that

$$M_t f(x)=(\boldsymbol{T}_t\check{f})|_S(x)\leq v_t(x).$$

From now on we shall assume that the fundamental system (T_t^0, K, π) is determined by $[X, k, \pi]$ and is H-regular or weakly H-regular. We shall assume further that

$$\sup_{x\in S}\int\pi(x, d\boldsymbol{y})\check{1}(\boldsymbol{y})\equiv K<\infty \tag{4. 83}$$

and

(4.84) $\qquad k\cdot G(\cdot\,;\,g)\in H_0(\text{resp. } \widetilde{H}_0)$ if $g\in H_0$ (resp. $\widetilde{H}_0$)

in the case when the fundamental system is H-regular (resp. weakly H-regular).

From (4.83) we have for every $g\in \boldsymbol{B}(S)$,

(4.85) $$\|G(\cdot\,;\,g)\|\leq K\cdot\|g\|.$$

Now, for given $f\in\boldsymbol{B}^*(S)$, define $\{u_n(t,x)\}_{n=0}^{\infty}$ successively by

(4.86) $$u_0(t,x)\equiv 0,$$

$$u_n(t,x)=T_t^0f(x)+\int_0^t\int_S K(x;\,dsdy)G(\cdot\,;\,u_{n-1}(t-s,\,\cdot)).$$

Then just as in the case of the S-equation, $u_n\uparrow u_\infty$, where u_∞ is the minimal solution of (4.79), and hence $u_\infty(t,\,\cdot)=M_tf$ by the above theorem. We shall now prove

(4.87) $$\|M_tf\|\leq e^{K\|k\|t}\|f\|.$$

For, if we assume

(4.88) $$\|u_n(t,\,\cdot)\|<\sum_{j=0}^{n}\frac{(K\cdot\|k\|t)^j}{j!}\|f\|,$$

then

$$\begin{aligned}0\leq u_{n+1}(t,x)&=T_t^0f(x)+\int_0^t T_s^0\{k\cdot G(\cdot\,;\,u_n(t-s,\,\cdot))\}ds\\ &\leqq\|f\|+\|k\|K\int_0^t\|u_n(s,\,\cdot)\|ds\\ &\leqq\|f\|+\|k\|\cdot K\cdot\int_0^t\sum_{j=0}^{n}\frac{(K\cdot\|k\|\cdot s)^j}{j!}\|f\|ds\\ &=\sum_{j=0}^{n+1}\frac{(K\|k\|t)^j}{j!}\|f\|.\end{aligned}$$

This proves (4.88) for every n and hence letting $n\to\infty$ we have (4.87). Now noting the following property of G,

(4.89) $$\|G(\cdot\,;\,g)-G(\cdot\,;\,h)\|\leq\|g-h\|,$$

we can repeat the same arguments as for the S-equation to obtain the following

Theorem 4.14. *Assume that the fundamental system* (T_t^0, K, π) *is H-regular or weakly H-regular and* (4.83) *is satisfied; then for given* $f\in \boldsymbol{B}(S)$ *there exists a unique solution* $u(t, x)\in \boldsymbol{B}(S)$ *of* (4.79) *and* $u(t, x)=M_tf(x)\equiv E_x[\check{f}(\boldsymbol{X}_t)]$.[33] M_t *satisfies*

$$\|M_tf\|\leq e^{\|k\|K t}\|f\|,\ f\in \boldsymbol{B}(S). \tag{4.90}$$

Further, (i) *if* (T_t^0, K, π) *is H-regular, then* M_t *is a strongly continuous semi-group on* H_0 *with the infinitesimal generator* L *such that* $D(L)=D(A_H)(=D(A_H^0))$[34]
and

$$\begin{aligned} Lu &= A_H^0u+k\cdot G(\cdot\,;\,u) \\ &= A_Hu+k\{G(\cdot\,;\,u)-u\}. \end{aligned} \tag{4.91}$$

(ii) *If* (T_t^0, K, π) *is weakly H-regular, then* M_t *is a weakly right-continuous semi-group on* $\widetilde{H}_0$ *with the weak infinitesimal generator* $\widetilde{L}$ *such that* $D(\widetilde{L})=D(\widetilde{A}_H)(=D(\widetilde{A}_H^0))$[35] *and*

$$\begin{aligned} \widetilde{L}u &= \widetilde{A}_H^0u+k\cdot G(\cdot\,;u) \\ &= \widetilde{A}_Hu+k\{G(\cdot\,;\,u)-u\}. \end{aligned} \tag{4.92}$$

Now consider for instance the case when $\pi(x, dy)=\delta_{[x,x]}(dy)$;[36] then $G(x; f)=2f(x)$ and hence $Lu=Au+ku$. By Kac's theorem

$$M_tf(x)=E_x[\exp(\int_0^t k(x_s)ds)f(x_t)], \tag{4.93}$$

where E_x is the expectation with respect to the process X. If $k\leq 0$ the Markov process corresponding to M_t is obtained from X by shortening the life time (cf. §0.1), while in the case $k\geq 0$ we must introduce creation of new particles and the branching process $\boldsymbol{X}$ seems to be one of the natural and nice models for the creation (cf. Knight [23] for another approach).

33) This implies, in particular, that $\boldsymbol{B}^1=\boldsymbol{B}(S)$.

34) $A_H(A_H^0)$ is the H-infinitesimal generator of $T_t(T_t^0)$.

35) $\widetilde{A}_H(\widetilde{A}_H^0)$ is the weak H-infinitesimal generator of $T_t(T_t^0)$.

36) $\delta_{[x,x]}(dy)$ is the unit measure on S at $[x, x]\in S^2$.

Finally we shall derive some equations for higher moments of ξ_t^f. For simplicity we shall assume (T_t^0, K, π) is regular and for any $f \in C(S)^+$

$$E_x[(\check{f})^p(X_t)] \equiv E_x[(\xi_t^f)^p] \in C(S)^+.$$

Set

$$u^{(p)}(t, x) = E_x[(\check{f})^p(X_t)]. \tag{4.94}$$

Now we shall introduce the following notations: let $(\alpha_n^{(i)})_{n=0}^{\infty}$, $i = 1, 2, \cdots$ be a countable family of sequences and define $P_n^m(\alpha_\cdot^{(\cdot)})$ by

$$\prod_{j=1}^{n}\left(\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\alpha_n^{(j)}\right) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left[\sum_{i=1}^{n}\alpha_n^{(i)} + P_n^m(\alpha_\cdot^{(\cdot)})\right]. \tag{4.95}$$

Clearly $P_n^m(\alpha_\cdot^{(\cdot)})$ is a polynomial in $\alpha_k^{(i)}$, $k=1, 2, \cdots, n-1$, $i=1, 2, \cdots, m$. For $\boldsymbol{y} \in \boldsymbol{S}$, $\boldsymbol{y} = [y_1, \cdots, y_m] \in S^m$, $m(\boldsymbol{y}) = m$ and

$$\begin{aligned} H_p(t, \boldsymbol{y}) &\equiv H_p(u^{(1)}(t, \cdot), u^{(2)}(t, \cdot), \cdots, u^{(p-1)}(t, \cdot))(\boldsymbol{y}) \\ &= P_p^{m(\boldsymbol{y})}(\alpha_\cdot^{(\cdot)}), \end{aligned} \tag{4.96}$$

where

$$\alpha_k^{(i)} = u^{(k)}(t, y_i), \quad i=1, 2, \cdots, m(\boldsymbol{y}), \quad k=1, 2, \cdots, p-1.$$

Theorem 4.15. *Under the assumptions above, we have*

$$\begin{aligned} u^{(p)}(t, x) &= M_t[f^p](x) \\ &\quad + \int_0^s M_{t-s}\left[k\int_{\boldsymbol{S}}\pi(\cdot\,;\, d\boldsymbol{y})H_p(s, \boldsymbol{y})\right](x)ds, \quad x \in S. \end{aligned} \tag{4.97}$$

Proof. It is sufficient to prove (4.97) for non-negative f. If we take $h \equiv 1$ in (4.75) we have

$$\begin{aligned} u^{(p)}(t, x) &= T_t^0[(f)^p](x) + \int_0^t T_{t-s}^0[kG(\cdot\,;\, u_s^{(p)})](x)ds \\ &\quad + \int_0^t T_{t-s}^0\left[k\int_{\boldsymbol{S}}\pi(\cdot\,;\, d\boldsymbol{z})H_p(s, \boldsymbol{z})\right](x)ds. \end{aligned} \tag{4.98}$$

Now put

$$v(t, x) = M_t[f^p](x) + \int_0^t M_{t-s}\left[k\cdot\int_{\boldsymbol{S}}\pi(\cdot\,;\, d\boldsymbol{y})H_p(s, \boldsymbol{y})\right](x)ds.$$

Combining this with

$$M_t[g](x) = T_t^0[g](x) + \int_0^t T_{t-s}^0[kG(\cdot\,;\, M_s(g))](x)ds,$$

we have

$$v(t, x)=T_t^0[f^p](x)+\int_0^t T_{t-s}^0\Big[k\int_S \pi(\cdot\,;\,dy)H_p(s, y)\Big](x)ds$$
$$+\int_0^t T_{t-s}^0[kG(\cdot\,;\,M_s[f^p])](x)ds$$
$$+\int_0^t\int_0^{t-s}\Big[T_{t-s-\theta}^0\Big[kG(\cdot\,;\,M_\theta\,[k\int_S \pi(\cdot\,;\,dy)H_p(s, y)])\Big]d\theta ds.$$

It is easy to see that

$$\int_0^t\int_0^{t-s} T_{t-s-\theta}^0\Big[kG\{\cdot\,;M_\theta[k\int_S \pi(\cdot\,;dy)H_p(s,y)]\}\Big](x)\,d\theta\,ds$$
$$=\int_0^t\int_s^t T_{t-u}^0\Big[kG\{\cdot\,;\,M_{u-s}[k\int_S \pi(\cdot\,;\,dy)H_p(s, y)]\}\Big](x)du\,ds$$
$$=\int_0^t ds\int_0^s T_{t-s}^0\Big[kG\{\cdot\,;\,M_{s-u}[k\int_S \pi(\cdot\,;\,dy)H_p(u, y)]\}\Big](x)\,du.$$

Hence

$$v(t, x)=T_t^0[f^p](x)+\int_0^t T_{t-s}^0\Big[k\int_S \pi(\cdot\,;\,dy)H_p(s, y)\Big](x)\,ds$$
$$+\int_0^t T_{t-s}^0\Big[kG(\cdot\,;\,M_s[p^f]+\int_0^s M_{s-u}[k\int_S \pi(\cdot\,;\,dy)H_p(u, y)]\,du)\Big](x)ds$$
$$=T_t^0[f^p](x)+\int_0^t T_{t-s}^0\Big[k\int_S \pi(\cdot\,;\,y)H_p(s, y)\Big](x)\,ds$$
$$+\int_0^t T_{t-s}^0[kG(\cdot\,;\,v_s)](x)\,ds.$$

Therefore we have

$$v(t, x)=T_t^0[f^p](x)+\int_0^t T_{t-s}^0\Big[k\int_S \pi(\cdot\,;\,dy)H_p(s, y)\Big](x)\,ds \tag{4.99}$$
$$+\int_0^t T_{t-s}^0[kG(\cdot\,;\,v_s)](x)\,ds.$$

Since the equation (4.99) has a unique solution in $\boldsymbol{C}(S)$, we have

$$v(t, x)=u^{(p)}(t, x)$$

which completes the proof.

The formula (4.97) permits us to obtain $u^{(p)}(t, x)$ successively though it is quite complicated even for $p=3$. For example $u^{(1)}(t, x)=M_tf(x)$, and

$$u^2(t, x) = M_t[f^2](x) + \int_0^t M_{t-s}\Big[k\int_S \pi(\cdot\,; dy)\sum_{i\neq j} M_s f(y_i) M_s f(y_j)\Big](x)\, ds.$$

In a similar way we can prove

$$\boldsymbol{E}_x[\check{f}(x_t)\check{g}(x_t)] = M_t(f\cdot g) + \int_0^t M_{t-s}\Big[k\int_S \pi(\cdot\, dy)\sum_{i\neq j} M_s f(y_i) M_s g(y_j)\Big](x)\, ds.$$

If, in particular, $\pi(x, dy) = \sum_{n=0}^{\infty} p_n \delta[\underbrace{x, \cdots, x}_{n}](dy)$ and $C \equiv \sum_{n=1}^{\infty} n(n-1)p^n < \infty$, then

$$E_x[\check{f}(x_t)\check{g}(x_t)] = M_t[fg](x) + C\int_0^t M_{t-s}[kM_s f M_s g](x)\, ds.$$

V. Transformations of branching Markov processes

In this chapter we shall consider transformations of branching Markoy processes; i.e., operations on a branching Markov process which yield a new branching Markov process. We shall discuss mainly the transformations by multiplicative functionals (cf. §0. 1 Definition 0. 8) and obtain, in particular, the condition on a multiplicative functional under which the transformed process will be a branching Markov process.

§5.1. Multiplicative functionals of branching type.

Let $\boldsymbol{X} = (\Omega, \mathcal{B}_t, 0 \leq t \leq \infty, \boldsymbol{P}_x, x \in \hat{S}, \boldsymbol{X}_t, \theta_t)$[1)] be a branching Markov process and $M_t(\omega)$ be an $\boldsymbol{N}_{t+0}$-multiplicative functional of $\boldsymbol{X}$. Unless otherwise stated we shall assume always

(5·1) $\quad \boldsymbol{E}_{\boldsymbol{x}}[M_t] \leq 1$, for every $\boldsymbol{x} \in \boldsymbol{S}$

and

(5. 2) $\quad \boldsymbol{P}_\partial[M_t = 1] = \boldsymbol{P}_\Delta[M_t = 1] = 1$, for every $t \geq 0$.

Also we shall assume that

1) We are assuming always $\bar{\mathcal{B}}_{t+0} = \mathcal{B}_t$.

(5·3) $\Omega = W \equiv$ the set of all right continuous path functions

w: $t \in [0, \infty) \to w(t) \in \widehat{S}$ such that if $w(t) = \partial (= \Delta)$ then $w(s) = \partial$ (resp. $= \Delta$) for all $s \geqq t$.

Let $W^{(n)}$ be the n-fold product of W and put

$$\widetilde{W} = \bigcup_{n=1}^{\infty} W^{(n)}: \text{ the sum of } W^{(n)}.$$

We define a mapping φ of $\widetilde{W}$ to W by

$$(\varphi \widetilde{w})(t), = \gamma(w^1(t), w^2(t), \cdots, w^n(t)), \ t \geq 0, \tag{5.4}$$

when $\widetilde{w} = (w^1, w^2, \cdots, w^n) \in W^{(n)}$, $w^j \in W$, $j = 1, 2, \cdots, n$, where γ is defined by (0.19).

Definition 5.1. A multiplicative functional M_t of $\boldsymbol{X}$ is said to be *of branching type* if it satisfies for any $n \geq 1$

$$M_t(\varphi \widetilde{w}) = \prod_{i=1}^{n} M_t(w^i), \ t \geqq 0, \ (\text{a.s. } \widetilde{\boldsymbol{P}}_{\boldsymbol{x}}, \forall \boldsymbol{x} \in S^{(n)}) \tag{5·5}$$

where $\widetilde{w} = (w^1, w^2, \cdots, w^{(n)}) \in W^{(n)}$ and

$$\widetilde{\boldsymbol{P}}_{\boldsymbol{x}} = \boldsymbol{P}_{x_1} \times \boldsymbol{P}_{x_2} \times \cdots \times \boldsymbol{P}_{x_n}, \ \boldsymbol{x} = (x_1, x_2, \cdots, x_n).$$

Theorem 5.1. *Let* $\boldsymbol{X}$ *be a branching Markov process,* M_t *be an* $\mathfrak{N}_t$*-multiplicative functional of* $\boldsymbol{X}$ *satisfying* (5.1) *and* (5.2) *and* $\boldsymbol{X}^M$ *be the* M_t*-subprocess of* $\boldsymbol{X}$. *Then the following statements are equivalent to each other*:

(i) X^M *is a branching Markov process,*

(ii) M_t *is a multiplicative functional of branching type.*

Proof. 1°) (i) → (ii). Suppose the M_t-subprocess[2] $\boldsymbol{X}^M = (\boldsymbol{X}_t, P_{\boldsymbol{x}}^M, W)$ is a branching Markov process. Then $\boldsymbol{X}^M$ has the property B.I, and hence for $0 \leq t_1 < t_2 < \cdots < t_p = t$ and $f_1, \cdots, f_p \in \boldsymbol{C}^*(S)$, we have

$$\boldsymbol{E}_{\boldsymbol{x}}^M [\prod_{j=1}^{p} \widehat{f}_j(\boldsymbol{X}_{t_j})] = \prod_{i=1}^{n} \boldsymbol{E}_{x_i}^M [\prod_{j=1}^{p} \widehat{f}_j(\boldsymbol{X}_{t_j})], \ \boldsymbol{x} = [x_1, x_2, \cdots, x_n]. \tag{5.6}$$

2) Cf. §0.1.

Also we have by the property B.I of $\boldsymbol{X}$,

$$\text{(5.7)} \qquad \boldsymbol{E}_{\boldsymbol{x}}^{M}[\prod_{j=1}^{p}\widehat{f}_j(\boldsymbol{X}_{t_j})] = \boldsymbol{E}_{\boldsymbol{x}}[\prod_{j=1}^{p}\widehat{f}_j(\boldsymbol{X}_{t_j})M_t]$$

$$= \boldsymbol{E}_{x_1}\times\boldsymbol{E}_{x_2}\times\cdots\times\boldsymbol{E}_{x_n}[\prod_{j=1}^{p}\widehat{f}_j(\boldsymbol{X}_{t_j}(\varphi\widetilde{w}))\cdot M_t(\varphi\widetilde{w})].$$

From (5.6) and (5.7) we have

$$\boldsymbol{E}_{x_1}\times\boldsymbol{E}_{x_2}\times\cdots\times\boldsymbol{E}_{x_n}[\prod_{j=1}^{p}\widehat{f}_j(\boldsymbol{X}_{t_j}(\varphi\widetilde{w}))\{M_t(\varphi\widetilde{w})-\prod_{i=1}^{n}M_t(w^i)\}]=0.$$

Since $\prod_{j=1}^{p}\widehat{f}_j(\boldsymbol{X}_{t_j}(\varphi(\widetilde{w})))$ generates $\sigma\{\widetilde{W}, \mathcal{B}(\widetilde{S});\ \varphi\widetilde{w}(s);\ s\leq t\}$, this proves (5.5), that is, M_t is a multiplicative functional of branching type.

2°) (ii)→(i). If M_t is a multiplicative functional of branching type, then noting that $\boldsymbol{X}_t$ has the property B.I we have

$$\boldsymbol{E}_{\boldsymbol{x}}^{M}[\widehat{f}(\boldsymbol{X}_t)] = \boldsymbol{E}_{\boldsymbol{x}}[\widehat{f}(\boldsymbol{X}_t)M_t]$$

$$= \boldsymbol{E}_{x_1}\times\cdots\times\boldsymbol{E}_{x_n}[\widehat{f}(\boldsymbol{X}_t(\varphi\widetilde{w}))M_t(\varphi\widetilde{w})]$$

$$= \boldsymbol{E}_{x_1}\times\cdots\times\boldsymbol{E}_{x_n}[\prod_{j=1}^{n}\widehat{f}(\boldsymbol{X}_t(w^j))\prod_{j=1}^{n}M_t(w^j)]$$

$$= \prod_{j=1}^{n}\boldsymbol{E}_{x_j}[\widehat{f}(\boldsymbol{X}_t)M_t]$$

$$= \prod_{j=1}^{n}\boldsymbol{E}_{x_i}^{M}[\widehat{f}(\boldsymbol{X}_t)],$$

which implies that the M_t-subprocess is a branching Markov process.

Remark 5.1. In Theorem 5.1 the assertion "(ii)→(i)" is true if M_t is an $\mathcal{N}_{t+0}$-multiplicative functional.

Definition 5.2. Let M_t be a multiplicative functional of $\boldsymbol{X}$. M_t is said to be *of branching type in the weak sense* if for any $n\geq 1$,

$$\text{(5.5)}' \qquad M_t(\varphi\widetilde{w}) = \prod_{j=1}^{n}M_t(w^j),\ 0\leqq t\leqq\tau(\varphi\widetilde{w})\ (\text{a.s. } \widetilde{P}_{\boldsymbol{x}},\ \boldsymbol{x}\in S^{(n)}).$$

Theorem 5.2. *Let* $\boldsymbol{X}$ *be a branching Markov process satisfying the conditions* (c. 1) *and* (c. 2) *of* §1. 2, *and* M_t *an* $\mathcal{N}_t$*-multiplicative functionl such that* M_t*-subprocess* $\boldsymbol{X}^M$ *of* $\boldsymbol{X}$ *satisfies* (c. 1)

and (c. 2). *Then the following statements are equivalent*:
(i) X^M *is a branching Markov process*,
(ii) M_t *is a multiplicative functional of branching type in the weak sense.*

Proof. (i)→(ii) is clear from the previous theorem since every multiplicative functional of branching type is of branching type in the weak sense. Assume conversely that M_t is of branching type in the weak sense. Let $\boldsymbol{x}=[x_1, x_2, \cdots, x_n]\in S^n$; then

$$\begin{aligned}
&\boldsymbol{E}^M_{\boldsymbol{x}}[\widehat{f}(\boldsymbol{X}_t);\ t<\tau]=\boldsymbol{E}_{\boldsymbol{x}}[\widehat{f}(\boldsymbol{X}_t)M_t;\ t<\tau]^{3)}\\
&=\boldsymbol{E}_{x_1}\times\boldsymbol{E}_{x_2}\times\cdots\times\boldsymbol{E}_{x_n}[\widehat{f}(\boldsymbol{X}_t(\varphi\widetilde{w}))\cdot M_t(\varphi\widetilde{w});\ t<\tau(\varphi\widetilde{w})]\\
&=\boldsymbol{E}_{x_1}\times\boldsymbol{E}_{x_2}\times\cdots\times\boldsymbol{E}_{x_n}[\prod_{j=1}^{n}\{\widehat{f}(\boldsymbol{X}_t(u^j))M_t(w^j)\cdot I_{[t<\tau(w^j)]}\}]\\
&=\prod_{j=1}^{n}\boldsymbol{E}_{x_j}[\widehat{f}(\boldsymbol{X}_t)M_t;\ t<\tau]\prod_{j=1}^{n}\boldsymbol{E}^M_{x_j}[\widehat{f}(\boldsymbol{X}_t);\ t<\tau],
\end{aligned}$$

which proves X^M has the property B. III (i). Quite similarly we can prove that X^M has the property B. III (ii). By Theorem 1. 2 d), X^M is a branching Markov process.

Remark 5.2. (ii)→(i) is true if M_t is an $\mathcal{N}_{t+0}$-multiplicative functional.

§5.2. Examples

Example 5.1. (*Harmonic transformation*). Let $f\in\boldsymbol{C}^*(S)^+$; assume that $e(\boldsymbol{x})=\lim_{t\to\infty}\boldsymbol{T}_t\widehat{f}(\boldsymbol{x})$ exists and $e(\boldsymbol{x})>0$ for every $\boldsymbol{x}\in\boldsymbol{S}$. Then

$$M_t(w)=\begin{cases}\dfrac{e(\boldsymbol{X}_t(w))}{e(\boldsymbol{X}_0(w))}, & \text{if } \boldsymbol{X}_0(w)\in\boldsymbol{S},\\ 1, & \text{if } \boldsymbol{X}_0(w)=\Delta\end{cases}\tag{5.6}$$

defines a multiplicative functional of branching type. In fact $e(\boldsymbol{x})=\widehat{e|_S}(\boldsymbol{x})$, and hence

3) This follows from the general formula: $\boldsymbol{P}^M_x[B;\ e_\Delta>t]=\boldsymbol{E}_x[M_t;\ B, e_\Delta>t]$, $\forall B\in\mathcal{B}_t$.

$$(5.7)\qquad M_t(\varphi\tilde{w})=\frac{e(\boldsymbol{X}_t(\varphi\tilde{w}))}{e(\boldsymbol{X}_0(\varphi\tilde{w}))}=\frac{\prod_{j=1}^{n}e(\boldsymbol{X}_t(w^j))}{\prod_{j=1}^{n}(e(\boldsymbol{X}_0(w^j))}=\prod_{j=1}^{n}M_t(w^j),$$

where $\tilde{w}=(w^1, w^2, \cdots, w^n)\in W^{(n)}$. If in particular

$$e_1(x)=\boldsymbol{P}_x[e_\partial<\infty]>0 \text{ or } e_2(x)=\boldsymbol{P}_x[e_\Delta=+\infty]>0,$$

then they define a multiplicative functional of branching type since $e_1(x)=\lim_{t\to\infty}\boldsymbol{T}_t\hat{0}(x)$ and $e_2(x)=\lim_{t\to\infty}\boldsymbol{T}_t\hat{1}(x)$.

Example 5.2. (*Killing of the non-branching part*). For $f\in\boldsymbol{B}(S)^+$, set

$$(5\cdot 8)\qquad M_t(w)=\begin{cases}\exp(-\int_0^t\check{f}(\boldsymbol{X}_s(w))ds), & \text{if } \boldsymbol{X}_0(w)\in\boldsymbol{S}\\ 1 & \text{, if } \boldsymbol{X}_0(w)=\Delta.\end{cases}$$

Then $M_t(w)$ is a contraction[4)] multiplicative functional of branching type since

$$M_t(\varphi\tilde{w})=\exp(-\sum_{j=1}^{n}\int_0^t\check{f}(\boldsymbol{X}_s(w^j))ds)$$
$$=\prod_{j=1}^{n}M_t(w^j),$$

where $\tilde{w}=(w^1, w^2, \cdots, w^n)\in W^{(n)}$.

It is easy to see that the non-branching part of $\boldsymbol{X}^M$ is the $e^{-\int_0^t f(x_s^0)ds}$-subprocess of the non-branching part of $\boldsymbol{X}$.

Example 5.3. (*Transformation of branching laws*).

Let $\boldsymbol{X}$ be an (X^0, π)-branching process such that the non-branching part X^0 is the $e^{-\int_0^t k(x_s)ds}$-subprocess of a conservative Hunt process $X=(x_t, P_x)$ on S, where $k\in\boldsymbol{B}(S)^+$. Let $f(x, y)$ be a function in $\boldsymbol{B}(S\times\hat{S})^+$ such that $\int_{\hat{S}}e^{f(x,y)}\pi(x, dy)=1$ for every $x\in S$. We define a kernel $n(x, dy)$ on $S\times\hat{\boldsymbol{S}}$ by

4) $M_t(w)$ is called a contraction multiplicative functional if $M_t(w)\leq 1$ for every t and w.

$$n(x, dy)=\sum_{i=0}^{n} k(x_i)\pi(x_i, dy_i)\times\prod_{j\neq i}\delta_{\{x_j\}}(dy_j)\circ\gamma^{-1}$$

where $\gamma: (y_1, y_2, \cdots, y_n)\rightarrow y$ is defined by (0.19). Define a kernel $n^*(x, dy)$ in the same way using the kernel $\pi^*(x, dy)\equiv e^{f(x,y)}\pi(x, dy)$ instead of $\pi(x, dy)$. Then since n^* is absolutely continuous with respect to n it is easy to see that there exists $f(x, y)$ which is an extension of $f(x, y)$ such that $f(x, y)\in \boldsymbol{B}(\boldsymbol{S}\times\widehat{\boldsymbol{S}})$ and $n^*(x, dy) =e^{f(x,y)}n(x, y)$. Now we shall define a multiplicative functional $M_t(w)$ of the process $\boldsymbol{X}$ by

$$M_t(w)=\exp\{\sum_{\tau_n\leq t} f(\boldsymbol{X}_{\tau_n-}, \boldsymbol{X}_{\tau_n})\}.$$[5]

Then it is clear that $M_t(w)\equiv 1$ if $\boldsymbol{X}_0=\partial$ or Δ, and we can show that it is a multiplicative functional of branching type in the weak sense such that $\boldsymbol{E}_x[M_t]=1$ for every x and $t\geq 0$. The M_t-subprocess X^M coincides with the (X^0, π^*)-branching Markov process. (Cf. [27] where the transformation of Lévy measures by multiplicative functionals is discussed).

§5. 3. Construction of a multiplicative functional of branching type.

Let $\boldsymbol{X}$ be an (X^0, π)-branching Markov process and m_t be a multiplicative functional of the non-branching part X^0 of $\boldsymbol{X}$. We shall construct a multiplicative functional M_t of branching type in the weak sense of the process $\boldsymbol{X}$ by piecing out m_t.

Let $W=\bigcup_{n=0}^{\infty} W_n$ where $W_n=\{w\in W;\ w(0)\in S^n\}$. Define a mapping φ from the n-fold product $W_1\times W_1\times\cdots\times W_1$ of W_1 to W_n by

(5. 9) $$(\varphi\widetilde{w})(t)=\gamma[w^1(t), w^2(t), \cdots, w^n(t)],$$

where $\widetilde{w}=(w^1, w^2, \cdots, w^n)\in W_1\times W_1\times\cdots\times W_1$.

Lemma 5.1. *Let $F(w)$ be a bounded $\mathcal{N}_\infty|_{W_1}$-measurable function*

5) $\{\tau_n\}$ is defined by (1. 8).

on W_1. *Then there exists one and only one* $\mathcal{N}_\infty|_{W_n}$*-measurable function* $\widetilde{F}$ *on* W_n *such that*

$$(5.10)\qquad \widetilde{F}(\varphi\widetilde{w})=\prod_{j=1}^{n}F(w^j)\quad\text{for}\quad \widetilde{w}=(w^1, w^2, \cdots, w^n).$$

Proof. It is sufficient to show that if $\varphi\widetilde{w}=\varphi\widetilde{w}'$ $(\widetilde{w}, \widetilde{w}'\in W_1\times\cdots\times W_n)$, then $\prod_{j=1}^{n}F(w^j)=\prod_{j=1}^{n}F(w'^j)$. But this is clearly true if $F(w)$ is of the from

$$F(w)=\sum_{k=1}^{n}a_k\prod_{i=1}^{m_k}\sum_{l=1}^{p_{ik}}C_{lik}\widehat{f}_{lik}(X_{t_i}(w)),$$

where $f_{lik}\in C^*(S)$, and hence by Lemma 0.2 it is true for all bounded $\mathcal{N}_\infty|_{W_1}$-meaurable function F.

Now let $X^0=\{W_1, \mathcal{N}_t|_{W_1}, X_t, t<\tau, P_x, x\in S\}$ be the non-branching part on S of X_t and m_t be the $\mathcal{N}_t|_{W_1}$-multiplicative functional of X^0 whose defining set is W_1'.[6] For $n\geq 0$ we extend m_t as follows: when $n\geq 1$, we put

$$(5.11)\qquad \begin{aligned}\widetilde{m}_t(\varphi\widetilde{w})&=\prod_{j=1}^{n}m_t(w^j),\quad\text{if}\quad t<\tau(\varphi\widetilde{w})\\ &=\widetilde{m}_{\tau(\varphi\widetilde{w})}(\varphi\widetilde{w}),\quad\text{if}\quad t\geq\tau(\varphi\widetilde{w})\end{aligned}$$

and when $n=0$, we put

$$\widetilde{m}_t(\varphi\widetilde{w})=1.$$

Then $\widetilde{m}_t$ is well defined as an $\mathcal{N}_\infty|_{W_n}$-measurable function by the previous lemma. As is easily seen, we can take $W'=\bigcup_{n=0}^{\infty}W_n'$, where $W_n'=\varphi(W_n'\times\cdots\times W_1')$, as a defining set of $\widetilde{m}_t$. We shall now define $M_t(w)$ as follows:

$$(5\cdot 12)\qquad \begin{aligned}M_t(w)&=\widetilde{m}_\tau(w)\cdot\theta_{\tau_1}\widetilde{m}_\tau(w)\cdots\theta_{\tau_{j-1}}\widetilde{m}_\tau(w)\cdot\widetilde{m}_{t-\tau_j}(w)(\theta_{\tau_j}w),\\ &\qquad\qquad\text{on}\quad w\in A_j^t,\quad j=0, 1, 2\cdots\\ &=\prod_{j=1}^{\infty}\theta_{\tau_j}\widetilde{m}_\tau(w),\quad\text{on}\quad w\in\{t\geq\tau_\infty\},\end{aligned}$$

where

6) Cf. §0.1.

(5. 13) $$\theta_{\tau_j}\widetilde{m}_\tau(w)=\widetilde{m}_a(\theta_{\tau_j}w),$$

where $a=\tau(\theta_{\tau_j}w)$ and

(5·14) $$A_j^t=\{w:\tau_j\leq t<\tau_{+1}\}.$$

Lemma 5.2. *M_t is $\mathcal{N}_{t+0}$-measurable.*

Proof. We first note that

(5. 15) $$M_t(w)=M_{\tau_i}(w)\cdot\widetilde{m}_{t-\tau_j(w)}(\theta_{\tau_j}w) \text{ on } A_j^t.$$

Then $M_{\tau_j}(w)$ is $\mathcal{N}_{\tau_j}$-measurable, and hence $M_{\tau_j}\cdot I_{A_j^t}$ is $\mathcal{N}_{t+0}$-measurable. Next we set

$$\tau_j^n(w)=\frac{m-1}{2^n} \text{ on } B_m=\left\{w:\frac{m-1}{2^n}<\tau_j(w)\leq\frac{m}{2^n}\right\};$$

then $\tau_j^n\uparrow\tau_j(n\to\infty)$ and hence $t-\tau_j^n\downarrow t-\tau_j$. Now $\widetilde{m}_{t-\tau_j^n}(\theta_{\tau_j}w)\cdot I_{B_m}$ $=\widetilde{m}_{t-\frac{m-1}{2^n}}(\theta_{\tau_j}w)\cdot I_{\{t<\tau_j+t-\frac{m-1}{2^n}\leq t+\frac{1}{2^n}\}}$. Since $\widetilde{m}_{t-\frac{m-1}{2^n}}(\theta_{\tau_j}w)$ is $\mathcal{N}_{\tau_j+t-\frac{m-1}{2^n}}$-measuable, $\widetilde{m}_{t-\tau_j^n}(\theta_{\tau_j}w)I_{B_m}$ is $\mathcal{N}_{t+\frac{1}{2^n}}$-measurable, and hence $\lim_{n\to\infty}\widetilde{m}_{t-\tau_n^j}(\theta_{\tau_j}w)$ $=\widetilde{m}_{t-\tau_j}(\theta_{\tau_j}w)$ is $\mathcal{N}_{t+0}$-measurable.

Lemma 5.3. *$M_t(w)$ is multiplicative*, i.e.,

(5. 6) $$M_{t+s}(w)=M_t(w)M_s(\theta_tw),\ w\in W'.$$

Proof. Since

$$M_s(\theta_tw)=M_{\tau_j(\theta_tw)}(\theta_tw)\cdot\widetilde{m}_{s-\tau_j(\theta_tw)}(\theta_{\tau_j}(\theta_tw)),\ \theta_tw\in A_j^s,$$

and

$$M_t(w)=M_{\tau_i}(w)\cdot\widetilde{m}_{t-\tau_i(w)}(\theta_{\tau_i}w),\ w\in A_i^t,$$

we have for $w\in A_i^t\cap\theta_t^{-1}(A_j^s)\cap W'$

(5·17)
$$M_t(w)\cdot M_s(\theta_tw)=M_{\tau_i}(w)\widetilde{m}_{t-\tau_i}(\theta_{\tau_i}w)M_{\tau_j(\theta_tw)}(\theta_tw)\widetilde{m}_{s-\tau_j(\theta_tw)}(\theta_{\tau_j}(\theta_tw))$$
$$=M_{\tau_i}(w)\widetilde{m}_{t-\tau_i}(\theta_{\tau_i}w)\widetilde{m}_{\tau(\theta_tw)}(\theta_tw)\theta_{\tau_1(\theta_tw)}\widetilde{m}_{\tau(\theta_tw)}(\theta_tw)\cdots\widetilde{m}_{s-\tau_j(\theta_tw)}(\theta_{\tau_j}(\theta_\tau w)).$$

If $w\in A_i^t\cap\theta_t^{-1}(A_j^s)\cap W'$, we have

$$(5.18)\qquad \begin{cases} \tau_k(\theta_t w)=\tau_{i+k}(w)-t, & k=1,2,\cdots \\ \theta_t w=\theta_{t-\tau_i}(\theta_{\tau_i}w), \end{cases}$$

and hence

$$\begin{aligned}(5.19)\qquad & \tilde{m}_{t-\tau_i}(\theta_{\tau_i}w)\tilde{m}_{\tau(\theta_t w)}(\theta_t w) \\ &= \tilde{m}_{t-\tau_i}(\theta_{\tau_i}w)\tilde{m}_{\tau_{i+1}-t}(\theta_{t-\tau_i}(\theta_{\tau_i}w)) \\ &= \tilde{m}_{\tau_{i+1}-\tau_i}(\theta_{\tau_i}w)=\theta_{\tau_i}\tilde{m}_\tau(w),\end{aligned}$$

$$\begin{aligned}(5.20)\qquad & \theta_{\tau_1(\theta_t w)}\tilde{m}_{\tau(\theta_t w)}(\theta_t w) \\ &= \tilde{m}_{\tau(v)}(\theta_{\tau_1(\theta_t w)}(\theta_t w)), \quad v=\theta_{\tau_1(\theta_t w)}(\theta_t w), \\ &= \tilde{m}_{\tau(v)}(\theta_{\tau_{i+1}w}), \qquad v=\theta_{\tau_{i+1}}w, \\ &= \theta_{\tau_{i+1}}\tilde{m}_t(w), \\ & \vdots \end{aligned}$$

Also for $w\in A_i^t\cap\theta_t^{-1}(A_j^s)\cap W'$, we have

$$(5.21)\qquad \begin{cases} s-\tau_j(\theta_t w)=t+s-\tau_{i+j}(w) \\ \theta_{\tau_j(\theta_t w)}(\theta_t w)=\theta_{\tau_j(\theta_t w)+t}w=\theta_{\tau_{i+j}}w, \end{cases}$$

and hence

$$(5.22)\qquad \tilde{m}_{s-\tau_j(\theta_t w)}(\theta_{\tau_j}(\theta_t w))=\tilde{m}_{t+s-\tau_{i+j}}(\theta_{\tau_{i+j}}w).$$

(5.17), (5.15), (5.20) and (5.22) imply

$$M_t(w)M_s(\theta_t w)=M_{t+s}(w), \quad w\in W'.$$

Remark 5.3. If $m_t\leq 1$ then $\tilde{m}_t\leq 1$ and hence $M_t\leq 1$.

Lemma 5.4. *If* $\boldsymbol{E}_x[m_\tau]=1$ *for every* $x\in S$, *then for every* n $\boldsymbol{E}_x[M_{t\wedge\tau_n}]=1$ *for* $x\in\hat{S}$.

Proof. Eirst it is clear that $\boldsymbol{E}_x[M_\tau]=1$ for every $x\in S$. Then $\boldsymbol{E}_x[M_{\tau_2}]=\boldsymbol{E}_x[M_{\tau_1+\tau_1(\theta_{\tau_1}w)}]=\boldsymbol{E}_x[M_{\tau_1}\boldsymbol{E}_{x_{\tau}}[M_{\tau_1}]]=1$, and repeating this we have $\boldsymbol{E}_x[M_{\tau_k}]=1$ for every k. Next we have

$$(5.23)\qquad M_{t\wedge\tau_n+\tau_1(\theta_{\tau_n\wedge t}w)}=\begin{cases} M_{\tau_{k+1}}, & \text{if } \tau_k\leq t<\tau_{k+1}, \ (k<n) \\ M_{\tau_{n+1}}, & \text{if } t\geq\tau_n, \end{cases}$$

and hence

$$
\begin{aligned}
&\boldsymbol{E}_x[M_{t\wedge\tau_n}] = \boldsymbol{E}_x[M_{t\wedge\tau_n}\boldsymbol{E}_{X_{\tau_n\wedge t}}[M_{\tau_1}]] \\
&= \boldsymbol{E}_x[M_{t\wedge\tau_n}M_{\tau_1(\theta_{\tau_n\wedge t}w)}(\theta_{\tau_n\wedge t}w)] \\
&= \boldsymbol{E}_x[M_{t\wedge\tau_n+\tau_1(\theta_{\tau_n\wedge t}w)}] \\
&= \sum_{k=1}^{n-1}\boldsymbol{E}_x[M_{\tau_{k+1}};\tau_k\leq t<\tau_{k+1}] + \boldsymbol{E}_x[M_{\tau_{n+1}};\ t\geq\tau_n].
\end{aligned}
$$

Also, if $k\leq n-1$,

$$
\begin{aligned}
&\boldsymbol{E}_x[M_{\tau_{k+1}};\ \tau_k\leq t<\tau_{k+1}] \\
&= \boldsymbol{E}_x[M_{\tau_{k+1}}\boldsymbol{E}_{x\tau_{k+1}}[M_{\tau_{n-k}}];\ \tau_k\leq t<\tau_{k+1}] \\
&= \boldsymbol{E}_x[M_{\tau_{k+1}}M_{\tau_{n-k}}(\theta_{\tau_{k+1}}w);\ \tau_k\leq t<\tau_{k+1}] \\
&= \boldsymbol{E}_x[M_{\tau_{n+1}};\ \tau_k\leq t<\tau_{k+1}].
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\boldsymbol{E}_x[M_{t\wedge\tau_n}] &= \sum_{k=0}^{n-1}\boldsymbol{E}_x[M_{\tau_{n+1}};\ \tau_k\leq t<\tau_{k+1}] + E[M_{\tau_{n+1}};\ t\geq\tau_n] \\
&= \boldsymbol{E}_x[M_{\tau_{n+1}}] = 1,
\end{aligned}
$$

which proves the lemma.

From this lemma we see that if $\boldsymbol{E}_x[M_\tau]=1$ for every $x\in S$, then $\boldsymbol{E}_x[M_t]\leq\varliminf_{n\to\infty}\boldsymbol{E}_x[M_{t\wedge\tau_n}]=1$ and $\boldsymbol{E}_x[M_t]=1$ if $\{M_{t\wedge\tau_n},\ n=0,1,2,\cdots\}$ is uniformly integrable. Summarizing we have the following

Theorem 5.3. *Let* $\boldsymbol{X}$ *be a branching Markov process and* $X^0 = \{W_1, \mathcal{N}_t|_{W_1},\ X_t,\ t<\tau, x\in S\}$ *be the non-branching part of* $\boldsymbol{X}$. *Let* m_t *be a multiplicative functional of* X^0 *satisfying either*

(i) $m_t\leq 1$

or

(ii) $\boldsymbol{E}_x[m_\tau]=1,\ x\in S$.

Then $M_t(w)$ *defined by* (5.12) *is an* $\mathcal{N}_{t+0}$*-multiplicative functional of* $\boldsymbol{X}$ *which is of branching type in the weak sense satisfying*

(i)′ $M_t\leq 1$

or

(ii)′ $\boldsymbol{E}_x[M_t]\leq 1,\ x\in S$

according as m_t *satisfies* (i) *or* (ii).

If further $\{M_{t\wedge\tau_n}, n=1,2,\cdots\}$ *is uniformly integrable, then we have in the case of* (ii)

(ii)'' $\boldsymbol{E}_x[M_t]=1,\ x\in S$.

§5.4. Transformation of drift

Let $X=(X_t, \mathcal{B}_t, P_x)$ be a Hunt process on S with a reference measure[7] and B_t be a continuous additive functional of the process X such that $F_x[B_t^2]<\infty$ and $E_x[B_t]=0$.[8] Then it is known that there exists a unique non-negative continuous additive functional $\langle B\rangle_t$ such that $E_x[B_t^2]=E_x[\langle B\rangle_t]$. Set

$$m_t=\exp\left\{B_t-\frac{1}{2}\langle B\rangle_t\right\}. \tag{5.24}$$

Lemma 5.5. *Let* σ *be a finite valued Markov time of* X *satisfying for every* $t>0$

$$\{t<\sigma\}\subset\{\sigma\leq t+\sigma(\theta_t w)\}.\text{[9]} \tag{5.25}$$

If $\sup_{x\in S}E_x[\langle B\rangle_\sigma]<\infty$, *then* $E_x[m_\sigma]=1$ *for every* $x\in S$.

Proof.[10] Set $\sigma_n=\inf\{t;\ |C_t|\geq n\}\wedge n,\ n=1,2,\cdots,$ where $C_t=B_t-\frac{1}{2}\langle B\rangle_t$; then we have

$$E_x[m_{\sigma\wedge\sigma_n}]=1,\ n=1,2,\cdots. \tag{5.26}$$

For, by a formula on stochastic integrals (cf. [27])

$$\begin{aligned} m_t-1=e^{C_t}-1&=\int_0^t m_s dB_s-\frac{1}{2}\int_0^t m_s d\langle B\rangle_s+\frac{1}{2}\int_0^t m_s d\langle B\rangle_s \\ &=\int_0^t m_s dB_s. \end{aligned}$$

Then, ncting $m_t\leq e^n$ for $t\leq\sigma\wedge\sigma_n$, we have $E_x[\int_0^{\sigma\wedge\sigma_n} m_s dB_s]=0$ proving (5.26). Next we shall prove

7) Cf. §0.1.

8) The class of such additive functionals was studied in [32].

9) If σ is a quasi-hitting time or $\sigma\equiv t$, (5.25) is clearly satisfied.

10) We have borrowed the essential part of the proof from Dynkin [6].

(5.27) $$\inf_{x\in S} E_x[m_\sigma] \equiv d > 0.$$

For, by the assumption $\sup_x E_x[\langle B\rangle_\sigma] < \infty$, we have

$$P_x[C_\sigma < -2k] \le P_x[|C_\sigma| > 2k] \le P_x[|B_\sigma| > k] + P_x\left[\frac{1}{2}\langle B\rangle_\sigma > k\right]$$

$$\le \frac{1}{k^2}E_x[B_\sigma^2] + \frac{1}{2k}E_x[\langle B\rangle_\sigma] = \left(\frac{1}{k^2}+\frac{1}{2k}\right)E_x[\langle B\rangle_\sigma] < \frac{1}{2}$$

for all x if k is sufficiently large. Then for all $x\in S$

$$E_x[m_\sigma] \ge E_x[m_\sigma;\ C_\sigma \ge -2k] \ge e^{-2k}P_x[C_\sigma \ge -2k]$$

$$= e^{-2k}(1 - P_x[C_\sigma < -2k]) \ge \frac{e^{-2k}}{2}$$

proving (5.27).

Now by (5.25) we have $\sigma \le \sigma_n + \sigma(\theta_{\sigma_n} w)$ and hence by the super-martingale inequality[11]

$$E_x[m_\sigma;\ \sigma_n \le \sigma] \ge E[m_{\sigma_n + \sigma(\theta_{\sigma_n} w)};\ \sigma_n \le \sigma]$$

$$= E_x[m_{\sigma_n} E_{X_{\sigma_n}}[m_\sigma];\ \sigma_n \le \sigma] \ge dE_x[m_{\sigma_n};\ \sigma_n \le \sigma].$$

Therefore we have, (noting $\sigma_n \uparrow \infty$),

(5.28) $$\lim_{n\to\infty} E_x[m_{\sigma_n};\ \sigma_n \le \sigma] = 0.$$

Then

$$1 = E_x[m_{\sigma\wedge\sigma_n}] = E_x[m_\sigma;\ \sigma < \sigma_n] + E_x[m_{\sigma_n};\ \sigma_n \le \sigma]$$

and by (5.28)

$$\lim_{n\to\infty} E_x[m_\sigma;\ \sigma < \sigma_n] = E_x[m_\sigma] = 1.$$

Now assume that the non-branching part X^0 of a (X^0, π)-branching Markov process $\boldsymbol{X}$ is equivalent to an e^{-A_t}-subprocess of a Hunt process $X = (X_t, \mathcal{B}_t, P_x)$, where A_t is a continuous non-negative additive functional of X. Then X^0 is equivalent to the process $\{\overline{X}_t, P_x\}$ defined by (0.12) and (0.13). By enlarging $\mathcal{B}_t$ if necessary we can assume that the life time $\overline{\zeta}$ defined by (0.12) is a $\mathcal{B}_t$-Markov

11) It is easy to see that $(m_t, \mathcal{B}_t)$ is a supermartingale for every P_x; we have $E_x[m_{t\wedge\sigma_n}]=1$ just as (5.26) and hence $E_x[m_t] \le \underline{\lim}\, E_x[m_{t\wedge\sigma_n}] = 1$ for every t. Thus $E_x[m_{t+s}|\mathcal{B}_s] = E_{X_s}[m_t]\cdot m_s \le m_s$ a.s.

time for which (5. 25) is easily verified. If further the condition

$$\sup_{x\in S} E_x[\langle B\rangle_\zeta]<\infty \tag{5.29}$$

is satisfied, then we have

$$E_x[m_\zeta]=1.$$

Now $m_{t\wedge\zeta}$ can be considered as a multiplicative functional of X^0 and applying Theorem 5. 3 we have a multiplicative functional M_t of $\boldsymbol{X}$. We shall call this M_t a *multiplicative functional of drift.*

Example 5.4. Let $X=\{x_t=(x_t^1, x_t^2, \cdots, x_t^N), P_x\}$ be an N-dimensional Brownian motion[12] and $A_t=\int_0^t k(x_s)ds$, where $k\in \boldsymbol{C}(S)$ such that $k(x)\geq c>0$. Let

$$B_t=\sum_{i=0}^{N}\int_0^t b_i(x_s)dx_s^i$$

where $b_i(x)$, $i=1, 2, \cdots, N$, are bounded continuous functions on R^N. Then $\langle B\rangle_t=\sum_{i=1}^{N}\int_0^t |b_i|^2(x_s)ds$. In this case the conditions (5. 29) can be easily verified, and hence we have a multiplicative functional of drift M_t for every branching Markov process $\boldsymbol{X}$ whose non-branching part is equivalent to X^0. The backward equation of $\boldsymbol{X}$ is given by

$$\frac{\partial u}{\partial t}=\frac{1}{2}\bar{\Delta}u+k(x)\cdot\{F(x;\, u)-u\},$$

while the backward equation of X^M is given by

$$\frac{\partial u}{\partial t}=\frac{1}{2}\bar{\Delta}u+\sum_{i=1}^{N}b_i\frac{\partial u}{\partial x_i}+k(x)(F(x;\, u)-u).$$

Thus M_t induces a drift.

§5.5. Another transformation.

The following transformation is a generalization of a well known transformation for a branching process of a single type $(S=\{a\})$,

12) We take as S the one-point compactification of R^N. cf. Chapter III Ex. 3(A).

(cf. Harris [8], p 14).

Let X be a branching Markov process with the semi-group $\boldsymbol{T}_t$ such that $q(x)<1$ for every $x\in S$ where

$$q(x)=\lim_{t\to\infty}\boldsymbol{T}_t\widehat{0}(x)=\boldsymbol{P}_x[e_\partial<\infty].^{13)} \tag{5.30}$$

Theorem 5.4. *There exists a (unique) branching semi-group $\widetilde{\boldsymbol{T}}_t$ (and hence a branching Markov process) such that*

$$\widetilde{\boldsymbol{T}}_t\widehat{f}|_S(x)=\frac{1}{1-q(x)}\{\boldsymbol{T}_t\widehat{(q+f(1-q))}|_S(x)-q(x)\}. \tag{5.31}$$

Proof. It is sufficient to to show that there exists a substochastic kernel $\mu_t(x,dy)$ on $S\times S$ such that the right-hand side of (5.31) is equal to $\int\widehat{f}(y)\mu_t(x,dy)$, since, then by Lemma 0.3 there exists a unique substochastic kernel $\widetilde{\boldsymbol{T}}_t(x,dy)$ on $S\times S$ such that

$$\int_S\widetilde{\boldsymbol{T}}_t(x,dy)\widehat{f}(y)=\widehat{\int_S\mu_t(\cdot,dy)\widehat{f}(y)}\,(x),$$

and the semi-group property of $\widetilde{\boldsymbol{T}}_t$ is obvious from (5.31). First we note $\boldsymbol{T}_t\hat{q}=\hat{q}$ since $\boldsymbol{T}_t\hat{q}=\lim_{s\to\infty}\boldsymbol{T}_t\boldsymbol{T}_s\widehat{0}=\lim_{s\to\infty}\widehat{\boldsymbol{T}_{t+s}\widehat{0}|_S}=\hat{q}$. Then

$$\begin{aligned}&\frac{1}{1-q(x)}\{\boldsymbol{T}_t(\widehat{q+f(1-q)})(x)-q(x)\}\\&=\frac{1}{1-q(x)}\int_S\boldsymbol{T}_t(x,dy)[\widehat{(q+f(1-q))}(y)-\hat{q}(y)]\\&=\frac{1}{1-q(x)}\int_S\boldsymbol{T}_t(x,dy)\{\sum_{y'\prec y}{}^*\hat{q}(y')\widehat{(1-q)}(y'')\widehat{f}(y'')\}.^{14)}\end{aligned}$$

But for fixed x and t,

$$\mu_t^x(g)=\int_S\boldsymbol{T}_t(x,dy)\{\sum_{y'\prec y}{}^*\hat{q}(y')\widehat{(1-q)}(y'')g(y'')\},\quad g\in\boldsymbol{B}(S)$$

defines clearly a non-negative linear functional on $\boldsymbol{B}(S)$ and hence

13) $q(x)$ is called the *extinction probability*.

14) For fixed $y\in S$, $y=[y_1,\cdots,y_n]$, we denote $y'\prec y$ if $y'=[y'_1,\cdots,y'_k]$, $k\leq n$, such that $y'_i=y_{l_i}$ for some l_i, $1\leq l_i\leq n$ and all l_i, $i=1,2,\cdots,k$ are different. $y''=[y''_1,\cdots,y''_{n-k}]$ is the remainder of y excluding y', i.e., y' and y'' define a partition of y. $\sum_{y'\prec y}^*$ denotes the sum (for fixed y) over all y' such that $y'\prec y$ and $y'\neq y$.

it is given by a non-negative Radon measure $\mu_t(x, dy)$. μ_t is a substochastic kernal on $S\times \boldsymbol{S}$ since

$$\int_S \mu_t(x, dy)\widehat{1}(y) = \frac{1}{1-q(x)}(\boldsymbol{T}_t\widehat{1}(x) - q(x)) \leq 1.$$

From (5. 31) we have $\widetilde{\boldsymbol{T}}_t\widehat{0}(x) = \dfrac{1}{1-q(x)}\{q(x) - q(x)\} = 0$; i.e., *for the transformed branching process the extinction probability is identically zero.*

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