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Supplement to my paper: Spherical functions on locally compact groups

By

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§ 1. Description of the problem

In our earlier paper [1], we studied the characterization of spherical functions on locally compact unimodular groups and obtained Proposition 1 in [1]. But, recently, the author obtained a stronger result which he wish to show in this paper.

At first, let's recall some notations in [1]. Let G be a locally compact unimodular group, and K a compact subgroup of G. We shall denote by $L(G)$ the algebra of all continuous functions on G with compact supports (the product is convolution product). We can topologize $L(G)$ in the usual way (see [1]). For every equivalence class δ of irreducible representations of K, put

$$L(\delta)=\{f\in L(G);\ \bar{\chi}_\delta * f=f*\bar{\chi}_\delta=f\}$$

where $\bar{\chi}_\delta=(\dim \delta)\ \overline{\mathrm{Tr}[\delta]}$. Moreover put

$$L^0(\delta)=\{f\in L(\delta); f^0=f\}$$

where $f^0(x)=\int_K f(kxk^{-1})dk$ (dk is the normalized Haar measure on K).

In [1], the author proved the following proposition: for every spherical function ϕ on G, we can find a finite-dimensional irreducible (continuous) representation $f \rightarrow U(f)$ of $L^0(\delta)$ such that

$$\int_G \phi(x)f(x)dx=(\dim \delta)\mathrm{Tr}[U(f)]$$

for all $f \in L^0(\delta)$. But, conversely, for every finite-dimensional irreducible representation $f \rightarrow U(f)$ of $L^0(\delta)$, does there exist a spherical function ϕ satisfying the above relation? This problem is not completely solved in [1]. The purpose of the present paper is to give an affirmative solution.

We shall denote by $T(\delta)$ the set of all equivalence classes of finite-dimensional irreducible representations of $L^0(\delta)$. If a representation $f \rightarrow U(f)$ of $L^0(\delta)$ belongs to $\tau \in T(\delta)$, we put

$$\mu_\tau(f)=(\dim \delta)\mathrm{Tr}[U(\bar{\chi}_\delta * f^0)]$$

for all $f \in L(G)$. Clearly μ_τ is a continuous linear functional on $L(G)$.

Let $\Phi_g(\delta)$ be the set of all spherical functions in the generalized sence of type δ (see [1, p. 74]), and $\Phi(\delta)$ the set of all spherical functions of type δ. If G is σ-compact, $\Phi_g(\delta)=\Phi(\delta)$ as is shown in [1].

Now, our aim is to prove the following

Theorem. *For every $\tau \in T(\delta)$, μ_τ is a function on G and $\mu_\tau \in \Phi_g(\delta)$, and $\tau \rightarrow \mu_\tau$ is a one-to-one mapping from $T(\delta)$ onto $\Phi_g(\delta)$. Moreover τ is p-dimensional if and only if μ_τ is of height p.*

§ 2. Proof of a proposition

We shall denote by ϵ_x the measure on G given by $f \rightarrow f(x)$, $f \in L(G)$.

Lemma 1. *If $f \in L(\delta)$, $\mu_\tau(f*g)=\mu_\tau(g*f)$ for all $g \in L(G)$.*

Proof. For $f \in L^0(\delta)$,

$$\mu_\tau(f*g)=(\dim \delta)\mathrm{Tr}[U(\bar{\chi}_\delta * f * g^0)]$$

$$=(\dim\delta)\mathrm{Tr}[U(f)U(\bar{\chi}_\delta * g^0)]$$

$$=(\dim\delta)\mathrm{Tr}[U(\bar{\chi}_\delta * g^0)U(f)]$$

$$=(\dim\delta)\mathrm{Tr}[U(\bar{\chi}_\delta *(g*f)^0)]$$

$$=\mu_\tau(g*f).$$

Therefore, for every $k\in K$ and $f\in L^0(\delta)$,

$$\mu_\tau((\epsilon_k*f)*g)=\mu_\tau((\epsilon_k*f*g)^0)=\mu_\tau((f*g*\epsilon_k)^0)$$
$$=\mu_\tau(f*(g*\epsilon_k))=\mu_\tau(g*\epsilon_k*f)=\mu_\tau(g*(\epsilon_k*f)).$$

Since $\{\epsilon_k*f;\ k\in K,\ f\in L^0(\delta)\}=\{f*\epsilon_k;\ k\in K,\ f\in L^0(\delta)\}$ is total in $L(\delta)$ [1, Lemma 14], the above equation implies $\mu_\tau(f*g)=\mu_\tau(g*f)$ for every $f\in L(\delta)$. q.e.d.

If we put $f'(x)=f(x^{-1})$, it is natural to denote by $f'*\mu_\tau(f\in L(\delta))$ the measure

$$L(G)\ni g \longrightarrow \mu_\tau(f*g).$$

Now we must prove the following key proposition.

Proposition. $\mathfrak{p}=\{f\in L(\delta);\ f'*\mu_\tau=0\}$ *is a closed regular maximal two-sided ideal in* $L(\delta)$ *such that*

$$\dim(L(\delta)/\mathfrak{p})<+\infty.$$

Proof. It is obvious that $\mathfrak{p}$ is closed. For $f\in\mathfrak{p}$, $g\in L(\delta)$, and $h\in L(G)$,

$$(g*f)'*\mu_\tau(h)=\mu_\tau(g*f*h)$$
$$=\mu_\tau(f*h*g)=(f'*\mu_\tau)(h*g)=0,$$

$$(f*g)'*\mu_\tau(h)=\mu_\tau(f*g*h)=(f'*\mu_\tau)(g*h)=0.$$

This implies that $g*f, f*g\in\mathfrak{p}$, i.e., $\mathfrak{p}$ is a two-sided ideal in $L(\delta)$. The regularity of $\mathfrak{p}$ follows from the existence of a function $u \in L^0(\delta)$ such that $U(u)=1$ (Burnside's theorem). To show that $\dim(L(\delta)/\mathfrak{p}) < +\infty$, we need some lemmas.

Denote by V the space on which linear operators $U(f)$, $f \in L^0(\delta)$, act. For every $k \in K$ and $v \in V$, we associate a V-valued continuous linear function

$$\Phi_{v,k}(f) = U((f*\epsilon_k)^0)v$$

on $L(\delta)$.

Lemma 2. *The set $\{\Phi_{v,k};\ k \in K,\ v \in V\}$ spans a finite-dimensional vector space W.*

Proof. Let $k \to D(k)$ be a unitary irreducible representation of K belonging to δ, and $d_{ij}(k)$ the matrix elements of $D(k)$. Since

$$\begin{aligned} f*\epsilon_k = f*\bar{\chi}_\delta*\epsilon_k &= f*\left\{(\dim\delta)\sum_{i,j=1}^{\dim\delta} d_{ij}(k)\bar{d}_{ij}\right\} \\ &= (\dim\delta)\sum_{i,j=1}^{\dim\delta} d_{ij}(k)(f*\bar{d}_{ij}) \end{aligned}$$

for every $f \in L(\delta)$ and $k \in K$, we have

$$\begin{aligned} \Phi_{v,k}(f) &= U((f*\epsilon_k)^0)v \\ &= (\dim\delta)\sum_{i,j=1}^{\dim\delta} d_{ij}(k)\,U((f*\bar{d}_{ij})^0)v \\ &= (\dim\delta)\sum_{i,j=1}^{\dim\delta} d_{ij}(k)\Phi_{v,e}(f*\bar{d}_{ij}) \quad (f*\bar{d}_{ij} \in L(\delta)!!) \end{aligned}$$

where e is the unit of G. Moreover $\Phi_{v+w,k} = \Phi_{v,k} + \Phi_{w,k}$ is obvious. From these facts, the lemma immediately follows. q.e.d.

For every $v \in V$ and $f \in L(\delta)$, let's define a V-valued continuous linear function $\Phi_{v,f}$ on $L(\delta)$ as

$$\Phi_{v,f}(g)=U((g*f)^0)v.$$

Clearly we have $\Phi_{v+w,f}=\Phi_{v,f}+\Phi_{w,f}$, $\Phi_{\lambda v,f}=\lambda\Phi_{v,f}=\Phi_{v,\lambda f}$ $(\lambda\in\mathbf{C})$, and $\Phi_{v,f+g}=\Phi_{v,f}+\Phi_{v,g}$.

Lemma 3. *$\Phi_{v,f}\in W$ for all $f\in L(\delta)$ and $v\in V$.*

Proof. Let X be the dense subspace of $L(\delta)$ spanned by $\{\epsilon_k*f;\ f\in L^0(\delta),\ k\in K\}$, and put

$$H_v=\{\Phi_{v,f};\ f\in L(\delta)\},$$
$$H'_v=\{\Phi_{v,f};\ f\in X\}.$$

By the pointwise convergence, H_v is a topological vector space. Since the linear mapping

$$L(\delta)\ni f\longrightarrow\Phi_{v,f}\in H_v$$

is continuous, H'_v is densely contained in H_v. On the other hand, for every $\epsilon_k*f\in X$, we have

$$\begin{aligned}\Phi_{v,\epsilon_k*f}(g)&=U((g*\epsilon_k*f)^0)v=U((g*\epsilon_k)^0*f)v\\&=U((g*\epsilon_k)^0)U(f)v=\Phi_{U(f)v,k}(g)\in W.\end{aligned}$$

This shows that $H'_v\in W$, and therefore H'_v is finite-dimensional. Consequently H_v must be also finite-dimensional and $H_v=H'_v\subset W$.
q.e.d.

By Lemma 3, we can define linear operators T_f, $f\in L(\delta)$, on W by

$$(T_f\Phi)(g)=\Phi(g*f)\qquad g\in L(\delta).$$

Moreover, $f\to T_f$ is a (continuous) representation of $L(\delta)$ on W. Using the notation

$$A \Longleftrightarrow B$$

to denote the equivalence of statements A and B,

$$\begin{aligned}
f\in\mathfrak{p} &\Longleftrightarrow \mu_\tau(f*g)=0 \quad \text{for every } g\in L(G)\\
&\Longleftrightarrow \mathrm{Tr}[U(\bar{\chi}_\delta*(f*g)^0)]=0 \quad \text{for every } g\in L(G)\\
&\Longleftrightarrow \mathrm{Tr}[U((\bar{\chi}_\delta*f*g*\bar{\chi}_\delta)^0)]=0 \quad \text{for every } g\in L(G)\\
&\Longleftrightarrow \mathrm{Tr}[U((f*g)^0)]=0 \quad \text{for every } g\in L(\delta)\\
&\Longleftrightarrow \mathrm{Tr}[U((f*g*\epsilon_k)^0)]=0 \quad \text{for every } g\in L^0(\delta) \text{ and } k\in K\\
&\Longleftrightarrow \mathrm{Tr}[U((\epsilon_k*f*g)^0)]=0 \quad \text{for every } g\in L^0(\delta) \text{ and } k\in K\\
&\Longleftrightarrow \mathrm{Tr}[U((\epsilon_k*f)^0)U(g)]=0 \quad \text{for every } g\in L^0(\delta) \text{ and } k\in K\\
&\Longleftrightarrow U((\epsilon_k*f)^0)=0 \quad \text{for every } k\in K\\
&\Longleftrightarrow U((\epsilon_{k'}*\epsilon_k*f)^0)U(g)=0 \quad \text{for every } k,\ k'\in K \text{ and } g\in L^0(\delta)\\
&\Longleftrightarrow U((\epsilon_k*f*g*\epsilon_{k'})^0)=0 \quad \text{for every } k,\ k'\in K \text{ and } g\in L^0(\delta)\\
&\Longleftrightarrow U((\epsilon_k*f*g)^0)=0 \quad \text{for every } k\in K \text{ and } g\in L(\delta)\\
&\Longleftrightarrow U((f*g*\epsilon_k)^0)=0 \quad \text{for every } k\in K \text{ and } g\in L(\delta)\\
&\Longleftrightarrow U(h)U((f*g*\epsilon_k)^0)=0 \quad \text{for every } k\in K,\ h\in L^0(\delta), \text{ and } g\in L(\delta)\\
&\Longleftrightarrow U((h*f*g*\epsilon_k)^0)=0 \quad \text{for every } k\in K,\ h\in L^0(\delta), \text{ and } g\in L(\delta)\\
&\Longleftrightarrow U((\epsilon_k*h*f*g)^0)=0 \quad \text{for every } k\in K,\ h\in L^0(\delta), \text{ and } g\in L(\delta)\\
&\Longleftrightarrow U((h*f*g)^0)=0 \quad \text{for every } h,\ g\in L(\delta)\\
&\Longleftrightarrow U((h*f*g)^0)v=0 \quad \text{for every } v\in V \text{ and } h,\ g\in L(\delta)\\
&\Longleftrightarrow T_f\Phi_{v,\,g}=0 \quad \text{for every } v\in V \text{ and } g\in L(\delta)\\
&\Longleftrightarrow T_f=0.
\end{aligned}$$

Thus $\dim(L(\delta)/\mathfrak{p})<+\infty$ is obvious. This completes the proof of the proposition.

§ 3. Proof of Theorem

The proof of Theorem is similar to that of Proposition 1 in [1]. Only difference is that we don't know at the beginning whether μ_τ is a function or not.

Let $\mathfrak{a}$ be a maximal left ideal in $L(\delta)$ containing $\mathfrak{p}$, then $\mathfrak{a}$ is closed since $\dim(L(\delta)/\mathfrak{p})<+\infty$. Therefore

$$\mathfrak{m}=\{f\in L(G);\ \bar{\chi}_\delta * g * f * \bar{\chi}_\delta \in \mathfrak{a} \text{ for all } g\in L(G)\}$$

is a closed regular maximal left ideal in $L(G)$, and $\mathfrak{H}=L(G)/\mathfrak{m}$ is a locally convex topological vector space with respect to the topology induced from $L(G)$. If we denote by L_x the linear operator on $\mathfrak{H}$ defined by

$$L_x\{f\}=\{\epsilon_x * f\}$$

where $\{f\}=f+\mathfrak{m}$, we obtain an algebraically irreducible representation of G on $\mathfrak{H}$ (the author does not know whether $\mathfrak{H}$ is complete or not). The space $\mathfrak{H}(\delta)$, the set of all vectors in $\mathfrak{H}$ transformed according to δ under $k\to L_k$, is identified with $L(\delta)/\mathfrak{a}$ (see [1]), and therefore $\dim \mathfrak{H}(\delta)<+\infty$. If $\dim \mathfrak{H}(\delta)=q\cdot\dim\delta$, we obtain a spherical function ϕ_δ in the generalized sence of type δ of height q. Then, there exists a q-dimensional irreducible representation $f\to V(f)$ of $L^0(\delta)$ such that (*i*) $V(f)=0$ is equivalent to $L_f\mathfrak{H}(\delta)=0$ where $L_f=\int_G L_x f(x)dx$, (*ii*) $\int_G \phi_\delta(x)f(x)dx=(\dim\delta)\mathrm{Tr}[V(f)]$ for all $f\in L^0(\delta)$. On the other hand, $U(f)=0$ implies $f\in\mathfrak{p}$, and therefore $L_f\mathfrak{H}(\delta)=0$. Thus $U(f)=0$ implies $V(f)=0$. From this fact, two representations $f\to U(f)$ and $f\to V(f)$ must be equivalent and

$$\int_G \phi_\delta(x)f(x)dx=(\dim\delta)\mathrm{Tr}[V(f)]=(\dim\delta)\mathrm{Tr}[U(f)]=\mu_\tau(f)$$

for every $f \in L^0(\delta)$. This implies $\mu_\tau = \phi_\delta \in \Phi_g(\delta)$. The latter half of the theorem is easily proved.

Remark. If G is σ-compact, we obtain a concrete one-to-one correspondence between $\Phi(\delta)$ and $T(\delta)$.

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Bibliography

[1] H. Shin'ya; Spherical functions on locally compact groups, J. Math. Kyoto Univ., 12-1 (1972), 55-85.