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A relative theory of Finsler spaces

Dedicated to Professor Jōyō Kanitani on the occasion of his eighty-eighth birthday

By

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We consider a Finsler space F^n equipped with a fundamental function $L(x, y)$. Let $g(x, y)$ be the determinant consisting of components $g_{ij}(x, y)$ of the fundamental tensor of F^n. We sometimes have experience giving us to understand some importance of the scalar $*L=Lg^{w/2}$ as it will be reported in §2. Thus it seems to the present author that a theory of Finsler spaces based on this scalar $*L(x, y)$ may come in useful. The main purpose of the present paper is to construct metrical Finsler connections from $*L(x, y)$.

§1. Relative fundamental functions

We shall deal with differential invariants, called relative tensor [9], in Finsler geometry also. At a supporting element (x, y) of a Finsler space F^n, a collection $T=(T^{i\cdots}_{j\cdots})$ which is given with respect to every coordinate system (x^i) is called a *relative tensor of weight* w, if the transformation law

$$\overline{T}^{a\cdots}_{b\cdots}=J^{-w}T^{i\cdots}_{j\cdots}\overline{X}^a_i\cdots\underline{X}^j_b\cdots \tag{1.1}$$

is satisfied under coordinate change $(x^i)\to(\bar{x}^a)$, where $J=\det(\overline{X}^a_i)$, $\overline{X}^a_i=\partial\bar{x}^a/\partial x^i$ and $\underline{X}^j_b=\partial x^j/\partial\bar{x}^b$. It is remarked that we also have $\bar{y}^a=\overline{X}^a_i y^i$, so $\overline{X}^a_i=\partial\bar{y}^a/\partial y^i$ and $\underline{X}^j_b=\partial y^j/\partial\bar{y}^b$.

Let $L(x, y)$ be the fundamental function of a Finsler space F^n and $g_{ij}(x, y)$ be components of the fundamental tensor of F^n. The fundamental tensor is an absolute tensor defined by $g_{ij}=(\dot{\partial}_i\dot{\partial}_j L^2)/2$; it obeys the transformation law $\bar{g}_{ab}=g_{ij}\underline{X}^i_a\underline{X}^j_b$. Thus $g=\det(g_{ij})$ satisfies $\bar{g}=J^{-2}g$, so that g is a relative scalar of weight two.

Throughout the present paper we shall assume g be positive. If we have to be concerned with a domain where g is negative, we are to treat $-g$ instead of g itself.

Definition 1. Consider a Finsler space F^n with a fundamental function $L(x, y)$ and let $g(x, y)$ be the determinant consisting of components $g_{ij}(x, y)$ of the fundamental tensor of F^n. For a real number w, a scalar $*L=Lg^{w/2}$ is called the *relative* *fundamental function of weight* w of F^n.

Corresponding to the ordinary (absolute) theory of Finsler spaces, we shall introduce following quantities with asterisk:

$$*l_i=\dot{\partial}_i{}^*L,\quad {}^*F=({}^*L)^2/2,\quad {}^*y_i=\dot{\partial}_i{}^*F\quad (={}^*L{}^*l_i),$$
$$*g_{ij}=\dot{\partial}_i\dot{\partial}_j{}^*F,\quad {}^*C_{ijk}=(\dot{\partial}_k{}^*g_{ij})/2.$$

By the well-known equation $C_i\ (=C^r_{ri})=\dot{\partial}_i g/2g$, we get ${}^*l_i=g^{w/2}(l_i+wLC_i)$. But we shall denote the ordinary C-tensor C_{ijk} by g_{ijk}, that is, $g_{ijk}=(\dot{\partial}_k g_{ij})/2$, to avoid the confusion. Thus g_i stands for $g^{jk}g_{ijk}$. In these notations we have

$$*l_i=g^{w/2}(l_i+wLg_i),$$
$$*g_{ij}=g^w\{g_{ij}+2w(g_iy_j+g_jy_i)+4w^2Fg_ig_j+2wF\dot{\partial}_jg_i\},$$
$$*C_{ijk}=g^w[g_{ijk}+4w^3Fg_ig_jg_k+wF\dot{\partial}_k\dot{\partial}_jg_i+wS_{(ijk)}\{(g_{ij}+2wF\dot{\partial}_jg_i)g_k$$
$$+(2wg_ig_j+\dot{\partial}_jg_i)y_k\}],$$

where $S_{(ijk)}$ denotes cyclic permutation of i, j, k and summation. These show somewhat complicated character of ${}^*g_{ij}$ and ${}^*C_{ijk}$ in comparison with g_{ij} and g_{ijk} [6].

The homogeneity property of ${}^*L(x, y)$ in y^i shows

$$*y_i={}^*g_{ij}y^j,\quad ({}^*L)^2={}^*g_{ij}y^iy^j,\quad {}^*C_{ijk}y^k=0.$$

From (1.1) it follows easily that $\dot{\partial}_kT^{i\cdots}_{j\cdots}$ are components of a relative tensor of the same weight with $T^{i\cdots}_{j\cdots}$. Thus ${}^*l_i$ is a relative covariant vector of weight w, and ${}^*y_i$, ${}^*g_{ij}$ and ${}^*C_{ijk}$ are relative tensors of weight $2w$. The relative tensor ${}^*g_{ij}$ is called the relative fundamental tensor of weight $2w$. It is remarked that ${}^*C^j_{ik}={}^*g^{jr}{}^*C_{irk}$ is an absolute tensor, because the reciprocal ${}^*g^{ij}$ of ${}^*g_{ij}$ is of weight $-2w$, and we have an absolute vector ${}^*C_k\ (={}^*C^r_{rk})=\dot{\partial}_k{}^*g/2{}^*g$.

§2. Relatively Riemannian spaces and relative isometries

A Finsler space with a fundamental function $L(x, y)$ is Riemannian, if and only if L^2 is a quadratic form of y^i. We shall generalize this character of Riemannian space from the standpoint of relative theory as follows:

Definition 2. A Finsler space is called *relatively Riemannian of weight* w, if $({}^*L)^2=L^2g^w$ is a quadratic form of y^i.

Therefore a Riemannian space is, of course, relatively Riemannian of arbitrary weight. On the other hand, a Finsler space is relatively Riemannian, iff components ${}^*g_{ij}$ of the relative fundamental tensor are functions of position x only, or the ${}^*C$-tensor ${}^*C_{ijk}$ vanishes identically.

Example 1. In a previous paper [6] the present author introduced a concept of the B^p-condition: A Finsler space is said to satisfy the B^p-condition, if $L^2g^{-1/p}$

is a quadratic form of y^i. That is, a Finsler space is relatively Riemannian of weight $-1/p$, iff the B^p-condition is satisfied. Consequently it is known [6] that if a two-dimensional Finsler space satisfies the *T-condition* ($T_{hijk}=0$), namely, *its main scalar I is a function of position x alone, then the space is relatively Riemannian of weight* $-1/2$. According to Berwald's theory [1], any non-Minkowski Berwald space of dimension two is relatively Riemannian of weight $-1/2$. Therefore we have examples of relatively Riemannian spaces which are not Riemannian.

Example 2. It was shown by the present author [6] that if a non-Riemannian Finsler space of three dimensions satisfying the *T-condition* is relatively Riemannian, its weight must be equal to $-1/(2I^2+1)$ or $-I^2/(2I^2+1)$, where the constant I is one of the main scalars (H, I, J).

Definition 3. If a change $L\to\bar{L}$ of the fundamental function L of a Finsler space (F^n, L) satisfies $^*\bar{L}=\tau{}^*L$ (i.e., $\bar{L}\bar{g}^{w/2}=\tau L g^{w/2}$) for some real number w and a scalar $\tau(x)$, this change is called *relatively conformal of weight* w and $\tau(x)$ is the conformal factor. In case of $\tau=1$, the change is called *relatively isometric.*

It is obvious from the above definition that if a change $L\to\bar{L}$ is relatively conformal of weight w and $\bar{L}$ is a Riemannian metric, the Finsler space with the metric L is relatively Riemannian of weight w.

We shall restrict our consideration to a Riemannian case of relatively conformal change $L\to\bar{L}$. That is, assume that both L and $\bar{L}$ are Riemannian. Differentiating $\bar{L}^2\bar{g}^w=\tau^2L^2g^w$ by y^i twice, we immediately get $\bar{g}_{ij}(x)\bar{g}^w=\tau^2 g_{ij}(x)g^w$, so that this change must be conformal in an ordinary sense. Conversely any conformal change is clearly a relatively conformal change of arbitrary weight, and in particular it is relatively isometric of weight $-1/n$ where n is the dimension. Therefore the concept of relatively conformal change becomes trivial in Riemannian case.

For a Finsler space F^n *a conformal change* $L\to\bar{L}=\tau(x)L$ *is a relative isometry of weight* $-1/n$. We, however, have relatively conformal changes which are not conformal, as the following examples show:

Example 3. We are concerned with a change $L\to\bar{L}$ of the metric of an n-dimensional Finsler space F^n where $\bar{L}=\bar{L}\ (L, \beta)$ is a positively homogeneous function of degree one of two variables L and $\beta=b_i(x)y^i$ [5]. Putting $\bar{L}_1=\partial\bar{L}/\partial L$, $\bar{L}_2=\partial\bar{L}/\partial\beta$, $\bar{L}_{11}=\partial\bar{L}_1/\partial L$, $\bar{L}_{12}=\partial\bar{L}_1/\partial\beta$, $\bar{L}_{22}=\partial\bar{L}_2/\partial\beta$ and

$$p=\bar{L}\bar{L}_1/L,\quad p_0=(\bar{L}_2)^2+\bar{L}\bar{L}_2,\quad p_1=(\bar{L}_1\bar{L}_2+\bar{L}\bar{L}_{12})/L,$$

$$p_2=\{(\bar{L}_1)^2+\bar{L}\bar{L}_{11}-\bar{L}\bar{L}_1/L\}/L^2,\quad b^2=g^{ij}b_ib_j,$$

$$U=p^2+pp_1\beta-\{p_0p_2-(p_1)^2\}\beta^2,$$

$$V=pp_0+\{p_0p_2-(p_1)^2\}L^2,$$

the determinant $\bar{g}$ is written as

$$\bar{g}=p^{n-2}(U+Vb^2)g. \tag{2.1}$$

The homogeneity property of $\bar{L}$ yields

$$p_0\beta + p_1 L^2 = \bar{L}\bar{L}_2, \quad p_1\beta + p_2 L^2 = 0. \tag{2.2}$$

We shall be concerned with special cases:

(I) It follows from (2.2) that $p_0 p_2 - (p_1)^2 = \bar{L}\bar{L}_2 p_2/\beta$, and we are easily led to

$$U = \bar{L}^3(\bar{L}_1/L - \bar{L}_{11})/L^2. \tag{2.3}$$

Therefore, a condition $U=0$ is expressed by the differential equation $\bar{L}_1/L = \bar{L}_{11}$. Integrating this equation, we easily get a Finsler metric $\bar{L}$ of Kropina type, which may be essentially written as $\bar{L} = L^2/\beta$. Then (2.1) becomes $\{\bar{L}^{2(n+1)}\bar{g}^{-1}\}2^{n-1}b^2 = L^{2(n+1)}g^{-1}$. Consequently this change $L \to \bar{L} = L^2/\beta$, which may be called a *Kropina change, is relatively conformal of weight* $-1/(n+1)$; in particular it is relatively isometric in case of $b^2 = 2^{1-n}$.

(II) On the other hand, (2.2) yields $p_0 p_2 - (p_1)^2 = \bar{L}\bar{L}_2(p_0\beta - \bar{L}\bar{L}_2)/L^4$, so that we get

$$V = \bar{L}^3\bar{L}_{22}/L^2. \tag{2.4}$$

Therefore a condition $V=0$ is equivalent to $\bar{L}_{22}=0$, which immediately leads us to a Finsler metric $\bar{L}$ of Randers type, which may be essentially written as $\bar{L} = L + \beta$. Then (2.1) becomes $\bar{L}^{n+1}\bar{g}^{-1} = L^{n+1}g^{-1}$. Therefore *any Randers change* $L \to \bar{L} = L + \beta$ *is relatively isometric of weight* $-2/(n+1)$.

As a consequence of Example 3, if we restrict ourselves to Riemannian L, we have

Theorem 1. *Let* $(M^n, L(x, dx))$ *be an n-dimensional Riemannian space with a metric* $L = \sqrt{g_{ij}(x)dx^i dx^j}$ *and let* $\beta = b_i(x)dx^i$ *be a differential one-form on* M^n. *Then the Kropina metric* L^2/β *on* M^n *and the Randers metric* $L+\beta$ *on* M^n *are relatively Riemannian of weight* $-1/(n+1)$ *and* $-2/(n+1)$ *respectively.*

§3. Covariant differentiations of relative tensors

We are concerned with a Finsler space F^n and a Finsler connection $F\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ on F^n [3]. The h-covariant derivative $g_{ij|k}$ of the fundamental tensor $g_{ij}(x, y)$ with respect to $F\Gamma$ is defined by

$$g_{ij|k} = \partial_k g_{ij} - 2g_{ijr}N^r_k - g_{rj}F^r_{ik} - g_{ir}F^r_{jk},$$

where g_{ijr} stands for $(\dot{\partial}_r g_{ij})/2$ as mentioned in §1. Therefore, as to $g = \det(g_{ij})$, it is seen that

$$\partial_i g (= g g^{rs}\partial_i g_{rs}) = g g^{rs} g_{rs|i} + 2g g_r N^r_i + 2g F_i, \tag{3.1}$$

where we put $g_r = g^{ij}g_{ijr}$ and $F_i = F^r_{ri}$. Since $\dot{\partial}_i g = 2g g_i$ is obvious, (3.1) is written $g_{|i} = g g^{rs} g_{rs|i}$, where we put

$$g_{|i} = \partial_i g - (\dot{\partial}_r g)N^r_i - 2g F_i, \tag{3.2}$$

which is a relative covariant vector of weight two, because $gg^{rs}g_{rs|i}$ is such a vector. The h-covariant derivative of g is thus defined as $g_{|i}$.

We shall deal with an arbitrary relative tensor $T^{i\cdots}_{j\cdots}$ of weight w. Introducing an absolute tensor $\overset{(a)}{T}{}^{i\cdots}_{j\cdots}=g^{-w/2}T^{i\cdots}_{j\cdots}$, we make its h-covariant derivative:

$$\begin{aligned}\overset{(a)}{T}{}^{i\cdots}_{j\cdots|k} &= \partial_k(g^{-w/2}T^{i\cdots}_{j\cdots})-\{\dot{\partial}_r(g^{-w/2}T^{i\cdots}_{j\cdots})\}N^r_k \\ &\quad +g^{-w/2}T^{r\cdots}_{j\cdots}F^i_{rk}+\cdots-g^{-w/2}T^{i\cdots}_{r\cdots}F^r_{jk}-\cdots \\ &= g^{-w/2}\{\partial_k T^{i\cdots}_{j\cdots}-(\dot{\partial}_r T^{i\cdots}_{j\cdots})N^r_k+T^{r\cdots}_{j\cdots}F^i_{rk}+\cdots \\ &\quad -T^{i\cdots}_{r\cdots}F^r_{jk}-\cdots\}-(w/2)g^{-w/2-1}\{\partial_k g-(\dot{\partial}_r g)N^r_k\}T^{i\cdots}_{j\cdots}.\end{aligned}$$

Thus, paying attention to (3.2), we observe that the quantities

$$\begin{aligned}T^{i\cdots}_{j\cdots|k} &= \partial_k T^{i\cdots}_{j\cdots}-(\dot{\partial}_r T^{i\cdots}_{j\cdots})N^r_k+T^{r\cdots}_{j\cdots}F^i_{rk}+\cdots \\ &\quad -T^{i\cdots}_{r\cdots}F^r_{jk}-\cdots-wT^{i\cdots}_{j\cdots}F_k\end{aligned}\tag{3.3}$$

are written as

$$T^{i\cdots}_{j\cdots|k}=g^{w/2}\overset{(a)}{T}{}^{i\cdots}_{j\cdots|k}+(w/2)(g_{|k}/g)T^{i\cdots}_{j\cdots},$$

which shows that $T^{i\cdots}_{\cdots|k}$ defines a relative tensor of weight w. This relative tensor defines the h-covariant derivative of $T^{i\cdots}_{j\cdots}$. It is noted that *the additional last term in* (3.3) *has the coefficient* $-w$ *and contains the contracted connection coefficients* F_k.

Next we consider the v-covariant differentiation of relative tensors. It has been remarked that the partial derivatives $\dot{\partial}_k T^{i\cdots}_{j\cdots}$ of a relative tensor $T^{i\cdots}_{j\cdots}$ directly yields a relative tensor. Therefore the ordinary procedure constructing v-covariant derivative, that is,

$$\dot{\partial}_k T^{i\cdots}_{j\cdots}+T^{r\cdots}_{j\cdots}C^i_{rk}+\cdots-T^{i\cdots}_{r\cdots}C^r_{jk}-\cdots$$

yields a relative tensor also. It has, however, remarked that (1.1) may be regarded as the transformation law under the change $(y^i)\to(\bar{y}^a)$. Further a procedure similar to the above leads to the facts that

$$g|_i=\dot{\partial}_i g-2gC_i, \qquad (C_i=C^r_{ri}),\tag{3.4}$$

is equal to $gg^{rs}g_{rs|i}$, and

$$T^{i\cdots}_{j\cdots}|_k=\dot{\partial}_k T^{i\cdots}_{j\cdots}+T^{r\cdots}_{j\cdots}C^i_{rk}+\cdots-T^{i\cdots}_{r\cdots}C^r_{jk}-\cdots-wT^{i\cdots}_{j\cdots}C_k\tag{3.5}$$

is written as

$$T^{i\cdots}_{j\cdots}|_k=g^{w/2}\overset{(a)}{T}{}^{i\cdots}_{j\cdots}|_k+(w/2)(g|_k/g)T^{i\cdots}_{j\cdots}.$$

In the following we shall regard (3.5) as defining the v-covariant differentiation of relative tensor.

§4. Relative metrical connections

The well-known metrical connection, called the Cartan connection, is uniquely defined by the system of five axioms from the absolute fundamental tensor $g_{ij}(x, y)$ [3]. We shall apply a similar system of axioms to determine a Finsler connection which is metrical with respect to a relative fundamental tensor $*g_{ij}(x, y)$.

Definition 4. Let $*L=Lg^{w/2}$ be a relative fundamental function of an n-dimensional Finsler space $(F^n, L(x, y))$ and suppose that F^n be regular with respect to $*L$, namely, $*g=\det(*g_{ij})$ of the relative fundamental tensor $*g_{ij}(x, y)$ of weight w do not vanish. If we have a Finsler connection $F\Gamma=(F^i_{jk}, N^i_j, C^i_{jk})$ satisfying the following five axioms, this connection is called the *relative Cartan connection of weight* w and denoted by $*C\Gamma$:

(I) $*g_{ij|k}=0$,
(II) $(h)h$-torsion tensor $T^i_{jk}=F^i_{jk}-F^i_{kj}=0$,
(III) deflection tensor $D^i_j=y^rF^i_{rj}-N^i_j=0$,
(IV) $*g_{ij}|_k=0$,
(V) $(v)v$-torsion tensor $S^i_{jk}=C^i_{jk}-C^i_{kj}=0$.

We shall consider how to construct $*C\Gamma$. For the later use we introduce relative Christoffel symbols with respect to x^i:

$$*\gamma_{ijk}=(\partial_k *g_{ij}+\partial_i *g_{jk}-\partial_j *g_{ki})/2 \tag{4.1}$$

and $*\gamma^j_{ik}=*g^{jr}*\gamma_{irk}$. Then we have

$$*\gamma_k\ (=*\gamma^r_{rk})=\partial_k *g/2*g. \tag{4.2}$$

Now the first axiom (I) is written

$$\partial_k *g_{ij}-2*C_{ijr}N^r_k-F_{ijk}-F_{jik}-2w*g_{ij}F_k=0, \tag{4.3}$$

where $F_{ijk}=*g_{jr}F^r_{ik}$ and $F_k=F^r_{rk}$. Contracting (4.3) by $*g^{ij}$ and paying attention to (4.2), we get

$$F_k=(*\gamma_k-N_k)/(nw+1), \tag{4.4}$$

where $N_k=*C_rN^r_k$ and we assume $nw+1\neq 0$.

Next we make $*\gamma_{ijk}$ paying attention to $\partial_k *g_{ij}$ in (4.3). On account of (II) we then get

$$\begin{aligned}F_{ijk}=*\gamma_{ijk}-*C_{ijr}N^r_k-*C_{jkr}N^r_i+*C_{kir}N^r_j\\ -w(*g_{ij}F_k+*g_{jk}F_i-*g_{ki}F_j).\end{aligned}$$

Substituting from (4.4), the above is written in the form

$$F_{ijk}=E_{ijk}-H_{ijr}N^r_k-H_{jkr}N^r_i+H_{kir}N^r_j, \tag{4.5}$$

where we put

$$E_{ijk} = {}^*\gamma_{ijk} - w({}^*g_{ij}{}^*\gamma_k + {}^*g_{jk}{}^*\gamma_i - {}^*g_{ki}{}^*\gamma_j)/(nw+1),$$

$$H_{ijk} = {}^*C_{ijk} - w{}^*g_{ij}{}^*C_k/(nw+1).$$

It is remarked that these quantities E_{ijk} and H_{ijk} are known, namely, these are directly obtained from ${}^*g_{ij}$.

Further we contract (4.5) by y^i and pay attention to (III). Then we get

$$N_{jk}\ (= {}^*g_{jr}N^r_k) = E_{0jk} - H_{jkr}N^r_0 + w({}^*y_jN_k - {}^*y_kN_j)/(nw+1). \tag{4.6}$$

The equations (4.6) constitute a system of linear equations with unkonwn N^i_j. Assume that we have got definite N^i_j by solving (4.6). Then F^j_{ik} are determined by (4.5), from which we observe

$$F_{ijk} + F_{jik} = \partial_k{}^*g_{ij} - 2{}^*C_{ijr}N^r_k - 2w{}^*g_{ij}({}^*\gamma_k - N_k)/(nw+1).$$

Contracting this by ${}^*g^{ij}$, we get (4.4), so that the above reduces to (4.3), i.e., (I). The other axioms (II) and (III) directly follow from (4.5) and (4.6).

Consequently only (4.6) remains to be solved in order to find (F^i_{jk}, N^i_j). To do so, we shall follow Schouten-Haantjes' method [8]. First, contracting (4.6) by y^k, we get

$$N_{j0} (= {}^*g_{jr}N^r_0) = E_{0j0} + 2w({}^*y_jy^r - {}^*F\delta^r_j)N_r/(nw+1). \tag{4.7}$$

Secondly we contract (4.6) by ${}^*C^j\ (= {}^*g^{jr}C_r)$, we get

$$N_k (= {}^*C_rN^r_k) = {}^*C^j\{E_{0jk} - H_{jkr}N^r_0 - w{}^*y_kN_j/(nw+1)\}. \tag{4.8}$$

Substituting from (4.7) for N^r_0 appearing in the right-hand side of (4.8), the resulting equation is written

$$\Delta^i_kN_i = (E_{0ik} - H_{ikr}E_{0s0}{}^*g^{rs}){}^*C^i, \tag{4.9}$$

where we put

$$\Delta^i_k = \delta^i_k + w{}^*C^r(\delta^i_r{}^*y_k - 2{}^*FH_{rks}{}^*g^{si})/(nw+1). \tag{4.10}$$

Therefore, if $\Delta = \det(\Delta^i_k)$ does not vanish, (4.9) gives N_i uniquely. Then (4.7) determines N_{j0} and (4.6) does N^i_j.

We now turn to finding C^i_{jk} by the axioms (IV) and (V). (IV) is written

$$\dot\partial_k{}^*g_{ij} - C_{ijk} - C_{jik} - 2w{}^*g_{ij}C_k = 0, \tag{4.11}$$

where $C_{ijk} = {}^*g_{jr}C^r_{ik}$. The Christoffel symbols $(\dot\partial_k{}^*g_{ij} + \dot\partial_i{}^*g_{jk} - \dot\partial_j{}^*g_{ki})/2$ with respect to y^i are nothing but ${}^*C_{ijk}$. Thus (4.11) gives

$$C_{ijk} = {}^*C_{ijk} - w({}^*g_{ij}C_k + {}^*g_{jk}C_i - {}^*g_{ki}C_j). \tag{4.12}$$

Contracting (4.11) by ${}^*g^{ij}$, we get $\dot\partial_k{}^*g/{}^*g = 2(nw+1)C_k$, namely,

$$C_k = {}^*C_k/(nw+1). \tag{4.13}$$

Substituting from (4.13) into (4.12), we have

(4.14) $$C_{ijk} = {}^*C_{ijk} - w({}^*g_{ij}{}^*C_k + {}^*g_{jk}{}^*C_i - {}^*g_{ki}{}^*C_j)/(nw+1).$$

Then (4.13) is only a consequence of (4.14).

Summarizing all the above, we have

Theorem 2. *We consider an n-dimensional Finsler space F^n which is regular with respect to $^*L$ of weight w. If $nw+1 \neq 0$ and the* $\det(\Delta^i_k)$ (*cf.* (4.10)) *does not vanish, we uniquely get the relative Cartan connection $^*C\Gamma$ of weight w.*

Remark 1. As it was above seen, the equality $nw+1=0$ is an obstacle to determining the relative Cartan connection. It is noteworthy that the weight of $^*g = \det({}^*g_{ij})$ is $2(nw+1)$, so that in this inconvenient case $^*g$ is an absolute scalar.

Remark 2. If a regular Finsler space is relatively Riemannian, Δ^i_k reduce to δ^i_k and the connection $^*C\Gamma$ is uniquely determined as $F_{ijk} = E_{ijk}$ and $C_{ijk}=0$. It is observed in (4.10) that even a weaker condition $^*C_i = 0$ leads to $\Delta^i_k = \delta^i_k$, but it is conjectured that $^*C_i = 0$ may give rise to $^*C_{ijk} = 0$ [2].

Further, if the space is relatively isometric of weight w to a Riemannian space, we have a Riemannian $\bar{L}$ such that $({}^*L)^2 = \bar{L}^2 \bar{g}^w$. Since the Riemannian connection defined by $\bar{L}$ is metrical, F^i_{jk} of $^*C\Gamma$ are nothing but the connection coefficients of this Riemannian connection.

§5. Torsions and curvatures of the relative Cartan connection

Let $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ be two Finsler spaces defined on a same differentiable n-manifold M^n and assume the change $L \to \bar{L}$ be relatively isometric of weight w. That is to say, $^*L = Lg^{w/2}$ of F^n is equal to $\bar{L}\bar{g}^{w/2}$ of $\bar{F}^n$. Therefore, if we can construct the relative Cartan connection $^*C\Gamma$ of weight w on F^n, it coincides with the relative Cartan connection $^*C\Gamma$ of weight w on $\bar{F}^n$. Thus $^*C\Gamma$ may be called an *invariant connection* of this relative isometry. Although there may be other invariant connections, this $^*C\Gamma$ will be the most important in our experience of the absolute theory.

Unfortunately this concept of invariant connection can not be applicable to conformal changes, because $w = -1/n$ in case of conformal change.

Further we obtain a concept of *invariant tensor field* under a relative isometry. The torsion and curvature tensors of an invariant connection are typical invariant tensors. Thus we are interested in the torsion and curvature tensors of $^*C\Gamma$.

From the axioms of $^*C\Gamma$ it follows immediately that the $(h)h$- and $(v)v$-torsion tensors T and S^1 vanish and the deflection tensor D vanishes also. Therefore the so-called D-condition [3] holds good, but the C_1- and C_2-conditions do not hold: From (4.14) we get

(5.1) $$C_{0jk} = C_{kj0} = w({}^*y_k{}^*C_j - {}^*y_j{}^*C_k)/(nw+1).$$

It is easily seen that this is equal to zero, iff $^*C_j = 0$, provided $w \neq 0$. Originally $D = 0$ is equivalent to $y^i|_j = 0$, while the failure of the C_1-condition gives rise to $y^i|_j \neq \delta^i_j$:

(5.2) $$y^i|_j=\delta^i_j+C^i_{0j}.$$

Next, $*C\Gamma$ is positively homogeneous, because F^i_{jk}, N^i_j and C^i_{jk} are (0), (1) and $(-1)p$-homogeneous respectively (cf. §23 of [3]). From the homogeneity and the D-condition, the equation (21.10) of [3] gives $P^i_{j0}=0$, and so (21.11′) of [3] shows

(5.3) $$P^i_{j0}=0, \quad P^i_{hj0}=-C^i_{hk|j}y^k=-(C^i_{h0})_{|j}.$$

The failure of the C_1-condition gives rise to one more noteworthy fact: $R^i_{0jk}=R^i_{jk}$ etc. do not hold, although these are familiar in the Cartan connection $C\Gamma$. That is, (19.9) and (19.10) of [3] together with the D-condition lead us to

(5.4) $$R^h_{0jk}=R^h_{jk}+C^h_{0r}R^r_{jk},$$

(5.5) $$P^h_{0jk}=P^h_{jk}+C^h_{0r}P^r_{jk}-C^h_{0k|j},$$

and (19.11) of [3] together with $S_1=0$ gives

(5.6) $$S^h_{0jk}=A_{(jk)}\{C^h_{rj}|_k y^r-C^h_{rk}C^r_{0j}\},$$

where $A_{(jk)}$ denotes interchange of j and k and subtraction.

It is remarked that a well-known simple form of S^h_{ijk} of $C\Gamma$ in terms of the C-tensor (cf. (29.6) of [3]) does not hold for $*C\Gamma$; we have the general form (21.6) of [3] only.

Now we shall be concerned with the most important property, metrical property of $*C\Gamma$. In general, let S be a relative scalar field of weight σ. Then, similarly to (3.2), we have

$$S_{|i}=\partial_i S-(\dot\partial_r S)N^r_i-\sigma SF_i.$$

Thus the direct computation leads to the commutation formula

(5.7) $$\begin{aligned} S_{|i|j}-S_{|j|i}&=-(\dot\partial_r S)R^r_{ij}-\sigma SK_{ij}\\ &=-S|_r R^r_{ij}-\sigma SR_{ij}, \end{aligned}$$

where R^r_{ij} is the $(v)h$-torsion tensor, $R_{ij}=R^r_{rij}$, $K_{ij}=K^r_{rij}=R_{ij}-C_sR^s_{ij}$. The Rund h-curvature tensor K^h_{kij} is defined by

(5.8) $$K^h_{kij}=A_{(ij)}\{\partial_j F^h_{ki}-(\dot\partial_r F^h_{ki})N^r_j+F^r_{ki}F^h_{rj}\}.$$

In particular, with respect to $*C\Gamma$, we have $*g_{|i}=*g|_i=0$. Thus, applying (5.7) to $*g$, we get

(5.9) $$R_{ij}=0.$$

Similarly we have

(5.10) $$S_{|i}|_j-S|_{j|i}=-S_{|r}C^r_{ij}-S|_r P^r_{ij}-\sigma SP^r_{rij},$$

(5.11) $$S|_i|_j-S|_j|_i=-\sigma SS^r_{rij}.$$

Thus we get

(5.12) $$P^r_{rij}=0, \qquad S^r_{rij}=0.$$

As a consequence of (5.9) and (5.12), the Ricci identities, commutation formulae, for a relative tensor field reduce to usual one:

$$X^h_{i|j|k}-X^h_{i|k|j}=X^r_i R^h_{rjk}-X^h_r R^r_{ijk}-X^h_i|_r R^r_{jk}, \quad etc.,$$

namely, the additional term $-wX^h_i R^r_{rjk}$ does not appear. Therefore we have

$$^*g_{hi|j|k}-{}^*g_{hi|k|j}=-{}^*g_{ri}R^r_{hjk}-{}^*g_{hr}R^r_{ijk}, \quad \text{etc.},$$

which yield

(5.13) $$R_{hijk}+R_{ihjk}=0, \quad P_{hijk}+P_{ihjk}=0,$$
$$S_{hijk}+S_{ihjk}=0.$$

§6. Relative Berwald spaces and relative Minkowski spaces

A Finsler space is called a Berwald (affinely connected) space, if F^i_{jk} of the Cartan connection $C\Gamma$ are functions of position x alone. We shall generalize this concept from the standpoint of the relative theory:

Difinition 5. If F^i_{jk} of the relative Cartan connection $^*C\Gamma=(F^i_{jk}, N^i_j, C^i_{jk})$ of a Finsler space F^n are functions of position x alone, F^n is called a *relative Berwald space of weight w.*

In general, according to (21.11′) of [3], the condition $F^i_{jk}=F^i_{jk}(x)$ is equivalent to

(6.1) $$P^h_{ijk}=-C^h_{ik|j}+C^h_{ir}P^r_{jk},$$

Now, considering $^*C\Gamma$, we contract (6.1) by y^i and compare the result with (5.5). Then we see $P^h_{jk}=0$, and so (6.1) reduces to $P_{ihjk}=-C_{ihk|j}$. Therefore (5.13) gives $(C_{ihk}+C_{hik})_{|j}=0$. Then, from (4.14) it follows that this equation is written

$$^*C_{ihk|j}-w{}^*g_{ih}{}^*C_{k|j}/(nw+1)=0.$$

Contraction by $^*g^{ih}$ leads to $^*C_{k|j}=0$, so that $^*C_{ihk|j}=0$. Thus (4.14) gives $C_{hijk}=0$.

Theorem 3. *A Finsler space is a relative Berwald space of weight w, iff the relative Cartan connection $^*C\Gamma$ of weight w satisfies $C_{hij|k}=0$ and $P^h_{ij}=0$.*

To prove the sufficiency, we first recall one of Bianchi identities: In this case (20.9) of [3] reduces to $P^h_{ijk}-P^h_{jik}=0$, from which we get

$$(P_{ihjk}-P_{jhik})+(P_{jihk}-P_{hijk})-(P_{hjik}-P_{ijhk})=0,$$

namely, $P_{ihjk}=0$ from (5.13). Thus (6.1) holds good.

Remark. It is well-known that a Finsler space is a Berwald space, iff $C_{hij|k}=0$ with respect to $C\Gamma$. In general we have $P^h_{ij}=\dot{\partial}_j N^h_i-F^h_{ji}$ (cf. (21.10) of [3]) and in case of $C\Gamma$ we have $P^h_{ij}=C^h_{ij|0}$.

Definition 6. If there exists a local coordinate system (x^i) such that $*L=Lg^{w/2}$ is a function of y^i alone, the Finsler space is called *relatively Minkowski of weight* w and (x_i) is adapted.

In this case we see $*g_{ij}=*g_{ij}(y)$, so that $\partial_k *g_{ij}=0$ in (4.3). Thus the uniqueness property of $*C\Gamma$ yields $F^i_{jk}=0$, $N^i_j=0$ from (4.3). Consequently the space must be relatively Berwald ($C_{hij|k}=0$, $P^h_{ij}=0$) and we have $R^h_{ijk}=0$ and $R^h_{ij}=0$ from general equations defining these tensors (cf. (21.8) and (21.9) of [3]).

Conversely, if $\Delta^h C=P^1=R^1=0$ and $R^2=0$ of $*C\Gamma$ as above, $\Delta^h C=P^1=0$ show the space is relatively Berwald, so that $F^i_{jk}=F^i_{jk}(x)$. Next $R^1=0$ and $R^2=0$ give $K^i_{hjk}=0$ from a general equation $R^i_{hjk}=K^i_{hjk}+C^i_{hr}R^r_{jk}$. Thus (5.8) yields $A_{(jk)}\{\partial_k F^i_{hj}+F^r_{hj}F^i_{rk}\}=0$. Therefore, similarly to the case of flat Riemannian space [4], it is easily shown that there exists such a local coordinate system $(\bar{x}^i)$ that $\bar{F}^i_{jk}=0$ in $(\bar{x}^i)$. Finally (III) of Definition 4 gives $\bar{N}^i_j=0$, so that (4.3) concludes $\partial_k *\bar{g}_{ij}=0$.

Theorem 4. *A Finsler space is relatively Minkowski of weight* w, *iff we have* $\Delta^h C=P^1=R^1=0$ *and* $R^2=0$ *of the relative Cartan connection of weight* w.

§7. Relative Berwald connections

In the absolute theory of Finsler spaces we sometimes refer to the Berwald connection $B\Gamma$, although $B\Gamma$ is not metrical and in consequence inconvenient to some theories. Recently T. Okada [7] gives a system of axioms by means of which $B\Gamma$ is uniquely determined. In this section we shall consider such a system of axioms to get a connection similar to $B\Gamma$.

Definition 7. If we have a homogeneous Finsler connection $*B\Gamma=(G^i_{jk}, G^i_j, 0)$ satisfying the following four axioms, $*B\Gamma$ is called a *relative Berwald connection of weight* w.

(I) $*L_{|i}=0$, $\quad (*L=Lg^{w/2})$,
(II) $(h)h$-torsion $T^i_{jk}=G^i_{jk}-G^i_{kj}=0$,
(III) deflection $D^i_j=y^r G^i_{rj}-G^i_j=0$,
(IV) $(v)hv$-torsion $P^i_{jk}=\dot{\partial}_k G^i_j-G^i_{kj}=0$.

The most characteristic perperty of a Finsler connection of Berwald type will be (IV) (cf. (21.10) of [3]).

We consider $*B\Gamma$. First, from (II) and (IV) we see $\dot{\partial}_k G^i_j=\dot{\partial}_j G^i_k$, so that we locally have functions $G^i(x, y)$ such that

$$G^i_j=\dot{\partial}_j G^i. \tag{7.1}$$

The functions G^i_j are $(1)p$-homogeneous from the assumption of homogeneity of $*B\Gamma$, so that G^i in (7.1) should be supposed to be $(2)p$-homogeneous.

Next, putting $*F=(*L)^2/2$, (I) is equivalent to $*F_{|i}=0$, i.e.,

$$\partial_i *F=*y_r G^r_i+2w*FG_i, \tag{7.2}$$

where $G_i=G^r_{ri}$. From (IV) and (7.1) we have $G_i=\dot{\partial}_i\dot{\partial}_r G^r$. Thus, if we put

$$G=\mathring{\partial}_r G^r, \tag{7.3}$$

G is (1)p-homogeneous and $G_i=\mathring{\partial}_i G$.

Further, we put

$$2A_i=y^r\mathring{\partial}_i\partial_r{}^*F-\partial_i{}^*F, \qquad A_{ij}=\mathring{\partial}_j A_i. \tag{7.4}$$

Then (7.2) leads to

$${}^*g_{ir}G^r=A_i-w({}^*y_iG-{}^*FG_i). \tag{7.5}$$

We consider A_{ij}. From (7.4) we easily get

$$A_{0j}=\partial_j{}^*F. \tag{7.6}$$

On the other hand, (7.5) gives

$$2{}^*C_{irj}G^r+{}^*g_{ir}G^r_j=A_{ij}-w({}^*g_{ij}G+{}^*y_iG_j-{}^*y_jG_i-{}^*FG_{ij}),$$

where we put $G_{ij}=\mathring{\partial}_i\mathring{\partial}_j G$. Substituting from (7.5) into G^r appearing in the first term, we get

$$\begin{aligned}{}^*g_{ir}G^r_j=A_{ij}-2{}^*C^r_{ij}A_r-w({}^*g_{ij}G+2{}^*F{}^*C^r_{ij}G_r\\ +{}^*y_iG_j-{}^*y_jG_i-{}^*FG_{ij}).\end{aligned} \tag{7.7}$$

Finally, contracting (7.7) by ${}^*g^{ij}$, we get

$${}^*g^{rs}A_{rs}-2{}^*C^rA_r=(nw+1)G+2w{}^*F({}^*C^rG_r)-w{}^*F({}^*g^{rs}G_{rs}). \tag{7.8}$$

Consequently, G^i and G^i_j are respectively written in the forms given by (7.5) and (7.7) in terms of G, its derivatives and known quantities. This G must satisfy (7.8), a differential equation of second order, provided $w\neq 0$.

Conversely, assume that we have a (1)p-homogeneous function G satisfying the differential equation (7.8) of second order. Then (7.5) and (7.7) give G^i and G^i_j respectively. At it is known from the procedure by which (7.7) was obtained from (7.5) and (7.8) was obtained from (7.7), we have $G^i_j=\mathring{\partial}_jG^i$ and $G=G^r_r$. Further (7.7) shows ${}^*y_rG^r_j=A_{0j}-2w{}^*FG_j$, namely, we get (7.2) in virtue of (7.6). Finally G^i_{jk} is defined as $\mathring{\partial}_jG^i_k$ and then the four axioms are all satisfied.

Theorem 5. *A relative Berwald connection* ${}^*B\Gamma$ *of weight* w *is defined by* G^i_j *in* (7.7) *and* $G^i_{jk}=\mathring{\partial}_jG^i_k$, *if and only if we have a* (1)p*-homogeneous function* G *satisfying* (7.8).

It will be difficult to examine the existence and uniqueness of solution of (7.8). It is noted that in the absolute case (7.8) reduces to $G={}^*g^{rs}A_{rs}-2{}^*C^rA_r$.

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