J. Math. Kyoto Univ. (JMKYAZ)
16-1 (1976) 113-115

Orientation reversing involution

By

Katsuo KAWAKUBO

(Communicated by Prof. Toda, Dec. 4, 1974)

§ 1. Introduction

In this note we discuss the following problem from the bordism point of view. "Which manifold admits an orientation reversing involution?"

Let Ω_* be the oriented cobordism group. Then we shall have

Theorem. *An element x of Ω_* has a representative which admits an orientation reversing involution if and only if x is a torsion element of Ω_*.*

The author wishes to thank Professor A. Hattori for suggesting him the problem above.

§ 2. Proof of the theorem

If an oriented closed manifold M^n admits an orientation reversing diffeomorphism, then $2[M^n]=0$ in Ω_n, that is $[M^n]$ is a torsion element of Ω_n. Therefore we have only to prove that each torsion element of Ω_n has a representative which admits an orientation reversing involution.

Let us first recall from Wall [4] that the algebra W_* of unoriented cobordism classes represented by a manifold with the first Stiefel-Whitney class w_1 reduced integral is a polynomial algebra over Z_2 on classes X_{2k-1}, X_{2k}, $X_{2^j}^2$ with k not a power of 2. There is a homomorphism ∂: $W_{n+1}\rightarrow\Omega_n$ obtained by sending the class of M into the class of the submanifold N of M dual to w_1, and Wall has shown that image ∂ is precisely the set of torsion elements of Ω_*.

Using Wall's representative manifolds, one has that the classes X_{2k-1} and $X_{2^j}^2$ contain orientable manifolds. It follows that $\partial X_{2^j}^2=0$ and $\partial X_{2k-1}=0$. Since ∂ is a derivation, for any monomial

$$Y=\prod_\alpha X_{2^{j_\alpha}}^2\cdot\prod_\beta X_{2k_\beta-1}\cdot\prod_\gamma X_{2k_\gamma},$$

one has that

$$\partial X=\prod_\alpha X_{2^{j_\alpha}}^2\cdot\prod_\beta X_{2k_\beta-1}\cdot\partial\{\prod_\gamma X_{2k_\gamma}\}.$$

Thus, it suffices to show that each class $\partial\{X_{2k_1}\cdots X_{2k_r}\}$ contains a representative manifold on which Z_2 acts in the desired fashion.

First consider the case $r=1$. Then $\partial X_{2k}=X_{2k-1}$ is representable by a Dold manifold $P(2m+1, 2n)$ (see [2]). The Dold manifold $P(2m+1, 2n)$ is obtained from $S^{2m+1}\times \boldsymbol{CP}(2n)$ by identifying the points

$$(z_0, \cdots, z_m, \eta_0, \cdots, \eta_{2n}) \quad\text{and}\quad (-z_0, \cdots, -z_m, \bar\eta_0, \cdots, \bar\eta_{2n})$$

where $z_i, \eta_j\in \boldsymbol{C}$. Define a map

$$T\colon S^{2m+1}\times \boldsymbol{CP}(2n)\to S^{2m+1}\times \boldsymbol{CP}(2n)$$

by

$$T(z_0, \cdots, z_m, \eta_0, \cdots, \eta_{2n})=(\bar z_0, z_1, \cdots, z_m, \eta_0, \cdots, \eta_{2n}).$$

This preserves identifications to give an involution

$$T\colon P(2m+1, 2n)\to P(2m+1, 2n).$$

Consider a neighborhood of a fixed point $(0, z_1, \cdots, z_m, \eta_0, \cdots, \eta_{2n})$ of T. Then T reverses the orientation locally. Since T is a diffeomorphism, we can easily deduce that T reverses the orientation globally.

Next we consider the case $r>1$. Recall that a representative for X_{2k} is obtained as follows: $P(2m+1, n)$ has an involution

$$\tau\colon P(2m+1, n)\to P(2m+1, n)$$

induced by the involution

$$(z_0, \cdots, z_m, \eta_0, \cdots, \eta_n)\to(z_0, \cdots, z_{m-1}, \bar z_m, \eta_0, \cdots, \eta_n).$$

Then let $Q(2m+1, n)$ be formed from $S^1\times P(2m+1, n)$ by identifying (t, u) and $(-t, \tau(u))$. As noted by Wall [4], $Q(2m+1, n)$ represents X_{2k} (for properly chosen m and n).

Let $N_i=Q(2m_i+1, n_i)$ represent X_{2k_i}, $i=1, \cdots, r$. Define an involution T_1 on N_1 by setting $T_1(t, z_0, \cdots, z_{m_1}, \eta_1, \cdots, \eta_{n_1})=(t, \bar{z}_0, z_1, \cdots, z_{m_1}, \eta_1, \cdots, \eta_{n_1})$. It is easy to see that T_1 is well-defined. Remark that even if $m_1=0$, the map T_1 is well-defined. Let π_i: $N_i \to S^1$ be the projection induced by $(t, u) \to t^2$. Then π_i is a fibration and realizes $w_1(N_i)$. Now let P: $N_1 \times \cdots \times N_r \to S^1$ be the map defined by $P(v_1, \cdots, v_r) = \pi_1(v_1) \cdots \pi_r(v_r)$. This is the composition of the bundle maps

$$\pi_1 \times \cdots \times \pi_r \colon N_1 \times \cdots \times N_r \to S^1 \times \cdots \times S^1$$

and

$$\mu \colon S^1 \times \cdots \times S^1 \to S^1$$

where μ is the multiplication. Thus P is transverse regular on $1 \in S^1$. In addition, P realizes w_1, so

$$V=\{(v_1, \cdots, v_r) \in N_1 \times \cdots \times N_r | \pi_1(v_1) \cdots \pi_r(v_r)=1\}$$

represents $\partial(X_{2k_1} \cdots X_{2k_r})$. This construction of V is due to Anderson [1] and Stong [3]. Obviously $T_1 \times id \times \cdots \times id$ induces an involution T on V.

Consider a neighborhood of a fixed point $(t(1), 0, \cdots, 0, 1, \eta_1(1), \cdots, \eta_{n_1}(1)) \times \cdots \times (t(r), z_1(r), \cdots, z_{m_r}(r), \eta_1(r), \eta_1(r), \cdots, \eta_{n_r}(r))$ of T where $t^2(1) \cdots t^2(r)=1$. Then T reverses the orientation locally.

Since T is a diffeomorphism, we can easily deduce that T reverses the orientation globally. Thus $\partial(X_{2k_1} \cdots X_{2k_r})$ is represented by V on which Z_2 acts in the desired manner. This completes the proof of the theorem.

DEPARTMENT OF MATHEMATICS
OSAKA UNIVERSITY

References

[1] Anderson, P. G., *Cobordism classes of squares of orientable manifolds*, Ann. of Math. **83** (1966), 47–53.

[2] Dold, A., *Vollständigkeit der Wu-schen Relationen zwischen den Stiefel-Whitneyschen Zahlen differenzierbarer Mannigfaltigkeiten*, Math. Z. **65** (1956), 200–206.

[3] Stong, R. E., *Stationary point free group actions*, Proc. A. M. S. **18** (1967), 1089–1092.

[4] Wall, C. T. C., *Determination of the cobordism ring*, Ann. of Math., **72** (1960), 292–311.