J. Math. Kyoto Univ. (JMKYAZ)
23-4 (1983) 695–706

Some characterizations of smoothness

By

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(Communicated by Prof. Nagata, April 19, 1982; revised Aug. 25, 1982)

Introduction

In [10], H. Matsumura asks

(I) what is the difference between smoothness and I-smoothness?

(II) when is a ring $A[[X_1,\dots, X_n]]/\mathfrak{a}$ smooth over A?

In the present paper, we study his problems (mainly Problem (II)) when A is a noetherian ring. Concerning Problem (II), we list up three problems:

(A) When is $A[[X_1,\dots, X_n]]$ smooth over A?

(B) When is $A[[X_1,\dots, X_n]]/\mathfrak{a}$ smooth over A in the case that $\mathfrak{a} \neq 0$?

In particular

(C) when is $(A, I)^\wedge$ smooth over A?

We can find some results about smoothness in some papers (cf. [1], [6], [8], [9] etc.). They are stated in terms of differential modules or cohomology modules. But the modules are not so easy to calculate in general. So, in this paper, we shall find other criteria of smoothness in the above three cases.

§1 consists of notation, terminology and preliminary results. Proposition 1.5 gives the well-known criteria of unramifiedness and smoothness in terms of differential modules. For the proof of the criterion of smoothness, we could not find a reference, so we prove it here. Lemma 1.6 is essentially due to Seydi (cf. [15, Th. 1.2]). Here we prove it more simply than he. Lemma 1.7 is due to Kunz (cf. [7, Lemma 2.4]).

§2 is devoted to study Problem (A). When A contains a field, we get the following result: if $A[[X_1,\dots, X_n]]$ is smooth over A for some $n>0$, then $ch(A)=p>0$ and A is finite over A^p. And the converse is also true.

In §3, we reduce Problem (B) to Problem (C) in some cases. Minutely speaking, if $B=A[[X_1,\dots, X_n]]/\mathfrak{a}$ is smooth over A, B is isomorphic to the completion of A with respect to an ideal of A.

In §4, we deal with Problem (C) when A contains a field. We consider the problem in two cases, that is, $ch(A)=0$ and $ch(A)>0$. As an application of the result of the latter case, we shall prove Kunz' theorem (cf. [9, (42.B) Th. 108]) in the different way from his.

In §5, first we construct an excellent DVR A containing a field of arbitrary characteristic such that $\hat{A}$ is smooth over A and $\hat{A} \not\cong A$. Next we study the quotient rings of smooth algebras. Here, we state I-smoothness in terms of a subset of the set of maximal ideals. Finally, concerning this, we shall give an example.

The writer wishes to express his hearty thanks to Prof. H. Matsumura for his kind suggestions, and to Prof. K. Watanabe who also gave him kind suggestions, in particular in Proposition 3.3.

§1. Notation, terminology and preliminary results

In this paper, all rings are commutative rings with unit element. When a ring A has only one maximal ideal $\mathfrak{m}$, we call the ring a local ring and denote it by $(A, \mathfrak{m})$ or $(A, \mathfrak{m}, k)$ where k is the residue field of A. When a ring A is a domain, the total quotient field is denoted by $Q(A)$. For $\mathfrak{p} \in \operatorname{Spec}(A)$, we denote $Q(A/\mathfrak{p})$ by $k(\mathfrak{p})$. For a ring A and an ideal I of A, we denote the I-adic completion of A by $(A, I)^{\wedge}$ and the henselization of A with respect to I by $(A, I)^h$.

Definition 1.1. Let $\boldsymbol{P}$ be a property concerning noetherian local rings. For example, $\boldsymbol{P}=$regular, normal or reduced. We say that a noetherian ring A is $\boldsymbol{P}$ when $A_{\mathfrak{p}}$ has the property $\boldsymbol{P}$ for all $\mathfrak{p} \in \operatorname{Max}(A)$. A ring homomorphism $A \to B$ is called a $\boldsymbol{P}$-homomorphism if it is flat and all its fibres are geometrically $\boldsymbol{P}$. A noetherian ring A is called a $\boldsymbol{P}$-ring if, for all $\mathfrak{p} \in \operatorname{Spec}(A)$, the canonical map $A_{\mathfrak{p}} \to \hat{A}_{\mathfrak{p}}$ is a $\boldsymbol{P}$-homomorphism. In particular, when $\boldsymbol{P}=$regular (resp. normal, resp. reduced), the ring A is called a G-ring (resp. a Z-ring, resp. a N-ring) (cf. [9] or [13, Def. (0.1)]).

Definition 1.2. A noetherian ring A is Nor-2 if, for every finitely generated A-algebra B, $\{\mathfrak{p} \in \operatorname{Spec}(B) \mid B_{\mathfrak{p}} \text{ is normal}\}$ is open in the Zariski topology. In particular, if a ring A is a N-ring which is Nor-2, then A is called a Nagata ring (cf. [6, (7.7.2)] and [9]).

Definition 1.3. A noetherian domain A is N–1 if the integral closure of A in $Q(A)$ is a finite A-module.

Definition 1.4. Let A be a (not necessarily noetherian) commutative ring, B an A-algebra, and I an ideal of B. We say that B is I-smooth (resp. I-unramified) over A if for any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow[g]{} & C/N \end{array}$$

where C is an A-algebra, N is an ideal of C such that $N^2=0$, and v is a ring homomorphism such that $v(I^n)=0$ for some n, there exists at least one (resp. at most one) homomorphism $\varphi\colon B \to C$ such that $f=\varphi \circ u$ and $v=g \circ \varphi$. If B is I-smooth and I-unramified over A, we say that B is I-etale over A. In particular, if $I=0$, we say shortly that B is smooth (resp. unramified, resp. etale) over A.

Now we can restate smoothness and unramifiedness in terms of differential modules:

Proposition 1.5. (i) (cf. [10, §25]) *B is unramified over A if and only if $\Omega_{B/A}=0$.*
(ii) *When A and B are noetherian, B is smooth over A if and only if $\Omega_{B/A}$ is a projective B-module and the ring homomorphism $A \to B$ is regular.*

Proof of (ii). "Only if". By [2, (1, 1)], we have that

(*) $A \to B$ is regular if and only if for every $P \in \mathrm{Spec}(B)$ and $\mathfrak{p}=P \cap A$, B_P is PB_P-smooth over $A_{\mathfrak{p}}$.

So if B is smooth over A, $A \to B$ is regular. And then $\Omega_{B/A}$ is a projective B-module by [9, (29.B) Lemma 1].

"If". By (*) and [1, Supplément Th. 30], $A \to B$ is regular if and only if $H_1(A, B, W)=0$ for every B-module W. So we have the conclusion by [8, Prop. 3.1.3].

Finally we state two lemmas.

Lemma 1.6. (cf. [15, Th. 1.2]). *Let $(A, \mathfrak{m}, k)$ be a noetherian local domain containing a field of characteristic $p>0$ such that $[k: k^p]<\infty$. Put $K=Q(A)$. Then if A is a Nagata ring, we have $[K: K^p]<\infty$. In particular A is a finite A^p-algebra.*

Proof. Let $\hat{A}$ be the $\mathfrak{m}$-adic completion of A. Then by Cohen's structure theorem, $\hat{A}$ is a homomorphic image of $k[[X_1,\dots, X_n]]$ where $X_1,\dots, X_n$ are variables over k. Since $[k: k^p]<\infty$, $\hat{A}$ is finite over $\hat{A}^p$. So if we put $L=Q(\hat{A}/\mathfrak{p})$ for any $\mathfrak{p} \in \mathrm{Min}(\hat{A})$, $[L: L^p]<\infty$. Now since A is a Nagata local domain, L is separable over K (cf. [9, (31.F)]). Thus by MacLane's theorem, L^p and K are linearly disjoint over K^p. So we have $[K: K^p] \leqq [L: L^p]<\infty$. Q. E. D.

Remark. In [14, Th. 2], Rotthaus also proves the above lemma simply, where she uses differential modules.

Lemma 1.7. (cf. [7, Lemma 2.4]). *Let A be a noetherian semi-local reduced ring containing a field of characteristic $p>0$. If A is a finite A^p-module, then A is analytically unramified.*

§ 2. Problem (A)

First of all, we consider the lemma which gives an answer to Problem (A) when A is a field:

Lemma 2.1. *Let k be a field, and $X=\{X_1,\dots, X_n\}$ be variables over k. Then the following are equivalent:*
(i) *$k[[X]]$ is smooth over k;*
(ii) *$ch(k)=p>0$ and $[k: k^p]<\infty$.*

Proof. (i)$\Rightarrow$(ii). For the sequence of ring homomorphisms $k \to k[X] \to k[[X]]$, we have the following exact sequence:

$$\Omega_{k[X]/k}\otimes_{k[X]}k[[X]]\xrightarrow{\varphi}\Omega_{k[[X]]/k}\longrightarrow\Omega_{k[[X]]/k[X]}\longrightarrow 0.$$

Since $k[[X]]$ is smooth over k and $k[[X]]$ is a local ring, $\Omega_{k[[X]]/k}$ is a free $k[[X]]$-module. Now $\Omega_{k[[X]]/k}\otimes_{k[[X]]}(k[[X]]/(X))\cong\bigoplus_{i=1}^{n}(k[[X]]/(X))dX_i$. Therefore $\Omega_{k[[X]]/k}\cong\bigoplus_{i=1}^{n}k[[X]]dX_i$. Thus φ is an isomorphism, and $\Omega_{k[[X]]/k[X]}=0$. So $\Omega_{k((X))/k(X)}=0$. Since $\mathrm{tr.\,deg}_{k(X)}k((X))=\infty$, we have $ch(k)=p>0$ and $(k((X)))^p[k(X)]=k((X))$. From this it follows easily that $[k: k^p]<\infty$.

(ii)⇒(i). Since $[k: k^p]<\infty$, we have $(k[[X]])^p[k[X]]=k[[X]]$. So $\Omega_{k[[X]]/k[X]}=0$. Moreover $k[X]\to k[[X]]$ is regular. Therefore $k[[X]]$ is smooth over $k[X]$. So $k[[X]]$ is smooth over k. Q.E.D.

Now we give an answer to Problem (A) when A contains a field.

Theorem 2.2. *Let A be a noetherian ring containing a field k. Then the following are equivalent*:

(i) *$A[[X_1,\dots,X_n]]$ is smooth over A for every $n\geq 1$;*

(ii) *$A[[X_1,\dots,X_n]]$ is smooth over A for some $n\geq 1$;*

(iii) *$ch(k)=p>0$ and A is a finite A^p-algebra.*

Proof. (iii)⇒(i). By Kunz' theorem (cf. [9, (42.A) Th.108]), A is a G-ring. So if we put $X=\{X_1,\dots,X_n\}$, $A[X]$ is a G-ring. Thus $A[X]\to A[[X]]$ is regular. Moreover by the assumption, $(A[[X]])^p[A[X]]=A[[X]]$. So $\Omega_{A[[X]]/A[X]}=0$. Thus $A[[X]]$ is smooth over $A[X]$, and also smooth over A.

(i)⇒(ii). Clear.

(ii)⇒(iii). Put $X=\{X_1,\dots,X_n\}$. For all $\mathfrak{m}\in\mathrm{Max}(A)$, $k(\mathfrak{m})[[X]]$ is smooth over $k(\mathfrak{m})$. So by Lemma (2.1), we have

$$(*)\qquad ch(k)=p>0\quad\text{and}\quad[k(\mathfrak{m}): k(\mathfrak{m})^p]<\infty.$$

In order to show that A is a finite A^p-algebra, we may assume that A is a domain by the same argument as in the proof of [15, Cor. (1.3)]. Put $\mathscr{S}=\{\mathfrak{p}\in\mathrm{Spec}(A)\mid A/\mathfrak{p}$ is not finite over $(A/\mathfrak{p})^p\}$. If $\mathscr{S}\neq\phi$, there exists a maximal element $\mathfrak{p}_0$ in $\mathscr{S}$. Then $\mathfrak{p}_0$ is not a maximal ideal by $(*)$. Put $B=A/\mathfrak{p}_0$. We shall show that B is finite over B^p. For the purpose we have only to show that $B^p[[X]][B]=B[[X]]$.

Take any non-unit $0\neq a\in B$ and put $\hat{B}=(B,(a))^\wedge$. Then $\hat{B}/(a)\cong B/(a)$. So by the definition of $\mathfrak{p}_0$ and Kunz' theorem, $\hat{B}/(a)$ is a Nagata ring. So by Marot's theorem (cf. [9, (41.D) Th.106]), $\hat{B}$ is a Nagata ring. By the way, since $B[[X]]$ is smooth over B and $\hat{B}\cong B[[X]]/(X_1-a,\dots,X_n-a)$, $\hat{B}$ is smooth over B by [9, (29.C) and (29.E)]. So if we put $S_a=1+(a)=\{1+ax\mid x\in B\}$, $S_a^{-1}B\to\hat{B}$ is faithfully flat and reduced. Since $\hat{B}$ is a Nagata ring, $S_a^{-1}B$ is also a Nagata ring by [11, (4.9)]. Thus by $(*)$ and Lemma 1.6, we have $[Q(B): Q(B)^p]<\infty$. Now let $\{B_\lambda\}$ be an inductive system of finite B^p-algebras such that $B_\lambda\subseteq B$ and $\varinjlim_\lambda B_\lambda=B$. Then $B^p[[X]][B]=\varinjlim_\lambda B_\lambda[[X]]$. Now we can show $\Omega_{B[[X]]/B[X]}=0$ in the same way as the proof of Lemma 2.1 (i)⇒(ii). So $Q(B^p[[X]][B])=Q(B[[X]])$. Thus for any $h\in B[[X]]$, we can write $h=g/f$ such that $f,g\in B_\lambda[[X]]$ for some λ. Then there exists $0\neq a\in B^p$ such that $h\in(B_\lambda)_a[[X]]$. On the other hand, since $[Q(B)$:

$Q(B)^p]<\infty$ and $(S_a^{-1}B)^p$ is a Nagata ring, there exists $\mu\geq\lambda$ such that $T_a^{-1}B_\mu=T_a^{-1}B$ where $T_a=S_a^p$. Therefore $h\in(B_\mu)_a[[X]]\cap(T_a^{-1}B_\mu)[[X]]=B_\mu[[X]]\subseteq B^p[[X]][B]$. So $B^p[[X]][B]=B[[X]]$ as wanted. This is a contradiction. Thus $\mathscr{I}=\phi$ and A is finite over A^p. Q.E.D.

Remark 2.3. (I) In general, if A is a noetherian ring containing a field of characteristic $p>0$ and if A is a finite A^p-module, then $\hat{A}=(A, I)^\wedge$ is etale over A. In fact, by [9, (28.P) Lemma], $\hat{A}^p[A]=\hat{A}$. So $\Omega_{\hat{A}/A}=0$. Since A is excellent, $A\to\hat{A}$ is regular. Thus $\hat{A}$ is etale over A.

(II) By the above theorem, we see easily that, if $A[[X_1,\dots, X_n]]$ is smooth over A for some $n\geq 1$ and if A contains a field, then $(S^{-1}A)[[X_1,\dots, X_n]]$ is also smooth over $S^{-1}A$ for every multiplicatively closed subset S of A.

Remark. To prove the above theorem (ii)$\Rightarrow$(iii), the referee advised the writer to prove $B^p[[X]][B]=B[[X]]$. He expresses his hearty thanks to him.

§3. Problem (B)

In this section, we show that Problem (B) can be reduced to Problem (C) in some cases.

Lemma 3.1 *Let k be a field, $X=\{X_1,\dots, X_n\}$ variables over k and $\mathfrak{a}$ an ideal of $k[[X]]$. Suppose that $k[[X]]/\mathfrak{a}$ is smooth over k. Then*

(i) *if $ch(k)=0$, or if $ch(k)=p>0$ and $[k: k^p]=\infty$, then $\mathfrak{a}=(X)$.*

(ii) *if $ch(k)=p>0$ and $[k: k^p]<\infty$, then there exists some $\varphi\in \mathrm{Aut}_k(k[[X]])$ such that $\varphi(\mathfrak{a})=(X_1,\dots, X_m)$ for some natural number $m\leq n$.*

Proof. Since $k[[X]]/\mathfrak{a}$ is regular over k, it is a regular local ring. So the lemma follows from Lemma 2.1. Q.E.D.

Lemma 3.2. *Let A be a noetherian ring, I an ideal of A and J an ideal of $\hat{A}=(A, I)^\wedge$ such that $J\subseteq\mathrm{rad}(\hat{A})$. If $\hat{A}/J$ is flat over A, then $J=0$.*

Proof. Consider the exact sequence $0\to J\to\hat{A}\to\hat{A}/J\to 0$. Put $\mathfrak{a}=\bigcap_{\substack{\mathfrak{m}\in\mathrm{Max}(A)\\ I\subseteq\mathfrak{m}}}\mathfrak{m}$. Since $\hat{A}/J$ is flat over A, $\mathrm{Tor}_1^A(\hat{A}/J, A/\mathfrak{a})=0$. So we have the exact sequence $0\to J/\mathfrak{a}J\to\hat{A}/\mathfrak{a}\hat{A}\xrightarrow{\varphi}\hat{A}/J+\mathfrak{a}\hat{A}\to 0$. Now since $I\subseteq\mathfrak{a}$, $\mathfrak{a}\hat{A}=\mathrm{rad}(\hat{A})$. So by our assumption $J\subseteq\mathrm{rad}(\hat{A})$, φ is an isomorphism. So $J=\mathfrak{a}J=\mathrm{rad}(\hat{A})J$. Therefore $J=0$ by NAK. Q.E.D.

Let A be a noetherian ring and $\mathfrak{p}\in\mathrm{Spec}(A)$. We say that $\mathfrak{p}$ satisfies SC if $k(\mathfrak{p})$ satisfies the condition of Lemma 3.1, (i). Moreover we define the natural map π: $A[[X_1,\dots, X_n]]\to A$ such that $\pi(X_i)=0$ for all i, that is, $\pi(f)$ $(f\in A[[X_1,\dots, X_n]])$ is the constant term of f.

Proposition 3.3. *Let A be a noetherian ring and $X=\{X_1,\dots, X_n\}$ be variables over A. Let $\mathfrak{a}$ be an ideal of $A[[X]]$ such that every $\mathfrak{m}\in\mathrm{Max}(A)$ containing $\pi(\mathfrak{a})$ satisfies SC. Then if $R=A[[X]]/\mathfrak{a}$ is smooth over A, $R\cong(A, \pi(\mathfrak{a}))^\wedge$.*

Proof. Put $S=1+\pi(\mathfrak{a})$. Then the elements of S are units in R. So $(S^{-1}A)[[X]]/\mathfrak{a}(S^{-1}A)[[X]]\cong R$ because the ring on the left side is the (X)-adic completion of $S^{-1}R\cong R$. Furthermore R is smooth over $S^{-1}A$ and $(A, \pi(\mathfrak{a}))^{\wedge}\cong(S^{-1}A, S^{-1}\pi(\mathfrak{a}))^{\wedge}$. Thus replacing A by $S^{-1}A$, we may assume that $\pi(\mathfrak{a})\subseteq \operatorname{rad}(A)$. Then for every $\mathfrak{m}\in \operatorname{Max}(A)$, $k(\mathfrak{m})[[X]]/\mathfrak{a}k(\mathfrak{m})[[X]]$ is smooth over $k(\mathfrak{m})$. Since the ideal $\mathfrak{m}$ satisfies SC, we have

$$(*)\qquad \mathfrak{a}+\mathfrak{m}A[[X]]=(X)+\mathfrak{m}A[[X]]$$

by Lemma 3.1. Now we define $M=\left\{(a_1,\dots, a_n)\ \middle|\ \begin{array}{l}\mathfrak{a}\ni\alpha+\sum_i a_iX_i+\beta\\ \text{where } \alpha,\ a_i\in A \text{ and } \beta\in(X)^2\end{array}\right\}$ $\subseteq A^n$. Then M is a finite A-module, and by $(*)$ we have $M+\mathfrak{m}A^n=A^n$ for every $\mathfrak{m}\in\operatorname{Max}(A)$. Thus $M=A^n$ by NAK. So there exists elements $f_i=\alpha_i+X_i+\beta_i\in\mathfrak{a}$ $(i=1,\dots, n)$ such that $\alpha_i\in A$ and $\beta_i\in(X)^2$. Then putting $Y_i=X_i+\beta_i$ $(i=1,\dots, n)$ and $Y=\{Y_1,\dots, Y_n\}$, we have $A[[X]]=A[[Y]]$ and $\mathfrak{a}\supseteq(Y_1+\alpha_1,\dots, Y_n+\alpha_n)$. Put $I=(\alpha_1,\dots,\alpha_n)\subset A$. Then there exists an ideal $J\subset\hat{A}=(A, I)^{\wedge}$ such that $R\cong\hat{A}/J$. Since $\mathfrak{a}\subseteq\operatorname{rad}(A[[X]])$, we have $J\subseteq\operatorname{rad}(\hat{A})$. Therefore $J=0$ by Lemma 3.2. Since $\mathfrak{a}=(Y_1+\alpha_1,\dots, Y_n+\alpha_n)$, we have $\pi(\mathfrak{a})=I$. Q.E.D

Remark. In the above proof, the module M is suggested to the writer by Prof. K. Watanabe.

§4. Problem (C)

In this section, we answer Problem (C) when the ring A contains a field k. For this purpose we consider two cases, that is, whether $ch(k)=0$ or not. First we deal with the case $ch(k)=0$.

Lemma 4.1. (cf. [5, Lemma 4]). *Let **P** be a property concerning noetherian local rings (cf. Def. 1.1). Assume that **P** satisfies the following conditions:*

(i) *Regular local rings are **P**.*

(ii) ***P** is stable under generalization.*

*Let A be a noetherian ring and I an ideal of A. Then if $A\to(A, I)^{\wedge}$ is a **P**-homomorphism, $(A, I)^h\to(A, I)^{\wedge}$ is also a **P**-homomorphism.*

Remark. Let **P** be one of the properties in [6, (7.3.8)], in particular **P**$=$ reduced or normal. Then **P** satisfies the above conditions.

The proofs of the following two lemmas are easy and we omit them.

Lemma 4.2. *Let A be a noetherian ring and I an ideal of A. If $A/\mathfrak{p}$ is I-adically complete for all $\mathfrak{p}\in\operatorname{Min}(A)$, then A is also I-adically complete.*

Lemma 4.3. *Let A be a noetherian ring and I an ideal of A. Put $\hat{A}=(A, I)^{\wedge}$. Then if $\hat{A}$ is smooth over A, $\hat{A}$ is etale over A.*

Theorem 4.4. *Let A be a noetherian ring containing a rational field, and let*

I be an ideal of A. Put $\hat{A}=(A, I)^{\wedge}$ and $A^h=(A, I)^h$. Then the following conditions are equivalent:

(i) *$\hat{A}$ is smooth over A*;

(ii) *$\hat{A}$ is etale over A*;

(iii) *$\hat{A}$ is unramified over A, and $A \to \hat{A}$ is a normal homomorphism*;

(iv) $A^h \cong \hat{A}$.

Proof. (i)$\Leftrightarrow$(ii): by Lemma 4.3. (ii)$\Rightarrow$(iii) is clear. Since A^h is etale over A, (iv)$\Rightarrow$(ii) is clear. Let us prove (iii)$\Rightarrow$(iv). Since $\hat{A}$ is unramified over A, $\hat{A}$ is unramified over A^h. And by Lemma 4.1, $A^h \to \hat{A}$ is normal. Thus we may suppose that $A=A^h$. Moreover, by Lemma 4.2, we may suppose that A is a domain. Then by [3, Th. 1], $\hat{A}$ is a domain. So since $\Omega_{\hat{A}/A}=0$ and $ch(A)=0$, $Q(\hat{A})$ is algebraic over $Q(A)$. Thus $\hat{A}$ is algebraic over A. So $\hat{A}=A$ by [3, Th. 1]. Q. E. D.

Remark. (I) By [5, Th. 3, Prop. 4 and Lemma 4], if $A/\mathfrak{p}$ is N-1 for all $\mathfrak{p} \in \operatorname{Min}(A)$, the above condition (iii) is equivalent to the following:

(iii)′ $\hat{A}$ is unramified over A, $\hat{A} \otimes_A k(\mathfrak{p})$ is normal for all $\mathfrak{p} \in \operatorname{Min}(A)$, and $A \to \hat{A}$ is a reduced homomorphism.

(II) In §5, we construct a DVR A containing a rational field such that $\hat{A}$ is smooth over A and $\hat{A} \not\cong A$.

Next we consider Problem (C) when the ring A contains a field of characteristic p.

Lemma 4.5. *Let A be a noetherian domain such that $ch(A)=p>0$, and let I be an ideal of A. Put $\hat{A}=(A, I)^{\wedge}$ and $K=Q(A)$. Suppose that $A \to \hat{A}$ is a reduced homomorphism. Then for all A^p-algebra R contained in K, $\hat{A}^p \otimes_{A^p} R \cong \hat{A}^p[R]$ where the map is induced by the canonical map $\hat{A}^p \otimes_{A^p} K \to Q(\hat{A})$.*

This can be proved easily by MacLane's theorem, so we omit the proof.

Remark 4.6. Let A be a noetherian domain and I an ideal of A. Let $\hat{A}$ denote the I-adic completion of A. If $\hat{A}$ is reduced, it follows easily that $Q(\hat{A}^p[A])=Q(\hat{A})^p[Q(A)] \subseteq Q(\hat{A})$.

Theorem 4.7. *Let A be a noetherian ring containing a field of characteristic $p>0$, and let I be an ideal of A. Put $A^h=(A, I)^h$, $\hat{A}=(A, I)^{\wedge}$, and $\bar{A}=$the homomorphic image of A in $\hat{A}$. Suppose that $A/\mathfrak{p}$ is* N-1 (*cf. Def.* 1.3) *for all* $\mathfrak{p} \in \operatorname{Min}(A)$. *Then the following conditions are equivalent*:

(i) *$\hat{A}$ is smooth over A*;

(ii) *$\hat{A}$ is etale over A*;

(iii) *$\hat{A}$ is unramified over A, and $A \to \hat{A}$ is normal*;

(iv) *$\hat{A}^p[\bar{A}]=\hat{A}$, and $A \to \hat{A}$ is reduced*;

(v) *$\hat{A}^p[A^h]=\hat{A}$, and $A \to \hat{A}$ is reduced.*

Proof. (i)$\Leftrightarrow$(ii): by Lemma 4.3. (ii)$\Rightarrow$(iii) and (iv)$\Rightarrow$(v) are clear.

(iii)$\Rightarrow$(iv): We have only to show the first part.

Case 1. A is a normal domain. In this case, since $A \to \hat{A}$ is normal and since $\hat{A}$ is reduced, the ring $\hat{A}^p \otimes_{A^p} A$ is normal by [6, (6.14.1)]. Thus $\hat{A}^p[A]$ is normal by

Lemma 4.5. Now, since $\hat{A}$ is unramified over A, $Q(\hat{A})$ is unramified over $Q(A)$. Moreover, since $\hat{A}$ is reduced, $Q(\hat{A})$ is a direct product of a finite number of fields of characteristic p. So we have $Q(\hat{A}^p[A])=Q(\hat{A})^p[Q(A)]=Q(\hat{A})$ by Remark 4.6. Since $\hat{A}$ is integral over $\hat{A}^p[A]$, we have $\hat{A}=\hat{A}^p[A]$ by [6, (6.14.1.1)].

Case 2. A is a reduced ring. Put $\mathrm{Min}(A)=\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$. Then we have the following injective ring homomorphisms:

$$A \overset{\varphi}{\hookrightarrow} B=\prod_{i=1}^{n} A/\mathfrak{p}_i \overset{\psi}{\hookrightarrow} \tilde{B}=\prod_{i=1}^{n} \widetilde{A/\mathfrak{p}_i}$$

where each $\widetilde{A/\mathfrak{p}_i}$ is the derived normal ring of $A/\mathfrak{p}_i$. Then φ is finite. And by our assumption, ψ is finite. Thus $\tilde{B}$ is a finite A-module. So by change of base, $\hat{\tilde{B}}=(\tilde{B}, I\tilde{B})^\wedge$ is unramified over $\tilde{B}$, and $\tilde{B}\to\hat{\tilde{B}}$ is normal. Therefore we have $\hat{\tilde{B}}^p[\tilde{B}]=\hat{\tilde{B}}$ by Case 1. Now since $\tilde{B}$ is finite over A, $\hat{\tilde{B}}=\hat{\tilde{B}}^p[\tilde{B}]$ is a finite extension of $\hat{A}^p[A]$. Since $\hat{\tilde{B}}$ is noetherian, $\hat{A}^p[A]$ is noetherian by the theorem of Eakin-Nagata. Moreover $\hat{A}$ is contained in $\hat{\tilde{B}}$, so $\hat{A}$ is finite over $\hat{A}^p[A]$. Now since $\hat{A}$ is unramified over A, $\Omega_{\hat{A}/A}=0$. Thus $\hat{A}=\hat{A}^p[A]$ by [6, 0_{IV} (21.1.7)].

Case 3. General case. By change of base, $\widehat{A_{red}}=(A_{red}, IA_{red})^\wedge$ is unramified over A_{red}, and $A_{red}\to\widehat{A_{red}}$ is normal. So by Case 2, we have $\hat{A}^p[\bar{A}]+nil(A)\hat{A}=\hat{A}$. Thus $\hat{A}^p[\bar{A}]=\hat{A}$.

(v)$\Rightarrow$(ii): A^h is etale over A, and $A^h\to\hat{A}$ is a reduced homomorphism by Lemma 4.1. So we suppose $A=A^h$ and prove $\hat{A}$ is etale over A. By our assumption, $\Omega_{\hat{A}/A}=\Omega_{\hat{A}/\hat{A}^p[A]}=0$. So we have only to show that $A\to\hat{A}$ is regular. For the purpose, we may assume that A is a domain. From now on, we shall show that if A is a domain satisfying the condition (v), then $\hat{A}$ is smooth over A.

Consider the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \hat{A} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \psi} \\ C & \xrightarrow{v} & C/N \end{array}$$

where C is a ring and N is an ideal of C such that $N^2=0$. Since $N^p=0$, there exists a homomorphism $f\colon C/N\to C^p$ such that for $x\in C$, $f(x \bmod N)=x^p$. On the other hand, since $\hat{A}$ is reduced, the Frobenius map $F\colon\hat{A}\to\hat{A}^p$ is an isomorphism. So we can define a homomorphism $g=f\circ\psi\circ F^{-1}\colon \hat{A}^p\to C$. Then it follows easily that $u|_{A^p}=g\circ\varphi|_{A^p}$ and $v\circ g=\psi|_{\hat{A}^p}$. So we have a homomorphism $h=g\otimes u\colon \hat{A}^p\otimes_{A^p}A\to C$. Now since $A\to\hat{A}$ is reduced and A is a domain, we have $\hat{A}^p\otimes_{A^p}A\cong\hat{A}^p[A]\cong\hat{A}$ by Lemma 4.5 and our assumption. It is easy to see that $u=h\circ\varphi$ and $\psi=v\circ h$. So h is a lifting of ψ over A. Therefore $\hat{A}$ is smooth over A. Q.E.D.

Remark. (I) In the above proof, we didn't use the condition N–1 except for (iii)$\Rightarrow$(iv).

(II) Let $(A, \mathfrak{m}, k)$ be a noetherian local ring containing a field of characteristic $p>0$. When $[k\colon k^p]<\infty$, smoothness of $\hat{A}$ over A is mentioned in some papers (cf. [2, (3.5)] and [16, (3.3)]).

(III) In §5, we construct an excellent DVR A of characteristic p such that $\hat{A}$ is smooth over A, $\hat{A}\not\cong A$ and A is not a finite A^p-module.

Corollary 4.8. *Let A be a noetherian normal Z-ring* (*cf. Def.* 1.3) *containing a field of characteristic $p>0$, and let I be an ideal of A. Then $\hat{A}=(A, I)^\wedge$ is smooth over A if and only if $Q(\hat{A}^p[A])=Q(\hat{A})$.*

As an application of Th. 4.7, we shall prove the following corollary due to E. Kunz without using a homological method (cf. [2, Th. 3.4]).

Corollary 4.9. (cf. [9, (42.B) Th. 108]). *Let A be a noetherian ring containing a field of characteristic $p>0$. Then if A is a finite A^p-module, A is a G-ring.*

Proof. We can assume that $(A, \mathfrak{m})$ is a local domain. By [16, Lemma 1.2], we have $\hat{A}=\hat{A}^p[A]$ where $\hat{A}=(A, \mathfrak{m})^\wedge$. Thus in order to show that $A\to\hat{A}$ is regular, by Th. 4.7 we have only to prove that $A\to\hat{A}$ is reduced. For this purpose, by [6, (7.6.4)] and [9, (31.E)] we have only to show that if A is a semi-local domain such that A is finite over A^p, then A is analytically unramified. This follows from Lemma 1.7. Q.E.D.

§5. Quotient rings of smooth algebras and examples.

Example 5.1. Concerning Problem (C), we are interested to construct a noetherian ring A and an ideal I of A which satisfy the following conditions:

(i) A contains a field k,

(ii) $A\not\cong\hat{A}=(A, I)^\wedge$,

(iii) $\hat{A}$ is smooth over A.

We distinguish three cases.

Case (I): $ch(k)=0$. In [12, (11.3) Ex. 3], it is shown that there exists a DVR A which satisfies the above three conditions. We shall sketch the construction.

Let k be a field of characteristic 0, X a variable over k, and B a transcendence base of $k((X))$ over $k(X)$. Put $A=k[[X]]\cap k(X)(B)$. Then $(A, (X))$ is a DVR and $\hat{A}\cong k[[X]]\cong A^h$. In particular, $\hat{A}$ is smooth over A.

Case (II): $ch(k)=p>0$, and A is a finite A^p-module. Let k be a field of characteristic p such that $[k: k^p]<\infty$, and let X be a variable over k. Put $A=k[X]$ and $I=(X)$. Then it follows easily that A and I satisfy the above three coditions. (cf. Remark 2.3)

Case (III): $ch(k)=p>0$, and A is not a finite A^p-module. Imitating the construction in Case (I), we shall construct a desirable example.

Let k be a field of characteristic p such that $[k: k^p]=\infty$, and let X be a variable over k. Then $k((X))$ is separable over $k(X)$. Let B be a p-basis of $k((X))$ over $k(X)$. Then $k((X))$ is separable over $k(X)(B)$ by [9, (38.E)]. Put $A=k[[X]]\cap k(X)(B)$. Then it follows easily that $(A, (X), k)$ is an excellent DVR such that $\hat{A}=(A, (X))^\wedge\cong k[[X]]$ and $A\not\cong\hat{A}$. Moreover, since $[k: k^p]=\infty$, A is not a finite A^p-module. Since $Q(\hat{A}^p[A])=k^p((X^p))[k(X)(B)]=k((X))=Q(\hat{A})$, $\hat{A}$ is smooth over A by Cor. 4.8. Therefore A is the example which we want.

Next we study quotient rings of smooth algebras.

For a noetherian ring A and a noetherian A-algebra B, we put $\mathscr{F}_A(B)=\{\mathfrak{m}\in \operatorname{Max}(B) \mid B_{\mathfrak{m}}$ is flat over $A\}$ and $\mathscr{R}_A(B)=\{\mathfrak{m}\in \operatorname{Max}(B) \mid B_{\mathfrak{m}}\otimes_A k(\mathfrak{m}\cap A)$ is geometrically regular over $k(\mathfrak{m}\cap A)\}$. And for an ideal $\mathfrak{a}$ of B, we put $Z(\mathfrak{a})=V(\mathfrak{a})\cap \operatorname{Max}(B)$.

Proposition 5.2. *Let A be a noetherian ring, R a noetherian smooth A-algebra and $\mathfrak{a}$ an ideal of R. Put $B=R/\mathfrak{a}$ and let I be an ideal of B. Then B is I-smooth over A if and only if $Z(I)\subseteq \mathscr{F}_A(B)\cap \mathscr{R}_A(B)$.*

Proof. Assume that $Z(I)\subseteq \mathscr{F}_A(B)\cap \mathscr{R}_A(B)$. For $P\in Z(I)$, we put $P\cap A=\mathfrak{p}$. Then, by [9, (39.C) Th. 93] and [6, 0_{IV} (19.7.1)], B_P is PB_P-smooth over $A_{\mathfrak{p}}$. Thus, by [9, (29.E) Th. 64], $(\mathfrak{a}/\mathfrak{a}^2)\otimes_B B_P\to \Omega_{R/A}\otimes_R B_P$ is left-invertible. So $(\mathfrak{a}/\mathfrak{a}^2)\otimes_B \bar{B}_{\bar{P}}\to \Omega_{R/A}\otimes_R \bar{B}_{\bar{P}}$ is left-invertible for every $\bar{P}\in \operatorname{Max}(\bar{B})$, where $\bar{B}=B/I$ and $\bar{P}=P/I$. Now $(\mathfrak{a}/\mathfrak{a}^2)\otimes_B\bar{B}$ is a finite $\bar{B}$-module, and since R is smooth over A, $\Omega_{R/A}\otimes_R\bar{B}$ is a projective $\bar{B}$-module. Thus, by [6, 0_{IV} (19.1.14)], $(\mathfrak{a}/\mathfrak{a}^2)\otimes_B \bar{B}\to\Omega_{R/A}\otimes_R\bar{B}$ is left-invertible. Therefore, by [9, (29.C) Th. 63], B is I-smooth over A. The converse follows easily from [6, 0_{IV} (19.7.1)] and [9, (39.C) Th. 93]. Q.E.D.

Remark. If B is an essentially finite type over A, B satisfies the condition of Proposition 5.2.

Corollary 5.3. *Let A and B be as in Proposition 5.2. Then the following conditions are equivalent:*

(i) *B is smooth over A.*

(ii) *B is flat over A, and $\mathscr{R}_A(B)=\operatorname{Max}(B)$.*

(iii) *B is regular over A.*

Now, for a noetherian ring A and an A-algebra B, we put $\mathscr{L}_A(B)=\{P\in \operatorname{Spec}(B) \mid B$ is P-smooth over $A\}$. It is clear that $\mathscr{L}_A(B)$ is stable under specialization. Moreover, if B is regular over A and B satisfies the condition of Proposition 5.2, then we have $\mathscr{L}_A(B)=\operatorname{Spec}(B)$ by Cor. 5.3, and $\mathscr{L}_A(B)$ is closed in $\operatorname{Spec}(B)$. But in general, $\mathscr{L}_A(B)$ is not necessarily closed in $\operatorname{Spec}(B)$ even if B is regular over A. We shall construct such an example.

First we show the following proposition which is also mentioned without proof in [2, (7.4)].

Proposition 5.4. *Let $(A, \mathfrak{m})$ be a noetherian local ring, $X=\{X_1,\dots, X_n\}$ variables over A and $\mathfrak{a}$ an ideal of $A[[X]]$. Then, if $\mathfrak{a}$ is generated by $A[[X]]$-regular sequence and if $B=A[[X]]/\mathfrak{a}$ is smooth over A, $A[[X]]$ is $\mathfrak{a}$-smooth over A. Conversely, if $\mathfrak{a}=\sum_i (X_i-a_i)A[[X]]$ for some elements $a_1,\dots, a_n\in\mathfrak{m}$ and if $A[[X]]$ is $\mathfrak{a}$-smooth over A, then $\hat{A}=(A, I)^\wedge$ is smooth over A for the ideal $I=\sum a_iA$.*

Proof. Let W be a B-module. Then for the sequence of ring homomorphisms $A\to A[[X]]\to B$, we have the following exact sequence: $H^1(A, B, W)\to H^1(A, A[[X]], W)\to H^2(A[[X]], B, W)$. If B is smooth over A, $H^1(A, B, W)=0$

by [1, *XVI*. Prop. 17]. And if $\mathfrak{a}$ is generated by a regular sequence, $H^2(A[[X]], B, W)=0$ by [1, *VI*. Th. 25]. Thus we have $H^1(A, A[[X]], W)=0$ for every B-module W. Therefore $A[[X]]$ is $\mathfrak{a}$-smooth over A by [1, *XVI*. Prop. 17].

Conversely, if $A[[X]]$ is $\mathfrak{a}$-smooth over A, $\Omega=\Omega_{A[[X]]/A}\otimes_{A[[X]]}(A[[X]]/\mathfrak{a})$ is a projective $A[[X]]/\mathfrak{a}$-module by [9, (29.B), Lemma 1]. Since $A[[X]]/\mathfrak{a}$ is a local ring, Ω is a free $A[[X]]/\mathfrak{a}$-module. Now it follows easily that $\Omega\otimes_{A[[X]]}(A[[X]]/\mathfrak{a}+(X))\cong\bigoplus_{i=1}^{n}(A[[X]]/\mathfrak{a}+(X))dX_i$. Therefore $\Omega\cong\bigoplus_{i=1}^{n}(A[[X]]/\mathfrak{a})dX_i$. Now since $\mathfrak{a}=\sum_i(X_i-a_i)A[[X]]$, $\mathfrak{a}/\mathfrak{a}^2$ is a free $A[[X]]/\mathfrak{a}$-module of rank n, and so the canonical homomorphism $\mathfrak{a}/\mathfrak{a}^2\to\Omega$ is represented by the unit matrix. Thus $\mathfrak{a}/\mathfrak{a}^2\cong\Omega$. Therefore by [9, (29.C), Th. 63], $\hat{A}\cong A[[X]]/\mathfrak{a}$ is smooth over A. Q. E. D.

Now let k be a field such that $ch(k)=p>0$ and $[k: k^p]=\infty$. For the field k, we construct an excellent DVR $(A, \mathfrak{m}, k)$ as in Example 5.1, Case (III). Let $a\in\mathfrak{m}$ and let X be a variable over A. Then $(A, (a))^{\wedge}\cong A[[X]]/(X-a)$ is smooth over A by Example 5.1, Case (III). So for every $a\in\mathfrak{m}$, $(X-a)\in\mathscr{L}_A(A[[X]])$ by Prop. 5.4. Thus $\mathscr{L}_A(A[[X]])$ is dense in Spec$(A[[X]])$. On the other hand, since A is not a finite A^p-module, $A[[X]]$ is not smooth over A by Th. 2.2. Thus $\mathscr{L}_A(A[[X]])\supsetneqq 0$. Moreover $A\to A[[X]]$ is regular because A is excellent. Therefore these rings A and $A[[X]]$ meet our expectations.

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