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$\mathscr{E}$-well posedness for hyperbolic mixed problems with constant coefficients

By

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Introduction. Hersh has made a characterization of hyperbolic mixed problems ([1], [2]), where it seems that there are some rough discussions, especially about analyticities. Recently, Shirota has also made its characterization by means of Lopatinski's determinants with some restrictions ([3]). In this paper, we deal with the same problem as Shirota without his restrictions.

Now we state our problems, assumptions and main results. We consider the mixed problem

$$(P)\begin{cases} A(D_t, D_x, D_y)u=f(t, x, y) & \text{for } t>0,\ x>0,\ y\in R^{n-1}, \\ B_j(D_t, D_x, D_y)u=g_j(t, y)(j=1,\dots, \mu) & \text{for } t>0,\ x=0,\ y\in R^{n-1}, \\ D_t^j u=h_j(x, y)(j=0, 1,\dots, m-1) & \text{for } t=0,\ x>0,\ y\in R^{n-1} \\ \left(D_t=\frac{1}{i}\frac{\partial}{\partial t},\ D_x=\frac{1}{i}\frac{\partial}{\partial x},\ D_y=\left(\frac{1}{i}\frac{\partial}{\partial y_1},\dots,\frac{1}{i}\frac{\partial}{\partial y_{n-1}}\right)\right), \end{cases}$$

where $\{A, B_j\}$ are differential operators of orders $\{m, r_j\}$ with constant coefficients and $\{f, g_j, h_j\}$ are given deta. We denote the principal parts of $\{A, B_j\}$ by $\{A_0, B_{j0}\}$. We assume

i) A is hyperbolic with respect to $(1, 0, 0)$, i.e.

$$\begin{cases} A_0(1, 0, 0)\neq 0, \\ A(\tau, \xi, \eta)\neq 0 \qquad \text{for } \operatorname{Im}\tau<-\gamma_0,\ (\xi, \eta)\in R^n, \end{cases}$$

ii) $A_0(0, 1, 0) \neq 0$,

iii) $B_{j0}(0, 1, 0) \neq 0$, $0 \leq r_j \leq m-1$, and $r_i \neq r_j$ if $i \neq j$.

Remark. Assumptions ii) and iii) can be removable, but here we assume them in order to avoid some troublesome discussions about adjoint problems.

We say that the problem (P) is $\mathscr{E}$-well posed, if for every $f \in \mathscr{E}(\overline{R_+^1 \times R_+^n})$, $g_j \in \mathscr{E}(\overline{R_+^1 \times R^{n-1}})$, $h_j \in \mathscr{E}(\overline{R_+^n})$ with compatibility conditions of infinite order there exists a unique solution $u \in \mathscr{E}(\overline{R_+^1 \times R_+^n})$, where $\mathscr{E}(X)$ means a Rréchet space of infinitely differentiable functions in X with semi-norms

$$|u|_{l,K} = \sum_{|\nu| \leq l} \sup_{x \in K} |D^\nu u(x)|,$$

where l is a positive integer and K is a compact set in X. Of course, if (P) is $\mathscr{E}$-well posed, then the mapping from data to solution becomes continuous.

Now from the assumption i), there exists no real zero or $A(\tau, \xi, \eta)$ with respect to ξ for $\operatorname{Im} \tau < -\gamma_0$, $\eta \in R^{n-1}$. Therefore we denote there

$$\begin{aligned} A(\tau, \xi, \eta) &= c \prod_{j=1}^{\mu} (\xi - \xi_j^+(\tau, \eta)) \prod_{j=1}^{m-\mu} (\xi - \xi_j^-(\tau, \eta)) \\ &= c A_+(\tau, \eta; \xi) A_-(\tau, \eta, \xi) \qquad (\operatorname{Im} \xi_j^\pm(\tau, \eta) \gtrless 0). \end{aligned}$$

Here we define Lopatinski's determinant of $\{A, B_j\}$ by

$$R(\tau, \eta) = \det\left(\frac{1}{2\pi i} \oint \frac{B_j(\tau, \xi, \eta) \xi^{k-1}}{A_+(\tau, \eta; \xi)} d\xi\right)_{j,k=1,\dots,\mu}$$

for $\operatorname{Im} \tau < -\gamma_0$ and $\eta \in R^{n-1}$. We say that $\{A, B_j\}$ satisfy Lopatinski's condition if there exists $\gamma_1 (\geq \gamma_0)$ such that

$$R(\tau, \eta) \neq 0 \qquad \text{for } \operatorname{Im} \tau < -\gamma_1 \text{ and } \eta \in R^{n-1}.$$

Remark. Let

$$\mathscr{R}(\tau, \eta; \xi_1,\dots, \xi_\mu)=\frac{1}{\prod\limits_{i<j}(\xi_i-\xi_j)} \det(B_i(\tau, \xi_j, \eta))_{i,j=1,\dots,\mu}$$

$$=\begin{vmatrix} B_1^{(\mu-1)}(\tau, \eta; \xi_1,\dots, \xi_\mu)\dots B_1^{(1)}(\tau, \eta; \xi_{\mu-1}, \xi_\mu)B_1^{(0)}(\tau, \eta; \xi_\mu) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ B_\mu^{(\mu-1)}(\tau, \eta; \xi_1,\dots, \xi_\mu)\dots B_\mu^{(1)}(\tau, \eta; \xi_{\mu-1}, \xi_\mu)B_\mu^{(0)}(\tau,\eta; \xi_\mu) \end{vmatrix},$$

where

$$B_i^{(0)}(\tau, \eta; \xi_\mu)=B_i(\tau, \xi_\mu, \eta),$$

$$B_i^{(j)}(\tau, \eta; \xi_{\mu-j}, \xi_{\mu-j+1},\dots, \xi_\mu)^{\bullet}$$

$$=\frac{B_i^{(j-1)}(\tau, \eta; \xi_{\mu-j},\dots, \xi_{\mu-1})-B_i^{(j-1)}(\tau, \eta; \xi_{\mu-j+1},\dots, \xi_\mu)}{\xi_{\mu-j}-\xi_\mu}$$

$$(j=1, 2,\dots, \mu-1),$$

then we have

$$R(\tau, \eta)=\mathscr{R}(\tau, \eta; \xi_1^+(\tau, \eta),\dots, \xi_\mu^+(\tau, \eta)) \qquad \text{for } \operatorname{Im}\tau<-\gamma_0,\ \eta\in R^{n-1}.$$

Our main result is

Theorem. *In order that* (P) *is* $\mathscr{E}$*-well posed, it is necessary and sufficient that*

i) $R(\tau, \eta)\neq 0 \qquad$ *for* $\operatorname{Im}\tau<-\gamma_1,\ \eta\in R^{n-1}$,

ii) $\tilde{R}_0(1, 0)\neq 0$,

where $\tilde{R}_0$ *is the pricipal part of* R, *which will be defined in* §3.

§1. Necessity of Lopatinski's condition for $\mathscr{E}$-well posedness.

Lemma 1.1. *In order that* (P) *is* $\mathscr{E}$*-well posed, it is necessary that there exists* $p>0$ *such that*

$$R(\tau, \eta)\neq 0 \qquad \text{for } \operatorname{Im}\tau<-p\{\log(1+|\tau|+|\eta|)+1\},\ \eta\in R^{n-1}.$$

Proof. 1) We assume that for any p there exist $\{\tau_p, \eta_p\}$ such that

$$|\tau_p|+|\eta_p| \xrightarrow[p\to\infty]{} \infty,$$

$$\operatorname{Im}\tau_p < -p\log(|\tau_p|+|\eta_p|),$$

$$R(\tau_p, \eta_p)=0.$$

Then there exist $(c_{1p},\dots, c_{\mu p})$ such that

$$\sum_{j=1}^{\mu}|c_{jp}|^2=1,$$

$$v_p(x)=\sum_{j=1}^{\mu} c_{jp}\frac{1}{2\pi i}\oint\frac{e^{ix\xi}\xi^{j-1}}{A_+(\tau_p, \eta_p; \xi)}d\xi\cdot(|\tau_p|+|\eta_p|)^{\mu-j},$$

and

$$\begin{cases} A(\tau_p, D_x, \eta_p)v_p(x)=0, \ x>0, \\ B_j(\tau_p, D_x, \eta_p)v_p(0)=0 \qquad (j=1,\dots, \mu). \end{cases}$$

Now we denote

$$u_p(t, x, y)=v_p(x)\cdot e^{it\tau_p+iy\eta_p},$$

then we have

$$\begin{cases} A(D_t, D_x, D_y)u_p=0 & \text{for } t>0, x>0, y\in R^{n-1}, \\ B_j(D_t, D_x, D_y)u_p=0 \quad (j=1,\dots, \mu) & \text{for } t>0, x=0, y\in R^{n-1}. \end{cases}$$

2) Let $p=1, 2,\dots$. Since $\sum_{j=1}^{\mu}|c_{jp}|^2=1$, there exists μ_0 such that

$$c_{\mu p}, c_{\mu-1 p},\dots, c_{\mu-\mu_0+1 p} \xrightarrow[p\to\infty]{} 0,$$

$$c_{\mu-\mu_0\mu} \not\longrightarrow 0,$$

therefore we have

$$c_{\mu-\mu_0 p_k} \xrightarrow[k\to\infty]{} c\neq 0$$

3) Now we denote

$$D_t^j u_p(0, x, y) = \tau_p^j v_p(x) e^{iy\eta_p} = h_{jp}(x, y).$$

Since

$$\sup_{x>0} |D_x^k v_p(x)| \le C_k(|\tau_p| + |\eta_p|)^k,$$

we have

$$\sup_{x>0\, y\in R^{n-1}} |D_x^k D_y^\nu h_{jp}(x, y)| = |\tau_p|^j |\eta_p|^{|\nu|} \sup_{x>0} |D_x^k v_p(x)|$$

$$\le C_k(|\tau_p| + |\eta_p|)^{j+k+|\nu|},$$

therefore we have

$$\sum_{j=0}^{m-1} |h_{jp}|_{\mathscr{B}^l(R_+^n)} \le C_l(|\tau_p| + |\eta_p|)^{m-1+l}$$

4) On the other hand, we have

$$\begin{pmatrix} v_p(0) \\ (|\tau_p| + |\eta_p|)^{-1} D_x v_p(0) \\ \vdots \\ (|\tau_p| + |\eta_p|)^{-\mu+1} D_x^{\mu-1} v_p(0) \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \iddots & \alpha_{1p} \\ 0 & \iddots & \iddots & \vdots \\ 1 & \alpha_{1p} & \cdots & \alpha_{\mu-1p} \end{pmatrix} \begin{pmatrix} C_{1p} \\ C_{2p} \\ \vdots \\ C_{\mu p} \end{pmatrix},$$

where

$$|\alpha_{jp}| = \left| \frac{1}{2\pi i} \oint \frac{\xi^{j+\mu-1}}{A_+(\tau_p, \eta_p; \xi)} d\xi (|\tau_p| + |\eta_p|)^{-j} \right| < C.$$

Then we have

$$(|\tau_{p_k}| + |\eta_{p_k}|)^{-\mu_0} D_x^{\mu_0} v_{p_k}(0)$$

$$= c_{\mu-\mu_0 p_k} + \alpha_{1p_k} c_{\mu-\mu_0+1, p_k} + \cdots + \alpha_{\mu_0 p_k} c_{\mu p_k} \xrightarrow[k\to\infty]{} c,$$

therefore

$$|D_x^{\mu_0} u_{p_k}(t, 0, 0)| = |D_x^{\mu_0} v_{p_k}(0) e^{it\tau_{p_k}}| = |D_x^{\mu_0} v_{p_k}(0)| e^{-t \operatorname{Im} \tau_{p_k}}$$

$$\geq |D_x^{\mu_0} v_{p_k}(0)| (|\tau_{p_k}| + |\eta_{p_k}|)^{p_k t} \geq \frac{c}{2} (|\tau_{p_k}| + |\eta_{p_k}|)^{p_k t + \mu_0} \quad (k \geq k_0),$$

which is a contradiction to $\mathscr{E}$-well posedness. Q.E.D.

Lemma 1.2. *If there exists* $p>0$ *such that*

$$R(\tau, \eta) \neq 0 \qquad \text{for } \operatorname{Im} \tau < -p\{\log(1+|\tau|+|\eta|)+1\},\ \eta \in R^{n-1},$$

then there exists $\gamma_1 > 0$ *such that*

$$R(\tau, \eta) \neq 0 \qquad \text{for } \operatorname{Im} \tau < -\gamma_1,\ \eta \in R^{n-1}.$$

Proof.
Let us denote

$$A(\tau, \xi, \eta) = c\{\xi^m + a_1(\tau, \eta)\xi^{m-1} + \cdots + a_m(\tau, \eta)\},$$

and

$$M_r = \{\tau, \eta, \xi_1, \dots, \xi_m;\ \sum_{i=1}^m \xi_i = -a_1(\tau, \eta), \dots, \prod_{i=1}^m \xi_i = (-1)^m a_m(\tau, \eta),$$

$$\operatorname{Im} \xi_1 > 0, \dots, \operatorname{Im} \xi_\mu > 0,$$

$$|\tau|^2 + |\eta|^2 \leq r^2,\ \operatorname{Im} \tau < -\gamma_0,\ \eta \in R^{n-1},$$

$$\mathscr{R}(\tau, \eta; \xi_1, \dots, \xi_\mu) = 0\}.$$

Then we have from Seidenberg's lemma ([4])

$$M_r \equiv \phi \quad \text{or} \quad \mu(r) = \sup_{M_r}\{-\operatorname{Im} \tau\} = Cr^a(1+o(1)) \ (r \to +\infty).$$

On the other hand, we have from the assumption

$$\mu(r) \leq p\{\log(1+r)+1\},$$

therefore we have $a \leq 0$. Q.E.D.

Here we have from lemmas 1.1, 1.2

Proposition 1.1. *In order that* (P) *is* $\mathscr{E}$*-well posed, it is necessary* *that* $\{A, B_j\}$ *satisfy Lopatinski's condition.*

§2 Hyperbolic functions.

In this section, we consider hyperbolic functions in general, laying aside the matter in hand. Let us say that $f(\xi)$ is a hyperbolic function with respect to $\xi_0(\in R^n-\{0\})$, if there exists an open connected cone $C(\subset R^n)$, where $\xi_0\in C$ and $\xi_0+C\subset C$, and the following conditions i) $\sim$iii) are satisfied:

i) there exists $\gamma_0>0$ such that $f(\xi)$ is holomorphic in $R^n-iC_{\gamma_0}$, where $C_{\gamma_0}=C+\gamma_0\xi_0$,

ii) there exists $f_0(\xi)\not\equiv 0$, which is holomorphic in $\boldsymbol{C}=\bigcup_{z\in C^1-\{0\}} z(R^n-iC)$,

$$f_0(z\xi)=z^h f_0(\xi) \qquad \text{for } z\in C^1,\ \xi\in \boldsymbol{C},$$

and

$$z^{-h}f(z\xi)-f_0(\xi)\xrightarrow[0<z\to+\infty]{} 0 \qquad \text{for } \xi\in R^n-iC_{\gamma_0},$$

whose convergence is locally uniform in $R^n-iC_{\gamma_0}$,

iii) $f(\xi)\neq 0$ for $\xi\in R^n-i\Theta_{\gamma_0}$, $(\Theta=\{\lambda\xi_0;\ \lambda>0\},\ \Theta_{\gamma_0}=\Theta+\gamma_0\xi_0)$

$f_0(\xi)\neq 0$ for $\xi\in C$.

Lemma 2.1. *Let us assume* i), ii) *in the above definition, and assume*

$$\begin{cases} f(\xi)\neq 0 & \text{for } \xi\in R^n-i\Theta_{\gamma_0},\\ f_0(\xi_0)\neq 0, \end{cases}$$

then we have

$$f_0(\xi)\neq 0 \qquad \text{for } \xi\in R^n-i\Theta.$$

Proof. It is sufficient to show that

$$f_0(\eta-i\lambda\xi_0)\neq 0 \qquad \text{for } \eta\in R^n,\ \mathrm{Re}\,\lambda>0.$$

Let us assume that there exist $\eta_0\in R^n$, $\mathrm{Re}\,\lambda_0>0$ such that $f_0(\eta_0-i\lambda_0\xi_0)=0$. Since

$$(-i\lambda)^{-h}f_0(\eta_0-i\lambda\xi_0)=f_0\left(i\frac{\eta_0}{\lambda}+\xi_0\right)\xrightarrow[|\lambda|\to+\infty]{}f_0(\xi_0),$$

and $f_0(\xi_0)\neq 0$, we have $f_0(\eta_0-i\lambda\xi_0)\not\equiv 0$, therefore

$$f_0(\eta_0-i\lambda\xi_0)=(\lambda-\lambda_0)^l\varphi(\lambda),\ \varphi(\lambda_0)\neq 0,$$

hence

$$|f_0(\eta_0-i\lambda\xi_0)|\geq c|\lambda-\lambda_0|^l \qquad \text{for } |\lambda-\lambda_0|\leq\delta.$$

Since

$$\mu^{-h}f(\mu\eta_0-i\mu\lambda\xi_0)\xrightarrow[\mu\to+\infty]{}f_0(\eta_0-i\lambda\xi_0)$$

uniformly for $|\lambda-\lambda_0|\leq\delta$. By Rouché's theoren, we have for any $\mu\geq\mu_0$, $f(\mu\eta_0-i\mu\lambda\xi_0)=0$ has l-roots with respect to λ within $|\lambda-\lambda_0|<\delta$, which is a contradiction to $f(\xi)\neq 0$ for $\xi\in R^n-i\Theta_{\gamma_0}$. Q.E.D.

Lemm 2.2. *Let $f(\xi)$ be a hyperbolic function with respect to ξ_0 with cone C. Then we have*

$$\begin{cases} f(\xi)\neq 0 & \text{for } \xi\in R^n-iC_{\gamma_0}, \\ f_0(\xi)\neq 0 & \text{for } \xi\in \boldsymbol{C}. \end{cases}$$

Proof. 1) Let $\eta_0\in R^n$, $\xi_0'\in C_{\gamma_0}$ be arbitrarily fixed, then $f(\eta_0-i\lambda\xi_0-i\mu\xi_0')$ is holomorphic in $\operatorname{Re}\lambda>\gamma_0$, $\operatorname{Re}\mu\geq 0$. It is sufficient to show that it is non-zero there. Since

$$\eta_0-i\lambda\xi_0-i\mu\xi_0'\in R^n-i\Theta_{\gamma_0} \qquad \text{for } \operatorname{Re}\lambda>\gamma_0,\ \operatorname{Re}\mu=0,$$

we have

$$f(\eta_0-i\lambda\xi_0-i\mu\xi_0')\neq 0 \qquad \text{for } \operatorname{Re}\lambda>\gamma_0,\ \operatorname{Re}\mu=0.$$

2) Since

$$\mu^{-h}f(\eta_0-i\lambda\xi_0-i\mu\xi_0')-f_0\left(\frac{\eta_0}{\mu}-i\frac{\lambda}{\mu}\xi_0-i\xi_0'\right)\xrightarrow[|\mu|\to+\infty]{}0$$

uniformly for $|\mu|>C_0|\lambda|$, and

$$f_0(-i\xi_0')=(-i)^h f_0(\xi_0')\neq 0,$$

then we have

$$f(\eta_0-i\lambda\xi_0-i\mu\xi_0')\neq 0 \qquad \text{for } \operatorname{Re}\lambda>\gamma_0,\ \operatorname{Re}\mu>0,\ |\mu|>C|\lambda|.$$

From 1), 2), we have that the number of zeros of $f(\eta_0-i\lambda\xi_0-i\mu\xi_0')$ with respect to μ in $\operatorname{Re}\mu>0$ is finite and independent of λ in $\operatorname{Re}\lambda>\gamma_0$.

3) Since

$$\lambda^{-h}f(\eta_0-i\lambda\xi_0-i\mu\xi_0')-f_0\left(\frac{\eta_0}{\lambda}-i\xi_0-i\frac{\mu}{\lambda}\xi_0'\right)\xrightarrow[|\lambda|\to+\infty]{}0$$

uniformly for $|\mu|<c|\lambda|$, and $f_0(-i\xi_0)=(-i)^h f_0(\xi_0)\neq 0$, then we have

$$f(\eta_0-i\lambda\xi_0-i\mu\xi_0')\neq 0 \qquad \text{for } \operatorname{Re}\lambda>\gamma_0,\ \operatorname{Re}\mu>0,\ |\mu|<c|\lambda|,\ |\lambda|>M.$$

4) We have

$$\mu^{-h}f(\eta_0-i\lambda\xi_0-i\mu\xi_0')-f_0\left(\frac{\eta_0}{\mu}-i\frac{\lambda}{\mu}\xi_0-i\xi_0'\right)\xrightarrow[|\lambda|\to+\infty]{}0$$

uniformly for $c|\lambda|\leqslant|\mu|\leqslant C|\lambda|$. Let $\lambda>\gamma_0$ and $\operatorname{Re}\mu>0$, then $\operatorname{Re}\dfrac{\lambda}{\mu}>0$, therefore $\left\{\dfrac{\lambda}{\mu}\xi_0+\xi_0'\right\}$ are contained some compact set in $(R^n-i\Theta)\cup(R^n+i\Theta)\cup C$, whenever $c\lambda\leqslant|\mu|\leqslant C\lambda$. Hence we have from lemma 2.1 that

$$\left|f_0\left(\frac{\lambda}{\mu}\xi_0+\xi_0'\right)\right|>\delta>0 \qquad \text{for } \lambda>\gamma_0,\ \operatorname{Re}\mu>0,\ c\lambda\leqslant|\mu|\leqslant C\lambda.$$

Therefore we have

$$f(\eta_0-i\lambda\xi_0-i\mu\xi_0')\neq 0 \qquad \text{for } \lambda>\gamma_0,\ \operatorname{Re}\mu>0,$$
$$c\lambda\leqslant|\mu|\leqslant C\lambda,\ \lambda>M'.$$

From 3), 4), we have

$$f(\eta_0 - i\lambda\xi_0 - i\mu\xi_0') \neq 0 \qquad \text{for } \lambda > \gamma_0, \operatorname{Re}\mu > 0,$$

$$\lambda > \max(M, M'). \qquad \text{Q.E.D.}$$

Finally, we state a result of the theory of Fourier-Laplace transform.

Lemma 2.3. *Let C be an open connectted cone. Let*

i) *$f(\xi)$ be holomorphic in $R^n - iC_{\gamma_0}$,*

ii) *for any compact set $K \subset C_{\gamma_0}$,*

$$|f(\xi)| \leqslant c_K(1+|\xi|)^{h_K} \qquad \textit{for } \xi \in R^n - iC_K,$$

where $C_K = \{\lambda\xi;\ \xi \in K, \lambda \geqslant 1\}$. Then $\bar{F}[f] \in \mathscr{D}'(R^n)$ is defined by

$$<\bar{F}[f], \varphi> = \int_{R^n - i\zeta_0} f(\xi)(\bar{F}[\varphi])(\xi)d\xi, \ \zeta_0 \in C_{\gamma_0}, \ \varphi \in \mathscr{D}(R^n),$$

and supp $\bar{F}[f] \subset C'$, where

$$C' = \{x \in R^n;\ x \cdot \xi \geqslant 0, \ \forall \xi \in C\}.$$

§3. Lopatinski's determinants.

Let us denote

$$\sigma(\xi, \eta) = \max_{A_0(\tau, \xi, \eta) = 0} \tau \qquad \text{for } (\xi, \eta) \in R^n,$$

$$\sigma(\eta) = \min_{\xi \in R^1} \sigma(\xi, \eta) \qquad \text{for } \eta \in R^{n-1},$$

$$\Gamma = \{(\tau, \xi, \eta) \in R^{n+1};\ \tau > \sigma(\xi, \eta)\},$$

$$\dot{\Gamma} = \{(\tau, \eta) \in R^n;\ \tau > \sigma(\eta)\},$$

and $\Gamma_{\gamma_0} = \Gamma + \gamma_0(1, 0, 0)$, $\dot{\Gamma}_{\gamma_0} = \dot{\Gamma} + \gamma_0(1, 0)$.

Lemma 3.1. *$R(\tau, \eta)$ is holomorphic in $R^n - i\dot{\Gamma}_{\gamma_0}$.*

Proof. Since

$$\begin{cases} A(\tau, \xi, \eta) \neq 0 & \text{for } \operatorname{Im}\tau < -\gamma_0,\ (\xi, \eta) \in R^n, \\ A_0(\tau, \xi, \eta) \neq 0 & \text{for } (\tau, \xi, \eta) \in \Gamma, \end{cases}$$

we have from lemma 2.2 that $A(\tau, \xi, \eta)$ is a hyperbolic function with respect to $(1, 0, 0)$ with cone Γ. Let us denote for $(\tau, \eta) \in \dot{\Gamma}$

$$\xi_{\max}(\tau, \eta) = \sup_{(\tau, \xi, \eta) \in \Gamma} \xi, \qquad \xi_{\min}(\tau, \eta) = \inf_{(\tau, \xi, \eta) \in \Gamma} \xi,$$

then

$$\Gamma = \{(\tau, \xi, \eta) \in R^{n+1};\ (\tau, \eta) \in \dot{\Gamma},\ \xi_{\min}(\tau, \eta) < \xi < \xi_{\max}(\tau, \eta)\},$$

therefore the zeros of $A(\tau, \xi, \eta)$ satisfy

$$\operatorname{Im}\xi_j^+(\tau, \eta) > \xi_{\max}(\operatorname{Im}\tau, \operatorname{Im}\eta) \qquad j = 1, \dots, \mu,$$

$$\operatorname{Im}\xi_j^-(\tau, \eta) < \xi_{\min}(\operatorname{Im}\tau, \operatorname{Im}\eta) \qquad j = 1, \dots, m-\mu$$

for $(\tau, \eta) \in R^n - i\dot{\Gamma}_{\gamma_0}$. Hence, denoting

$$A_+(\tau, \eta; \xi) = \prod_{j=1}^{\mu} (\xi - \xi_j^+(\tau, \eta)) = \xi^\mu + a_1^+(\tau, \eta)\xi^{\mu-1} + \cdots + a_\mu^+(\tau, \eta),$$

$\{a_j^+(\tau, \eta)\}$ are holomorphic in $R^n - i\dot{\Gamma}_{\gamma_0}$. Q.E.D.

Lemma 3.2. *Let K be a compact set in $R^n - i\dot{\Gamma}$, then there exists $\lambda_K > 0$ such that*

$$R(\lambda\tau, \lambda\mu) = \lambda^h\{R_0(\tau, \eta) + \frac{1}{\lambda}R_1(\tau, \eta) + \frac{1}{\lambda^2}R_2(\tau, \eta) + \cdots\}$$

$$\left(h = \sum_{j=1}^{\mu} r_j - \frac{\mu(\mu-1)}{2}\right),$$

whose convergence is uniform in $K \times \{\lambda \in C^1;\ |\lambda| > \lambda_K\}$, where

i) $\{R_j(\tau, \eta)\}$ *are holomorphic in* $\dot{\boldsymbol{\Gamma}} = \bigcup_{z \in C^1 - \{0\}} z(R^n - i\dot{\Gamma})$,

ii) $R_j(\lambda\tau, \lambda\mu) = \lambda^{h-j}R_j(\tau, \eta)$ *for* $(\tau, \eta) \in \dot{\boldsymbol{\Gamma}}$, $\lambda \in C^1 - \{0\}$.

Remark. Let

$$\begin{cases} R_0(\tau, \eta) \equiv R_1(\tau, \eta) \equiv \cdots \equiv R_{k-1}(\tau, \eta) \equiv 0, \\ R_k(\tau, \eta) \not\equiv 0, \end{cases}$$

then we denote $R_k(\tau, \eta) = \tilde{R}_0(\tau, \eta) \qquad (h_0 = h - k)$.

Proof. We denote

$$A(\tau, \xi, \eta) = A_0(\tau, \xi, \eta) + A_1(\tau, \xi, \eta) + \cdots + A_m,$$

where $A_k(\tau, \xi, \eta)$ is a homogeneous polynomial of order $m-k$. Then

$$A(\lambda\tau, \lambda\xi, \lambda\eta) = \lambda^m \left\{ A_0(\tau, \xi, \eta) + \frac{1}{\lambda} A_1(\tau, \xi, \eta) + \cdots + \frac{1}{\lambda^m} A_m \right\}.$$

Now let us introduce a real parameter ν and consider

$$A_{(\nu)}(\tau, \xi, \eta) = A_0(\tau, \xi, \eta) + \nu A_1(\tau, \xi, \eta) + \cdots + \nu^m A_m.$$

Since $A_{(\nu)}(\tau, \xi, \eta) = \nu^m A\left(\frac{1}{\nu}\tau, \frac{1}{\nu}\xi, \frac{1}{\nu}\eta\right)$, $A_{(\nu)}(\tau, \xi, \eta) \not\equiv 0$ for $(\tau, \xi, \eta) \in R^{n+1} - i\Gamma_{\gamma_0}$ and $0 < \nu \leqslant 1$, therefore the zeros of $A_{(\nu)}(\tau, \xi, \eta)$ are in the analogous situation to those of $A(\tau, \xi, \eta)$, which we denote by $\{\zeta_j^+(\tau, \eta; \nu)\}_{j=1,\dots,\mu}$, $\{\zeta_j^-(\tau, \eta; \nu)\}_{j=1,\dots,m-\mu}$:

$$\operatorname{Im} \zeta_j^+(\tau, \eta; \nu) > \xi_{\max}(\operatorname{Im}\tau, \operatorname{Im}\eta),$$

$$\operatorname{Im} \zeta_j^-(\tau, \eta; \nu) < \xi_{\min}(\operatorname{Im}\tau, \operatorname{Im}\eta)$$

$$\text{for } (\tau, \eta) \in R^n - i\dot{\Gamma}_{\gamma_0},\ 0 \leqslant \nu \leqslant 1.$$

Then, denoting

$$\prod_{j=1}^{\mu} (\xi - \zeta_j^+(\tau, \eta; \nu)) = \xi^\mu + b_1^+(\tau, \eta; \nu)\xi^{\mu-1} + \cdots + b_\mu^+(\tau, \eta; \nu),$$

we have that $\{b_j^+(\tau, \eta; \nu)\}$ are holomorphic in $\{R^n - i\dot{\Gamma}_{\gamma_0}\} \times \{0 \leqslant \nu \leqslant 1\}$. Since

$$\xi_j^+(\lambda\tau, \lambda\eta) = \lambda\zeta_j^+\left(\tau, \eta; \frac{1}{\lambda}\right),\quad a_j^+(\lambda\tau, \lambda\eta) = \lambda^j b_j^+\left(\tau, \eta; \frac{1}{\lambda}\right)$$

for $(\tau, \eta) \in R^n - i\dot{\Gamma}_{\gamma_0}$ and $\lambda \geqslant 1$, we have

$$a_j^+(\lambda\tau, \lambda\eta) = \lambda^j \{a_{j0}^+(\tau, \eta) + \frac{1}{\lambda} a_{j1}^+(\tau, \eta) + \frac{1}{\lambda^2} a_{j2}^+(\tau, \eta) + \cdots\},$$

uniformly in $K \times \{\lambda \in C^1; |\lambda| > \lambda_K\}$ (K: any compact set in $R^n - i\Gamma_{\gamma_0}$). it is easily shown that $a_{jk}^+(\tau, \eta)$ are holomorphic in $R^n - i\Gamma_{\gamma_0}$, and

$$a_{jk}^+(\lambda\tau, \lambda\eta) = \lambda^{j-k} a_{jk}^+(\tau, \eta) \qquad \text{for } (\tau, \eta) \in R^n - i\dot{\Gamma} \text{ and } \lambda \geqslant 1,$$

therefore the homogeneous extensions of $a_{jk}^+(\tau, \eta)$ into $\dot{\Gamma}$ are also holomorphic there. Q.E.D.

Proposition 3.1. *Let the problem (P) be $\mathscr{E}$-well posed, then we have* $\tilde{R}_0(1, 0) \neq 0$.

Proof. Let $\tilde{R}_0(1, 0) = 0$ and $\tilde{R}_0(\tau_0, \eta_0) \neq 0$. From the expansion of R, we have

$$R(s + t\tau_0, t\eta_0) = s^{h_0} \left\{ \tilde{R}_0\left(1 + \frac{t}{s}\tau_0, \frac{t}{s}\eta_0\right) + \frac{1}{s}\tilde{R}_1\left(1 + \frac{t}{s}\tau_0, \frac{t}{s}\eta_0\right) + \cdots \right\}$$

$$\text{for } \left|\frac{t}{s}\right| < 1, \qquad \left|\frac{1}{s}\right| < c.$$

Now since

$$\tilde{R}_0(1 + w\tau_0, w\eta_0) + z\tilde{R}_1(1 + w\tau_0, w\eta_0) + \cdots$$

is holomorphic at $(w, z) = (0, 0)$, its zeros are given by

$$w(z) = \sum_{j=1}^{\infty} c_j z^{\frac{j}{q}} \qquad (|z| < c_0),$$

which become zero at $z = 0$. Set

$$t(s) = sw\left(\frac{1}{s}\right) \qquad \left(\left|\frac{1}{s}\right| < c_0\right),$$

then we have

$$|t(s)| \geqslant C|s|^{1-\frac{1}{q}},$$

$$R(s+t(s)\tau_0,\ t(s)\eta_0)=0.$$

Now set

$$s_p=-ip \quad (p\colon \text{integr},\ p\geqslant p_0)$$

and

$$\tau_p=s_p+t(s_p)\tau_0,\ \ \eta_p=t(s_p)\eta_0$$

then we have

$$|\eta_p|\leqslant C|\operatorname{Im}\tau_p|^{1-\frac{1}{q}},$$

$$R(\tau_p,\ \eta_p)=0,$$

and zeros of $A_+(\tau_p,\ \eta_p;\ \xi)$ have positive imaginary parts. Here we are in an analogous situation to that in the proof of lemma 1.1, that is, we can find $\{c_{jp}\}_{j=1,\dots,\mu,p\geq p_0}$ such that

$$\sum_{j=1}^{\mu}|c_{jp}|^2=1,\ v_p(x)=\sum_{j=1}^{\mu}c_{jp}\frac{1}{2\pi i}\oint\frac{e^{ix\xi}\xi^{j-1}}{A_+(\tau_p,\ \eta_p;\ \xi)}d\xi(|\tau_p|+\ |\eta_p|)^{\mu-j},$$

$$A(\tau_p,\ D_x,\ \eta_p)v_p(x)=0,\ B_j(\tau_p,\ D_x,\ \eta_p)v_p(0)=0 \qquad (j=1,\dots,\ \mu),$$

and then $u_p(t,\ x,\ y)=v_p(x)e^{it\tau_p+iy\eta_p}$ will lead us to a contradiction to $\mathscr{E}$-well posedness. Q.E.D.

Example 1. Let

$$\begin{cases} A=\tau^2-\xi^2-\eta^2, \\ B=\xi+\tau+1, \end{cases}$$

then

$$R(\tau,\ \eta)=\xi_+(\tau,\ \eta)+\tau+1, \qquad R_0(\tau,\ \eta)=\xi_+(\tau,\ \eta)+\tau,$$

therefore

$$R(\tau, \eta)=0 \Longleftrightarrow \tau=-\frac{\eta^2+1}{2}$$

and

$$R_0(\tau, \eta) \not\equiv 0, \quad R_0(1, 0)=0.$$

Example 2. Let

$$\begin{cases} A = (\tau^2-\xi^2-\eta^2)^2, \\ B_1 = \xi, \\ B_2 = \xi^2+\tau^2-\eta^2+1, \end{cases}$$

then

$$R(\tau, \eta)=1, \quad R_0(\tau, \eta)=0.$$

§4. Existence theorems.

Hereafter we assume

$$\begin{cases} \text{i)} \quad R(\tau, \eta) \neq 0 \qquad \text{for } \operatorname{Im} \tau < -\gamma_1, \ \eta \in R^{n-1}, \\ \text{ii)} \quad \tilde{R}_0(1, 0) \neq 0. \end{cases}$$

Let us denote

$$\sigma_0(\eta)=\begin{cases} \sigma(\eta) & \text{if } \tilde{R}_0(\tau, \eta) \neq 0 \qquad \text{for } \tau > \sigma(\eta), \\ \sup\limits_{\tilde{R}_0(\tau, \eta)=0, \tau>\sigma(\eta)} & \text{otherwise}, \end{cases}$$

for $\eta \in R^{n-1}$, and

$$\dot{\Sigma}=\{(\tau, \eta) \in R^n; \ \tau > \sigma_0(\eta)\}.$$

Example. Let

$$A=\tau^2-\xi^2-|\eta|^2, \quad B=\xi+b\cdot\eta \qquad (b \in R^{n-1}),$$

then

$$R(\tau, \eta)=\xi^{+}(\tau, \eta)+b\cdot\eta,$$

$$\sigma_0(\eta)=\begin{cases}\{|\eta|^2+(b\cdot\eta)^2\}^{\frac{1}{2}} & \text{if } b\cdot\eta>0,\\ |\eta| & \text{if } b\cdot\eta\leq 0.\end{cases}$$

Remark. From lemma 2.1, we have

$$\tilde{R}_0(\tau, \eta)\neq 0 \qquad \text{for } \operatorname{Im}\tau\neq 0,\ \eta\in R^{n-1}.$$

Therefore, since $\tilde{R}_0(\tau, \eta)$ is analytic on $(\tau, \eta)\in\dot{\Gamma}$, $\sigma_0(\eta)$ is continuous.

Since $\dot{\Sigma}\subset\dot{\Gamma}$ and $\dot{\Sigma}+(1, 0)\subset\dot{\Sigma}$, we have from lemma 2.2

Lemma 4.1. *$R(\tau, \eta)$ is a hyperbolic function with respect to $(1, 0)$ with cone $\dot{\Sigma}$.*

Lemma 4.2. *Let K be a compact set in $\dot{\Sigma}_{\gamma_1}$, then there exist $c_K>0$, $a_K>0$ such that*

$$|R(\tau, \eta)|\geq c_K(|\tau|+|\eta|)^{-a_K} \qquad \text{for } (\tau, \eta)\in R^n-iC_K,$$

where

$$C_K=\{(\lambda\tau, \lambda\eta);\ (\tau, \eta)\in K,\ \lambda\geqslant 1\}.$$

Proof. Let $(\tau_0, \eta_0)\in\Sigma_{\gamma_1}$, then there exists $\varepsilon>0$ such that

$$U_\varepsilon(\tau_0, \eta_0)=\{(\tau, \eta)\in R^n;\ |\tau-\tau_0|^2+|\eta-\eta_0|^2<\varepsilon^2\}\subset\dot{\Sigma}_{\gamma_1},$$

therefore

$$\lambda U_\varepsilon(\tau_0, \eta_0)\subset\dot{\Sigma}_{\gamma_1} \qquad \text{for } \lambda\geq 1.$$

Now let us denote

$$\mathscr{D}_r(\tau_0, \eta_0;\varepsilon)=\{\tau=\tau'-i\lambda\tau'',\ \eta=\eta'-i\lambda\eta'';\ (\tau', \eta')\in R^n,\ (\tau'', \eta'')\in U_\varepsilon(\tau_0, \eta_0)$$
$$\lambda\geq 1,\ |\tau|^2+|\eta|^2\leq r^2\}\subset R^n-i\dot{\Sigma}_{\gamma_1},$$

and

$$\tilde{\mathscr{D}}_r(\tau_0, \eta_0; \varepsilon) = \{(\tau, \eta) \in \mathscr{D}_r(\tau_0, \eta_0; \varepsilon), (\xi_1, \dots, \xi_m) \in C^m;$$

$$\Sigma \xi_i = -a_i(\tau, \eta), \sum_{i<j} \xi_i \xi_j = a_2(\tau, \eta), \dots, \Pi \xi_i = (-1)^m a_m(\tau, \eta)$$

$$\operatorname{Im} \xi_1 \geq \operatorname{Im} \xi_2 \geq \cdots \geq \operatorname{Im} \xi_m\},$$

where

$$A(\tau, \xi, \eta) = c\{\xi^m + a_1(\tau, \eta)\xi^{m-1} + \cdots + a_m(\tau, \eta)\}.$$

Since

$$\operatorname{Im} \xi_j^+(\tau, \eta) > \operatorname{Im} \xi_K^-(\tau, \eta) \qquad \text{for } (\tau, \eta) \in R^n - i\dot{\Gamma}_{\gamma_0},$$

we have

$$\mathscr{R}(\tau, \eta; \xi_1, \dots, \xi_\mu) = R(\tau, \eta) \qquad \text{in } \tilde{\mathscr{D}}_r(\tau_0, \eta_0; \varepsilon),$$

therefore

$$\mathscr{R}(\tau, \eta; \xi_1, \dots, \xi_\mu) \neq 0 \qquad \text{in } \tilde{\mathscr{D}}_r(\tau_0, \eta_0; \varepsilon).$$

From Seidenberg's lemma, we have

$$\sup_{\tilde{\mathscr{D}}_r} \frac{1}{|\mathscr{R}(\tau, \eta; \xi_1, \dots, \xi_\mu)|^2} = \sup_{\mathscr{D}_r} \frac{1}{|R(\tau, \eta)|^2}$$

$$= Cr^\alpha(1 + o(1)) \text{ as } r \longrightarrow +\infty. \qquad \text{Q.E.D.}$$

Let us denote

$$\mathscr{P}_k(\tau, \eta; \xi) = \frac{1}{i} \sum_{j=1}^{\mu} \frac{R_{jk}(\tau, \eta)}{R(\tau, \eta)} \frac{\xi^{j-1}}{A_+(\tau, \eta; \xi)},$$

where

$$R_{jk}(\tau, \eta)\text{: } (k, j)\text{-cofactor of} \left(\frac{1}{2\pi i} \oint \frac{B_j(\tau, \xi, \eta)\xi^{k-1}}{A_+(\tau, \eta; \xi)} d\xi \right)_{j,k=1,\dots,\mu}.$$

Then the solution in $H^m(R_+^1)$ of the problem

$$(\hat{P})\begin{cases} A(\tau, D_x, \eta)\hat{u}(x)=0, & x>0, \\ B_j(\tau, D_x, \eta)\hat{u}(0)=\hat{g}_j, & j=1,\dots,\mu, \end{cases}$$

is given by

$$\hat{u}(x)=\sum_{k=1}^{\mu}\frac{1}{2\pi}\oint e^{ix\xi}\mathscr{P}_k(\tau, \eta; \xi)d\xi\hat{g}_k, \quad x>0.$$

Now let us denote

$$\Sigma=\{(\tau, \xi, \eta)\in\Gamma; (\tau, \eta)\in\dot{\Sigma}\},$$

then we have

Lemma 4.3.

i) *$\mathscr{P}_j(\tau, \eta; \xi)$ are holomorphic in $R^{n+1}-i\Sigma_{\gamma_1}$,*

ii) *let K be a compact set in Σ_{γ_1}, then there exist $c_K>0$, $a_K>0$ suct that*

$$|\mathscr{P}_j(\tau, \eta; \xi)|\leq c_K(|\tau|+|\eta|+|\xi|)^{a_K} \quad \textit{for } (\tau, \xi, \eta)\in R^{n+1}-iC_K,$$

where

$$C_K=\{(\lambda\tau, \lambda\xi, \lambda\eta); (\tau, \xi, \eta)\in K, \lambda\geq 1\}.$$

From lemma 4.3 and lemma 2.3 we have $P_j=\bar{F}[\mathscr{P}_j]\in\mathscr{D}'(R^{n+1})$ and $\operatorname{supp}[P_j]\subset\Sigma'$. Therefore, if $g_j\in\mathscr{D}'(R^n)$ and $\operatorname{supp}[g_j]\subset\dot{\Sigma}'$, $u=\sum_{j=1}^{\mu}P_j*\{g_j\otimes\delta_x\}\in\mathscr{D}'(R^{n+1})$ and $\operatorname{supp}[u]\subset\Sigma'$, moreover if $g_j\in\mathscr{E}(R^n)$, then $u\in\mathscr{E}(\overline{R_+^{n+1}})$. Now we denote for a set S in R^{n+1}

$$\Gamma'_S=\Gamma'+S, \quad \Sigma'_S=\Sigma'+\dot{\Gamma}'_S, \quad K_S=\Gamma'_S\cup\Sigma'_S,$$

then we have

Proposition 4.1. *Let the supports of data be contained in S, then there exists a solution of (P), whose support is contained in K_S. Moreover, let the data be infinitely differentiable with compatibility conditions, there exists an infinitely differentiable solution of (P).*

§ 5. Adjoint problems.

At first, we shall construct an adjoint system of $\{A, B_j\}$ as follows. Here we may assume that $A_0(0, 1, 0)=1$ and $B_{j_0}(0, 1, 0)=1$ $(j=1,\dots, \mu)$ without loss of generality. We denote

$$B_j = D_x^{r_j} \qquad (j=\mu+1,\dots, m),$$

where $\{r_j\}_{j=\mu+1,\dots,m}$ is the complement of $\{r_j\}_{j=1,\dots,\mu}$ in $\{0, 1,\dots, m-1\}$, and denote

$$\begin{pmatrix} B_1(\tau, \xi, \eta) \\ \vdots \\ B_m(\tau, \xi, \eta) \end{pmatrix} = B(\tau, \eta) \begin{pmatrix} 1 \\ \xi \\ \vdots \\ \xi^{m-1} \end{pmatrix},$$

then we have

$$\det B(\tau, \eta) = \operatorname{sgn}\begin{pmatrix} 1 & \cdots & m \\ r_1 & \cdots & r_m \end{pmatrix}.$$

Now we denote

$$\frac{A(\tau, \xi, \eta) - A(\tau, \bar{\xi}, \eta)}{\xi - \bar{\xi}} = \sum_{j=1}^{m} B_j(\tau, \xi, \eta) B'_j(\tau, \bar{\xi}, \eta),$$

that is,

$$\begin{pmatrix} B'_1(\tau, \xi, \eta) \\ \vdots \\ B'_m(\tau, \xi, \eta) \end{pmatrix} = {}^t\{B^{-1}(\tau, \eta)\} \begin{pmatrix} & 0 & & 1 \\ & & \cdot^{\cdot^{\cdot}} & a_1(\tau, \eta) \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ 1 & a_1(\tau, \eta) & \cdots & a_{m-1}(\tau, \eta) \end{pmatrix} \begin{pmatrix} 1 \\ \xi \\ \vdots \\ \xi^{m-1} \end{pmatrix}$$

$$= B'(\tau, \eta) \begin{pmatrix} 1 \\ \xi \\ \vdots \\ \xi^{m-1} \end{pmatrix},$$

where

$$A(\tau,\xi,\eta)=\xi^m+a_1(\tau,\eta)\xi^{m-1}+\cdots+a_m(\tau,\eta).$$

Here denote

$$\begin{pmatrix} B_1^*(\tau,\xi,\eta) \\ \vdots \\ B_m^*(\tau,\xi,\eta)\end{pmatrix}=\begin{pmatrix} \overline{B_1'}(\tau,\xi,\eta) \\ \vdots \\ \overline{B_m'}(\tau,\xi,\eta)\end{pmatrix},\quad A^*(\tau,\xi,\eta)=\bar{A}(\tau,\xi,\eta),$$

and we say that $\{A^*(D_t,D_x,D_y);\ B_{\mu+1}^*(D_t,D_x,D_y),\ \dots,\ B_m^*(D_t,D_x,D_y)\}$ is an adjoint system of $\{A(D_t,D_x,D_y);\ B_1(D_t,D_x,D_y),\dots,B_\mu(D_t,D_x,D_y)\}$.

Since we have

$$A(\tau,\xi,\eta)=A_+(\tau,\eta;\xi)A_-(\tau,\eta;\xi),$$

$$A_+(\tau,\eta;\xi)=\xi^\mu+a_1^+(\tau,\eta)\xi^{\mu-1}+\cdots+a_\mu^+(\tau,\eta),$$

$$A_-(\tau,\eta;\xi)=\xi^{m-\mu}+a_1^-(\tau,\eta)\xi^{m-\mu-1}+\cdots+a_{m-\mu}^-(\tau,\eta)$$

$$\text{for } \operatorname{Im}\tau<-\gamma_1,\ \eta\in R^{n-1}$$

we denote, (τ,η) being fixed,

$$A(\xi)\ =A(\tau,\xi,\eta),$$

$$A_\pm(\xi)=A_\pm(\tau,\eta;\xi),$$

$$\bar{A}_+(\xi)=\xi^m+\overline{a_1^+(\tau,\eta)}\xi^{\mu-1}+\cdots+\overline{a_\mu^+(\tau,\eta)},$$

$$\bar{A}_-(\xi)=\xi^{m-\mu}+\overline{a_1^-(\tau,\eta)}\xi^{m-\mu-1}+\cdots+\overline{a_{m-\mu}^-(\tau,\eta)},$$

and

$$(u,v)=\int_0^\infty u(x)\overline{v(x)}dx,$$

$$<u,v>=u(0)\cdot\overline{v(0)},$$

then

$$(A(D_x)u, v)-(u, \bar{A}(D_x)v)$$

$$=i\sum_{k=0}^{m-\mu-1} a_k^-\{<D_x^{m-\mu-k-1}A_+(D_x)u, v>+\cdots+<A_+(D_x)u, D_x^{m-\mu-k-1}v>\}$$

$$+i\sum_{k=0}^{\mu-1} a_k^+\{<D_x^{\mu-k-1}u, \bar{A}_-(D_x)v> +\cdots+<u, D_x^{\mu-k-1}\bar{A}_-(D_x)v>\}.$$

Here we denote

$$U=\begin{pmatrix} u(0) \\ D_x u(0) \\ \vdots \\ D_x^{\mu-1}u(0) \\ A_+(D_x)u(0) \\ D_x A_+(D_x)u(0) \\ \vdots \\ D^{m-\mu-1}A_+(D_x)u(0) \end{pmatrix}, \quad V=\begin{pmatrix} D_x^{\mu-1}\bar{A}_-(D_x)v(0) \\ D_x^{\mu-2}\bar{A}_-(D_x)v(0) \\ \vdots \\ \bar{A}_-(D_x)v(0) \\ D_x^{m-\mu-1}v(0) \\ D_x^{m-\mu-2}v(0) \\ \vdots \\ v(0) \end{pmatrix},$$

$$\mathscr{A}=\begin{pmatrix} 1 & & & & & & \\ a_1^+ & \ddots & & & & 0 & \\ \vdots & \ddots & \ddots & & & & \\ a_{\mu-1}^+ & \cdots & a_1^+ & 1 & & & \\ & & & & 1 & & \\ & 0 & & & a_1^- & \ddots & \\ & & & & \vdots & \ddots & \ddots \\ & & & & a_{m-\mu-1}^- & \cdots\, a_1^- & 1 \end{pmatrix}=\begin{pmatrix} \mathscr{A}_+ & 0 \\ 0 & \mathscr{A}_- \end{pmatrix},$$

then we have

$$(A(D_x)u, v)-(u, \bar{A}(D_x)v)=iV^*\mathscr{A}U.$$

Let us denote

$$\begin{pmatrix} B_1(D_x)u(0) \\ \vdots \\ B_\mu(D_x)u(0) \\ B_{\mu+1}(D_x)u(0) \\ \vdots \\ B_m(D_x)u(0) \end{pmatrix}=\begin{pmatrix} \beta_{11}\cdots\cdots\beta_{1\mu} & \beta_{1\mu+1}\cdots\cdots\beta_{1m} \\ \vdots \qquad \vdots & \vdots \qquad \vdots \\ \beta_{\mu 1}\cdots\cdots\beta_{\mu\mu} & \beta_{\mu\mu+1}\cdots\cdots\beta_{\mu m} \\ \beta_{\mu+1\,1}\cdots\beta_{\mu+1\,\mu} & \beta_{\mu+1\,\mu+1}\cdots\beta_{\mu+1\,m} \\ \vdots \qquad \vdots & \vdots \qquad \vdots \\ \beta_{m1}\cdots\cdots\beta_{m\mu} & \beta_{m\mu+1}\cdots\cdots\beta_{mm} \end{pmatrix}\begin{pmatrix} u(0) \\ \vdots \\ D_x^{\mu-1}u(0) \\ A_+(D_x)u(0) \\ \vdots \\ D_x^{m-\mu-1}A_+(D_x)u(0) \end{pmatrix}$$

$$=\begin{pmatrix} \mathscr{B}_{11} & \mathscr{B}_{12} \\ \mathscr{B}_{21} & \mathscr{B}_{22} \end{pmatrix} U=\mathscr{B}U,$$

then we have

i) $\det \mathscr{B}=\operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & m \\ r_1 & r_2 & \cdots & r_m \end{pmatrix}$,

ii) $R=(-1)^{\frac{\mu(\mu-1)}{2}} \det \mathscr{B}_{11}$,

because i) follows from

$$B=\mathscr{B}\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ a_\mu^+ \cdots\cdots & a_1^+ & & 1 & & \\ & \ddots & \ddots & & \ddots & \\ & & a_\mu^+ \cdots\cdots & a_1^+ & & 1 \end{pmatrix},$$

and

$$\det B=\operatorname{sgn}\begin{pmatrix} 1 & \cdots & m \\ r_1 & \cdots & r_m \end{pmatrix},$$

and ii) follows from

$$R=\begin{vmatrix} \beta_{11} \cdots\cdots \beta_{1\mu} \\ \vdots \qquad \vdots \\ \beta_{\mu 1} \cdots\cdots \beta_{\mu\mu} \end{vmatrix} \cdot \begin{vmatrix} \frac{1}{2\pi i}\oint \frac{1}{A_+(\xi)}d\xi & \cdots\cdots & \frac{1}{2\pi i}\oint \frac{\xi^{\mu-1}}{A_+(\xi)}d\xi \\ \vdots & & \\ \frac{1}{2\pi i}\oint \frac{\xi^{\mu-1}}{A_+(\xi)}d\xi & \cdots\cdots & \frac{1}{2\pi i}\oint \frac{\xi^{2\mu-2}}{A_+(\xi)}d\xi \end{vmatrix}.$$

Now let us denote

$$V^*\mathscr{A}U=(\overline{\mathscr{B}'}V)^*\cdot\mathscr{B}U$$

where

$${}^t(\mathscr{A}\mathscr{B}^{-1})=\mathscr{B}',$$

then we have

$$\begin{pmatrix} B'_1(\xi) \\ \vdots \\ B'_\mu(\xi) \\ B'_{\mu+1}(\xi) \\ \vdots \\ B'_m(\xi) \end{pmatrix} = \mathscr{B}' \begin{pmatrix} \xi^{\mu-1}A_-(\xi) \\ \vdots \\ A_-(\xi) \\ \xi^{m-\mu-1} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \mathscr{B}'_{11} & \mathscr{B}'_{12} \\ \mathscr{B}'_{21} & \mathscr{B}'_{22} \end{pmatrix} \begin{pmatrix} \xi^{\mu-1}A_-(\xi) \\ \vdots \\ A_-(\xi) \\ \xi^{m-\mu-1} \\ \vdots \\ 1 \end{pmatrix}.$$

Let us denote

$$R' = \begin{vmatrix} \frac{1}{2\pi i}\oint \frac{B'_{\mu+1}(\xi)}{A_-(\xi)} d\xi & \cdots\cdots & \frac{1}{2\pi i}\oint \frac{B'_{\mu+1}(\xi)\xi^{m-\mu-1}}{A_-(\xi)} d\xi \\ \vdots & & \vdots \\ \frac{1}{2\pi i}\oint \frac{B'_m(\xi)}{A_-(\xi)} d\xi & \cdots\cdots\cdots & \frac{1}{2\pi i}\oint \frac{B'_m(\xi)\xi^{m-\mu-1}}{A_-(\xi)} d\xi \end{vmatrix},$$

then we have

iii) $R' = \det \mathscr{B}'_{22}$,

because

$$R' = \det \mathscr{B}'_{22} \begin{vmatrix} \frac{1}{2\pi i}\oint \frac{\xi^{m-\mu-1}}{A_-(\xi)} d\xi & \cdots\cdots & \frac{1}{2\pi i}\oint \frac{\xi^{2(m-\mu-1)}}{A_-(\xi)} d\xi \\ \vdots & & \vdots \\ \frac{1}{2\pi i}\oint \frac{1}{A_-(\xi)} d\xi & \cdots\cdots\cdots & \frac{1}{2\pi i}\oint \frac{\xi^{m-\mu-1}}{A_-(\xi)} d\xi \end{vmatrix}$$

$$= \det \mathscr{B}'_{22}.$$

Now we denote

$$\mathscr{B}^{-1} = \begin{pmatrix} \mathscr{H}_{11} & \mathscr{H}_{12} \\ \mathscr{H}_{21} & \mathscr{H}_{22} \end{pmatrix},$$

then

$${}^t\mathscr{B}' = \begin{pmatrix} \mathscr{A}_+ & \\ & \mathscr{A}_- \end{pmatrix} \begin{pmatrix} \mathscr{H}_{11} & \mathscr{H}_{12} \\ \mathscr{H}_{21} & \mathscr{H}_{22} \end{pmatrix} = \begin{pmatrix} \mathscr{A}_+\mathscr{H}_{11} & \mathscr{A}_+\mathscr{H}_{12} \\ \mathscr{A}_-\mathscr{H}_{21} & \mathscr{A}_-\mathscr{H}_{22} \end{pmatrix},$$

therefore we have

$$\det \mathscr{B}'_{22} = \det \mathscr{H}_{22}.$$

On the other hand, we have

$$\begin{pmatrix} \mathscr{B}_{11} & \mathscr{B}_{12} \\ & \begin{matrix} 1 & & \\ & \ddots & \\ & & \end{matrix} \\ 0 & \qquad 1 \end{pmatrix} \begin{pmatrix} \mathscr{H}_{11} & \mathscr{H}_{12} \\ & \\ \mathscr{H}_{21} & \mathscr{H}_{22} \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ \mathscr{H}_{21} & \mathscr{H}_{22} \end{pmatrix},$$

therefore we have

$$\det \mathscr{B}_{11} \cdot \frac{1}{\det \mathscr{B}} = \det \mathscr{H}_{22}.$$

Hence we have

iv) $\det \mathscr{B}'_{22} = \dfrac{\det \mathscr{B}_{11}}{\det \mathscr{B}}.$

Here we have from i)$\sim$iv)

Lemma 5.1.

$$R'(\tau, \eta) = \operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & m \\ r_1 & r_2 & \cdots & r_m \end{pmatrix} (-1)^{\frac{\mu(\mu-1)}{2}} R(\tau, \eta).$$

Now let us consider

$$(\tilde{P}^*) \begin{cases} A^*(-D_t, -D_x, -D_y)u = f & \text{for } t>0,\ x<0,\ y \in R^{n-1}, \\ B_j^*(-D_t, -D_x, -D_y)u = g_j & (j=\mu+1, \dots, m) \\ & \text{for } t>0,\ x=0,\ y \in R^{n-1}, \\ D_t^j u = u_j \ (j=0, 1, \dots, m-1) & \text{for } t=0,\ x<0,\ y \in R^{n-1}. \end{cases}$$

We remark that

$$A^*(-\tau, -\xi, -\eta) = \overline{A(-\bar{\tau}, -\bar{\xi}, -\bar{\eta})} = \overline{A_+(-\bar{\tau}, -\bar{\eta}; -\bar{\xi})}\,\overline{A_-(-\bar{\tau}, -\bar{\eta}; -\bar{\xi})},$$

$$B_j^*(-\tau, -\xi, -\eta) = \overline{B'_j(-\bar{\tau}, -\bar{\xi}, -\bar{\eta})},$$

where zeros of $\overline{A_+(-\bar{\tau}, -\bar{\eta}; -\bar{\xi})}$ with respect to ξ have positive imaginary parts and zeros of $\overline{A_-(-\bar{\tau}, -\bar{\eta}; -\bar{\xi})}$ with respect to ξ have negative imaginary parts for $(\tau, \eta) \in R^n - i\dot{\Theta}_{\gamma_0}$. Here we denote

$$R^*(\tau, \eta) = \det\left(\frac{1}{2\pi i}\oint \frac{\overline{B'_j(-\bar{\tau}, -\bar{\xi}; -\bar{\eta})\xi^{k-1}}}{\overline{A_-(-\bar{\tau}, -\bar{\eta}; -\bar{\xi})}} d\xi\right),$$

$$j = \mu+1, \dots, m$$

$$k = 1, \dots, m-\mu$$

then we have

Lemma 5.2.

$$R^*(\tau, \eta) = (-1)^{m-2\mu} \operatorname{sgn}\begin{pmatrix} 1 & \cdots & m \\ r_1 & \cdots & r_m \end{pmatrix} \overline{R(-\bar{\tau}, -\bar{\eta})}.$$

Proof.

$$\overline{R^*(\tau, \eta)} = \det\left(\frac{1}{2\pi i}\oint \frac{B'_j(-\bar{\tau}, -\bar{\xi}, -\bar{\eta})\bar{\xi}^{k-1}}{A_-(-\bar{\tau}, -\bar{\eta}; -\bar{\xi})} d\bar{\xi}\right)$$

$$= (-1)^{\frac{(m-\mu)(m-\mu-1)}{2}} \det\left(\frac{1}{2\pi i}\oint \frac{B'_j(-\bar{\tau}, \xi, -\bar{\eta})\xi^{k-1}}{A_-(-\bar{\tau}, -\bar{\eta}; \xi)} d\xi\right)$$

$$= (-1)^{\frac{(m-\mu)(m-\mu-1)}{2}} R(-\bar{\tau}, -\bar{\eta}).$$

Corollary. *$R^*(\tau, \eta)$ is hyperbolic function with respect to* $(1, 0)$ *with cone* $\dot{\Sigma}$.

In fact, since $(-\bar{\tau}, -\bar{\eta}') \in R^n - i\dot{\Sigma}_{\gamma_0}$ is equivalent to $(\tau, \eta) \in R^n - i\dot{\Sigma}_{\gamma_0}$, $\overline{R(-\bar{\tau}, -\bar{\eta})}$ is holomorphic in $R^n - i\dot{\Sigma}_{\gamma_0}$.

Here we have

Proposition 5.1. *Let the supports of data be contained in* S, *then there exists a solution of* $(\tilde{P}^*)$, *whose support is contained in* K_S.

Let us denote

$$(P^*)\begin{cases} A^*(D_t, D_x, D_y)u=f & \text{for } t<0,\ x>0,\ y\in R^{n-1}, \\ B_j^*(D_t, D_x, D_y)u=g_j \ \ (j=\mu+1,\dots, m) & \text{for } t<0,\ x=0,\ y\in R^{n-1}, \\ D_t^j u=u_j \ \ (j=0, 1,\dots, m-1) & \text{for } t=0,\ x>0,\ y\in R^{n-1}, \end{cases}$$

then we have

Corollary 1. *Let the supports of data be contained in* S, *then there exists a solution of* (P^*), *whose support is contained in* $-K_{-S}$.

From this corollary, we have

Corollary 2. *A solution of* (P) *in* S *depends the data in* $-K_{-S}$.

Here we have the theorem stated in the introduction.

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