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The explosion problem of branching stable processes

By

Michio SHIMURA

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Introduction

Ito-McKean [4] proved that the semi-linear parabolic equation

(1-a) $$\frac{\partial u}{\partial t}(t, x)=\frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, x)+|x|^{\gamma} u(t, x)\{u(t, x)-1\},$$

(1-b) $$0 \leq u(t, x) \leq 1,$$

(1-c) $$u(0, x)=1, \qquad 0<t, \ -\infty<x<\infty,$$

has no solution except a trivial one $u \equiv 1$ if $0<\gamma \leq 2$, and that if $\gamma>2$, it has a non-trivial one in addition.

The purpose of this paper is to consider a similar problem in the following form:

(2-a) $$\frac{\partial u}{\partial t}(t, x)=\Lambda_{\alpha} u(t, x)+k(x) u(t, x)\{u(t, x)-1\},$$

(2-b) $$0 \leq u(t, x) \leq 1,$$

(2-c) $$u(0, x)=1, \qquad 0<t, \ -\infty<x<\infty,$$

where Λ_{α} is the infinitesimal operator of a one-dimensional symmetric stable process with index $\alpha(0<\alpha<2)$, i.e. $\Lambda_{\alpha}=-\left(-2^{-1} \frac{\partial^2}{\partial x^2}\right)^{\alpha / 2}$ and k is a non-negative continuous unbounded function on R.

One of the essential difficulties arising in the present case is caused by the discontinuities of sample paths of a stable process. To overcome the difficulties, it is necessary to formulate the problem on the basis of the general theory of braching Markov processes developed in Ikeda-Nagasawa-Watanabe [2].

§1 is devoted to preparatory consideration on branching processes. In §2 find §3, conditions will be discussed for (non-) explosion of branching stable processes in connection with (uniqueness) non-uniqueness problem of the semi-linear parabolic equation (2–a, b, c), where $k(x)$ will be taken to be $|x|^\gamma$ or $\log(1+|x|^\gamma)$ $(\gamma>0)$.

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§1. Preliminary

1.1 Let S be a compact metric space, S^n the n-fold product of S, $\mathbf{S}=\bigcup_{n=0}^{\infty} S^n$ the topological sum of S^n, where $S^0=\{\partial\}$, ∂ an extra point, and $\hat{\mathbf{S}}=\mathbf{S}\cup\{\Delta\}$ the one-point compactification of $\mathbf{S}$.

Let B(S) be the space of all bounded measurable functions on S. and B(**S**) be the space of all bounded measurable functions on **S** which vanish at Δ. The spaces $B_1(S)$ and $B_1^+(S)$ are defined as follows:

$$B_1(S)=\{f\in B(S);\ \|f\|\leq 1\}^{1)},$$

$$B_1^+(S)=\{f\in B_1(S);\ f\geq 0\}.$$

$B_1(\mathbf{S})$ and $B_1^+(\mathbf{S})$ are defined similarly.

For $f\in B_1(S)$, a function $\hat{f}\in B_1(\mathbf{S})$ is defined by

$$\hat{f}(\mathbf{x})\equiv\begin{cases}1, & \text{if } \mathbf{x}=\partial,\\ f(x_1)\cdots f(x_n), & \text{if } \mathbf{x}=(x_1,\cdots,x_n),\\ 0, & \text{if } \mathbf{x}=\Delta.\end{cases} \tag{1}$$

For a function f on S, a function $\check{f}$ on $\hat{\mathbf{S}}$ is defined by

1) $\|f\|=\sup_{x\in S}|f(x)|$.

$$(2)\qquad \check{f}(\mathbf{x})=\begin{cases} 0, & \text{if } \mathbf{x}=\partial \text{ or } \Delta, \\ f(x_1)+\cdots+f(x_n), & \text{if } \mathbf{x}=(x_1,\cdots,x_n). \end{cases}$$

1.2 A Markov process $\mathbf{X}=(\mathbf{W}, \mathbf{X}_t, \mathbf{P}_x)$ on $\mathbf{S}$ is called a *branching Markov process* if it is a strong Markov process with right continuous sample paths, and if its semi-group $(\mathbf{T}_t)_{t\geq 0}$[2] on $B(\hat{\mathbf{S}})$ has the *branching property* (B):

$$(\mathrm{B})\qquad \mathbf{T}_t\hat{f}(\mathbf{x})=\widehat{\mathbf{T}_t f_{|S}}(\mathbf{x}), \qquad \mathbf{x}\in\hat{S}, f\in B_1(S).$$

We define some random variables concerning a branching Markov process $\mathbf{X}$ as follows:

$$(3)\qquad Z_t^E(\mathbf{w})=\check{I}_E(\mathbf{X}_t(\mathbf{w})), \qquad E\in\mathscr{B}(S)^{3)},$$

and especially we denote $Z_t(\mathbf{w})=Z_t^S(\mathbf{w})$. $Z_t^E(\mathbf{w})$ stands for the total number of particles in the set E.

Put

$$(4)\qquad \tau_0(\mathbf{w})=0,$$

$$\tau_1(\mathbf{w})=\tau(\mathbf{w})=\inf\{t,\ Z_t(\mathbf{w})\neq Z_0(\mathbf{w})\}^{4)},$$

$$\tau_n(\mathbf{w})+\tau_{n-1}(\mathbf{w}^+_{\tau(\mathbf{w})})^{4)}, \qquad n=2, 3,\cdots,$$

$$(5)\qquad e_\triangle(\mathbf{w})=\inf\{t;\ \mathbf{X}_t(\mathbf{w})=\triangle\}.$$

Clearly the n-th *branching time* τ_n and the *explosion time* e_Δ are Markov times for the branching Markov process. And we can easily show that Δ is trap, i.e.

$$(6)\qquad \mathbf{P}_\mathbf{x}\{e_\Delta=\infty \ \text{ or } \ \mathbf{X}_t=\Delta,\ e_\Delta\leq t\}=1, \qquad \mathbf{x}\in\mathbf{S}$$

(Ikeda-Nagasawa-Watanabe [2], I).

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2) $\mathbf{T}_t f(\mathbf{x})=\mathbf{E}_\mathbf{x}[f(\mathbf{X}_t)]$, where the right-hand side is the expectation of $f(\mathbf{X}_t)$ with respect to $\mathbf{P}_\mathbf{x}$. Every semi-group associated with a Markov process is defined similarly.

3) $\mathscr{B}(S)$ is the topological σ-field of a topological space S.

4) When $\{\,''\,\}=\phi$, define $\tau(\mathbf{w})=\infty$, and when $\tau_{n-1}(\mathbf{w})=\infty$, define $\tau_n(\mathbf{w})=\infty$,

Definition 1. When $\mathbf{P}_{\mathbf{x}}\{e_\Delta=\infty\}<1$ *on* $\mathbf{S}$, a branching Markov process is said to be *explosive*, and when $\mathbf{P}_{\mathbf{x}}\{e_\Delta=\infty\}=1$ *on* $\mathbf{S}$, *non-explosive.*

1.3 *Branching Markov processes that we shall treat from now on will be supposed to satisfy the following condition*:

(C.1) Let $\mathbf{X}^0_{|S}$ be the *non-branching part* on S of a branching Markov process $\mathbf{X}$. $\mathbf{X}^0_{|S}$ is equivalent to the $e^{-\varphi_t}$-subprocess $\dot{X}=(W, \dot{X}_t, \zeta, P_x)$ of a conservative strong Markov process $X=(W, X_t, P_x)$ on S which is right continuous and has left limit. Here φ_t is given by

$$\varphi_t(w)=\int_0^t k(X_s(w))ds \tag{7}$$

where k is a non-negative measurable function on S. We shall call the function k killing rate of a branching Markov process.

(C.2) The branching law of a process $\mathbf{X}$ is given by a stochastic kernel $\pi(x, \Gamma)$ on $S\times\mathbf{S}$[5].

We shall call the process with (C.1) and (C.2) (X, k, π)-branching Markov process. *Given a Markov process* X, *then there exists a* (X, k, π)*-branching Markov process* (Ikeda-Nagasawa-Watanabe [2], II).

1.4 Next we shall introduce M-equation and S-equation of a branching Markov process $\mathbf{X}$. Let $(\mathbf{T}^0_t)_{t\geq 0}$ be the semi-group of the non-branching part $\mathbf{X}^0$ of $\mathbf{X}$, and a non-negative kernel $\Psi(\mathbf{x}, ds, dy)$ on $\mathbf{S}\times[0, \infty)\times\mathbf{S}$ be given by

$$\Psi(\mathbf{x}, ds, d\mathbf{y})=\mathbf{P}_x\{\tau\in ds, \mathbf{X}_\tau\in d\mathbf{y}\}. \tag{8}$$

The linear integral equation on $B(\mathbf{S})$

$$u(t, \mathbf{x})=\mathbf{T}^0_t f(\mathbf{x})+\int_0^t\int_{\mathbf{S}}\Psi(\mathbf{x}, ds, d\mathbf{y})u(t-s, \mathbf{y}), \qquad t>0, \mathbf{x}\in\mathbf{S} \tag{9}$$

will be called *M-equation* (*of an initial data* $f\in B(\mathbf{S})$). It is easy

5) A stochastic kernel $\Pi(x, \Gamma)$ on $S\times\mathbf{S}$ is a kernel such that for each $x\in S$, $\Pi(x, \cdot)$ is a probability measure on $\mathbf{S}$, and for each $\Gamma\in\mathscr{B}(\mathbf{S})$, $\Pi(\cdot, \Gamma)$ is a measurable function on S.

to see that $\mathbf{T}_t f(\mathbf{x})$ is a solution of the M-equation.

Let us take $\hat{f}, f \in B_1(S)$ as an initial data of the M-equation and restrict in on S, then we obtain the following non-linear equation on $B_1(S)$, which will be called *S-equation (of an initial data f)*

$$(10) \qquad u(t,\ x) = \dot{T}_t f(x) + \int_0^t \int_S K(x;\ ds,\ dy) F(y;\ u(t-s,\ \cdot)), \qquad t>0,\ x \in S,$$

where $(\dot{T}_t)_{t\geq 0}$ is the semi-group of the process $\dot{X}$, K is a non-negative kernel on $S \times [0, \infty) \times S$, F is a non-linear mapping on $B_1(S)$ into $B_1(S)$ defined as follows:

$$(11) \qquad K(x;\ ds,\ dy) = P_x\{\zeta \in ds,\ \dot{X}_{\zeta-} \in dy\}^{6)},$$

$$(12) \qquad F(x;f) = \int_{\mathbf{S}} \pi(x,\ d\mathbf{y}) \hat{f}(\mathbf{y}).$$

$u(t,\ x) = \mathbf{T}_t \hat{f}(x)$ is a solution of S-equation (10).
Because

$$(13) \qquad \mathrm{K}(x;\ ds,\ dy) = \dot{P}_x\{\dot{X}_s \in dy\} k(y) ds,$$

the S-equation (10) can be written as

$$(10') \quad u(t,\ x) = \dot{T}_t f(x) + \int_0^t \dot{T}_s[k(\cdot)F(\cdot\ ;\ u(t-s,\ *))](x) ds, \qquad t>0,\ x \in S.$$

From now on, we assume that initial data of the M-equation and the S-equation belong to $B_1^+(\mathbf{S})$ and $B_1^+(S)$ respectively. Moreover if $u(t, \mathbf{x}) \in B_1^+([0, \infty) \times \mathbf{S})$ satisfies (9), we shall call it a solution of M-equation (9), and if $u(t, x) \in B_1^+([0, \infty) \times S)$ satisfies (10), a solution of S-equation (10). We shall call $\underline{u}(t, \mathbf{x})$ $(\bar{u}(t, \mathbf{x}))$ the minimal (maximal) solution fo the M-equation, iff

$$\underline{u}(t,\ \mathbf{x}) \leqq u(t,\ \mathbf{x}) \quad (\bar{u}(t,\ \mathbf{x}) \geqq u(t,\ \mathbf{x})) \qquad \text{on } [0,\ \infty) \times \mathbf{S}$$

for every solution $u(t, \mathbf{x})$ of the M-equation. The minimal solution of the S-equation is defined similarly.

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6) $\dot{X}_{\zeta-} = \lim_{\varepsilon \searrow 0} \dot{X}_{\zeta-\varepsilon}$.

Lemma 1. *Let* $\underline{u}=\underline{u}(t, \mathbf{x})$ *and* $\bar{u}=\bar{u}(t, x)$ *be the minimal and maximal solutions of the M-equation respectively, then* $\underline{u}$ *and* $\bar{u}$ *are given by*

$$\underline{u}(t, \mathbf{x})=\mathbf{T}_t f(\mathbf{x}),$$

$$\bar{u}(t, \mathbf{x})=\mathbf{T}_t f(\mathbf{x})+\mathbf{P}_{\mathbf{x}}\{e_\Delta<t\}.$$

Lemma 2. (i) *Let* $u(t, x)$ *be a solution of S-equation* (10) *of initial data* f, *then* $\hat{u}(t, \mathbf{x})$ *is a solution of M-equation* (9) *of initial data* $\hat{f}$.

(ii) $\mathbf{T}_t\hat{1}(\mathbf{x})=\mathbf{P}_{\mathbf{x}}\{e_\Delta>t\}=\widehat{\mathbf{P}.\{e_\Delta>t\}_{|S}}(\mathbf{x})$

(iii) $\mathbf{P}_{\mathbf{x}}\{e_\Delta=\infty\}=\widehat{\mathbf{P}.\{e_\Delta=\infty\}_{|S}}(\mathbf{x})$

(iv) $\mathbf{P}_{\mathbf{x}}\{e_\Delta=\infty\}$ *is* $\mathbf{T}_t$*-harmonic function, that is for* $t>0$,

$$\mathbf{T}_t[\mathbf{P}.\{e_\Delta=\infty\}](\mathbf{x})=\mathbf{P}_{\mathbf{x}}\{e_\Delta=\infty\} \qquad on\ \hat{\mathbf{S}}.$$

Remark. By (iii) of lemma 2, explosion (non-explosion) of a branching Markov process is eqivalent to the condition $\mathbf{P}_x\{e_\Delta=\infty\}<1$ *on* S ($\mathbf{P}_x\{e_\Delta=\infty\}=1$ *on* S).

Lemma 3. *Let* $\underline{u}=\underline{u}(t, x)$ *be the minimal solution of the S-equation, then* $\underline{u}$ *is given by*

$$\underline{u}(t, x)=\mathbf{T}_t\hat{f}(x).$$

(cf. Ikeda-Nagasawa-Watanabe [2]).

Remark. When $f\in\mathcal{D}(\mathcal{G})$, and k is a continuous function on S, $u(t, x)=\mathbf{T}_t\hat{f}(x)$ is the minimal solution of

$$\frac{\partial u}{\partial t}(t, x)=\mathcal{G}u(t, x)+k(x)\{F(x; u(t, \cdot))-u(t, x)\},$$

(14) $$0\leq u(t, x)\leq 1,$$

$$u(0, x)=f(x), \qquad 0<t,\ x\in S.$$

where $\mathscr{G}$ is the infinitesimal operator of a Markov process X. (cf. Ikeda-Nagasawa-Watanabe [2]).

1.5

Theorem 1. *The following two statements are equivalent.*

(i) *For every initial data, the S-equation has a unique solution.*

(ii) *The branching Markov process is non-explosive.*

Theorem 1 is a direct consequence of lemma 1, 2 and 3, and the proof is omitted.

Theorem 2. *Let the non-linear mapping F defined by* (12) *satisfies the following condition*:

$$|F(x;f)-F(x;g)| \leq N|f(x)-g(x)| \qquad on\ S, \tag{15}$$

for $f, g \in B_1^+(S)$, where N is a positive constant. If

$$E_x[e^{(N-1)\varphi_t}] < \infty \qquad on \quad S, \tag{16}$$

for some $t_0 > 0$, then the branching Markov process is non-explosive.

Proof. First we remark that because φ_t is non-decreasing in t,

$$E_x[e^{(N-1)\varphi_t}] < \infty \qquad on\ [0, t_0] \times S. \tag{16'}$$

Let us prove the uniqueness of solution of the S-equation up to t_0 under the assumption (15) and (16). Let u_1 and u_2 be two solutions and set

$$\mathscr{W}(t, x) = |u_1(t, x) - u_2(t, x)|.$$

Then $\mathscr{W}$ satisfies

$$0 \leq \mathscr{W}(t, x) \leq N \int_0^t \dot{T}_s[k(\cdot)\mathscr{W}(t-s, \cdot)](x)ds,$$

because of (15). Moreover we have

$$0 \leq \mathscr{W}(t, x) \leq N^n E_x\left[e^{-\varphi_t} \sum_{i=n}^{\infty} \frac{\varphi_t^i}{i!}\right] \qquad n = 1, 2, \cdots, \tag{17}$$

by induction. When $0<N<1$, the right hand side of (17) tends to zero, while in the case of $N\geq 1$ since

$$N^n E_x\left[e^{-\varphi_t}\sum_{i=n}^{\infty}\frac{\varphi_t^i}{i!}\right]\leq E_x\left[e^{-\varphi_t}\sum_{i=n}^{\infty}\frac{(N\varphi_t)^i}{i!}\right]$$

and

$$\infty > E_x[e^{(N-1)\varphi_t}]\geq E_x\left[e^{-\varphi_t}\sum_{i=n}^{\infty}\frac{(N\varphi_t)^i}{i!}\right]\searrow 0 \qquad (n\to\infty),$$

on $[0, t_0]\times S$ according to (16′), we have

$$\mathscr{W}(t, x)=|u_1(t, x)-u_2(t, x)|=0 \qquad on\ [0, t_0]\times S.$$

From the uniqueness of solution of the S-equation up to t_0, we have $\mathbf{P}_x\{e_\Delta>t_0\}=1$ *on* S. Moreover by the Markov property, we have

$$\mathbf{P}_x\{e_\Delta>nt_0\}=1 \qquad on\ \ S,\ n=1, 2,\cdots,$$

and

$$\mathbf{P}_x\{e_\Delta=\infty\}=\lim_{n\to\infty}\mathbf{P}\{e_\Delta>nt_0\}=1 \qquad on\ S,$$

that is, the branching Markov process is non-explosive.

§2. Non-explosion of branching stable processes

Let $X=(W, X_t, P_x)$ be a one-dimensional symmetric stable process with index $\alpha(0<\alpha<2)$ and k be a non-negative continuous unbounded function on R and π be the stochastic kernel on $R\times(\bigcup_{n=0}^{\infty}R^n)$ given by

(1) $$\pi(x, dy)=\delta_{(x,x)}(dy), \qquad x\in R,\ dy\subset \mathbf{S}=\bigcup_{n=0}^{\infty}R^n.$$

We call the (X, k, π)-branching Markov process (α-) *branching stable process.*

In the following, two different kinds of functions will be considered as killing rates:

(i) $|x|^\gamma$, (ii) $\log(1+|x|^\gamma)$

where γ is any positive constant.

For the functions of the second type (ii), we have

Theorem 3. *The branching stable process is non-explosvie when* $k(x)=\log(1+|x|^{\gamma})$ $(\gamma>0)$.

Appling theorem 2, the theorem follows from

Proposition. *If* $k(x)=\log(1+|x|^{\gamma})$, *then*

$$E_x[e^{\varphi_t}]<\infty \qquad \text{on } R, \text{ for } 0\leq t<\frac{\alpha}{\gamma},$$

$$E_x[e^{\varphi_t}]=\infty \qquad \text{on } R, \text{ for } t>\frac{\alpha}{\gamma}.$$

To prove the proposition we need

Lemma 4. *Let* $P(t, x)$ *be the probability density of a symmetric stable process with index* α, *that is*

$$P(t, x)=(2\pi)^{-1}\int_{-\infty}^{\infty}\cos(xz)\exp\{-t|z|^{\alpha}\}dz.$$

Then for $c>0$,

(2) $$P(t, x)=c^{1/\alpha}P(ct, c^{1/\alpha}x),$$

and for $t>0$,

(3) $$\lim_{|x|\to\infty}|x|^{1+\alpha}P(t, x)=tv(\alpha),$$

where $v(\alpha)$ *is some positive constant depending only on index* α.

(2) *is well known. A proof of* (3) *is found in Polya* [6].

Corollary to Lemma 4. *For a symmetric stable process with index* α,

(4) $$E_x[|X_t|^{\gamma}]=\int_{-\infty}^{\infty}|y|^{\gamma}P(t, x-y)dy<\infty, \qquad 0\leq\gamma<\alpha,$$
$$=\infty, \qquad \gamma\geqq\alpha.$$

Proof of proposition. From the spatial homogeneity of stable processes and from the form of the additive functional φ_t, it is enough to prove the convergence or divergence of $E[e^{\varphi_t}]$[7].

First we note $E[e^{\varphi_0}]=E[e^0]=1$.

7) For abbreviation, we denote E_0 by E, and P_0 by P.

Next, for $0<t<\frac{\alpha}{\gamma}$

$$E[e^{\varphi_t}]=E\left[\exp\left\{t\int_0^t \log(1+|X_s|^\gamma)\frac{ds}{t}\right\}\right]$$

$$\leq E\left[\int_0^t \exp\{t\log(1+|X_s|^\gamma)\}\frac{ds}{t}\right],$$

where we used the Jensen's inequality for convex functions. Changing the order of integration and using the space-time transformation of stable processes:

$$(5) \qquad X_t \overset{d}{=} t^{1/\alpha}X_1, \qquad \text{in } P^{8)},$$

we have

$$E[e^{\varphi_t}]\leq\int_0^t E(1+s^{\gamma/\alpha}|X_1|^\gamma)^t\frac{ds}{t}$$

$$=\int_0^t s^{\gamma t/\alpha}E[|X_1|^{\gamma t}]\frac{ds}{t}=(\gamma t/\alpha+1)^{-1}t^{\gamma t/\alpha}E[|X_1|^{\gamma t}].$$

Therefore if $\gamma t<\alpha$, then by the corollary to lemma 4 we have $E[e^{\varphi_t}]<\infty$.

Now suppose $t>\frac{\alpha}{\gamma}$ [9].

Let N be any integer and take $t_0=\frac{\alpha}{\gamma N}$, then

$$E[e^{\varphi_{(N+1)t_0}}]=E[e^{\varphi_{t_0}(w)}e^{\varphi_{Nt_0}(w_{t_0}^+)}]\geq E[E_{X_{t_0}}[e^{\varphi_{Nt_0}}]]$$

$$\geq\sum_{n=1}^{\infty}E\left[n\leq X_{t_0}<n+1\,;E_{X_{t_0}}\left[\sup|X_s-X_0|\leq 1\,;\exp\left\{\int_0^{Nt_0}\log(1+|X_s|^\gamma)ds\right\}\right]\right]$$

$$\geqq\sum_{n=1}^{\infty}(1+|n-1|^\gamma)^{Nt_0}E[n\leq X_t<n+1\,;\,P_{X_{t_0}}\{\sup_{0\leq S\leq Nt_0}|X_s-X_0|\leq 1\}].$$

Put $Q=P_y\{\sup_{0\leq S\leq Nt_0}|X_s-X_0|<1\}$, then by the spatial homogeneity of stable processes, Q is independent of y and N (note $Nt_0=\frac{\alpha}{\gamma}$). There-

8) This indicates that the two random variables X_t and $t^{1/\alpha}X_1$ have the same distribution with respect to P.

9) The proof given here is due to Prof. M. Motoo.

for we have the following inequality

$$E[e^{\varphi(N+1)t_0}]\geq Q\sum_{n=1}^{\infty}(1+|n-1|^{\gamma})^{Nt_0}P\{n\leq X_{t_0}<n+1\}$$

$$=Q\sum_{n=1}^{\infty}(1+|n-1|^{\gamma})^{Nt_0}P\{nt_0^{-1/\alpha}\leq X_1<(n+1)t_0^{-1/\alpha}\}$$

$$>Q\sum_{n=M}^{\infty}(1+|n-1|^{\gamma})^{Nt_0}\int_{n^{-1/\alpha}}^{(n+1)^{-1/\alpha}}2^{-1}v(\alpha)x^{-1-\alpha}dx,$$

where M is an integer statisfying

$$P(1, x)\geq 2^{-1}v(\alpha)x^{-1-\alpha}, \qquad \text{for } x\geq Mt^{-1-\alpha}.$$

Existence of such M is guranteed by (3) of lemma 4. Therefore we have

$$E[e^{\varphi(N+1)t_0}]>(\text{const.})\sum_{n=M}^{\infty}\frac{\{1+(n-1)^{\gamma}\}^{Nt_0}}{(n+1)^{\alpha}}\cdot\frac{1}{n+1}.$$

Since $\gamma Nt_0=\alpha$,

$$\frac{\{1+(n-1)^{\gamma}\}^{Nt_0}}{(n+1)^{\alpha}}\to 1, \qquad (n\to\infty),$$

and hence

$$E[e^{\varphi(N+1)t_0}]=E[e^{\varphi(1+1/N)\alpha/\gamma}]=\infty, \qquad N=1, 2,\cdots.$$

Because φ_t is non-decreasing in t, we have for $t>\frac{\alpha}{\gamma}$, $E[e^{\varphi_t}]=\infty$. This completes the proof of the proposition, and of theorem 3.

§3. Explosion of branching stable processes

In this section, following the idea of Ito-McKean [4], we shall consider an explosion condition of branching stable processes when $k(x)=|x|^{\gamma}$.

Let $G(\cdot, f)$ be a mapping on $B_1^{+}(R)$ to $B_1^{+}(R)$, and k be a locally bounded non-negative measurable function on R. Consider an integral equation of an initial data $f\in B_1^{+}(R)$

(1) $$u_t(x)=T_tf(x)-\int_{0+}^{t}T_s[k(\cdot)G(\cdot : u_{t-s})](x)ds,$$

where $(T_t)_{t\geq 0}$ is the semi-group of a symmetric stable process and $u_t(x)$ stands for $u(t, x)$.

Lemma 5. *Every solution u_t of the equation* (1) (*if exists*) *is a continuous function on R for every $t>0$.*

Proof. First we note that $T_tf(\cdot)$ is a continuous function on R because of the strongly Feller property of a symmetric stable process. Next we set

$$I(t, x)=\int_{0+}^{t}dsT_s[k(\cdot)G(\cdot : u_{t-s})](x)$$

$$=\int_{0+}^{t}ds\int_{-\infty}^{\infty}dyP(s, x-y)k(y)G(y: u_{t-s}),$$

and we shall prove the continuity of $I(t, \cdot)$. Let T be any positive constant. Taking positive number N sufficiently large, we have from lemma 4,

$$\frac{P(t, x-y)}{P(t, y)}=\frac{P(1, t^{-1/\alpha}(x-y))}{P(1, t^{-1/\alpha}y)}$$

$$\cong|xy^{-1}-1|^{-1-\alpha}\leq 2^{1+\alpha}, \qquad 0<t\leq T, |x|\leq N, |y|\geq 2N.$$

Therefore we have

(2) $$P(t, x-y)\leq 2^{1+\alpha}P(t, y), \qquad 0<t\leq T, |x|\leq N, |y|\geq 2N.$$

Devide $I(t, x)$ into two parts

$$I(t, x)=\int_{0+}^{t}ds\int_{-2N}^{2N}dyP(s, x-y)k(y)G(y: u_{t-s})$$

$$+\int_{0+}^{t}ds\int_{\{|y|>2N\}}dyP(s, x-y)k(y)G(y: u_{t-s})=I_1(t, x)+I_2(t, x).$$

Because the probability density $P(s, \cdot)$ is continuous and k and $G(\cdot : u_{t-s})$ are bounded on $[-2N, 2N]$, $I_1(t, \cdot)$ is continuous. For the proof of the continuity of $I_2(t, \cdot)$, define a function $\Psi_t(s, y)$ by

$$\Psi_t(s, y)=2^{1+\alpha}P(s, y)k(y)G(y: u_{t-s}), \qquad 0<s<t\leq T.$$

It is easy to see that $\Psi_t(s, y)$ is a non-negative integrable function on $(0, t]\times R$. In fact, because

$$\int_{0+}^{t} ds\int_{-\infty}^{\infty} dy\Psi_t(s, y)=2^{1+\alpha}\int_{0+}^{t} ds\int_{-\infty}^{\infty} dyP(s, y)k(y)G(y: u_{t-s})$$

$$=2^{1+\alpha}\{T_tf(0)-u_t(0)\}<\infty.$$

And using (2), we have

$$0\leq \text{ the integrand of } I_2(t, x)\leq \Psi_t(s, y),$$

$$|x|\leq N, \ |y|\geq 2N, \ 0<s<t\leq T.$$

Therefore, using the theorem of Lebesgue, we have the continuity of $I_2(t, \cdot)$ on $[-N, N]$, and hence on R because N is arbitrary large number.

Because $u_t(x)=T_tf(x)-I(t, x)$, the assertion of this lemma is now proved.

Corollary to lemma 5. *Every solution u_t of the S-equation of a branching stable process is a continuous function on R. Here the S-equation is of the form*

$$(3) \qquad u_t(x)= \dot{T}_tf(x)+\int_{0+}^{t} \dot{T}_s[ku_{t-s}^2](x)ds$$

$$= T_tf(x)-\int_{0+}^{t} T_s[k\{u_{t-s}-u_{t-s}^2\}](x)ds$$

Lemma 6. (0–1 *law of the explosion probability*)
Let **X** *be an α-branching stable process* $(1\leq\alpha\leq 2)$, *then*

$$\mathbf{P}_x\{e_\Delta=\infty\}\equiv 0 \quad \textit{or } 1, \textit{ on } R.$$

Proof. In the case $\alpha=2$, a proof is found in Ito-McKean [4].

For a proof of the case $1 \leq \alpha \leq 2$, define hitting times, j_U of the symmetric stable process and j_U of the branching stable process by

$$j_U = \inf\{t>0;\ X_t \in U\},$$

$$j_U = \inf\{0<t<e_\Delta;\ \widehat{(1-I_U)}(\mathbf{X}_t)=0\},$$

where U is an open set in R. It is known (cf. e.g., McKean [5]) that for $1 \leq \alpha \leq 2$ and for $U \neq \phi$, $P_x\{j_U<\infty\}=1$, *on* R, then we can show for α-branching stable process $(1 \leq \alpha \leq 2)$,

$$\mathbf{P}_x\{j_U=\infty,\ e_\Delta=\infty\}=0, \qquad on\ R. \tag{4}$$

Let U and V are arbitrary open intervals in R. Using (4), we have

$$\begin{aligned}\mathbf{P}_x\{e_\Delta=\infty\} &= \mathbf{P}_x\{\mathbf{j}_V<\infty,\ e_\Delta=\infty\} = \mathbf{P}_x\{\mathbf{j}_V<\infty,\ e_\Delta(\mathbf{w}^+_{\mathbf{j}_V})=\infty\} \\ &= \mathbf{E}_x[\mathbf{j}_V<\infty,\ \mathbf{P}_{\mathbf{X}(\mathbf{j}_V)}\{e_\Delta=\infty\}] \leq \mathbf{P}_x\{\mathbf{j}_V<\infty\}\cdot \sup_{y\in V}\mathbf{P}_y\{e_\Delta=\infty\} \\ &\leq \sup_{y\in V}\mathbf{P}_y\{e_\Delta=\infty\}.\end{aligned} \tag{5}$$

Taking the supremum of the left hand side of (5) in U, we have

$$\sup_{x\in U}\mathbf{P}_x\{e_\Delta=\infty\} \leq \sup_{y\in V}\mathbf{P}_y\{e_\Delta=\infty\}.$$

Because U and V are arbitrary open intervals, we have

$$\sup_{x\in U}\mathbf{P}_x\{e_\Delta=\infty\}=c \tag{6}$$

where c is a constant independent of U. Because of (iv) of lemma 2, $\mathbf{P}_x\{e_\Delta=\infty\}$ is a stationary solution of the S-equation, and $\mathbf{P}_x\{e_\Delta=\infty\}$ is a continuous function on R on account of corollary to lemma 5. Therefore we have

$$\mathbf{P}_x\{e_\Delta=\infty\}=c, \qquad on\ R. \tag{7}$$

Because

$$c=\mathbf{P}_x\{\tau<\infty,\ e_\Delta=\infty\}=\mathbf{P}_x\{\tau<\infty,\ e_\Delta(w^+_\tau)=\infty\}$$

$$= \mathbf{E}_x[\tau < \infty ; \mathbf{P}_{\mathbf{X}_\tau}\{e_\Delta = \infty\}] = c^2,$$

we conclude $c=0$ or 1 which completes the proof.

Lemma 7. *In the case* $k(x)=|x|^\gamma$,

$$E_x[e^{\varphi_t}]=\infty, \qquad \textit{on } R, \textit{ fort } >0.$$

Remark. In this case, theorem 2 gives no information whether the branching stable process is explosive or not.

Proof. Let N be an integer satisfying $N\gamma \geq \alpha$, then

$$E[e^{\varphi_t}] \geq \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{N-1}}^t ds_N E[|X_{s_1}|^\gamma |X_{s_2}|^\gamma \cdots |X_{s_N}|^\gamma]$$

because

$$e^{\varphi_t} = \sum_{n=0}^\infty \frac{1}{n!}\varphi_t^n \geq \frac{1}{N!}\varphi_t^N = \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{N-1}}^t ds_N |X_{s_1}|^\gamma |X_{s_2}|^\gamma \cdots |X_{s_N}|^\gamma.$$

However we have

(8) $\quad E[|X_{s_1}|^\gamma |X_{s_2}|^\gamma \cdots |X_{s_N}|^\gamma]$

$$= E[|X_{s_1}|^\gamma |(X_{s_2}-X_{s_1})+X_{s_1}|^\gamma \cdots |(X_{s_N}-X_{s_{N-1}})+\cdots+(X_{s_2}-X_{s_1})|^\gamma$$

$$= \int_{-\infty}^\infty P(s_2-s_1, y_2)dy_2 \cdots \int_{-\infty}^\infty P(s_N-s_{N-1}, y_N)dx_N \int_{-\infty}^\infty |y_1|^\gamma |y_1$$

$$+y_2|^\gamma \cdots |y_1+y_2+\cdots+y_N|^\gamma P(s_1, y_1)dy_1,$$

Because of $|y_1|^\gamma |y_1+y_2|^\gamma \cdots |y_1+y_2+\cdots+y_N|^\gamma = 0(|y_1|^{N\gamma})$, $\quad (|y_1|\to\infty)$ and of corollary to lemma 4, the right hand side of (8) diverges, which proves $E[e^{\varphi_t}]=\infty$.

Here we shall give the main theorem of this section.

Theorem 4. *Let* $\mathbf{X}$ *be an* α*-branching stable process* $(1<\alpha\leq 2)$ *with killing rate* $k(x)=|x|^\gamma$, *then we have*

(i) *In the case* $1<\alpha<2$, *the branching stable process is explosive* *with probability* 1 *when* $\gamma>2\alpha/(\alpha-1)$.

(ii) (Ito-McKean [4]) *In the case* $\alpha=2$, *the branching Brownian process is non-explosive when* $\gamma\leq 2$, *and explosive with probability* 1 *when* $\gamma>2$.

Before proving theorem 4, we shall prepare several lemmas. Define two events A_e and A_∞ of the branching stable process by

$$A_e=\{\mathbf{w}\,;\,\mathbf{X}_0(\mathbf{w})=x,\ e_\Delta(\mathbf{w})<\infty\},$$

$$A_\infty=\bigcup_{n=1}^{\infty}\{\mathbf{w}\,;\,\mathbf{X}_0(\mathbf{w})=x,\ \lim_{k\to\infty}\inf_{0\leq t<n}\hat{I}_{(-\infty,K)}(\mathbf{X}_t(\mathbf{w}))=0\},$$

for $x\in R$.

Lemma 8. *For* A_∞ *and* A_e *defined above,*

$$\mathbf{P}_x\{A_\infty\backslash A_e\}=0 \qquad \textit{on } R.$$

Proof.

$$A_\infty\backslash A_e=\bigcup_{n=1}^{\infty}\{\mathbf{X}_0=x,\ e_\Delta=\infty,\ \lim_{K\to\infty}\inf_{0\leq t<n}\hat{I}_{(-\infty,K)}(\mathbf{X}_t)=0\} \tag{9}$$

$$=\bigcup_{n=1}^{\infty}\ \bigcup_{m=0}^{\infty}\{\mathbf{X}_0=x,\ \tau_m\leq n<\tau_{m+1},\ \lim_{K\to\infty}\inf_{0\leq t<n}\hat{I}_{(-\infty,K)}(\mathbf{X}_t)=0\}.$$

Using the conservativity of stable processes, we have

$$\mathbf{P}_x\{\mathbf{X}_0=x,\ \tau_m\leq n<\tau_{m+1},\ \lim_{K\to\infty}\inf_{0\leq t<n}\hat{I}_{(-\infty,K)}(\mathbf{X}_t)=0\}=0, \tag{10}$$

for $n=1, 2,\cdots; m,=0, 1, 2,\cdots$.
From (9) and (10), we have the assertion.

Let $p=(W, p_t; p_0=0)$ be a Poisson process with parameter 1, and p and the symmetric stable process X be independent. Now define a random time ζ of X by

$$\zeta(w)=\inf\{t>0;\ p(\varphi_t(w),\ w)\neq 0\}, \tag{11}$$

which is the life time of the $e^{-\varphi_t}$-subprocess $\dot{X}$ of X.
By the general theory of construction of branching Markov processes (Ikeda-Nagasawa-Watanabe [2], II), we can construct a branching stable process by piecing out the process $\dot{X}$ by the instantaneous distribution $\delta(X_{\zeta-}, X_{\zeta-})(dy)$. We need the fact that the symmetric stable process is continuous at ζ as is proved in the following lemma.

Lemma 9. *Let X be a symmetric stable process, then*

$$X_{\zeta-}(w)=X_{\zeta}(w) \qquad a.s.(P_x),\ x\in R.$$

Proof.[10] Set $Y_t=X(\varphi_t^{-1})$, where φ_t^{-1} is the inverse function of φ_t, that is $\varphi_t^{-1}=\sup\{u;\ \varphi_u\leq t\}$. Then by the general theory of random time change of a Markov process, the process $Y=(W, Y_t, P_x)$ is a standard Markov process.

Define $\xi=\inf\{t>0;\ p_t=1\}$. Because φ_t is a continuous and strictly increasing function in t, then we have

$$P_x\{X_{\zeta-}\neq X_{\zeta}\}=P_x\{Y_{\xi-}\neq Y_{\xi}\}=\int_0^{\infty}P_x\{Y_{t-}\neq Y_t\}e^{-t}dt.$$

Because the process Y is standard, it has no fixed discontinuites and then

$$P_x\{Y_{t-}\neq Y_t\}=0, \qquad for\ t>0.$$

Thus we have $P_x\{X_{\zeta-}\neq X_{\zeta}\}=0$ and we have proved lemma 9.

Remark. It is easy to see that the above proof can be applied for a wider class of Markov processes.

Next, we define Markov times $\mathbf{j}_y^x$ and $\mathbf{j}_y$ of the branching stable process, and j_y^x and j_y of the symmetric stable process. For $x<y$,

$$\mathbf{j}_y^x=\inf\{0<t<e_{\Delta};\ \hat{I}_{[x,y]}(\mathbf{X}_t)=0\},$$

$$\mathbf{j}_y=\lim_{x\to-\infty}\mathbf{j}_y^x,$$

10) The author's original proof was lengthy. The proof given here was suggested by Prof. S. Watanabe.

$$j_y^x = \inf\{t>0;\ I_{[x,y]}(X_t)=0\},$$

$$j_y = \lim_{x\to-\infty} j_y^x.$$

Lemma 10. *For $x<y$ and $z\in R$,*

$$\mathbf{P}_z\{\mathbf{j}_y^x<\infty,\ Z^{R\setminus[x,y]}(\mathbf{j}_y^x)\geq 2\}=0.$$

Proof.

$$(12)\quad \mathbf{P}_z\{\mathbf{j}_y^x<\infty,\ Z^{R\setminus[x,y]}(\mathbf{j}_y^x)$$
$$\geq 2\}=\mathbf{P}_z\{\mathbf{j}_y^x=0,\ Z_0^{R\setminus[x,y]}\geq 2\}+\sum_{n=0}^{\infty}\mathbf{P}_z\{\tau_n<\mathbf{j}_y^x\leq\tau_{n+1},\ Z^{R\setminus[x,y]}(\mathbf{j}_y^x)\geq 2\}.$$

Obviously the first term of the right hand side of (12) is zero, and for the second term,

$$(13)\quad \mathbf{P}_z\{\tau_n<\mathbf{j}_y^x\leq\tau_{n+1},\ Z^{R\setminus[x,y]}(\mathbf{j}_y^x)\geq 2\}$$
$$=\mathbf{E}_z[\tau_n<\mathbf{j}_y^x;\ \mathbf{P}_{\mathbf{X}(\tau_n))}\{\mathbf{j}_y^x<\tau,\ Z^{R\setminus[x,y]}(\mathbf{j}_y^x)\geq 2\}]$$
$$+\mathbf{E}_z[\tau_n<\mathbf{j}_y^x;\ \mathbf{P}_{\mathbf{X}(\tau_n)}\{\mathbf{j}_y^x=\tau,\ Z^{R\setminus[x,y]}(\mathbf{j}_y^x)\geq 2\}]$$

In the right hand side of (13), every coordinate of $\mathbf{X}(\tau_n)$ is in the interval $[x, y]$ because of the condition $\tau_n<\mathbf{j}_y^x$. Then by lemma 9, the second term is equal to zero. For the first term, because of the mutual independence of every branch before the branching time τ and no fixed discontinuities of a stable process, it is equal to zero.
Therefore we have

$$\mathbf{P}_z\{\tau_n<\mathbf{j}_y^x\leq\tau_{n+1},\ Z^{R\setminus[x,y]}(\mathbf{j}_y^x)\geq 2\}=0,\qquad n=1,2,\cdots,$$

which completes the proof.

Lemma 11. *For $x<y$ and $t>0$,*

$$\mathbf{P}_x\{\mathbf{j}_y>t\}\leq P_x\{j_y>t\}.$$

Proof. First we define a sequence of random times of the process X by

$$\zeta_0(w)=0,\quad \zeta_1(w)=\zeta(w),\quad \zeta_n(w)=\zeta(w)+\zeta_{n-1}(w^+_{\zeta(w)}),$$
$$n=2,3,\cdots.$$

Then,

$$\mathbf{P}_x\{\mathbf{j}_y>t\}=\mathbf{P}_x\{\mathbf{j}_y>t,\ \tau>t\}+\mathbf{P}_x\{\mathbf{j}_y>t\geq\tau\}$$
$$=P_x\{j_y>t,\ \zeta>t\}+\mathbf{E}_x[\tau\leq t,\ \tau<\mathbf{j}_y;\ \mathbf{P}_{\mathbf{X}_\tau}\{0\leq t-u<\mathbf{j}_y\}_{|u=\tau}]$$
$$=P_x\{j_y>t,\ \zeta>t\}+\int_{\{\zeta\leq t,\zeta<j_y\}}dP_x\mathbf{P}_{(X_{\zeta-},X_{\zeta-})}\{\mathbf{j}_y>t-u\}_{|u=\zeta}$$
$$\leq P_x\{j_y>t,\ \zeta>t\}+\int_{\{\zeta\leq t,\zeta<j_y\}}dP_x\mathbf{P}_{X_{\zeta-}}\{\mathbf{j}_y>t-u\}_{|u=\zeta}.$$

In the last step we used the structure of the branching measure (Ikeda-Nagasawa-Watanabe [2], I). Using lemma 9, we have

$$\mathbf{P}_x\{\mathbf{j}_y>t\}\leq P_x\{j_y>t,\ \zeta>t\}+\int_{\{\zeta\leq t,\zeta<j_y\}}dP_x\mathbf{P}_{X_\zeta}\{\mathbf{j}_y>t-u\}_{|u=\zeta}. \tag{14}$$

By induction of (14), it is easy to prove

$$\mathbf{P}_x\{\mathbf{j}_y>t\}\leq P_x\{j_y>t,\ \zeta_n>t\}+\int_{\{\zeta_n\leq t,\zeta_n<j_y\}}dP_x\mathbf{P}_{X_\zeta}\{\mathbf{j}_y>t-u\}_{|u=\zeta_n},\qquad n=1,2,\cdots. \tag{15}$$

Define $\zeta_\infty=\lim_{n\to\infty}\zeta_n$, then from (15)

$$\mathbf{P}_x\{\mathbf{j}_y>t\}\leq P_x\{j_y>t,\ \xi_\infty>t\}+P_x\{\xi_\infty\leq t\}.$$

Because $P_x\{\zeta_\infty<\infty\}=0$, we have the assertion of the lemma.

Lemma 12. *For $x<z<y$ and $t>0$, the following inequalities hold:*

$$\mathbf{P}_z\{Z=n\}\leq 2^{n-1}\sum_{i=0}^{n-1}P_z\{\zeta_i\leq(j_y^x\wedge t)<\zeta_{i+1}\},\qquad n=1,2,\cdots,$$

where $\{\zeta_i\}$ is the family of random times defined in lemma 11, and

Z is a random variable of $\mathbf{X}$ *defined by*

$$Z = Z(\mathbf{j}_y^x \wedge t).$$

For a proof of lemma 12, the structure of branching measure plays the essential role but the proof is ommitted here.

Lemma 13. *Let X be a symmetric stable process with index less than 2, then*

$$P\{\sup_{0 \le s \le t} |X_s| \ge 1\} \asymp t \qquad (t \searrow 0)^{11)}. \tag{16}$$

Proof. First we decompose the symmetric stable process X into mutually independent symmetric Lévy processes $X^{(1)}$ and $X^{(2)}$ as follows:

$$X_t^{(1)} = \lim_{n \to \infty} \int_{\{u: n^{-1} \le |u| \le 2\}} u N([0, t], du),$$

$$X_t^{(2)} = \int_{\{u: |u| > 2\}} u N([0, t], du),$$

where $\{N(A, B);\ A \in \mathscr{B}([0, \infty)),\ B \in \mathscr{B}(R \setminus \{0\})\}$ is family of Poisson random measures (cf. Ito [3]). Next we define random times j and ξ concerning to the processes $X^{(1)}$ and $X^{(2)}$ respectively by

$$j = \inf\{t > 0;\ |X_t^{(1)}| > 1\},$$

$$\xi = \inf\{t > 0;\ X_t^{(2)} \neq X_{t-}^{(2)}\}.$$

Then by the properties of $X^{(1)}$ and $X^{(2)}$ and by the definition of j and ξ,

$$\begin{aligned} P\{\sup_{0 \le s \le t} |X_s| < 1\} &= P\{j > t,\ \xi > t\} + P\{\sup_{0 \le s \le t} |X_s| < 1, j \le t\} \\ &\quad + P\{\sup_{0 \le s \le t} |X_s| < 1, j > t,\ \xi \le t\}. \end{aligned} \tag{17}$$

11) $f(t) \asymp t\ (t \searrow 0) \Longleftrightarrow 0 < \underline{\lim}_{t \searrow 0} t^{-1} f(t) \le \overline{\lim}_{t \searrow 0} t^{-1} f(t) < \infty.$

In the right-hand side of (17), the second term and the third term are obviously zero, and hence

$$P\{\sup_{0\leq s\leq t}|X_s|<1\}=P\{j>t,\ \xi>t\}$$
$$=P\{j>t\}\cdot P\{\xi>t\}$$
$$=1-P\{j\leq t\}-P\{\xi\leq t\}+P\{j\leq t\}\cdot P\{\xi\leq t\}.$$

Here we used the mutual independence of $X^{(1)}$ and $X^{(2)}$. And we have

$$(18)\qquad P\{\sup_{0\leq s\leq t}|X_s|\geq 1\}$$
$$=P\{j\leq t\}+P\{\xi\leq t\}-P\{j\leq t\}\cdot P\{\xi\leq t\}.$$

Now we shall investigate $P\{j\leq t\}$ and $P\{\xi\leq t\}$ when $t\searrow 0$.

$$(19)\qquad P\{j\leq t\}=P\{\sup_{0\leq s\leq t}|X_s^{(1)}|\geq 1\}\leq \mathrm{Var}(X_t^{(1)})=\frac{2^{1-\alpha}\cdot c}{2-\alpha}\cdot t.$$

Kolmogorov's inequality implies
Because $N([0,\ t],\ [-2,\ 2]^c)$ is a Poisson random variable with mean $(ct/2^\alpha\alpha)$,

$$(20)\qquad P\{\xi\leq t\}=1-P\{N([0,\ t],\ [-2,\ 2]^c)=0\}$$
$$=1-\exp\{-ct/2^\alpha\alpha\}\asymp t \qquad (t\searrow 0).$$

Combining (18), (19) and (20), we obtain (16).

Finally we shall introduce some sequences of numbers $\{h_n\}$, $\{H_n\}$, $\{t_n\}$ and a sequence of events $\{B_n\}$ of the branching stable process. We assume that they satisfy the following conditions:

$$(21)\qquad 0<h_n\searrow 0\ (n\to\infty),\ \text{and}\ \sum_{n=1}^{\infty}h_n=\infty,$$

$$H_0=0\ \text{and}\ H_n=\sum_{i=1}^{n}h_i, \qquad n=1,2,\cdots.$$

$$(22)\qquad 0<t_n\searrow 0\ (n\to\infty),\ \text{and}\ t=\sum_{n=1}^{\infty}t_n<\infty.$$

(23) $B_n=\{\mathbf{w};\ \mathbf{X}_0=0,\ \mathbf{j}_{H_1}\leq t_1$ and $Y_1=\Phi(\mathbf{X}(\mathbf{j}_{H_1}))$,

for a branch starting from Y_1 at $\mathbf{j}_{H_1}$

$\mathbf{j}_{h_2+Y_1}\leq t_2$ and $Y_2=\Phi(\mathbf{X}(\mathbf{j}_{Y_1+h_2}))$, $\cdots\cdots\cdots$,

for a branch starting from Y_{n-1} at $\mathbf{j}_{Y_{n-2}+h_{n-1}}$

$\mathbf{j}_{Y_{n-1}+h_n}\leq t_n$ and $Y_n=\Phi(\mathbf{X}(\mathbf{j}_{Y_{n-1}+h_n}))\}$,

$n=1, 2,\cdots,$

where Φ is a mapping from $\bigcup_{n=1}^{\infty} R^n$ to R defined by

$$\Phi(\mathbf{x})=\max\{x_i;\ 1\leq i\leq n\}, \qquad \text{if} \quad \mathbf{x}=(x_1,\cdots, x_n)\in R^n.$$

Obviously $\{B_n\}$ is a decreasing sequence of events, then we can define a sevent $B_\infty=\lim_{n\to\infty} B_n$. Clearly $B_\infty\subseteq A_\infty\subseteq A_e$. Therefore by lemma 6, if we show $\mathbf{P}_0(B_\infty)>0$, we can conclude $\mathbf{P}_x(A_e)=1$ *on* R, i.e. the branching stable process explodes with probability 1. Now we have finished the preparation.

Proof of theorem 4. First of all we shall calculate $\mathbf{P}_0(B_n)$ using the strong Markov property of the process.

$$\mathbf{P}_0(B_n)=\mathbf{E}_0[\mathbf{j}_{H_1}\leq t_1;\ \mathbf{E}_{\Phi(\mathbf{X}(\mathbf{j}_{H_1}))}[\mathbf{j}_{Y_1+h_2}\leq t_2$$
$$;\ \mathbf{E}_{\Phi(\mathbf{X}(\mathbf{j}_{Y_1+h_2}))}[\mathbf{j}_{Y_2+h_3}\leq t_3;\ \cdots$$
$$;\ \mathbf{P}_{\Phi(\mathbf{X}(\mathbf{j}_{Y_{n-2}+h_{n-1}}))}\{\mathbf{j}_{Y_{n-1}+h_n}\leq t_n\}|_{Y_{n-1}=\Phi(\mathbf{X}(\mathbf{j}_{Y_{n-2}+h_{n-1}}))}\cdots$$
$$]|_{Y_2=\Phi(\mathbf{X}(\mathbf{j}_{Y_1+h_2}))}]|_{Y_1=\Phi(\mathbf{X}(\mathbf{j}_{H_1}))}].$$

Then we have

$$\mathbf{P}_0(B_n)\geq\prod_{i=1}^{n}\inf_{H_{i-1}\leq Y}\mathbf{P}_Y\{\mathbf{j}_{Y+h_i}\leq t_i\},$$

and taking $n\to\infty$,

$$\mathbf{P}_0(B_\infty)\geq\prod_{i=1}^{\infty}\inf_{H_{i-1}\leq Y}\mathbf{P}_Y\{\mathbf{j}_{Y+h_i}<t_i\}. \tag{24}$$

For $Y \in R$ and $h, t > 0$,

$$\text{(25)} \quad \mathbf{P}_Y\{\mathbf{j}_{Y+h} \leq t\} \geq \mathbf{P}_Y\left\{\mathbf{j}_{Y+h}(\mathbf{w}^+(\mathbf{j}_{Y+h}^{Y-h} \wedge \tfrac{t}{2})) \leq \frac{t}{2}\right\}$$

$$= 1 - \mathbf{E}_Y\left[\mathbf{P}_{\mathbf{X}(\mathbf{j}_{Y+h}^{Y-h} \wedge \frac{t}{2})}\left\{\mathbf{j}_{Y+h} > \frac{t}{2}\right\}\right]$$

$$\geq 1 - \mathbf{E}_Y\left[P_{Y-h}\left\{j_{Y+h} > \frac{t}{2}\right\}^{Z-1}\right] = 1 - \mathbf{E}_Y\left[P\left\{j_{2h} > \frac{t}{2}\right\}^{Z-1}\right],$$

by lemma 10 and 11, where $Z = Z\left(\mathbf{j}_{Y+h}^{Y-h} \wedge \frac{t}{2}\right)$. Then from (24) and (25),

$$\mathbf{P}_0(B_\infty) \geq \prod_{i=1}^{\infty}\left\{1 - \sup_{H_{i-1} \leq Y} \mathbf{E}_Y\left[P\left\{j_{2h_i} > \frac{t_i}{2}\right\}^{Z_i - 1}\right]\right\},$$

where $Z_i = Z\left(\mathbf{j}_{Y+h_i}^{Y-h_i} \wedge \frac{t_i}{2}\right)$.

Set $I = \sum_{i=1}^{\infty} \sup_{H_{i-1} \leq Y} \mathbf{E}_Y[P\{j_{2h_i} > t_i/2\}^{Z_i - 1}]$. In order to prove $\mathbf{P}_0(B_\infty)$ it is sufficient to show $I < \infty$. Let positive numbers h and t be such that $0 < q = P\{j_{2h} > t/2\} < 1/2$ $(q' = 1 - 2q > 0)$. Then by lemma 12, we have

$$\mathbf{E}_Y[P\{j_{2h} > t/2\}^{Z-1}] = \sum_{n=0}^{\infty} q^n \mathbf{P}_Y\{Z = n+1]$$

$$\leq \sum_{0=n}^{\infty} (2q)^n \sum_{i=0}^{n} P_Y\left\{\zeta_i \leq j_{Y+h}^{Y-h} \wedge \frac{t}{2} < \zeta_{i+1}\right\}$$

$$= \frac{1}{q'} \sum_{i=0}^{\infty} (2q)^i P_Y\left\{p\left(\varphi j_{Y+h}^{Y-h} \wedge \frac{t}{2}\right) = i\right\}$$

$$= \frac{1}{q'} \sum_{i=0}^{\infty} E_Y\left[(i!)^{-1}\left\{2q\varphi\left(j_{Y+h}^{Y-h} \wedge \frac{t}{2}\right)\right\}^i \exp\left\{-\varphi\left(j_{Y+h}^{Y-h} \wedge \frac{t}{2}\right)\right\}\right]$$

$$= \frac{1}{q'} E_Y\left[\exp\left\{-q'\varphi\left(j_{Y+h}^{Y-h} \wedge \frac{t}{2}\right)\right\}\right]$$

$$= \frac{1}{q'} E_Y\left[j_{Y+h}^{Y-h} \leq \frac{t}{2};\ \exp\left\{-q'\varphi(j_{Y+h}^{Y-h})\right\}\right]$$

$$+ \frac{1}{q'} E_Y\left[j_{Y+h}^{Y-h} > \frac{t}{2};\ \exp\left\{-q'\varphi\left(\frac{t}{2}\right)\right\}\right].$$

Therefore

$$I\le\sum_{i=1}^{\infty}\sup_{H_{i-1}\le Y}\frac{1}{q_i'}E[\exp\{-q_i'|Y-h_i|^{\gamma}j_{h_i}^{-h_i}\}]$$

$$+\sum_{i=1}^{\infty}\sup_{H_{i-1}\le Y}\frac{1}{q_i'}\exp\left\{-q_i|Y-h_i|^{\gamma}\frac{t_i}{2}\right\}$$

$$\le\sum_{i=1}^{\infty}\frac{1}{q_i'}E[\exp\{-q_i|H_{i-1}-h_i|^{\gamma}j_{h_i}^{-h_i}\}]$$

$$+\sum_{i=1}^{\infty}\frac{1}{q_i'}\exp\left\{-q_i/H_{i-1}-h_i|^{\gamma}\frac{t_i}{2}\right\}$$

$$=I_1+I_2,$$

where $q_i=P\{j_{2h_i}>t_i/_2\}$, $q_i'=1-2q_i$ and positive numbers h_i and t_i are choosen as $0<q_i<1/2$.

Now take $\{h_i\}$ and $\{t_i\}$ of (21) and (22) in the following way.

(21′) $$h_i=ci^{-a},\ i=1,2,\cdots,\ 0<a<1,\ c>0.$$

(22′) $$t_i=i^{-b},\ i=0,1,2,\cdots,\ b>1.$$

Using the space-time transformation of stable processes ((5), §2), we have

$$q_i=P\{\sup_{0\le s\le t_i/2}X_s<2h_i\}=P\{\sup_{0\le s\le 1}X_s<(2^{1+1/\alpha}c)i^{b/\alpha-a}\}.$$

Take a, b and c of (21′) and (22′) as

(27) $b/\alpha-a=0$, c; sufficiently small constant,

then we can take $q_i(i=1,2,\cdots)$ to be a constant independent of i and $0<q<1/2$. For such $\{h_i\}$ and $\{t_i\}$, we have

(28) $$I_1'=\frac{1}{q'}\sum_i|H_{i-1}-h_i|^{\gamma}\int_0^{\infty}\exp\{-q|H_{i-1}-h_i|^{\gamma}u\}\cdot P\{j_{h_i}^{-h_i}<u\}du$$

$$=\frac{1}{q'}\sum_i\int_0^{\infty}\exp\{-q'|H_{i-1}-h_i|^{\gamma}h_i^{\alpha}u\}\cdot d_uP\{\sup_{0\le s\le u}|X_s|>1\}$$

$$\le\frac{1}{q'}\sum_i\int_0^{\infty}\exp\{-K_1i^{(1-a)\gamma-b}u\}d_uP\{\sup_{0\le s\le u}|X_s|>1\}$$

(29) $$I'_2 = \frac{1}{q'}\sum_i \exp\{-q\,|H_{i-1}-h_i|^\gamma t_i/2\} \le \frac{1}{q'}\sum_i \exp\{-K_2 i^{(1-a)\gamma-b}\},$$

where $q'=1-2q>0$ and summations of right hand sides are taken over all sufficiently large i, and K_1 and K_2 are some positive constants. Using lemma 13 and the Abelian theorem (Widder [7]) for the estimate of I'_1, we have

(28′) $$I'_1 \le (\textit{positive constant}) \sum_i i^{-(1-a)\gamma-b}.$$

Therefore suppose a, b and c satisfy (27) and

(30) $$(1-a)\gamma-\mathrm{b}>1,$$

then $I'_1<\infty$ and $I'_2<\infty$ hold by (28′) and (29), $I<\infty$ hold by (26), which implies $\mathbf{P}_0(B_\infty)>0$. It is easy to show that in the case $1<\alpha<2$, when $\gamma>2\alpha/(\alpha-1)$ we can choose a and b as they satisfy (27) and (30). Now we have completed the proof of (i) of theorem 4.

A proof of (ii) is found in Ito-McKean [4].

Remark. The author has not succeeded in proving that if $2\alpha/(\alpha-1)$ is critical or not for explosion in the case $1<\alpha<2$, and does not know if there is positive γ for which α-branching stable process explodes in the case $0<\alpha\le 1$.

Tokyo Institute of Technology[12)]

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12) The author's address is: Foundations of Mathematical Sciences, Tokyo University of Education.