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On the meromorphic differentials with an infinite number of polar singularities on open Riemann surfaces

By

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Introduction

In the present paper we shall study the meromorphic differential (or its inegral) with an infinite number of polar singularities on an open Riemann surface R.

According to H. Behnke and K. Stein [4] there exist always meromorphic functions with given divisor unless its carrier clusters on R, but it is also desirable to have the close analogue to the classical Abel's theorem and in fact this problem has been investigated by many authors in the case of finite divisor, that is, its carrier consists of a finite number of points on R (Accola [1], Ahlfors-Sario [2], Kusunoki [7], Mizumoto [10], Ota [12], Rodin [13], Yoshida [22]). While, in 1954 R. Bader [3] has studied the Schottky-Ahlfors differentials and on R of class O_{HB}, under some restrictions, given a necessary condition for the existence of meromorphic function whose divisor is exactly the given infinite divisor. In §I we shall deal with the Abel's theorem in the case related to infinite divisor. There we discuss the existence of multiplicative function wth given infinite divisor and some other properties, and treat its expression in terms of normal integrals. The expression obtained there may be regarded as a generalization of that in the classical theory to open Riemann surfaces and then the Abel's theorem follows from the condition for single-valuedness.

On the other hand, as for the Riemann-Roch theorem, some similar formulations to classical one have been obtained (Kusunoki [6], [7], Mizumoto [10], Ota [12], Rodin [14], Royden [15], Shiba [16], Yoshida [22]). In §II we shall be concerned with the case attached to the infinite divisor. Under some restrictions it will be shown that a meromorphic differential is expressed by the series of normal differentials and with the help of this expression, we can find that there is a duality relation between the vector space of functions and that of differentials which are multiples of some divisor, respectively. If genus and divisor are both finite, then it is seen that this may be regarded as an analogue of classical Riemann-Roch theorem.

Throughout this paper, the method used here is mainly, so-called, that of contour integration and it will be seen that the third kind of normal integral and Riemann's relation play a fundamental role as in the classical case (Osgood [11], Weyl [21]).

§I. Multiplicative functions

1. We shall consider an open Riemann surface R and denote its genus by $g\ (0 \leqq g \leqq +\infty)$. Let $\{\Omega_n\}_{n=1,2,\dots}$ be a canonical exhaustion of R and $\{A_i, B_i\}_{i=1,\dots,k(n),\dots}$ be a canonical homology basis with respect to $\{\Omega_n\}$ such that $\{A_i, B_i\}_{i=1,\dots,k(n)}$ form a canonical homology basis of $\Omega_n\ (\mathrm{mod}\ \partial\Omega_n)$ and $A_i \times B_j = \delta_{ij}$, $A_i \times A_j = B_i \times B_j = 0$.[1] (Ahlfors-Sario [2]).

On R there exists a system of diffferentials which is similar to that of the normalized differentials in the classical theory and consists of the following three kinds of differentials:

(I) *The first kind of normal differentials* $dw_i\ (i=1, 2,\dots)$; dw_i is square integrable analytic semiexact and $\int_{A_j} dw_i = \delta_{ij}$.

(II) *The second kind of normal differentials* dY_{p^n} (n; positive integer); (i) dY_{p^n} is holomorphic except at p and $dY_{p^n} = \Big(-\frac{n}{z^{n+1}}$

1) We note, throughout this paper, the intersection number of two cycles A, B is taken such that $A \times B$ has the positive sign when A crosses B from right to left as in Ahlfors-Sario [2]. Hence it has the opposite sign to that in Osgood [11], Schiffer-Spencer [18], and Weyl [21].

+regular term)dz at p. (ii) the norm of dY_{p^n} is finite outside of an arbitrary neighborhood of p. (iii) dY_{p^n} is semiexact on R and A-periods of it vanish.

(III) *The third kind of normal differential* $d\Pi_{p,q}$; (i) $d\Pi_{p,q}$ has a simple pole with residue 1 (-1) at p (q), respectively and is holomorphic elsewhere. (ii) The norm of $d\Pi_{p,q}$ is finite outside of an arbitrary neighborhood of p and q. (iii) $d\Pi_{p,q}$ is semiexact on R with a slit joinning p and q, and all A-periods of it vanish (Virtanen [19], [20], Kusunoki [8], Sainouchi [16]).

In general these differentials do not always exist uniquely and so we shall suppose that R satisfies a metrical condition as defined in **2.**

2. For our purpose we introduce a coordinate on R. We take mutually disjoint annuli D_n^i $(i=1, 2,\dots, m(n))$ each of which includes exactly one contour γ_n^i of $\partial\Omega_n=\bigcup_{i=1}^{m}\gamma_n^i$. Let $D_n=\bigcup_{i=1}^{m}D_n^i$ and assume that D_n $(n=1, 2,\dots)$ are disjoint each other. We denote by ν_n^i (resp. $\hat{\nu}_n$) *the harmonic modulus* of D_n^i (resp. D_n), namely, for instance, $\hat{\nu}_n$ is defined by $2\pi/d_n$ where d_n is the flux $\int_{c_n}{}^*du_n$ $(c_n=\partial D_n\cap\Omega_n)$ of the harmonic function u_n on D_n which is $=0$ on c_n and $=1$ on ∂D_n-c_n. It follows that $\sum_i\frac{1}{\nu_n^i}=\frac{1}{\hat{\nu}_n}$. Define a function u on $\bigcup_{n=1}^{\infty}D_n$ such that

$$u=\sum_{i=1}^{n-1}\hat{\nu}_i+\hat{\nu}_n u_n \qquad \text{on } D_n\ (n=1, 2,\dots),$$

then $u+iv$ (v; conjugate of u) maps $\bigcup_{n=1}^{\infty}D_n$ conformally onto a strip domain; $0<u<R'=\sum_{n=1}^{\infty}\hat{\nu}_n$, $0<v<2\pi$.

Now we shall suppose that the series $\sum_{n=1}^{\infty}\min_i \nu_n^i$ is divergent, then the following lemmas are obtained (Kobori-Sainouchi [5], Kusunoki-Sainouchi [9]).

Lemma 1. *Let* φ_j $(j=1, 2)$ *be two meromorphic differentials such that* $\|\varphi_j\|_{\cup D_n}<+\infty$ *and for all sufficiently large* n $\int_{\gamma_n^i}\varphi_j=0$ $(i=1, 2,\dots, m(n))$, *then there exists a canonical exhaustion* $\{\Omega_{n'}\}$ $(\subset$

$\{\Omega_n\}$) *such that* $\lim_{n'\to\infty}\int_{\partial\Omega_{n'}}\Phi_1\varphi_2=0$, *where* Φ_1 *is an integral of* φ_1 *defined separately on each contour* γ_n^i *of* $\partial\Omega_{n'}$.

Lemma 2. *The Riemann's bilinear relation*

$$(\omega, {}^*\sigma)=\sum_k\left(\int_{A_k}\omega\int_{B_k}\bar{\sigma}-\int_{A_k}\bar{\sigma}\int_{B_k}\omega\right) \qquad \text{(a finite sum)}$$

holds for two $\omega, \sigma\in\Gamma_{hse}$ *having only a finite number of non-vanishing A-periods.*

With the help of lemma 1 we know that if $\sum_n \min_i v_n^i$ is divergent, there is one and only one system of normal differentials on R.

3. In the classical theory there are some relations between the normal integrals (Osgood [11], Schiffer-Spencer [18]). If R satisfies $\sum_n \min_i v_n^i=+\infty$, then from the above lemmas it is seen that the following relations are valid on R. Later on, we shall use some of them.

Proposition 1. *Let* $\int_{B_j}dw_i=\tau_{ij}=\tau'_{ij}+i\tau''_{ij}$ (τ'_{ij}, τ''_{ij}; *real*), *then the matrix* (τ_{ij}) *is symmetric and for any positive integer* p $(\tau''_{ij})_{i,j=1,\dots,p}$ *is negative definite.*

Proof. Since an analytic differential is orthogonal to an anti-analytic differential, we have

$$(dw_i, {}^*\overline{dw_j})=0\,.$$

While, the lemma 2 yields

$$(dw_i, {}^*\overline{dw_j})=\sum_k\left(\int_{A_k}dw_i\int_{B_k}dw_j-\int_{B_k}dw_i\int_{A_k}dw_j\right)=\tau_{ji}-\tau_{ij}\,.$$

Next we put $dw=\sum_{i=1}^{p}\xi_i dw_i$ (ξ_i; complex number), then

$$\|dw\|^2=\sum_{i,j=1}^{p}\xi_i\bar{\xi}_j(dw_i, dw_j)=-i\sum_{i,j=1}^{p}\xi_i\bar{\xi}_j(\bar{\tau}_{ij}-\tau_{ij})=-2\sum_{i,j=1}^{p}\xi_i\bar{\xi}_j\tau''_{ij}.$$

Thus we see that $(\tau''_{ij})_{i,j=1,\dots,p}$ is negative definite. q.e.d.

Let $w_k = \int dw_k$ and $w_k = \sum_{n=0}^{\infty} \frac{1}{n!} w_k^{(n)}(p) z^n$ at p, where p corresponds to $z=0$ in terms of a local variable which maps a parameter disk U onto $|z|<1$.

Proposition 2. *The B_k-period of dY_{p^n} and $d\Pi_{p,q}$ is given by the formula, respectively,*

$$\int_{B_k} dY_{p^n} = 2\pi i \frac{w_k^{(n)}(p)}{(n-1)!}$$

and

$$\int_{B_k} d\Pi_{p,q} = 2\pi i(w_k(q) - w_k(p)) = 2\pi i \int_p^q dw_k,$$

where the integral on the right is taken over a path from p to q which lies in $R_0 = R - \bigcup_i \{A_i, B_i\}$.

Proof. From lemma 1 we see that there is a canonical exhaustion $\{\Omega_{n'}\}$ such that

$$\lim_{n'\to\infty} \int_{\partial\Omega_{n'}} w_k dY_{p^n} = 0.$$

By the Stokes formula and orthogonality $\Gamma_a(\Omega_{n'} - U) \perp \bar{\Gamma}_a(\Omega_{n'} - U)$ we have

$$\begin{aligned} 0 &= (dw_k, {}^*\overline{dY}_{p^n})_{\Omega_{n'}-U} \\ &= \sum_{A_j, B_j \subset \Omega_{n'}} \left(\int_{A_j} dw_k \int_{B_j} dY_{p^n} - \int_{B_j} dw_k \int_{A_j} dY_{p^n} \right) - \int_{\partial(\Omega_{n'}-U)} w_k dY_{p^n} \\ &= \int_{B_k} dY_{p^n} + \left(\int_{\partial U} - \int_{\partial\Omega_{n'}} \right) w_k dY_{p^n}. \end{aligned}$$

Since $\int_{\partial U} w_k dY_{p^n} = -2\pi i \frac{w_k^{(n)}(p)}{(n-1)!}$, as n' tends to ∞, we obtain the first formula.

Next let V be a simply connected neighborhood containing p and

q $(V \subset \Omega_{n'})$ and we apply to $(dw_k, *\overline{d\Pi_{p,q}})_{\Omega_{n'}-V}$ the same way as above, then we get the second formula. q.e.d.

Proposition 3. (*the law of interchange of argument*) *Let* $\Pi_{s,t}^{p,q} = \Pi_{s,t}(p) - \Pi_{s,t}(q) = \int_q^p d\Pi_{s,t}$, *where the integral is taken over a path from* q *to* p *which lies in* R_0 *with a slit joinning* s *and* t, *then*

$$\Pi_{s,t}^{p,q} = \Pi_{p,q}^{s,t}$$

Proof. We denote by U (resp. V) a simply connected neighborhood containing s and t (resp. p and q) $(U \cap V = \emptyset,\ U,\ V \subset \Omega_{n'})$. Considering $d\Pi_{s,t}^{a,b}$ as a differential of a, we have

$$0 = \left(d\Pi_{s,t}^{a,b}, \overline{d\Pi_{p,q}^{a,b}}\right)_{\Omega_{n'}-U-V}$$

$$= \left(\int_{\partial U} + \int_{\partial V} - \int_{\partial \Omega_{n'}}\right) \Pi_{s,t}^{a,b} d\Pi_{p,q}^{a,b}.$$

Since $\Pi_{s,t}^{a,b}\Pi_{p,q}^{a,b}$ is single valued in $\Omega_{n'} - U - V - \bigcup_{i=1}^{k(n')} \{A_i, B_i\}$

$$\int_{\partial U} d(\Pi_{s,t}^{a,b}\Pi_{p,q}^{a,b}) = 0,$$

hence

$$\int_{\partial U} \Pi_{s,t}^{a,b} d\Pi_{p,q}^{a,b} = -\int_{\partial U} \Pi_{p,q}^{a,b} d\Pi_{s,t}^{a,b} = -2\pi i(\Pi_{p,q}^{s,b} - \Pi_{p,q}^{t,b}) = -2\pi i \Pi_{p,q}^{s,t}.$$

While,

$$\int_{\partial V} \Pi_{s,t}^{a,b} d\Pi_{p,q}^{a,b} = 2\pi i \Pi_{s,t}^{p,q}.$$

As $n' \to \infty$, we have the desired result. q.e.d.

By the application of the same way as before to $(dY_{p^n}, *\overline{d\Pi_{s,t}^{p,q}})_{\Omega_{n'}-U-W}$ and $(dY_{p^n}, \overline{*dY_{q^m}})_{\Omega_{n'}-W-W'}$ where W (resp. W') is a neighborhood of p (resp. q), we obtain the following

Proposition 4. *Let* $Y_{p^n}^{s,t} = Y_{p^n}(s) - Y_{p^n}(t) = \int_t^s dY_{p^n}$, *then*

$$\frac{1}{(n-1)!}\frac{\partial^n \Pi^{p,q}_{s,t}}{\partial p^n} = -Y^{s,t}_{p^n} = \int_s^t dY_{p^n} \qquad ,$$

$$\frac{1}{(m-1)!}\frac{d^m Y_{p^n}(q)}{dq^m} = \frac{1}{(n-1)!}\frac{d^n Y_{q^m}(p)}{dp^n} \qquad .$$

4. Let $\mathscr{M}$ be a class of multiplicative functions on R such that each function $f(p)$ belonging to $\mathscr{M}$ has the following properties:

1) there exists an integer n_0 such that for all $n\ (\geqq n_0)$ and i

$$\int_{\gamma_n^i} d\log f = 0,$$

where γ_n^i $(i=1,\dots, m(n))$ are components of $\partial\Omega_n$.

2) The zeros and poles of $f(p)$ are contained in $R-\bigcup_{n=1}^{\infty}\bar{D}_n$.

From 1) it is seen that f has equally many zeros and poles, counted with multiplicities, in Ω_n $(n\geqq n_0)$.

Now let δ be a finite or infinite devisor on R whose carrier lies in $R-\bigcup_{n=1}^{\infty}\bar{D}_n$ and δ_n its restriction to Ω_n. We assume that $\deg\delta_n=0$. Here let us remark that whenever δ is given, then exists always a single-valued meromorphic function $f(p)\ (\in\mathscr{M})$ with given divisor δ. Because let $\delta=\frac{\Pi a_i}{\Pi b_i}$ and denote by $d\varphi$ a meromorphic differential which has simple poles of residue $1\ (-1)$ at $a_i\ (b_i)\ (i=1, 2,\dots,)$, respectively. Adding to $d\varphi$ an appropriate holomorphic differential, we can get meromorphic differential $d\psi$ such that $\int_{A_j} d\psi=2\pi i n_j$, $\int_{B_j} d\psi=2\pi i m_j$ and $\int_{\gamma_n^i} d\psi=0$ (n_j, m_j; integers). Set $\psi=\int d\psi$ and $f=\exp\psi$, then f is the single-valued meromorphic function with given divisor and $\int_{\gamma_n^i} d\log f=0$ (*cf.* Behnke and Stein [4]). However, if the functions are restricted in a suitable manner we will show that it is possible to derive a close analogue of Abel's classcial theorem. For this purpose, as a restriction on f, if R satisfies $\inf_n \min_i \nu_n^i>0$ we shall require that

(A) $$\|d\log f\|_{\cup D_n} < +\infty ,$$

otherwise

(B) $$\sup_n \frac{\|d \log f\|_{D_n}}{\min_i v_n^i} < +\infty .$$

At first we have to prove the following

Lemma 3. *Let dw be a meromorphic differential such that $\|dw\|_{\cup D_n} < +\infty$ and $\int_{\gamma_n^i} dw = 0$ for sufficiently large n. If ω is an other meromorphic differential such that $\int_{\gamma_n^i} \omega = 0$ for sufficiently large n and $\sup_n \frac{\|\omega\|_{D_n}}{\min_i v_n^i} < +\infty$ (or $\|\omega\|_{\cup D_n} < +\infty$ whenever $\inf_n \min_i v_n^i >$ 0), then*

$$\int_{\partial\Omega_n} \left(\int dw\right)\omega \longrightarrow 0 \quad (n \longrightarrow \infty)$$

and

$$\int_{\partial\Omega_n} \Pi_{s,t}^{p,q} \omega \Longrightarrow 0 \qquad (n \longrightarrow \infty),$$

where the latter integral converges to zero uniformly with respect to (s, t) on every compact subset of $(R_0 - \{q\}) \times (R_0 - \{q\}) - \{(s, t) | s = t\}$.

Proof. Let us prove that if $\sup_n \frac{\|\omega\|_{D_n}}{\min v_n^i} < +\infty$, then $\int_{\partial\Omega_n} \Pi_{s,t}^{p,q} \omega \Rightarrow 0$ $(n \to \infty)$. The remaining part of lemma is shown analogously.

We assume that s and t lie in Ω_{n_1} and consider the integral

$$L(r, s, t) = \left| \int_{u=r} \Pi_{s,t}^{p,q} \omega \right|, \qquad r \in \mathrm{I}_n = [\sum_{i=1}^{n-1} \hat{v}_i, \sum_{i=1}^{n} \hat{v}_i] \qquad (n > n_1)$$

and put $L(r_n, s, t) = \min_{r \in \mathrm{I}_n} L(r, s, t)$. Since $\partial\Omega_n$ is homologous to the level curve $\{p \in R | u(p) = r_n\}$ we have

$$\left| \int_{\partial\Omega_n} \Pi_{s,t}^{p,q} \omega \right| = L(r_n, s, t) \quad \text{and} \quad L(r_n, s, t) \leqq \sum_{i=1}^{m} \int_{\gamma_r^i} |d\Pi_{s,t}^{p,q}| \int_{\gamma_r^i} |\omega| ,$$

where $m = m(n)$ and γ_r^i denote the level curves $\{p \in R | u(p) = r\}$ contained in D_n^i. Let $\omega = adu + bdv$, then by the Schwarz's inequality

$$\int_{\gamma_r^i}|d\Pi_{s,t}^{p,q}|\cdot\int_{\gamma_r^i}|\omega|=\int_{\gamma_r^i}\left|\frac{\partial\Pi_{s,t}^{p,q}}{\partial v}\right|dv\cdot\int_{\gamma_r^i}|b|dv$$

$$\leqq 2\pi\frac{\hat{v}_n}{v_n^i}\left(\int_{\gamma_r^i}\left|\frac{\partial\Pi_{s,t}^{p,q}}{\partial v}\right|^2dv\cdot\int_{\gamma_r^i}|b|^2dv\right)^{1/2}.$$

Summing up from 1 to m it follows

$$L(r_n,s,t)\leqq 2\pi\,\hat{v}_n\frac{1}{\min\limits_i v_n^i}\left(\int_0^{2\pi}\left|\frac{\partial\Pi_{s,t}^{p,q}}{\partial v}\right|^2dv\int_0^{2\pi}|b|^2dv\right)^{1/2}.$$

By integrating with respect to $t\in I_n$ we obtain

$$L(r_n,s,t)\leqq\frac{2\pi}{\min\limits_i v_n^i}\|d\Pi_{s,t}^{p,q}\|_{D_n}\|\omega\|_{D_n}. \tag{1}$$

If (s,t) lies on compact subset of $(R_0-\{q\})\times(R_0-\{q\})-\{(s,t)|s=t\}$, then it is possible to choose a n'_0 such that for any positive ε

$$\|d\Pi_{s,t}^{p,q}\|_{D_n}<\varepsilon\qquad(n\geqq n'_0),$$

where by continuity of $\|d\Pi_{s,t}^{p,q}\|_{R-\Omega_{n_1}}$ with respect to (s,t) n'_0 is independent on (s,t) (cf. proposition 3 and lemma 1). Hence

$$\left|\int_{\partial\Omega_n}\Pi_{s,t}^{p,q}\omega\right|\leqq 2\pi\frac{\|\omega\|_{D_n}}{\min\limits_i v_n^i}\varepsilon\qquad(n\geqq n'_0).$$

Thus the proof is complete.

Remark. From (1) we have

$$\sum_{n=1}^{\infty}L(r_n,s,t)^2\left(\frac{\min v_n^i}{\|\omega\|_{D_n}}\right)^2\leqq 4\pi^2\|d\Pi_{s,t}^{p,q}\|^2_{\bigcup_n D_n}<+\infty,$$

hence if $\sum\limits_{n=1}^{\infty}\left(\frac{\min v_n^i}{\|\omega\|_{D_n}}\right)^2=+\infty$, then

$$\varliminf_{n\to\infty}L(r_n,s,t)=0,$$

and so there is a canonical exhaustion $\{\Omega_{n'}\}$ $(\subset\{\Omega_n\})$ such that

$$\lim_{n'\to\infty}\int_{\partial\Omega n'} \Pi_{s,t}^{p,q}\omega = 0\,.$$

But in this case $\{n'\}$ may be depend on (s, t).

5. For the divisor $\delta_n = \frac{a_1 \dots a_{l(n)}}{b_1 \dots b_{l(n)}}$ let us denote by γ_j a singular 1-chain in Ω_n such that $\partial\gamma_j = b_j - a_j$ and set $c(n) = \sum_{j=1}^{l(n)} \gamma_j$.

Theorem 1. *Suppose that* $\sum_n \min_i v_n^i = +\infty$. *In order to exist a single-valued meromorphic fucntion* $f(p)\,(\in \mathscr{M})$ *such that* $f(p)$ *satisfies* (B) *and its divisor is exactly* δ *it is necessary and sufficient that the following conditions are filled*:

1) $$\lim_{n\to\infty}\Big(\int_{c(n)} dw_i + \sum_{j=1}^{k(n)} n_j \tau_{ij}\Big) = m_i \qquad (i=1, 2,\dots)$$

where n_j *and* m_j *are integers.*

2) *the sequence of functions*

$$F_n(s) = \sum_{m=1}^{l(n)} \Pi_{a_m, b_m}^{s,t} + 2\pi i \sum_{j=1}^{k(n)} n_j \int_t^s dw_j$$

converges uniformly on every compact subset of $R_0 - \bigcup_i \gamma_i$ *and its limit function* $F(s)$ *satisfies* (B) *where* $t\,(\neq a_k, b_k)$ *is fixed in* R.

Proof. Take a simply connected neighborhood U_m containing a_m and b_m ($U_i \cap U_j = \emptyset$ $(i \neq j)$, $U_m \subset \Omega_n$ $(m=1,\dots, l(n))$) and set $\int_{A_j} d\log f = 2\pi i n_j$, $\int_{B_j} d\log f = 2\pi i m_j$, then

$$0 = (dw_i, \overline{{}^*d\log f})_{\Omega_n - \bigcup_{m=1}^{l} U_m}$$

$$= \sum_{A_j, B_j \subset \Omega_n}\Big(\int_{A_j} dw_i \int_{B_j} d\log f - \int_{A_j} d\log f \int_{B_j} dw_i\Big) - \int_{\partial(\Omega_n - \cup U_m)} w_i d\log f$$

$$= 2\pi i(m_i - \sum_{j=1}^{k(n)} n_j \tau_{ij}) + \Big(\sum_{m=1}^{l}\int_{\partial U_m} - \int_{\partial U_n}\Big) w_i d\log f\,.$$

By the residue theorem we have

$$\sum_{m=1}^{l}\int_{\partial Um} w_i d\log f = 2\pi i \sum_{m=1}^{l} (w_i(a_m) - w_i(b_m)) = -2\pi i \int_{c(n)} dw_i\,.$$

Therefore, as n tends to infinity, by lemma 3 we have 1).

Next let V be a simply connected neighborhood of s and t ($V \subset \Omega_n$, $V \cap U_i = \emptyset$), then

$$0 = (d\Pi_{s,t}^{p,q}, \overline{{}^*d\log f})_{\Omega_n - \cup_m U_m - V}$$

$$= \sum_{A_j, B_j \subset \Omega_n} \left(\int_{A_j} d\Pi_{s,t}^{p,q} \int_{B_j} d\log f - \int_{A_j} d\log f \int_{B_j} d\Pi_{s,t}^{p,q} \right)$$

$$+ \left(\int_{\partial V} + \sum_{m=1}^{l} \int_{\partial U_m} - \int_{\partial \Omega_n} \right) \Pi_{s,t}^{p,q} d\log f,$$

Since $\int_{\partial V} \Pi_{s,t}^{p,q} d\log f = -\int_{\partial V} \log f d\Pi_{s,t}^{p,q} = -2\pi i \int_t^s d\log f = -2\pi i \log f(s)/f(t)$,

$$\int_{\partial U_m} \Pi_{s,t}^{p,q} d\log f = 2\pi i\, \Pi_{s,t}^{a_m, b_m} = 2\pi i\, \Pi_{a_m, b_m}^{s,t} \qquad \text{(by proposition 3)}$$

and

$$\int_{B_j} d\Pi_{s,t}^{p,q} = 2\pi i \int_s^t dw_j \qquad \text{(by proposition 2)},$$

we have

$$F_n(s) = \sum_{m=1}^{l(n)} \Pi_{a_m, b_m}^{s,t} + 2\pi i \sum_{j=1}^{k(n)} n_j \int_t^s dw_j = \log f(s)/f(t) + \frac{1}{2\pi i} \int_{\partial \Omega_n} \Pi_{s,t}^{p,q}\, d\log f.$$

On letting n tend to ∞, by lemma 3 we obtain 2).

Conversely, the A_i-period of $F_n(s)$ is equal to $2\pi i n_i$, while by proposition 2 the B_i-period is equal to $2\pi i (\int_{c(n)} dw_i + \sum_{j=1}^{k} n_j \tau_{ij})$ and $\int_{\gamma_n^i} dF_n = 0$, and so by 1) A_i and B_i-period of $F(s)$ is equal to $2\pi i n_i$ and $2\pi i m_i$, respectively and $\int_{\gamma_n^i} dF = 0$. Finaly set $f(s) = \exp F(s)$, then it is seen that $f(s)$ is a desired function. q.e.d.

Remark. If R satisfies $\inf_n \min_i v_n^i > 0$, then for a function satisfying (A) in place of (B) the corresponding statement is valid, moreover in this case the desired function is uniquely expressed by $\exp F(s) = \lim_{n \to \infty} \exp \left(\sum_{m=1}^{l(n)} \Pi_{a_m, b_m}^{s,t} + 2\pi i \sum_{j=1}^{k(n)} n_j \int_t^s dw_j \right)$ up to a multiplicative constant.

Indeed, let g be an other meromorphic function with same properties as f, then $\log|f/g|$ is harmonic on R and $\|d\log|f/g|\|_{\underset{n}{\cup}D_n}<+\infty$, hence by lemma 1 we get $\|d\log|f/g|\|=0$ and so $g=Cf$ (C; constant).

The classical Abel's theorem follows from the following

Corollary. *Let* $\inf_n \min_i v_n^i>0$ *and* $\delta=\frac{a_1\dots a_l}{b_1\dots b_l}$ *be a finite divisor on* R *with finite genus* g. *Then there exists a meromorphic function* $f(\in\mathscr{M})$ *such that* $\|d\log f\|_{\underset{n}{\cup}D_n}<+\infty$ *and its divisor is exactly* δ *if and only if*

$$\int_c dw_i+\sum_{j=1}^{g} n_j\tau_{ij}=m_i \qquad (i=1,\dots,g),$$

where n_i *and* m_i *are integers and* c *is a finite chain* $\sum_{i=1}^{l}\gamma_i$ $(\partial\gamma_i=b_i-a_i)$.

As for the existence of a multiplicative meromorphic function on R, from the proof of theorem it is seen that the following result holds.

Theorem 1′. *Suppose that* $\inf_n \min_i v_n^i>0$. *In order to exist a multiplicative function* $m(p)\,(\in\mathscr{M})$ *such that* 1) $m(p)$ *satisfies* (A) 2) *its divisor is exactly* δ 3) *its multiplicators with respect to* A_j *are* $\exp 2\pi i\chi_j$, *where* χ_j *are complex numbers, it is necessary and sufficient that the sequence of functions*

$$M_n(s)=\sum_{m=1}^{l(n)}\Pi^{s,t}_{a_m,b_m}+2\pi i\sum_{j=1}^{k(n)}\chi_j\int_t^s dw_j$$

converges uniformly on every compact subset of $R_0-\bigcup_i\gamma_i$ *and its limit function* $M(s)$ *satisfies* $\|dM\|_{\underset{n}{\cup}D_n}<+\infty$, *where* $t\,(\neq a_k,b_k)$ *is fixed. If the condition is filled a desired function is expressed by* $\lim_{n\to\infty}\exp M_n(s)$.

6. In this section we shall derive a condition for uniform convergence of $\{M_n(s)\}$.

Theorem 2. *$\{M_n(s)\}$ converges uniformly on every compact subset of $R_0-\bigcup_i \gamma_i$ if and only if for each N $\lim_{m,n\to\infty} \|dM_m-dM_n\|_{D_N}=0$.*

Proof. It is easy to show "only if" part, thus we shall prove the "if" part. Take a $r\,(r\in I_N=[\sum_{i=1}^{N-1}\hat{v}_i,\ \sum_{i=1}^{N} v_i])$ and let $\Omega_N(r)$ be the relatively compact domain bounded by level curve $\{p|u(p)=r\}$ and $\Omega'_N=\Omega_N(\sum_{i=1}^{N-1} v_i)$. For sufficiently large m and n, dM_m-dM_n has no singularities and non-vanishing A-periods in $\Omega_N(r)$ and so

$$\|dM_m-dM_n\|^2_{\Omega_N(r)}=i\int_{U(p)=r}(M_m-M_n)\overline{d(M_m-M_n)}\,.$$

By the same way as we did in the proof of lemma 3 we obtain

$$\|dM_m-dM_n\|^2_{\Omega'_N}\leqq\|dM_m-dM_n\|^2_{\Omega_N(r)}$$

$$\leqq\sum_{i=1}^{m(N)}\Big(\int_{\gamma_r^i}|dM_m-dM_n|\Big)^2$$

$$\leqq 2\pi\frac{1}{\min\limits_i v_N^i}\|dM_m-dM_n\|^2_{D_N}\,.$$

Thus we can conclude in usual way that $M_n(s)$ converges uniformly on every compact subset in Ω'_N. q.e.d.

Corollary. *If $\sum_{m=1}^{\infty}\|d\Pi^{s,t}_{a_m,b_m}\|_{D_N}$ and $\sum_{j=1}^{\infty}|\chi_j|\|dw_j\|_{D_N}$ are convergent for each N, then $\{M_n(s)\}$ converges uniformly on every compact subset in $R_0-\bigcup_i\gamma_i$. Moreover if $\sum_{m=1}^{\infty}\|d\Pi^{s,t}_{a_m,b_m}\|_{\cup D_n}<+\infty$ and $\sum_{j=1}^{\infty}|\chi_j|\|dw_j\|_{\cup D_n}<+\infty$, then $\|dM\|_{\cup D_n}<+\infty$.*

Because, for $m>n$ we have

$$\|dM_m-dM_n\|_{D_N}\leqq\sum_{i=l(n)+1}^{l(m)}\|d\Pi^{s,t}_{a_i,b_i}\|_{D_N}+2\pi\sum_{j=k(n)+1}^{k(m)}|\chi_j|\,\|dw_j\|_{D_N},$$

and the result follows from theorem 2.

Also it is seen that if $\lim_{p\to\infty}\sum_{i,j=1}^{p}\chi_i\bar{\chi}_j\tau''_{ij}$ converges, then $w'=\sum_{j=1}^{\infty}\chi_j\int_t^s dw_j$ converges uniformly on every compact subset in R_0 and

$\|dw'\| < +\infty$. In fact, put $dw'_p = \sum_{i=1}^{p} \chi_i dw_i$, then

$$\|dw'_p\|^2 = -2i \sum_{i,j=1}^{p} \chi_i \bar{\chi}_j \tau''_{ij}$$

and the result is again obtained from theorem 2 and the completeness of Γ_{ase}.

7. Here let R be an arbitrary open Riemann surface. In the remaining sections in this § we shall give a formulation of the Abel's theorem by use of an another system of differentials, so-called, a system with real normalization. In the case of real normalization, according to the behavior of differential near ideal boundary, various systems have been considered (Kusunoki [7], Mizumoto [10], Shiba [16], Yoshida [22]) However, for our purpose, it is enough to use the canonical differentials as follows (Kusunoki [7]):

(I) *The first kind of canonical differentials* φ_{A_i}, φ_{B_i} $(i=1, 2,\dots)$; (i) φ_{A_i} and φ_{B_i} are semiexact canonical (ii) $\mathrm{Re}\int_{B_j}\varphi_{A_i}=\delta_{ij}$, $\mathrm{Re}\int_{A_j}\varphi_{B_i}=-\delta_{ij}$, $\mathrm{Re}\int_{A_j}\varphi_{A_i}=\mathrm{Re}\int_{B_j}\varphi_{B_i}=0$.

(II) *The second kind of canonical differentials* ψ_{p^n}, $\tilde{\psi}_{p^n}$ $(n\geqq 1)$; (i) ψ_{p^n} (resp. $\tilde{\psi}_{p^n}$) has a pole at p such that its singular part is $-\frac{n}{z^{n+1}}dz$ (resp. $-\frac{in}{z^{n+1}}dz$) (ii) they are semiexact canonical and A, B-periods of their real parts all vanish.

(III) *The third kind of canonical differentials* $\varphi_{p,q}$, $\tilde{\varphi}_{p,q}$; (i) $\varphi_{p,q}$ (resp. $\tilde{\varphi}_{p,q}$) has two simple poles such that its singular part is $\frac{1}{z}dz$ (resp. $\frac{i}{z}dz$) at p and $-\frac{1}{z}dz$ (resp. $-\frac{i}{z}dz$) at q (ii) they are semiexact canonical and A, B-periods of their real part all vanish.

It should be noted that these three kinds of differentials exist uniquely. Also it follows from definition that they are semiexact and square integrable outside of a compact subset of R.

There are some relations between their integrals as in the classical case (Weyl [21]):

(2)
$$\int_C \psi_{p^n} = -2\pi i \frac{1}{(n-1)!} \mathrm{Re} \frac{d^n \Phi_C}{dp^n}, \quad \int_C \tilde{\psi}_{p^n} = 2\pi i \frac{1}{(n-1)!} \mathrm{Jm} \frac{d^n \Phi_C}{dp^n}$$
$$\int_C \varphi_{p,q} = 2\pi i \, \mathrm{Re} \int_q^p \varphi_C, \quad \int_C \tilde{\varphi}_{p,q} = -2\pi i \, \mathrm{Jm} \int_q^p \varphi_C,$$

where C is a cycle A_j or B_j and Φ_C is the integral of φ_C.

(3)
$$\mathrm{Re}\, \Phi_{s,t}^{p,q} = \mathrm{Re}\, \Phi_{p,q}^{s,t}, \quad \mathrm{Jm}\, \tilde{\Phi}_{s,t}^{p,q} = \mathrm{Jm}\, \tilde{\Phi}_{p,q}^{s,t}, \quad \mathrm{Jm}\, \Phi_{s,t}^{p,q} = -\mathrm{Re}\, \tilde{\Phi}_{p,q}^{s,t},$$

where $\Phi_{s,t}^{p,q} = \Phi_{s,t}(p) - \Phi_{s,t}(q) = \int_q^p \varphi_{s,t}$, $\tilde{\Phi}_{s,t}^{p,q} = \int_q^p \tilde{\varphi}_{s,t}$ and in the last equality the path of integration from q to p dose not intersect the path from t to s.

(4)
$$\mathrm{Re}\, \Psi_{p^n}^{s,t} = -\frac{1}{(n-1)!} \mathrm{Re} \frac{\partial^n \Phi_{s,t}^{p,q}}{\partial p^n}, \quad \mathrm{Jm}\, \Psi_{p^n}^{s,t} = \frac{1}{(n-1)!} \mathrm{Re} \frac{\partial^n \tilde{\Phi}_{s,t}^{p,q}}{\partial p^n}$$
$$\mathrm{Re}\, \tilde{\Psi}_{p^n}^{s,t} = \frac{1}{(n-1)!} \mathrm{Jm} \frac{\partial^n \Phi_{s,t}^{p,q}}{\partial p^n}, \quad \mathrm{Jm}\, \tilde{\Psi}_{p^n}^{s,t} = -\frac{1}{(n-1)!} \mathrm{Jm} \frac{\partial^n \tilde{\Phi}_{s,t}^{p,q}}{\partial p^n}$$

where $\Psi_{p^n}^{s,t} = \Psi_{p^n}(s) - \Psi_{p^n}(t) = \int_t^s \psi_{p^n}$ and $\tilde{\Psi}_{p^n}^{s,t} = \int_t^s \tilde{\psi}_{p^n}$.

(5)
$$\frac{1}{(n-1)!} \mathrm{Re} \frac{d^n \Psi_{p^m}(q)}{dq^n} = \frac{1}{(m-1)!} \mathrm{Re} \frac{d^m \Psi_{q^n}(p)}{dp^m}$$
$$\frac{1}{(n-1)!} \mathrm{Jm} \frac{d^n \Psi_{p^m}(q)}{dq^n} = \frac{1}{(m-1)!} \mathrm{Re} \frac{d^m \tilde{\Psi}_{q^n}(p)}{dp^m}$$
$$\frac{1}{(n-1)!} \mathrm{Jm} \frac{d^n \tilde{\Psi}_{p^m}(q)}{dq^n} = \frac{1}{(m-1)!} \mathrm{Jm} \frac{d^m \tilde{\Psi}_{q^n}(p)}{dp^m}.$$

These formulas can be obtained easily from the following lemma, with a slight modification, by the same way as in **3** and so we shall omit their proofs.

Lemma 4. (Kusunoki [7]) *Let $df_j = du_j + idv_j$ $(j=1, 2)$ be any two semiexact canonical differentials such that u_j are single-valued and regular outside of a compact subset B, then for every dividing curve $C \subset R - B$ we have*

$$\mathrm{Jm}\int_C f_1 df_2 = 0\,.$$

8. Theorem 3. *The necessary and sufficient condition for the existence of a single-valued meromorphic function $f(p)$ $(\in \mathscr{M})$ scuh that $f(p)$ satisfies* (B) *and its divisor is exactly δ is that the following conditions are filled*:

1) $$\lim_{n\to\infty} \mathrm{Re}\int_{C(n)} \varphi_{A_i} = -n_i,\ \lim_{n\to\infty} \mathrm{Re}\int_{C(n)} \varphi_{B_i} = -m_i \qquad (i=1, 2,\ldots)$$

where n_i and m_i are integers.

2) *The sequence of functions*

$$G_n(s) = \mathrm{Re}\sum_{m=1}^{l(n)} \Phi^{s,t}_{a_m, b_m}$$

converges uniformly on every compact subset of $R_0 - \cup\gamma_i$ and exp $\lim\limits_{n\to\infty}(G_n + i^*G_n)$ *satisfies* (B), *where $t\,(\neq a_k, b_k)$ is fixed in R. If the conditions are filled the desired function is always expressed by* $C\lim\limits_{n\to\infty}\exp\sum\limits_{m=1}^{l(n)} \Phi^{s,t}_{a_m, b_m}$ (C; constant).

Proof. Let $\int_{A_i} d\log f = 2\pi i n_i, \int_{B_i} d\log f = 2\pi i m_i$ and take a simply connected neighborhood U_m containing a_m, b_m $(U_m \subset \Omega_n,\ U_i \cap U_j = \emptyset,\ i \neq j)$ and set $U = \bigcup\limits_{m=1}^{l(n)} U_m$, then

$$\begin{aligned} 0 &= (\varphi_{A_i}, \overline{{}^*d\log f})_{\Omega_n - U} \\ &= \mathrm{Jm}\sum_{A_j, B_j \subset \Omega_n}\Big(\int_{A_j}\varphi_{A_i}\int_{B_j} d\log f - \int_{A_j} d\log f\int_{B_j}\varphi_{A_i}\Big) \\ &\quad + \mathrm{Jm}\Big(\int_{\partial U} - \int_{\partial\Omega_n}\Big)\Phi_{A_i} d\log f, \end{aligned}$$

By the residue theorem we have

$$\mathrm{Jm}\int_{\partial U}\Phi_{A_i} d\log f = -2\pi\,\mathrm{Re}\int_{C(n)}\varphi_{A_i}\,.$$

While, by the same way as in lemma 3 we are able to prove

$$\mathrm{Jm}\int_{\partial\Omega_n}\Phi_{A_i}d\log f\longrightarrow 0 \qquad (n\longrightarrow\infty),$$

hence

$$\lim_{n\to\infty}\mathrm{Re}\int_{C(n)}\varphi_{A_i}=-n_i\,.$$

Analogously we get

$$\lim_{n\to\infty}\mathrm{Re}\int_{C(n)}\varphi_{B_i}=-m_i\,.$$

Next let V be a simply connected neighborhood of s and t ($V\subset\Omega_n$, $V\cap U_i=\emptyset$), then

$$\begin{aligned}0&=(d\Phi^{p,q}_{s,t},\ \overline{*d\log f})_{\Omega_n-U-V}\\&=\mathrm{Jm}\sum_{A_jB_j\subset\Omega_n}\left(\int_{A_j}d\Phi^{p,q}_{s,t}\int_{B_j}d\log f-\int_{A_j}d\log f\int_{B_j}d\Phi^{p,q}_{s,t}\right)\\&\quad+\mathrm{Jm}\left(\int_{\partial U}+\int_{\partial V}-\int_{\partial\Omega_n}\right)\Phi^{p,q}_{s,t}d\log f.\end{aligned}$$

Since

$$\mathrm{Jm}\int_{\partial V}\Phi^{p,q}_{s,t}\,d\log f=-2\pi\log|f(s)/f(t)|$$

and

$$\mathrm{Jm}\int_{\partial U}\Phi^{p,q}_{s,t}d\log f=2\pi\,\mathrm{Re}\sum_{m=1}^{l(n)}\Phi^{a_m,b_m}_{s,t}=2\pi\,\mathrm{Re}\sum_{m=1}^{l(n)}\Phi^{s,t}_{a_m,b_m},$$

we obtain

$$\log|f(s)/f(t)|=\mathrm{Re}\sum_{m=1}^{l(n)}\Phi^{s,t}_{a_m,b_m}-\frac{1}{2\pi}\mathrm{Jm}\int_{\partial\Omega_n}\Phi^{p,q}_{s,t}d\log f\,. \tag{6}$$

By the same way as we did in the proof of lemma 3 we know that if n tends to infinity $\mathrm{Jm}\int_{\partial\Omega_n}\Phi^{p,q}_{s,t}\,d\log f$ converges to zero uniformly on every compact subset of $R_0-\cup\gamma_i$. Thus the uniform convergence of $G_n(s)$ follows from (6) and we have

(7) $$\log|f(s)|=\log|f(t)|+\lim_{n\to\infty}G_n(s).$$

Conversely, set $A_n(s)=\sum_{m=1}^{l(n)}\Phi^{s,t}_{a_m,b_m}$ and $A(s)=\lim_{n\to\infty}A_n(s)$, then by (2) and 1) we have

$$\int_{A_j}dA=\lim_{n\to\infty}\int_{A_j}dA_n=2\pi i\lim_{n\to\infty}\sum_{m=1}^{l(n)}\mathrm{Re}\int_{b_m}^{a_m}\varphi_{A_j}=-2\pi i\lim_{n\to\infty}\mathrm{Re}\int_{C(n)}\varphi_{A_j}$$
$$=2\pi i n_j$$

and

$$\int_{B_j}dA=2\pi i m_j.$$

While $\int_C dA=0$ for any dividing cycle $C\subset R-\cup\gamma_i$. Now set $f(s)=\exp A(s)$, then $f(s)$ is a desired function. Let $g(s)$ be a meromorphic function with same properties as $f(s)$, then by (7) we get $\log|f(s)/g(s)|=\log|f(t)/g(t)|$ for arbitrary point s and so $g(s)=Cf(s)$. q.e.d.

Remark. 1) From the proof of theorem we know that *the necessary and sufficient condition for the existence of a multiplicative function* $n(p)$ $(\in\mathscr{M})$ *such that* i) $n(p)$ *satisfies* (B) ii) *its divisor is exactly* δ iii) *the absolute values of multiplicators with respect to* A_j (resp. B_j) *are* $\exp 2\pi\chi_{A_j}$ (resp. $\exp 2\pi\chi_{B_j}$) (χ_{A_j}, χ_{B_j}; real) *is that the sequence of functions*

$$N_n(s)=\mathrm{Re}\sum_{m=1}^{l(n)}\Phi^{s,t}_{a_m,b_m}+2\pi\sum_{j=1}^{k(n)}(\chi_{B_j}\mathrm{Re}\int_t^s\varphi_{A_j}-\chi_{A_j}\mathrm{Re}\int_t^s\varphi_{B_j})$$

converges uniformly on every compact subset of $R_0-\cup\gamma_i$ *and* $\exp\lim_{n\to\infty}(N_n(s)+i^*N_n(s))$ *satisfies* (B).

2) As for the uniform convergence of $\{N_n(s)\}$ the result corresponding to theorem 2 can be obtained, also.

3) If R satisfies $\inf_n\min_i v_n^i>0$, then for a function satisfying (A) in place of (B) the corresponding statement is valid, also.

From the above remark 3) it is possible to prove

Propositonm 5. *Suppose that* $\inf_n \min_i v_n^i > 0$, *then*

$$\varphi_{A_i} = \lim_{n\to\infty} \sum_{j=1}^{k(n)} \left(\int_{A_j} \varphi_{A_i}\right) dw_j ,$$

$$\varphi_{p,q} = d\Pi_{p,q} + 2\pi i \lim_{n\to\infty} \sum_{j=1}^{k(n)} \left(\mathrm{Re} \int_q^p \varphi_{A_j}\right) dw_j ,$$

$$dw_i = \lim_{n\to\infty} \sum_{j=1}^{k(n)} \left(\mathrm{Re} \int_{B_j} dw_i\right) \varphi_{A_j} - \varphi_{B_i} ,$$

$$d\Pi_{p,q} = \varphi_{p,q} + 2\pi \lim_{n\to\infty} \sum_{j=1}^{k(n)} \left(\mathrm{Jm} \int_q^p dw_j\right) \varphi_{A_j} .$$

Proof. $\exp \Phi_{A_i}$ and $\exp \Phi_{p,q}^{s,t}$ (t; fixed) satisfy the condition of theorem 1′ and have $\frac{1}{2\pi i}\int_{A_j} \varphi_{A_i}$ and $\frac{1}{2\pi i}\int_{A_j} d\Phi_{p,q}^{s,t}$ as χ_j, respectively. Hence by theorem 1′ we have

$$\Phi_{A_i}(s) = \Phi_{A_i}(t) + \lim_{n\to\infty} \sum_{j=1}^{k(n)} \left(\int_{A_j} \varphi_{A_i}\right) \int_t^s dw_j$$

$$\Phi_{p,q}^{s,t} = \Pi_{p,q}^{s,t} + \lim_{n\to\infty} \sum_{j=1}^{k(n)} \left(\int_{A_j} d\Phi_{p,q}^{s,t}\right) \int_t^s dw_j .$$

While, by (2)

$$\int_{A_j} d\Phi_{p,q}^{s,t} = 2\pi i \,\mathrm{Re} \int_q^p \varphi_{A_j} ,$$

and so the expressions for φ_{A_i} and $\varphi_{p,q}$ are obtained.

Next $\exp w_i$ and $\exp \Pi_{p,q}^{s,t}$ (t; fixed) satisfy the condition in the above remark 1). It is easy to see that $\chi_{A_j} = \frac{1}{2\pi}\delta_{ij}$, $\chi_{B_j} = \frac{1}{2\pi} \mathrm{Re} \int_{B_j} dw_i$ for $\exp w_i$ and $\chi_{A_j} = 0$, $\chi_{B_j} = \frac{1}{2\pi} \mathrm{Re} \int_{B_j} d\Pi_{p,q}^{s,t}$ for $\exp \Pi_{p,q}^{s,t}$. Hence we have

$$\mathrm{Re}\, w_i(s) = \mathrm{Re}\, w_i(t) + \lim_{n\to\infty} \sum_{j=1}^{k(n)} \left(\mathrm{Re} \int_{B_j} dw_i\right) \mathrm{Re} \int_t^s \varphi_{A_j} - \mathrm{Re} \int_t^s \varphi_{B_i}$$

$$\mathrm{Re}\, \Pi_{p,q}^{s,t} = \mathrm{Re}\, \Phi_{p,q}^{s,t} + \lim_{n\to\infty} \sum_{j=1}^{k(n)} \left(\mathrm{Re} \int_{B_j} d\Pi_{p,q}^{s,t}\right) \mathrm{Re} \int_t^s \varphi_{A_j} .$$

While, by proposition 2

$$\mathrm{Re}\int_{B_j} d\Pi^{s,t}_{p,q} = \mathrm{Re}\,(2\pi i \int_p^q dw_j) = 2\pi\,\mathrm{Jm}\int_q^p dw_j .$$

Thus the latter two expressions can be obtained by letting the operator $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)dz$ operate to $\mathrm{Re}\,w_i$ and $\mathrm{Re}\,\Pi^{s,t}_{p,q}$, respectively.

q.e.d.

From the remark of lemma 3 it follows that if $\sum_{n=1}^{\infty}\left(\frac{\min_i v^i_n}{\|d\log f\|_{D_n}}\right)^2$ is divergent, then there exists a canonical exhausion $\{\Omega_{n'}\}$ $(\subset\{\Omega_n\})$ such that

$$|f(s)| = |f(t)| \lim_{n'\to\infty} \exp(\mathrm{Re}\sum_{m=1}^{l(n')} \Phi^{s,t}_{a_m, b_m})$$

where $t\ (\neq a_m, b_m)$ is fixed in R. Particulary, let us set $\delta = 1$, then we find

Proposition 6. *Let R be an arbitrary open Riemann surface and $f(p)$ be a non-constant holomorphic function on R such that $\int_{\gamma^i_n} d\log f = 0$ and $f(p)$ has no zeros in R, then for any canonical exhaustion $\{\Omega_n\}$ the series*

$$\sum_{n=1}^{\infty}\left(\frac{\min_i v^i_n}{\|d\log f\|_{D_n}}\right)^2$$

is always convergent.

From the above propostition we know at once that if $f(p)$ is a non-constant holomorphic function on an arbitrary open Riemann surface R, then for any canonical exhaustion the series $\sum_{n=1}^{\infty}\left(\frac{\min v^i_n}{\|df\|_{D_n}}\right)^2$ is always convergent.

Corollary. *Let D be a simply connected domain and $f(p)$ be a non-constant holomorhic function which is defined in D and has no zeros there, then for any exhaustion $\{\Omega_n\}$ of D the series $\sum_{n=1}^{\infty}\left(\frac{\hat{v}_n}{\|d\log f\|_{D_n}}\right)^2$ is always convergnt, where we assume that each Ω_n is simply connected.*

§ II. Additive functions

1. Let $\delta'=\Pi a_\alpha^{\nu_\alpha}$ and $\delta''=\Pi b_\alpha^{\mu_\alpha}$ be two finite or infinite integral divisors whose supports are disjoint each other and are contained in $R-\bigcup_n \bar{D}_n$. Let us denote by $\delta'_n=a_1^{\nu_1}\dots a_{l(n)}^{\nu_{l(n)}}$ (resp. $\delta''_n=b_1^{\mu_1}\dots b_{m(n)}^{\mu_{m(n)}}$) the restriction to Ω_n of δ' (resp. δ'') and we show the following

Theorem 4. *Suppose that* $\inf_n \min_i \nu_n^i>0$. *The necessary and sufficient condition for the existence of a meromorphic differential dw such that* 1) *dw is a multiple of* $1/\delta''$ 2) *its singular part is* $\sum_{k=1}^{\mu_i}\frac{b_{ik}}{z^k}dz$ *at* b_i 3) *for all n and i* $\int_{\gamma_n^i}dw=0$ *and* $\int_{A_j}dw=w(A_j)$ 4) $\|dw\|_{\cup D_n}<+\infty$ *is that the sequence of functions*

$$H_n(s)=\sum_{i=2}^{m(n)} b_{i1}\Pi_{b_i,b_1}^{s,t}-\sum_{i=1}^{m(n)}\sum_{k=2}^{\mu_i}\frac{b_{ik}}{k-1}Y_{b_i^{k-1}}^{s,t}+\sum_{j=1}^{k(n)}w(A_j)\int_t^s dw_j$$

converges uniformly on every compact subset in $R_0-\bigcup_i\{b_i\}$ *and its limit function $H(s)$ satisfies* $\|dH\|_{\cup D_n}<+\infty$, *where $w(A_j)$ are given complex numbers and* $t\,(\neq b_i)$ *is fixed. If the conditions are filled, the desired differential is uniquely expressed by dH.*

Proof. At first it follows from 3) that for each n

$$\sum_{i=1}^{m(n)} b_{i1}=0. \tag{1}$$

Now let V and U_i be neighborhoods of s, t and b_1, b_i respectively ($V\cap U_i=\emptyset$, V, $U_i\subset\Omega_n$), then

$$\begin{aligned}0&=(d\Pi_{s,t}^{p,q},\overline{{}^*dw})_{\Omega_n-V-\bigcup_{i=2}^m U_i}\\&=\sum_{A_j,B_j\subset\Omega_n}\left(\int_{A_j}d\Pi_{s,t}^{p,q}\int_{B_j}dw-\int_{A_j}dw\int_{B_j}d\Pi_{s,t}^{p,q}\right)\\&\quad+\left(\int_{\partial(\bigcup_{i=2}^m U_i)}+\int_{\partial V}-\int_{\partial\Omega_n}\right)\Pi_{s,t}^{p,q}dw.\end{aligned}$$

Set $W=\int dw$, then

$$\int_{\partial V} \Pi_{s,t}^{p,q} dw = -\int_{\partial V} W d\Pi_{s,t}^{p,q} = -2\pi i(W(s) - W(t)).$$

While

$$\int_{B_j} d\Pi_{s,t}^{p,q} = -2\pi i \int_t^s dw_j$$

and

$$\int_{\partial(\bigcup_{i=2}^m U_i)} \Pi_{s,t}^{p,q}\, dw = 2\pi i \text{ (Residue sum of } \Pi_{s,t}^{p,q}\, dw \text{ in } \bigcup_{i=2}^m U_i)$$

$$= 2\pi i \sum_{i=1}^m \sum_{k=2}^{\mu_i} \frac{b_{ik}}{(k-1)!} \frac{\partial^{k-1} \Pi_{s,t}^{p,q}}{\partial p^{k-1}} (b_i)$$

$$= 2\pi i \sum_{i=1}^m \left[b_{i1} \Pi_{s,t}^{b_i,q} - b_{i1} \Pi_{s,t}^{b_1,q} - \sum_{k=2}^{\mu_i} \frac{b_{ik}}{k-1} Y_{b_i^{k-1}}^{s,t} \right]$$

(by (1) and Proposition 4)

$$= 2\pi i \sum_{i=2}^m b_{i1} \Pi_{b_i,b_1}^{s,t} - 2\pi i \sum_{i=1}^m \sum_{k=2}^{\mu_i} \frac{b_{ik}}{k-1} Y_{b_i^{k-1}}^{s,t}$$

(by proposition 3)

Thus we have

$$W(s) = W(t) + H_n(s) - \frac{1}{2\pi i} \int_{\partial \Omega_n} \Pi_{s,t}^{p,q}\, dw$$

and, on going to the limit, by lemma 3 we obtain the desired result. The converse follows easily from (1) and the expression of dH. q.e.d.

Remark. If R satisfies $\sum_n \min_i v_n^i = +\infty$, the statement of theorem 4 is valid whenever the property 4) of dw is replaced by $\sup_n \frac{\|dw\|_{D_n}}{\min_i v_n^i} < +\infty$.

Let $f(s)$ be an additive function which is a multiple of $1/\delta'$. If the singular part of f at a_i is $\sum_{k=1}^{\nu_i} \frac{a_{ik}}{z^k}$, then that of df is $\sum_{k=1}^{\nu_i} (-k) \frac{a_{ik}}{z^{k+1}} dz$ and so we have

Corollary 1. *Suppose that* $\inf_n \min_i \nu_n^i > 0$. *The necessary and sufficient condition for the existence of an additive function such that* 1) f *is a multiple of* $1/\delta'$ 2) *its singular part is* $\sum_{k=1}^{\nu_i} \frac{a_{ik}}{z^k}$ *at* a_i 3) $\int_{\gamma_n^i} df = 0$ *and* $\int_{A_j} df = F(A_j)$ 4) $\|df\|_{\cup D_n} < +\infty$ *is that the sequence of functions*

$$A_n(s) = \sum_{i=1}^{l(n)} \sum_{k=1}^{\nu_i} a_{ik} Y_{a_i^k}^{s,t} + \sum_{j=1}^{k(n)} F(A_j) \int_t^s dw_j$$

converges uniformly on every compact subset in $R_0 - \bigcup_i \{a_i\}$ *and its limit function* $A(s)$ *satisfies* $\|dA\|_{\cup D_n} < +\infty$. *If the conditions are filled* $A(s)$ *is the unique desired function up to an additive constant.*

Here it is to be pointed out that there exist always additive functions f with properties 1)$\sim$3) in corollary 1, but in order that f has more one property 4) some restrictions must be imposed on a_{ik} and $F(A_j)$ and in fact such a condition is shown in corollary 1. For example, now let $\nu_i = 1$ $(i = 1, 2, \ldots)$, $l(n) = n$ and $F(A_j) = 0$ $(j = 1, 2, \ldots)$. It is possible to choose a_{i1} $(i = 1, 2, \ldots)$ such that $\sum_{i=1}^{\infty} |a_{i1}| \|dY_{a_i}\|_{\cup D_n} < +\infty$. We set $A_n(s) = \sum_{i=1}^{n} a_{i1} Y_{a_i}^{s,t}$, then $f(s) = C + \lim_{n\to\infty} A_n(s)$ (C; constant) is the unique desired additive function. And so it is possible to restate corollary 1 as follows: An additive function f with properties 1)$\sim$3) has the finite norm over $\bigcup_n D_n$ if and only if a_{ik} and $F(A_j)$ can be chosen such that $A_n(s)$ converges uniformly to $A(s)$ with finite norm over $\bigcup_n D_n$ and $A(s) = f$ up to an additive constant.

Remark. The existence condition of a single-valued meromorphic function with properties 1), 2) and 4) is as follows: i) *the uniform convergence of* $B_n(s) = \sum_{i=1}^{l(n)} \sum_{k=1}^{\nu_i} a_{ik} Y_{a_i^k}^{s,t}$ *and* $\|dB\|_{\cup D_n} < +\infty$ $(B(s) = \lim_{n\to\infty} B_n(s))$

$$\lim_{n\to\infty} \sum_{i=1}^{l(n)} \sum_{k=1}^{\nu_i} \frac{a_{ik}}{(k-1)!} w_j^{(k)}(a_i) = 0 \qquad (j = 1, 2, \ldots)$$

where $w_j = \int dw_j = \sum_{k=0}^{\infty} \frac{1}{k!} w_j^{(k)}(a_i) z^k$ *at* a_i.

Indeed, the single-valuedness of $B(s)$ is equivalent to $\int_{B_j} dB = \lim_{n\to\infty} \int_{B_j} dB_n = 0$ $(j=1, 2,\dots)$. While, from proposition 2 it is seen that $\int_{B_j} dB_n = 2\pi i \sum_{i=1}^{l(n)} \sum_{k=1}^{\nu_i} \frac{a_{ik}}{(k-1)!} w_j^{(k)}(a_i)$.

Corollary 2. *Suppose that* $\inf_n \lim_i \nu_n^i > 0$, *then an arbitrary square integrable analytic semiexact differential* dw *is uniquely expressed by*

$$dw = \lim_{n\to\infty} \sum_{A_j \subset \Omega_n} \left(\int_{A_j} dw \right) dw_j ,$$

where the right hand side converges uniformly on every compact subset in R.

If $\inf_n \min_i \nu_n^i > 0$, then from corollary 2 it is seen that a canonical differntial is uniquely expressed in terms of normal differentials, for example

$$\psi_{p^m} = dY_{p^m} - \frac{2\pi i}{(m-1)!} \lim_{n\to\infty} \sum_{A_j \subset \Omega_n} \left(\mathrm{Re} \frac{d^n \Phi_{A_j}}{dp^n} \right) dw_j .$$

Indeed, since $\psi_{p^m} - dY_{p^m} \in \Gamma_{ase}$ we obtain

$$\psi_{p^m} - dY_{p^m} = \lim_{n\to\infty} \sum_{A_j \subset \Omega_n} \left[\int_{A_j} (\psi_{p^m} - dY_{p^m}) \right] dw_j .$$

From (2) in §I and $\int_{A_j} dY_{p^m} = 0$ the result follows at once.

2. Here we suppose that $\inf_n \min_i \nu_n^i > 0$ and let us consider the following four vector spaces in the complex number field:

$\hat{\mathfrak{M}}(1/\delta')$; The vector space consisting of additive functions f such that 1) f is a multiple of $1/\delta'$ 2) $\int_{A_j} df = 0$ $(j=1, 2,\dots)$, $\int_{\gamma_n^i} df = 0$ $(n=1, 2,\dots, i=1, 2,\dots, m(n))$ 3) $\|df\|_{\cup D_n} < +\infty$.

According to the corollary of theorem 4 such a f is expressed uniquely by

$$(2) \qquad f = \lim_{n\to\infty} \sum_{i=1}^{l(n)} \sum_{k=1}^{\nu_i} a_{ik} Y_{a_i^k}^{s,t} + C \qquad (C;\ \text{constant}),$$

where the right hand side is uniformly convergent and its differential has the finite norm over $\cup D_n$.

$M(1/\delta)$; The vector space of single-valued meromorphic functions $g(p)$ ($\in \hat{M}(1/\delta')$) which are multiples of δ''.

$W(1/\delta'')$; The vector space consisting of meromorphic differentials dw such that 1) dw is a multiple of $1/\delta''$ 2) $\int_{\gamma_n^i} dw=0$ 3) $\|dw\|_{\cup D_n}<+\infty$.

From theorem 4 it is seen that such a dw is expressed by

$$(3)\qquad dw=\lim_{n\to\infty}\left[\sum_{i=2}^{m(n)} b_{i1}d\Pi_{b_i,b_1}-\sum_{i=1}^{m(n)}\sum_{k=2}^{\mu_i}\frac{b_{ik}}{k-1}dY_{b_i^{k-1}}+\sum_{j=1}^{k(n)} w(A_j)dw_j\right],$$

where the right hand side is uniformly convergent and has the finite norm over $\cup D_n$.

$W(\delta)$; The vector space of meromorphic differentials dw ($\in W(1/\delta'')$) which are multiples of δ'.

Lemma 5. *Suppose that* $\inf_n \min_i v_n^i>0$, *then for* $f\in \hat{M}(1/\delta')$ *and* $dw\in W(1/\delta'')$

$$\lim_{n\to\infty}\sum_{a_i\subset\Omega_n}\operatorname{Res}_{a_i} f dw=0$$

is equivalent to

$$\lim_{n\to\infty}\left[\sum_{b_i\subset\Omega_n}\operatorname{Res}_{b_i} f dw-\frac{1}{2\pi i}\sum_{A_j,B_j\subset\Omega_n} w(A_j)\int_{B_j} df\right]=0\,.$$

Proof. Let U_i be a neighborhood of a_i ($U_i\subset\Omega_n$, $U_i\cap U_j=\emptyset$ $(i\neq j)$) and V_i a simply connected neighborhood of b_i, b_1 ($V_i\subset\Omega_n$, $V_i\cap U_j=\emptyset$), then

$$\begin{aligned}0&=(df,\overline{{}^*dY})_{\Omega_n-\bigcup_{i=1}^{l}U_i-\bigcup_{i=1}^{m}V_i}\\&=\sum_{A_j,B_j\subset\Omega_n}\left(\int_{A_j}df\int_{B_j}dw-\int_{A_j}dw\int_{B_j}df\right)\\&\quad+\left(\int_{\partial(\bigcup_i U_i)}+\int_{\partial(\cup V_i)}-\int_{\partial\Omega_n}\right)fdw\end{aligned}$$

$$= -\sum_{A_j, B_j \subset \Omega_n} w(A_j)\int_{B_j} df + 2\pi i \sum_{a_i \subset \Omega_n} \operatorname{Res}_{a_i} f dw$$

$$+ 2\pi i \sum_{b_i \subset \Omega_n} \operatorname{Res}_{b_i} f dw - \int_{\partial\Omega_n} f dw .$$

As n tends to infinity, the equivalence follows from lemma 3. q.e.d.

If $f \in \hat{M}(1/\delta')$ and $dw \in W(1/\delta'')$ are expressed by (2) and (3) respectively, then

$$(4) \quad \begin{aligned} &\sum_{a_i \subset \Omega_n} \operatorname{Res}_{a_i} f dw = \sum_{a_i \subset \Omega_n} \sum_{k=1}^{\nu_i} \frac{a_{ik}}{(k-1)!} w^{(k)}(a_i) \\ &\sum_{b_i \subset \Omega_n} \operatorname{Res}_{b_i} f dw = \sum_{b_i \subset \Omega_n} \sum_{k=1}^{\mu_i} \frac{b_{ik}}{(k-1)!} f^{(k-1)}(b_i) \qquad \left(\sum_{i=1}^{m(n)} b_{i1} = 0\right). \end{aligned}$$

Thus from lemma 5 we see that

$$\lim_{n\to\infty} \sum_{a_i \subset \Omega_n} \sum_{k=1}^{\nu} \frac{a_{ik}}{(k-1)!} w^{(k)}(a_i) = 0$$

is equivalent to

$$\lim_{n\to\infty} \left[\sum_{b_i \subset \Omega_n} \sum_{k=1}^{\mu_i} \frac{b_{ik}}{(k-1)!} f^{(k-1)}(b_i) - \frac{1}{2\pi i} \sum_{A_j, B_j \subset \Omega_n} w(A_j)\int_{B_j} df \right] = 0$$

$$\left(\sum_{i=1}^{m(n)} b_{i1} = 0\right).$$

Now let us put $<f, dw> = \lim_{n\to\infty} \sum_{a_i \subset \Omega_n} \operatorname{Res}_{a_i} f dw$ for $f \in \hat{M}(1/\delta')$ and $dw \in W(1/\delta'')$, then $<f, dw>$ may be divergent for some f, dw and so we shall consider the subspaces of $\hat{M}(1/\delta')$ and $W(1/\delta'')$ as follows:

$\hat{M}_0(1/\delta') = \{f \in \hat{M}(1/\delta) | <f, dw> \text{ is convergent for all } dw \in W(1/\delta'')\}$,

$W_0(1/\delta'') = \{dw \in W(1/\delta'') | <f, dw> \text{ is convergent for all } f \in \hat{M}(1/\delta')\}$,

$M_0(1/\delta) = M(1/\delta) \cap \hat{M}_0(1/\delta')$,

$W_0(\delta) = W(\delta) \cap W_0(1/\delta'')$.

It follows from the proof of lemma 5 that the convergence of $<f, dw>$

is equivalent to that of $\lim_{n\to\infty}\left[\sum_{b_i\subset\Omega_n}\operatorname{Res}_{b_i} f dw - \frac{1}{2\pi i}\sum_{A_j,B_j\subset\Omega_n} w(A_j)\int_{B_j} df\right]$, hence from (2), (3) and (4) it is seen that $\hat{M}_0(1/\delta')$ and $W_0(1/\delta'')$ contains an infinite number of linearly independent elements, respectively.

Now, according to the method due to Kusunoki [6], we can find

Theorem 5. *Suppose that* $\inf_n \min_i v_n^i > 0$, *then*

$$\dim(W_0(1\backslash\delta'')\backslash W_0(\delta)) = \dim(\hat{M}_0(1\backslash\delta')\backslash M_0(1\backslash\delta))$$

where we suppose it is granted that each of both sides is infinite.

Proof. At first we shall prove that

$$W_0(\delta) = \{dw \in W_0(1/\delta'') | <f, dw> = 0, \qquad \forall f \in \hat{M}_0(1/\delta')\}$$

$$M_0(1/\delta) = \{f \in \hat{M}_0(1/\delta') | <f, dw> = 0, \qquad \forall dw \in W_0(1/\delta'')\}.$$

Take a $dw \in W_0(\delta)$, then it is a multiple of δ' and so

$$w'(a_i) = \cdots\cdots = w^{(v_i)}(a_i) = 0 \qquad \text{at} \quad a_i\ (i=1, 2,\ldots).$$

It follows from (4) that $<f, dw> = 0$ for all $f \in \hat{M}_0(1/\delta')$. Conversely, if $dw \in W_0(1/\delta'')$ satisfies $<f, dw> = 0$ for all $f \in \hat{M}_0(1/\delta')$, then we choose $a_{ik} Y_{a_i}^k\ (a_{ik} \neq 0)$ as f. From (4) it is seen that $w^{(k)}(a_i) = 0\ (k = 1,\ldots, v_i,\ i=1, 2,\ldots)$, hence $dw \in W_0(\delta)$.

Next take a $f \in M_0(1/\delta)$, then f is a multiple of δ'' and single-valued function, and so $f(b_i) = \cdots = f^{(\mu_i - 1)}(b_i) = 0$ at $b_i\ (i=1, 2,\ldots)$ and $\int_{B_j} df = 0$ $(j=1, 2,\ldots)$, hence from lemma 5 it follows that $<f, dw> = 0$ for all $dw \in W_0(1/\delta'')$. Conversely, let $f \in \hat{M}_0(1/\delta')$ satisfy $<f, dw> = 0$ for all $dw \in W_0(1/\delta'')$. We take $b_{i1} d\Pi_{b_i, b_1} - \sum_{k=2}^{\mu_i} \frac{b_{ik}}{k-1} dY_{b_i}^{k-1}\ (i \neq 1)$ as dw, then by lemma 5 we have

$$<f, dw> = \sum_{k=1}^{\mu_i} \frac{b_{ik}}{(k-1)!} f^{(k-1)}(b_i) - b_{i1} f(b_1) = 0$$

and so $f'(b_i) = \cdots = f^{(\mu_i - 1)}(b_i) = 0$ $f(b_i) = f(b_1)$ $(i \neq 1)$.

At b_1 set $dw=\sum_{k=2}^{\mu_1}\frac{b_{1k}}{k-1}dY_{b_1^{k-1}}$, then

$$\sum_{k=2}^{\mu_1}\frac{b_{1k}}{(k-1)!}f^{(k-1)}(b_1)=0,$$

Hence $f'(b_1)=\cdots=f^{(\mu_1-1)}(b_1)=0$. We take the function $f-f(b_1)$ and denote it by f again, then f becomes to a multiple of $1/\delta$. Finally, take dw_j as dw, then $\int_{B_j}df=0$, and so it is seen that f is a single-valued function belonging to $M_0(1/\delta)$. Thus it follows that the latter equality is valid. The remaining part of theorem follows from the well known algebraic fact associated to the bilinear form $<f, dw>$ (cf. for example Yoshida [22]).

Remark. If R has finite genus and δ is finite, then the dimension of $W_0(1/\delta'')=W(1/\delta'')$ and $\hat{M}_0(1/\delta')=\hat{M}(1/\delta')$ are easily calculated and we are able to find an analogue of the classical Riemann-Roch theorem.

3. Here let R be an arbitrary open Riemann surface. We shall express the meromorphic diffferentials treated in preceeding sections in terms of canonical differentials. Since we make use of the similar way to that in theorem 4 we shall don't enter into detail.

Theorem 6. *There exists a meromorphic differential dv such that 1) dv is a multiple of $1/\delta''$ 2) its singular part is*

$$\sum_{k=1}^{\mu_i}\frac{b'_{ik}+ib''_{ik}}{z^k}dz \quad at\ b_i \quad (b'_{ik}, b''_{ik};\ real,\ \sum_{i=1}^{m(n)}(b'_{i1}+ib''_{i1})=0)$$

3) $\mathrm{Re}\int_{A_j}dv=v(A_j)'$, $\mathrm{Re}\int_{B_j}dv=v(B_j)'$ *and* $\int_{\gamma_n^i}dv=0$ 4) $\sup_n\frac{\|dv\|_{D_n}}{\min_i v_n^i}<+\infty$
if and only if

i) *the sequence of functions*

$$P_n(s)=\mathrm{Re}\Big[\sum_{i=2}^{m(n)}(b'_{i1}\Phi^{s,t}_{b_i,b_1}+b''_{i1}\tilde{\Phi}^{s,t}_{b_i,b_1})$$

$$-\sum_{i=1}^{m(n)}\sum_{k=2}^{\mu_i}\frac{1}{k-1}(b'_{ik}\Psi^{s,t}_{b_i^{k-1}}+b''_{ik}\tilde{\Psi}^{s,t}_{b_i^{k-1}})\Big]$$

$$+\sum_{j=1}^{k(n)}\left[v(B_j)'\operatorname{Re}\int_t^s\varphi_{A_j}\operatorname{Re}\int_t^s\varphi_{A_j}-v(A_j)'\operatorname{Re}\int_t^s\varphi_{B_j}\right]$$

convergens uniformly on every compact subset in $R_0-\bigcup_i\{b_i\}$

ii) *the limit function* $P(s)$ *satisfies* $\sup_n\frac{\|dP\|_{D_n}}{\min v_n^i}<+\infty$. *If the conditions are filled, then* dv *is expressed by* $\lim_{n\to\infty}(dP_n+i^*dP_n)$.

Proof. Let V and $U_i\,(i=1, 2,\dots, m(n))$ be simply connected neighborhoods of s, t and b_i, b_1 respectively, then

$$0=(d\Phi_{s,t}^{p,q},\overline{^*dv})_{\Omega_n-V-\bigcup_{i=1}^{m}U_i}$$

$$=2\pi\sum_{A_j,B_j\subset\Omega_n}\left(\operatorname{Re}\int_t^s\varphi_{A_j}\cdot\operatorname{Re}\int_{B_j}dv-\operatorname{Re}\int_{A_j}dv\cdot\operatorname{Re}\int_t^s\varphi_{B_j}\right)$$

$$+\operatorname{Jm}\left(\int_{\partial(\cup U_j)}+\int_{\partial V}-\int_{\partial\Omega_n}\right)\Phi_{s,t}^{p,q}dv\,.$$

By using of (3), (4) in §I and $\sum_{i=1}^{m}b_{i1}=\sum_{i=1}^{m}(b'_{i1}+ib''_{i1})=0$ we have

$$\operatorname{Jm}\sum_{i=1}^{m(n)}\int_{\partial U_i}\Phi_{s,t}^{p,q}dv=2\pi\operatorname{Re}\sum_{i=1}^{m}\sum_{k=1}^{\mu_i}\frac{b'_{ik}+ib''_{ik}}{(k-1)!}\frac{\partial^{k-1}\Phi_{s,t}^{p,q}}{\partial p^{k-1}}(b_i)$$

$$=2\pi\operatorname{Re}\sum_{i=2}^{m(n)}(b'_{i1}\Phi_{b_i,b_1}^{s,t}+b''_{i1}\tilde{\Phi}_{b_i,b_1}^{t,s})$$

$$-2\pi\operatorname{Re}\sum_{i=1}^{m}\sum_{k=2}^{\mu_i}\frac{1}{k-1}(b'_{ik}\Psi_{b_i^{k-1}}^{s,t}+b''_{ik}\tilde{\Psi}_{b_i^{k-1}}^{s,t})\,.$$

While, $\operatorname{Jm}\int_{\partial V}\Phi_{s,t}^{p,q}\,dv=-2\pi\operatorname{Re}\int_t^s dv$, hence

$$\operatorname{Re}\int_t^s dv=P_n(s)-\frac{1}{2\pi}\operatorname{Jm}\int_{\partial\Omega_n}\Phi_{s,t}^{p,q}\,dv\,.$$

Thus, on going to the limit, we get the above mentioned result.

Corollary 1. *The necessary and sufficient condition for the existence of an additive function such that* 1) f *is a multiple of* $1/\delta'$ 2) *its* *singular part is* $\sum_{k=1}^{\nu_i} \frac{a'_{ik}+ia''_{ik}}{z^k}$ *at* a_i (a'_{ik}, a''_{i1}; *real*) 3) $\mathrm{Re}\, f$ *is single-valued and* $\mathrm{Jm}\int_{\gamma_n^i} df=0$ 4) $\sup_n \frac{\|df\|_{D_n}}{\min \nu_n^i} < +\infty$ *is the uniform convergence of*

$$M_n(s)=\mathrm{Re}\sum_{a_i\subset\Omega_n}\sum_{k=1}^{\nu_i}(a'_{ik}\Psi^{s,t}_{a_ik}+a''_{ik}\tilde{\Psi}^{s,t}_{a_ik})$$

on every compact subset in $R_0-\bigcup_i\{a_i\}$ *and* $\sup_n \frac{\|dM\|_{D_n}}{\min \nu_n^i} < +\infty$, *where* $M(s)=\lim_{n\to\infty} M_n(s)$. *If the conditions are filled, then* $\mathrm{Re} f$ *is expressed by* $M(s)+C$ (C; *real constant*).

Remark. 1) From (2) in §I it is seen that *f is single-valued if and only if*

$$\lim_{n\to\infty}\sum_{a_i\subset\Omega_n}\sum_{k=1}^{\nu_i}\frac{1}{(k-1)!}\left[a'_{ik}\mathrm{Re}\frac{d^k\Phi_{B_j}}{dp^k}(a_i)-a''_{ik}\mathrm{Jm}\frac{d^k\Phi_{B_j}}{dp^k}(a_i)\right]=0$$

$$(j=1, 2,\ldots).$$

2) If R satisfies $\inf_n \min_i \nu_n^i>0$, the statement of theorem 6 (resp. Corollary) is valid whenever 4) is replaced by $\|dv\|_{\cup D_n}<+\infty$ (resp. $\|df\|_{\cup D_n}<+\infty$).

Corollary 2. *Suppose that* $\inf_n \min_i \nu_n^i>0$, *then* $dw\ (\in\Gamma_{ase})$ *is uniquely expressed by*

$$dw=\lim_{n\to\infty}\sum_{A_j,B_j\subset\Omega_n}\left[\left(\mathrm{Re}\int_{B_j}dw\right)\varphi_{A_j}-\left(\mathrm{Re}\int_{A_j}dw\right)\varphi_{B_j}\right].$$

Proposition 7. *If f is a single-valued meromorphic function such f satisfies* 1), 2) *in corollary* 1 *and* $\sup_n \frac{C_n(f)^2}{\hat{\nu}_n}<+\infty$, *then*

$$\mathrm{Re} f(s)=\mathrm{Re} f(t)+\lim_{n\to\infty} M_n(s),$$

where $C_n(f)=\max_{p\in D_n} f(p)$.

In fact, from the proof of theorem 6 it follows that

$$\text{(5)} \qquad \mathrm{Re}\int_t^s df = M_n(s) - \frac{1}{2\pi}\,\mathrm{Jm}\int_{\partial\Omega_n} \Phi_{s,t}^{p,q} df\,.$$

By the same way as in the proof of (1) in lemma 3, with a slight modification, it is possible to show that

$$\text{(6)} \qquad \left|\int_{\partial\Omega_n} \Phi_{s,t}^{p,q}\, df\right|^2 \leqq 2\pi \frac{C_n(f)^2}{\hat{v}_n} \|d\Phi_{s,t}^{p,q}\|_{D_n}^2\,.$$

Thus, as n tends to ∞, we obtain the above mentioned result.

q.e.d.

Proposition 8. *Let R be an arbitrary open Riemann surface and $f(p)$ be a non-constant holomorphic function on R, then for any canonical exhaustion $\{\Omega_n\}$ the series $\sum_{n=1}^{\infty} \frac{\hat{v}_n}{C_n(f)^2}$ is always convergent, where $C_n(f) = \max_{p \in \bar{D}_n} |f(p)|$.*

Proof. If $\sum_{n=1}^{\infty} \frac{\hat{v}_n}{C_n(f)^2}$ is divergent, then from (6) it is seen that there is a canonical exhaustion $\{\Omega_{n'}\}(\subset\{\Omega_n\})$ such that

$$\lim_{n'\to\infty}\int_{\partial\Omega_{n'}} \Phi_{s,t}^{p,q} df = 0,$$

and so from (5) we have $\mathrm{Re}f(s) = \mathrm{Re}f(t)$, where t is fixed in R. Hence $f(s)$ becomes a constant. q.e.d.

4. Finally we shall briefly mention about a formulation of Riemann-Roch theorem in terms of canonical differentials and integrals. Let R be an arbitrary open Riemann surface. If the series

$$f = C + \lim_{n\to\infty} \sum_{a_i \subset \Omega_n} \sum_{k=D}^{\nu_i} (a'_{ik}\Psi_{a_ik} + a''_{ik}\tilde{\Psi}_{a_ik})$$

converges uniformly on every compact subset in $R_0 - \bigcup_i \{a_i\}$ and $\|df\|_{\cup D_n} < +\infty$, then we denote by $\hat{N}(1/\delta')$ the real vector space consisting of such functions, where a'_{ik}, a''_{ik} are real and C is complex. Clearly $\hat{N}(1/\delta')$ is non trivial. Let $V(1/\delta'')$ be the vector space of meromorphic differentials dv such that 1) dv is a multiple of $1/\delta''$ 2) $\int_{\gamma_n^i} dv = 0$

$(i=1,\dots, m(n),\ n=1, 2,\dots)$ 3) $\sup\dfrac{\|dv\|_{D_n}}{\min v_n^i}<+\infty$, From theorem 6 it follows that $dv\ (\in V(1/\delta''))$ is expressed by

$$dv=\lim_{n\to\infty}\Bigg[\sum_{i=2}^{m(n)}(b'_{i1}\varphi_{b_1,b_1}+b''_{i1}\tilde{\varphi}_{b_ib_1})-\sum_{i=1}^{m(n)}\sum_{k=2}^{\mu_i}\frac{1}{k-1}(b'_{ik}\psi_{b_i^{k-1}}+b''_{ik}\tilde{\psi}_{b_i^{k-1}})$$

$$+\sum_{A_j,B_j\subset\Omega_n}\left(\left(\mathrm{Re}\int_{B_j}dv\right)\varphi_{A_j}-\left(\mathrm{Re}\int_{A_j}dv\right)\varphi_{B_j}\right)\Bigg],$$

where the right hand side is uniformly convergent on every compact subset in $R_0-\bigcup_i\{b_i\}$ and b'_{ik}, b''_{ik} are real. Here we suppose that $V(1/\delta'')\neq\{0\}$. If R satisfies $\inf_n\min_i v_n^i>0$ it is obvious that $V(1/\delta'')\neq\{0\}$. Then by the same way as in the proof of lemma 5 it is seen that for $f\in\hat{N}(1/\delta')$ and $dv\in V(1/\delta'')$

$$\lim_{n\to\infty}\Bigg\{\mathrm{Re}\Big(\sum_{a_i\subset\Omega_n}\mathrm{Res}_{a_i}fdv+\sum_{b_i\subset\Omega_n}\mathrm{Res}_{b_i}fdv\Big)$$

$$+\frac{1}{2\pi}\sum_{A_j,B_j\subset\Omega_n}\left[\left(\mathrm{Re}\int_{B_j}dv\right)\left(\mathrm{Jm}\int_{A_j}df\right)-\left(\mathrm{Re}\int_{A_j}dv\right)\left(\mathrm{Jm}\int_{B_j}df\right)\right]\Bigg\}=0$$

and so $\lim_{n\to\infty}\mathrm{Re}\sum_{a_i\subset\Omega_n}\mathrm{Res}_{a_i}fdv=0$ is equivalent to

$$\lim_{n\to\infty}\Bigg\{\sum_{A_j,B_j\subset\Omega_n}\left[\left(\mathrm{Re}\int_{B_j}dv\right)\left(\mathrm{Jm}\int_{A_j}df\right)-\left(\mathrm{Re}\int_{A_j}dv\right)\left(\mathrm{Jm}\int_{B_j}df\right)\right]$$

$$+2\pi\,\mathrm{Re}\Big(\sum_{b_i\subset\Omega_n}\mathrm{Res}_{b_i}fdv\Big)\Bigg\}=0\,.$$

While, it is seen that

$$\mathrm{Re}\sum_{a_i\subset\Omega_n}\mathrm{Res}_{a_i}fdv=\sum_{i=1}^{l(n)}\sum_{k=1}^{\nu_i}\frac{1}{(k-1)!}$$

$$[a'_{ik}(\mathrm{Re}\,v^{(k)}(a_i)-a''_{ik}(\mathrm{Jm}\,v^{(k)}(a_i))]$$

$$\mathrm{Re}\sum_{b_i\subset\Omega_n}\mathrm{Res}_{b_i}fdw=\sum_{i=1}^{m(n)}\sum_{k=1}^{\mu_i}\frac{1}{(k-1)!}$$

$$[b'_{ik}(\mathrm{Re}\,f^{(k-1)}(b_i)-b''_{ik}(\mathrm{Jm}\,f^{(k-1)}(b_i))]\,.$$

Now we put

$$\ll f, dv \gg = \lim_{n\to\infty} \operatorname{Re} \sum_{a_i \subset \Omega_n} \operatorname{Res}_{a_i} f dv$$

and define four vector spaces in the real number field.

$\hat{N}_0(1/\delta'') = \{f \in \hat{N}(1/\delta') | \ll f, dv \gg$ is convergent for all $dv \in V(1/\delta'')\}$

$V_0(1/\delta'') = \{dv \in V(1/\delta'') | \ll f, dv \gg$ is convergent for all $f \in \hat{N}(1/\delta')\}$

$N_0(1/\delta) = \{f \in \hat{N}_0(1/\delta') | f$ is single-valued and is a multiple of $\delta''\}$

$V_0(\delta) = \{dv \in V_0(1/\delta'') | dv$ is a multiple of $\delta'\}$.

We make use of $\ll f, dv \gg$ as a bilinear functional on $\hat{N}_0(1/\delta') \times V_0(1/\delta'')$ and follow up the same process as in the proof of theorem 5, then we obtain

$$\dim_R(V_0(1/\delta'')/V_0(\delta)) = \dim_R(\hat{N}_0(1/\delta')/N_0(1/\delta))$$

where $\dim_R$ indicates real dimension.

Similarly, if we replace the norm condition $\|df\|_{\cup D_n} < +\infty$ in the definition of $\hat{N}(1/\delta')$ by $\sup_n \frac{\|df\|_{D_n}}{\min_i v_n^i} < +\infty$ and that of $V(1/\delta'')$ by $\|dv\|_{\cup D_n} < +\infty$ and start from them, then we have the corresponding result,

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