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On fibre bundles and their homotopy groups

Dedicated to Professor Atuo Komatu on the occasion
of his 60th birthday

By

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1. Introduction

The group of problems associated with vector fields on spheres has long attracted the attention of topologists. Now that the original problems have mostly been cleared up it seems appropriate to consider generalizations. One of these has been studied in [4] but the results are mainly confined to what might be described as the stable version of the problem. The purpose of this note is to prepare for further investigations outside the stable range.

Let H be a subgroup of a topological group G. The action of H on G by left translation determines an action of H on the factor space $F=G/H$ of right cosets. Note that

$$x\cdot e=e \qquad (x\in H), \tag{1.1}$$

where $e\in F$ denotes the coset of the neutral element. Let B be a space and let E be a fibre bundle over B with group H and fibre F. In view of (1.1) there exists a canonical cross-section. Let G' be a subgroup of G such that $H\subset G'\subset G$. Associated with E we have a bundle X over B with fibre $Y=G/G'$. We regard E as a bundle over X with fibre $F'=G'/H$, in the obvious way. Consider the following diagram, where σ denotes the inclusion of the fibre in the total space, and ρ denotes the projection of the total space on the base.

$$\begin{array}{ccccc} F' & \xrightarrow{1} & F' & & \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} & & \\ F & \xrightarrow{\sigma} & E & \xrightarrow{\rho} & B \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle 1} \\ Y & \xrightarrow[\sigma]{} & X & \xrightarrow[\rho]{} & B \end{array}$$

Under what conditions does there exist a cross-section of E over X?

Suppose that there exists such a cross-section $h: X \to E$. By restriction h determines a cross-section $g: Y \to F$. We describe g, or its vertical homotopy class, as the *type* of h. Let $f: B \to E$ denote the canonical cross-section of E over B. Then $\rho f: B \to X$ is the canonical cross-section of X over B. We describe $h: X \to E$ as *proper* if

(1.2) $$h\rho f = f.$$

Our approach will be to take a cross-section $g: Y \to F$ and then ask whether E admits a proper cross-section over X of type g.

(Suppose, originally, that E is a bundle with group G' rather than H. The canoical cross-section $f': B \to X$ is defined. If $h: X \to E$ is a cross-section then we can reduce the group from G' to H in such a way that $hf': B \to E$ is canonical and so h is proper.)

When the bundle structure is trivial there exist proper cross-sections of arbitrary type. In general, however, the answer to our question depends on the choice of g. Assuming that the spaces concerned satisfy the hypotheses of an appropriate version of the covering homotopy theorem, the answer depends on the vertical homotopy class of g, rather than on g itself.

An automorphism ψ of H transforms H-bundles into H-bundles, in the standard way. Thus E, X are transformed into ψ_*E, ψ_*X, say. If E admits a proper cross-section over X of a certain type then ψ_*E admits a proper cross-section over ψ_*X of the same type.

In what follows we shall mainly be concerned with the case where the base space is a sphere or other suspended space. We show in §2 how the original problem can be converted into a more tractable problem in homotopy theory. Obstructions are obtained which can be expressed in terms of the mixed products of McCarty [6]. These give necessary conditions for the existence of a proper cross-section of given type. When the fibre Y and base B of X are both spheres there is just one obstruction, whose vanishing is both necessary and sufficient for the existence of a proper cross-section. The corresponding classification problem has been solved by Barcus and Barratt [2], in this case.

2. The main theorem

Let A be a CW-complex with base-point a_0. We form the (reduced) cone CA and suspension SA in the usual way. Thus A is embedded in CA as the base of the cone, and SA is obtained from CA by collapsing A.

Let G be a Lie group with closed subgroup H and factor space $F=G/H$. Given a map u: $A\to H$ we construct a space E from the union of $CA\times F$ and F by identifying $(a, x)\in A\times F$ with $u(a)\cdot x$. We define $\rho=\pi 1\theta^{-1}$, as shown below, where 1 is left projection and π, θ are the identification maps.

$$\begin{array}{ccc} CA\times F & \xrightarrow{\theta} & E \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \rho} \\ CA & \xrightarrow[\pi]{} & SA \end{array}$$

By method described in §3 [5] we can give E the structure of a fibre bundle over SA with projection ρ, fibre F and group H. Moreover E belongs to the class of bundle determined by u in the standard classification (see §18 of [7]), and the canonical cross-section is the map f which makes the following diagram commutative, where $i(c)=(c, e)$ $(c\in CA)$.

$$\begin{tikzcd} CA \arrow[r, "\pi"] \arrow[d, "i"'] & SA \arrow[d, "f"] \\ CA\times F \arrow[r, "\theta"'] & E \end{tikzcd}$$

Now let G' be a closed subgroup of G such that $H\subset G'\subset G$. Let τ: $H\rightarrow G'$ be the inclusion and let ρ: $F\rightarrow Y$ be the natural projection where $Y=G/G'$ as before. Let X be constructed in the same way as E except that u is replaced $v=\tau u$: $A\rightarrow G'$, and F is replaced by Y. Then a fibration

(2.1) $$\rho=\varphi(1\times\rho)\theta^{-1}$$

is defined, as shown in the following diagram, where φ denotes the identification map

$$\begin{tikzcd} CA\times F \arrow[r, "\theta"] \arrow[d, "1\times\rho"'] & E \arrow[d, "\rho"] \\ CA\times Y \arrow[r, "\varphi"'] & X \end{tikzcd}$$

Let g: $Y\rightarrow F$ be a cross-section and let λ, μ be the maps determined by θ, φ, respectively, as shown in the following diagram.

$$\begin{tikzcd} A\times Y \arrow[r, "v\times 1"] \arrow[d, "1\times g"'] & G'\times Y \arrow[r, "\mu"] & Y \arrow[d, "g"] \\ A\times F \arrow[r, "u\times 1"] \arrow[d, "1\times\rho"'] & H\times F \arrow[r, "\lambda"] & F \arrow[d, "\rho"] \\ A\times Y \arrow[r, "v\times 1"'] & G'\times Y \arrow[r, "\mu"'] & Y \end{tikzcd}$$

Note that the lower rectangle is commutative, and so is the whole square with the centre omitted. In the upper rectangle the two routes from $A\times Y$ to F agree on the subspace

$$A\vee Y=A\times e\cup a_0\times Y.$$

The purpose of this section is to prove

Theorem 2.2. *There exists a proper cross-section of E over X, of type g, if and only if*

$$\lambda(u\times g)\simeq g\mu(v\times 1) \text{ rel } A\vee Y.$$

We first show that the existence of the homotopy implies the existence of a cross-section. Recall that the identification map θ agrees with $\lambda(u\times 1)$ on $A\times F$ and so agrees with $g\mu(v\times 1)$ on $A\times e$. Hence there exists a map ψ', as shown below, which agrees with θ on $CA\times e$ and with $g\mu(v\times 1)$ on $A\times Y$.

$$\begin{tikzcd} CA\times e^1\cup A\times F \arrow[r, "\theta'"] \arrow[d, "1\times\rho"'] & E \arrow[d, "\rho"] \\ CA\times e^1\cup A\times Y \arrow[r, "\varphi'"'] \arrow[ur, "\psi'"] & X \end{tikzcd}$$

In the diagram θ', φ' are the restrictions of the identification maps. The definition of ψ' is such that both triangles are commutative. Now

$$\theta'(1\times g):\ CA\times e^1\cup A\times Y\to E$$

can be extended over $CA\times Y$ by $\theta(1\times g)$ so that

$$\rho\theta(1\times g)=\varphi(1\times\rho)(1\times g)=\varphi,$$

by (2.1). Suppose that

$$\lambda(u\times g)\simeq g\mu(v\times 1) \text{ rel } A\vee Y.$$

Then $\theta'(1\times g)\simeq\psi'$, rel $CA\vee Y$, by a vertical homotopy (with respect to $\rho: E\to B$). Hence, by the covering homotopy theorem, there exists an extension

$$\psi: CA \times Y \to E$$

of ψ' such that $\rho\psi = \varphi$. Since φ is non-singular outside $A \times Y$ the map

$$h = \psi\varphi^{-1}: X \to E$$

is well-defined, and constitutes a proper cross-section of type g.

Conversely, suppose that there exists such a cross-section h. Then $h\varphi$ and $\theta(1 \times g)$, as shown below, agree on $CA \vee Y$.

$$\begin{array}{ccc} CA \times Y & \xrightarrow{\varphi} & X \\ {\scriptstyle 1\times g}\downarrow & & \downarrow{\scriptstyle h} \\ CA \times F & \xrightarrow[\theta]{} & E \end{array}$$

Let σ: $F \to E$ denote the inclusion. We have the homotopy

$$k_t: A \times Y \to E \qquad (t \in I)$$

of $\sigma\lambda(u \times g)$ into $\sigma g\mu(v \times 1)$,

which is defined by

$$\begin{aligned} k_t(a, Y) &= \theta\ ((a, 1-2t), gY) && (t \leqslant 1/2), \\ &= h\varphi((a, 2t-1), Y) && (t \geqslant 1/2), \end{aligned}$$

where $a \in A$, $Y \in Y$. We will deform k_t into a homotopy l_t which keeps $A \vee Y$ fixed and which does not move $A \times Y$ out of the fibre. Consider the deformation

$$H_s: A \times I \times Y \to X \qquad (s \in I)$$

which is given by

$$\begin{aligned} H_s(a, t, y) &= \varphi((a, 1-2t), y) && (2t \leqslant 1+s), \\ &= \varphi((a, s), y) && (1-s \leqslant 2t \leqslant 1+s), \\ &= \varphi((a, 2t-1), y) && (2t \geqslant 1+s). \end{aligned}$$

We lift $H_s | A \times I \times e$ to

$$K_s\colon A\times I\times e\to E$$

where K_s is given by the same formula as $H_s|A\times I\times e$ except that φ is replaced by θ. Write $k_t(a, y)=k\ (a, t, y)$. Then k agrees with K_0 on $A\times I\times e$, and ρk agrees with H_0 on $A\times I\times Y$, by (2.1). By the covering homotopy theorem we can lift H_s to a deformation

$$L_s\colon A\times I\times Y\to E,$$

extending K_s, such that $L_0=k$ and such that L_s is stationary when H_s is stationary. We define $l_t(a, y)=l(a, t, y)$, where $l=L_1$. Since H_s is stationary on $A\times \dot{I}\times Y$, so is L_s. In particular l agrees with k on $A\times \dot{I}\times Y$, and so l_t, like k_t, constitutes a homotopy of $\sigma\lambda(u\times g)$ into $\sigma g\mu(v\times 1)$. Similarly l_t, like k_t, keeps $a_0\times Y$ fixed. On $A\times I\times e$, l is given by K_1, which is constant. Hence l_t keeps Axe fixed, as well as $a_0\times Y$, and so constitutes a homotopy rel $A\vee Y$. Finally $\rho l=H_1$, which is independent of t, and so l_t is vertical. Since l_t begins in the fibre it remains in the fibre throughout. This completes the proof of (2.2).

3. The mixed product

Let F be a space, with basepoint e, and let H be a topological transformation group acting on F. We denote the transform of $x\in F$ under $g\in H$ by $g\cdot x$. Suppose that

$$g\cdot e=e \qquad (g\in H). \tag{3.1}$$

Then the *mixed product* $\langle\alpha, \beta\rangle\in\pi_{p+q}(F)$ of an element $\alpha\in\pi_p(H)$ with an element $\beta\in\pi_q(F)$ can be defined as follows. Take representatives

$$u\colon S^p\to H,\ \ v\colon S^q\to F$$

of α, β, respectively, and let

$$h, k\colon\ S^p\times S^q\to F$$

be the maps given by

$$h(\xi, \eta) = u(\xi) \cdot v(\eta),$$
$$k(\xi, \eta) = v(\eta),$$

where $\xi \in S^p$, $\eta \in S^q$. Since h and k agree on $S^p \vee S^q$, by (3.1), their separation element $d(h, k) \in \pi_{p+q}(F)$ is defined. We write $d(h, k) = \langle \alpha, \beta \rangle$, after checking that the element is independent of the choice of representatives. For the basic properties of this product, such as linearity in both variables, the reader is referred[1] to [6].

As an illustration take $F = H$, acting on itself by conjugation. The action satisfies (3.1) and it is easy to show that the mixed product in this case is equivalent to the ordinary Samelson product in the homotopy groups of H.

The main example, for present purposes, is when we have a topological group G containing H as a subgroup. The action of H on G, by left translation, determines an action of H on the factor space G/H of left cosets, satisfying (3.1), and so the mixed product pairs $\pi_p(H)$ with $\pi_p(G/H)$ to $\pi_{p+q}(G/H)$. Among the properties of this pairing the main ones we need are as follows.

Let ψ be an automorphism of G which maps H into itself. Then ψ induces automorphisms of H and G/H, which we also denote by ψ, and the mixed product satisfies

$$\psi_* \langle \alpha, \beta \rangle = \langle \psi_* \alpha, \psi_* \beta \rangle, \tag{3.2}$$

where $\alpha \in \pi_p(H)$, $\beta \in \pi_q(G/H)$.

Let G' be a subgroup of G such that $H \subset G' \subset G$. Consider the inclusion τ: $H \to G'$ and the inclusion and projection

$$\frac{G'}{H} \xrightarrow{\sigma} \frac{G}{H} \xrightarrow{\rho} \frac{G}{G'} .$$

If $\alpha \in \pi_p(H)$ then it follows at once from the definition that

$$(3.3) \quad \begin{cases} \text{a)} \ \sigma_* \langle \alpha, \beta' \rangle = \langle \alpha, \sigma_* \beta' \rangle & (\beta' \in \pi_q(G'/H)), \\ \text{b)} \ \rho_* \langle \alpha, \beta \rangle = \langle \tau_* \alpha, \rho_* \beta \rangle & (\beta \in \pi_q(G/H)), \end{cases}$$

1) McCarty works in a somewhat different framework but his proofs cover everything needed here.

as indicated in the following diagram, where the verticals are mixed product pairings of appropriate type.

$$\begin{array}{ccccc}
\pi_p(H)\otimes\pi_q(G'/H) & \xrightarrow{1\otimes\sigma^*} & \pi_p(H)\otimes\pi_q(G/H) & \xrightarrow{\tau_*\otimes\rho_*} & \pi_p(G')\otimes\pi_q(G/G') \\
\downarrow & & \downarrow & & \downarrow \\
\pi_{p+q}(G'/H) & \xrightarrow[\sigma_*]{} & \pi_{p+q}(G/H) & \xrightarrow[\rho_*]{} & \pi_{p+q}(G/G')
\end{array}$$

An important example is when G acts transitively on S^q with G' as stability subgroup so that G/G' can be identified with S^q. Then

$$\langle\alpha', \iota_q\rangle = J\alpha' \ (\alpha'\in\pi_p(G')), \tag{3.4}$$

where $\iota_q\in\pi_q(S^q)$ is the class of the identity and where

$$J\colon \pi_p(G)\to\pi_{p+q}(S^q)$$

has its usual meaning.

4. The obstruction

Having defined the mixed product we now return to the situation of §2. Recall that $g\colon Y\to F$ is a cross-section and so, for $r\geqslant 2$, we have a direct sum decomposition

$$\pi_r(F')\oplus\pi_r(Y)\approx\pi_r(F),$$

given by σ_* on the first summand and by g_* on the second.

As before, let E be the bundle over $B=SA$ with fibre F defined by $u\colon A\to H$, and let X be the bundle over B with fibre Y defined by $v=\tau u\colon A\to G'$. We fibre E over X as in (2.1). Consider the mixed product

$$\langle u_*\alpha, g_*\beta\rangle\in\pi_{p+q}(F),$$

where $\alpha\in\pi_p(A)$, $\beta\in\pi_q(Y)$. The component of this product in the second summand is $\langle v_*\alpha, \beta\rangle$, by (3.3b). We now show that the component in the first summand is an obstruction to the existence of a proper cross-section of E over X, of type g.

For convenience of reference we repeat the main diagram studied in §2.

$$\begin{tikzcd} A\times Y \arrow[r, "v\times 1"] \arrow[d, "1\times g"'] & G'\times Y \arrow[r, "\mu"] & Y \arrow[d, "g"] \\ A\times F \arrow[r, "u\times 1"'] & H\times F \arrow[r, "\lambda"'] & F \end{tikzcd}$$

If we replace λ, μ by the right projections λ', μ' then

(4.1) $\qquad \lambda'(u\times g)=g\mu'(v\times 1).$

Let ξ: $S^p\to A$, η: $S^q\to Y$ be representatives of α, β, respectively. The maps

$$\lambda(u\times g)(\xi\times\eta),\ g\mu(v\times 1((\xi\times\eta):\ S^p\times S^q\to A$$

agree on $S^p\vee S^q$, and so their separation element is defined. I assert that

(4.2) $$d(\lambda(u\times g)(\xi\times\eta),\ g\mu(v\times 1)(\xi\times\eta))$$
$$=\langle u_*\alpha, g_*\beta\rangle-g_*\langle v_*\alpha, \beta\rangle.$$

For by definition of the mixed product we have

$$\langle u_*\alpha, g_*\beta\rangle=d(\lambda(u\times g)(\xi\times\eta),\ \lambda'(u\times g)(\xi\times\eta)),$$
$$\langle v_*\alpha, \beta\rangle=d(\mu(v\times 1)(\xi\times\eta),\ \mu'(v\times 1)(\xi\times\eta)).$$

By naturality

$$g_*\langle v_*\alpha, \beta\rangle=d(g\mu(v\times 1)(\xi\times\eta),\ g\mu'(v\times 1)(\xi\times\eta))$$
$$=d(g\mu(v\times 1)(\xi\times\eta),\ \lambda'(u\times g)(\xi\times\eta)),$$

by (4.1). Hence (4.2) follows at once from the addition formula for the separation element.

Combining (4.2) with the first part of (2.2) we obtain

Theorem 4.3. *Suppose that E admits a proper cross-section over X of type q. Then*

$$\langle u_*\alpha, g_*\beta\rangle=g_*\langle v_*\alpha, \beta\rangle$$

for all elements $\alpha \in \pi_p(A)$, $\beta \in \pi_q(Y)$.

Now suppose that $A = S^p$, $Y = S^q$. We take ξ and η to be the identity maps, and so can omit them from (4.2). Then

$$\lambda(u \times g) \simeq g\mu(v \times 1) \text{ rel } S^p \vee S^q,$$

if, and only if,

$$d(\lambda(u \times g), g\mu(v \times 1)) = 0.$$

Hence, using both parts of (2.2) we obtain

Theorem 4.4. *Let* $B = S^{p+1}$, $Y = S^q$. *Let* $\theta \in \pi_p(H)$, $\tau_* \theta \in \pi_p(G')$ *denote the classes of* u, v, *and let* $\gamma \in \pi_q(Y)$ *be the class of a cross-section. Then* E *admits a proper cross-section over* X *of type if, and only if,*

$$\langle \theta, \gamma \rangle = \gamma_* \langle \tau_* \theta, \iota_q \rangle.$$

The situation can be interpreted as follows. Use the canonical cross-section to embed S^{p+1} in X, so that the intersection with S^q is the basepoint. The complement of $S^{p+1} \vee S^q$ is a $(p+q+1)$-cell and so we can give X the structure of a complex, as shown in §3 of [5]. The cross-sections f: $S^{p-1} \to E$ and g: $S^q \to F$ determine a cross-section of E over $S^{p+1} \vee S^q$. By classical theory the only obstruction to extending this over the whole of X is an element of $\pi_{p+q}(F')$. Although it is unnecessary to do so for present purposes it can be shown, as might be expected, that this element is given by

$$\sigma_*^{-1}(\langle \theta, \gamma \rangle - \gamma_* \langle \tau_* \theta, \iota_q \rangle)$$

where $\gamma \in \pi_q(Y)$ is the class of g.

5. Frame-bundles

We adopt the same conventions as in [3] for the Stiefel manifolds and the various constructions associated with them. Thus O_m

denotes the group of orthogonal $m\times m$ matrices. We write $V_{m,k} = O_m/O_{m-k}$ $(1\leqslant k\leqslant m)$, where O_{m-k} denotes the subgroup in which the last k rows and columns (apart from the diagonal) are trivial. We represent points of $V_{m,k}$ by orthogonal matrices with m rows and k columns, so that the projection $\rho\colon O_m\rightarrow V_{m,k}$ is defined by taking the last k columns of the $m\times m$ matrix.

Let ξ be an R^m-bundle over B. Let ξ_k $(k=1, 2, \cdots)$ denote the associated bundle of orthonormal k-frames. Consider the fibration $\rho\colon \xi_k\rightarrow\xi_1$ given by taking the last vector of each k-frame to form a 1-frame. In [4] we have studied the problem of whether this fibration admits a cross-section. Let us suppose that the group of ξ has been reduced from O_m to O_{m-k}. We take

$$(G, G', H) = (O_m, O_{m-1}, O_{m-k})$$

in the general problem, so that

$$(F', F, Y) = (V_{m-1,k-1}, V_{m,k}S^{m-1}).$$

Thus our diagram now takes the following form

$$\begin{array}{ccccc}
V_{m-1,k-1} & \xrightarrow{1} & V_{m-1,k-1} & & \\
{\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\sigma} & & \\
V_{m,k} & \xrightarrow{\sigma} & \xi_k & \xrightarrow{\sigma} & B \\
{\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\rho} & & \big\downarrow{\scriptstyle 1} \\
S^{m-1} & \xrightarrow[\sigma]{} & \xi_1 & \xrightarrow[\rho]{} & B
\end{array}$$

Most of the results of [4] can be translated into the present frame-work, with suitable modifications. However, the only application of the methods of [4] we shall make is to prove

Theorem 5.1. *Let B be a finite complex such that dim $B\leqslant m-2k$. Let ξ be an R^m-bundle over B such that ξ_k admits a cross-sectionover ξ_1. Then ξ_k admits a cross-section over ξ_1 of any given type.*

The dimensionality restriction ensures that the group of ξ can be reduced from O_m to O_{m-k} and that all such reductions are equivalent. Hence every cross-section of ξ_k over ξ_1 is equivalent to a proper cross-section and so this condition can be ignored. By hypothesis ξ admits A_k-structure, in the terminology of [4]. By (1.6) of [4], the elements of the group $\widetilde{K}_R(B)$ which are representable by bundles with A_k-structure form a subgroup. Hence there exists a bundle η such that the Whitney sum $\xi\oplus\eta$ is trivial and such that η admits A_k-structure. Take any cross-section of η_k over η_1 and let $\beta\in\pi_{n-1}(V_{n,k})$ be its type, where $n=\dim\eta$. Let $\alpha\in\pi_{m-1}(V_{m,k})$ be the given class of cross-section. Then the intrinsic join

$$\alpha*\beta\in\pi_{m+n-1}(V_{m+n,k})$$

is also the class of a cross-section. Now $(\xi\oplus\eta)_k$ admits a cross-section over $(\xi\oplus\eta)_1$ of any given type, since $\xi\oplus\eta$ is trivial. In particular $(\xi\oplus\eta)_k$ admits a cross-section over $(\xi\oplus\eta)_1$ of type $\alpha*\beta$, and so it follows from the proof of (1.6) of [4] that ξ_k admits a cross-section over ξ_1 of a type α, as asserted.

Outside the stable range, as we shall see, there may exist a cross-section of one type but not of another. Before we look at some particular cases there is one more result of a general nature which it is convenient to mention here.

In O_m let d denote the diagonal matrix with -1 in the last place and $+1$ elsewhere. Conjugation by d leaves elements of O_{m-k} fixed ($k\geqslant 1$) and so induces a map $c\colon V_{m,k}\to V_{m,k}$. The effect of c is to change the sign of the last row and the last column. We prove

Theorem 5.2. *Let ξ be an R^m-bundle over B with group reduced from O_m to O_{m-k}. Suppose that ξ_k admits a proper cross-section over ξ_1 of type $\alpha\in\pi_{m-1}(V_{m,k})$. Then ξ_k admits a proper cross-section over ξ_1 of type-$c_*\alpha$.*

Since d commutes with the elements of O_{m-k}, the group of ξ, it follows that d determines a bundle map $\beta\colon \xi=\xi$ which in turn determines bundle maps β_1, β_k as shown below.

$$\begin{tikzcd} \xi_k \arrow[r, "\beta_k"] \arrow[d, "\rho"'] & \xi_k \arrow[d, "\rho"] \\ \xi_1 \arrow[r, "\xi_1"'] & \xi_1 \end{tikzcd}$$

If $h: \xi_1 \to \xi_k$ is a proper cross-section then so is $\beta_k h \beta_1^{-1}$. To determine the relation between the types we examine the case when B is a point. The result is as in (5.2).

In the next section we determine c_* in case $k=2$. When m is even and $m>4$ there are two classes of cross-sections of $V_{m,2}$. We shall calculate the action of c_* on these two classes and as a result we shall obtain

Corollary 5.3. *Let $m>4$ and $m\equiv 0$ mod 4. Let ξ be an R^m-bundle with group reduced from O_m to O_{m-2}. Let ξ_2 admits a cross-section over ξ_1 of one type then ξ_2 admits a cross-section over ξ_1 of the other.*

We shall also give an example to show that (5.3) can break down when $m\equiv 2$ mod 4.

6. Row and column operations

The Stiefel manifold $V_{m,k}$ consists of orthogonal matrices with m rows and k columns. Let

$$p, q: V_{m,k} \to V_{m,k}$$

denote the operations of changing the sign of a row, column respectively. The map, of course, depends on the choice of row or column, but the free homotopy class is independent of this choice, as pointed out in §1 of [3]. Basepoints can be ignored when $m \geqslant k+2$, since the manifold is simply-connected, and then the induced automorphisms

$$p_*, q_*: \pi_r(V_{m,k}) \to \pi_r(V_{m,k})$$

are defined without ambiguity. The automorphism c_* of $\pi_r(V_{m,k})$ is defined, as at the end of the last section, for $m \geqslant k$: when

$m \geqslant k+2$ we have the relation

$$c_* = p_* q_* . \tag{6.1}$$

The purpose of the present section is to determine these automorphisms when m is even and $k=2$.

Let $\beta_m \in \pi_{m-1}(V_{m,2})$ (m even) denote the homotopy class of the cross-section g_m: $S^{m-1} \to V_{m,2}$ whose value on

$$x = (x_1, \cdots, x_m) \in S^{m-1}$$

is the transpose of the matrix

$$\begin{pmatrix} -x_2 x_1 \cdots -x_m x_{m-1} \\ x_1 x_2 \cdots \quad x_{m-1} x_m \end{pmatrix}$$

Let $\alpha_m \in \pi_{m-2}(V_{m,2})$ denote the class of the inclusion of the $(m-2)$-sphere $V_{m-1,1} \subset V_{m,2}$. Then every element of $\pi_r(V_{m,2})$ can be expressed uniquely in the form

$$\alpha_m \circ \theta + \beta_m \circ \phi , \tag{6.2}$$

where $\theta \in \pi_r(S^{m-2})$, $\phi \in \pi_r(S^{m-1})$.

Clearly c_* leaves α_m fixed. We shall prove that

$$\begin{cases} c_* \beta_m = -\beta_m & (m \equiv 2 \bmod 4), \\ \quad\;\;\; = -\beta_m + \alpha_m \circ \eta & (m \equiv 0 \bmod 4), \end{cases} \tag{6.3}$$

where η generates $\pi_{m-1}(S^{m-2})$. In view of (6.2) this determines c_* in all dimensions.

Our proof of (6.3) will be by induction on m, raising the value by multiples of 4. The result is elementary when $m=2$. Consider the case $m=4$. The automorphism c acts on O_4 so that $\rho c = c \rho$, where ρ: $O_4 \to V_{4,2}$. The group $\pi_3(O_4)$ is freely generated by elements α, β such that

$$\rho_* \alpha = \alpha_4 \circ \eta, \quad \rho_* \beta = \beta_4 ,$$

where $\eta \in \pi_3(S^2)$ denotes the Hopf class. Now $c_* \beta = \alpha - \beta$, by (22.7) of [7], and so

$$c_*\beta_4 = c_*\rho_*\beta = \rho_* c_*\beta = \rho_*\alpha - \rho_*\beta = \alpha_4\circ\gamma - \beta_4 .$$

This proves (6. 3) when $m=4$. Recall that c acts on O_4 through conjugation by a diagonal matrix of determinant -1. Hence

$$c \simeq c' \colon O_4 \to O_4$$

where c' acts through conjugation by the non-trivial element of O_1. Now $\rho c' = p\rho$, where p changes the sign of the first row of $V_{4,2}$, and so $p\rho \simeq \rho c = c\rho$. Since $\beta_4 = \rho_*\beta$ it follows that

(6. 4) $$p_*\beta_4 = c_*\beta_4, \quad q_*\beta_4 = \beta_4 ,$$

To establish (6. 3) in general we use the intrinsic join structure. It is easy to check that

(6. 5) $$\begin{cases} \text{a)} \quad \alpha_{m+n} = \beta_n * \alpha_m , \\ \text{b)} \quad \beta_{m+n} = \beta_n * \beta_m , \end{cases}$$

where m, n are even and where $m \geqslant 4$ in (6. 5a). Take $n=4$ in these relations. Directly from the definition of the intrinsic join we have that

$$\begin{aligned} c_*\beta_{m+4} = c_*(\beta_4 * \beta_m) &= (q_*\beta_4) * (c_*\beta_m) \\ &= \beta_4 * (c_*\beta_m), \end{aligned}$$

by (6. 4). Suppose now that (6. 3) is true for some value of m. We express $\beta_4 * (c^*\beta_m)$ in standard form, using (6. 5) and the elementary properties of the intrinsic join. The result is (6. 3) with $m+4$ in place of m. By induction, therefore, (6. 3) is established. A similar argument shows that

(6. 6) $$p_*\beta_m = -\beta_m + \alpha_m \circ \eta ,$$

for all even values of m such that $m \geqslant 4$.

Let m be even and let $m \geqslant 6$. Then the only elements of $\pi_{m-1}(V_{m,2})$ which are representable by cross-sections are β_m and $\beta_m + \alpha_m \circ \eta = \beta'_m$, say By (6. 3), the automorphism $-c_*$ leaves these elements fixed when $m \equiv 2 \bmod 4$, and interchanges them when $m \equiv 0 \bmod 4$. This last result is the one needed to complete the proof of

(5. 3).

As another application, consider euclidean bundles over the circle S^1. In each dimension there is one class of orientable bundle (the trivial class) and one class of non-orientable bundle. A representative ξ of the latter, with dim $\xi = m$, can be constructed as follows. Let $a \in O_1$ be the non-trivial element, which acts on R^m by changing the sign of the first coordinate. Then ξ is formed from the cylinder $R^m \times I$ by identifying $(x, 0)$ with $(ax, 1)$ for all $x \in R^m$. The bundle structure of ξ, over $S^1 = I/\dot{I}$, is defined in the obvious way. Now a acts on $V_{m,k}$, through O_{m-k}, by changing the sign of the first row. We construct ξ_k from $V_{m,k} \times I$ by identifying $(y, 0)$ with $(ay, 1)$ for all $y \in V_{m,k}$. From first principles (without using (2.2)) a cross-section of ξ_k over ξ_1, say of type g: $S_m^{-1} \rightarrow V_{m,k}$, determines a free homotopy of g into g', where

$$g'x = ag(xa^{-1}) \qquad (x \in S^{m-1}).$$

Let m be even and let $m \geqslant k+2$. Then the existence of such a free homotopy is equivalent to the condition $p_* \beta = -\beta$, where $\beta \in \pi_{m-1}(V_{m,k})$ is the class of g. When $k=2$ we obtain a contradiction from this whatever the choice of β, by using (6. 6). Therefore ξ_k does not admit a cross-section over ξ_1 for $k=2$, and hence does not do so for $k>2$.

These results enable us to compute the mixed product with $\beta_6 \in \pi_5(V_{6,2})$ of an arbitrary element $\theta \in \pi_p(O_4)$. I assert that $\langle \theta, \beta_6 \rangle = \beta_6 \circ E\theta'$ for some element $\theta' \in \pi_{p+4}(S^4)$. In fact $\theta' = J\theta$, as shown in §3, but this will not be used. First take $p=3$ and consider the generators $\alpha, \beta \in \pi_3(O_4)$, as before. We can pull β back to $\pi_3(U_2)$ and β_6 back to $\pi_5(U_3/U_2)$. Hence we can pull $\langle \beta, \beta_6 \rangle$ back to $\pi_8(U_3/U_2)$, and the assertion follows at once in this case. Now $c_* \beta = \alpha - \beta$, and $c_* \beta_6 = -\beta_6$, by (6. 3). Therefore

$$\begin{aligned} \langle \alpha, \beta_6 \rangle &= \langle \beta, \beta_6 \rangle + \langle c_* \beta, \beta_6 \rangle \\ &= \langle \beta, \beta_6 \rangle - c_* \langle \beta, \beta_6 \rangle. \end{aligned}$$

Since the assertion is true for $\theta=\beta$ it follows that the assertion is ture for $\theta=\alpha$. By naturality the assertion remains true if we compose either of these generators with an element of $\pi_{l}(S^{3})$. Since an arbitrary element $\theta\in\pi_{p}(O_{4})$ can be expressed in the form

$$\alpha\circ\varphi+\beta\circ\psi \qquad\qquad (\varphi,\psi\in\pi_{p}(S^{3}))$$

it follows that $\langle\theta, \beta_{6}\rangle=\beta_{6}\circ E\theta'$, as asserted. Consider the direct sum decomposition

$$\pi_{r}(V_{6,2})\approx\pi_{r}(S^{4})\oplus\pi_{r}(S^{5})$$

determined by β_{6}. We have shown that $\langle\theta, \beta_{6}\rangle$ has no component in the first summand and so, as an application of (4. 4), we obtain

Theorem 6. 7. *Let ξ be an R^{6}-bundle over S^{p+1}, with group reduced from O_{6} to O_{4}. Then ξ_{2} admits a proper cross-section over ξ_{1} of type β_{6}.*

7. Variation of the obstruction

We focus our attention on the case of R^{n}-bundles over S^{p+1}, with group reduced from O_{n} to O_{n-k}, for $k\geqslant 2$. If $\theta\in\pi_{p}(O_{n-k})$ then

$$\langle\tau_{*}\theta, \iota_{n-1}\rangle=J\tau_{*}\theta=E^{k-1}J\theta ,$$

by (3. 4) and the properties of the J-homomorphism. The obstruction to the existence of a proper cross-section of type γ, as defined in §4, can therefore be written as

(7. 1) $$\psi(\theta, \gamma)=\langle\theta, \gamma\rangle-\gamma\circ E^{k-1}J\theta ,$$

where $\gamma\in\pi_{n-1}(V_{n,k})$. Clearly the obstruction is linear in θ. What happens when we vary the choice of γ?

Suppose that $V_{n,k}$ admits a cross-section over S^{n-1}. Then

$$\sigma_{*}: \pi_{r}(V_{n-1,k-1})\rightarrow\pi_{r}(V_{n,k})$$

is a monomorphism. If $\gamma_{i}(i=0, 1)$ is the class of a cross-section then $\gamma_{0}-\gamma_{1}=\sigma_{*}\alpha$, where $\alpha\in\pi_{n-1}(V_{n-1,k-1})$. By (3. 3a) we have

$$\langle \theta, \gamma_0 \rangle - \langle \theta, \gamma_1 \rangle = \langle \theta, \sigma_* \alpha \rangle = \sigma_* \langle \theta, \alpha \rangle,$$

and since $k \geqslant 2$ we have

$$\gamma_0 \circ E^{k-1} J\theta - \gamma_1 \circ E^{k-1} J\theta = (\sigma_* \alpha) \circ E^{k-1} J\theta = \sigma_* (\alpha \circ E^{k-1} J\theta).$$

By (7. 1), therefore,

$$\psi(\theta, \gamma_0) - \psi(\theta, \gamma_1) = \sigma_* \psi'(\theta, \alpha), \tag{7.2}$$

where $\psi'(\theta, \alpha) = \langle \theta, \alpha \rangle - \alpha \circ E^{k-1} J\theta$.

In particular, take $k=2$ and $n \geqslant 6$, and take $\alpha = \eta$, the generator of $\pi_{n-1}(S^{n-2})$. We have

$$\langle \theta, \eta \rangle = \langle \theta, \iota_{n-2} \rangle \circ E^p \eta = J\theta \circ E^p \eta,$$

by (3. 4), and so

$$\psi'(\theta, \eta) = J\theta \circ E^p \eta - \eta \circ E J\theta. \tag{7.3}$$

For example take $\theta = \beta \in \pi_3(O_4)$, as in §6, so that $J\theta = \nu \in \pi_7(S^4)$, where ν denotes the Hopf class. Then (7. 3) gives

$$\psi'(\beta, \eta) = \nu \circ E^3 \eta - \eta \circ E\nu.$$

Now $\pi_8(S^4) = Z_2 + Z_2$ (see [8]), with one summand generated by $\nu \circ E^3 \eta$ and the other by $\eta \circ E\nu$. Therefore $\psi'(\beta, \eta) \neq 0$ in this case, and so we obtain the following conclusion from (4. 3) and (7. 2). Consider the Hopf fibration of S^7 over S^4, as a 3-sphere bundle. Take ξ to be the Whitney sum of the associated R^4-bundle and a trivial R^2-bundle. Note that ξ admits almost-complex structure. In $\pi_s(V_{6,2})$ there are two classes of cross-section. Our conclusion is that ξ_2 admits a cross-section over ξ_1 of one type but not of the other.

The methods we have been using seem inadequate for the determination of the obstruction beyond a few special cases. In a subsequent paper a different method will be used to compute the mixed product, and hence the obstruction, in many more cases.

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