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On the equations of bioconvective flow

By

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1. Introduction

The purpose of this paper is to study some mathematical questions related to the equations of bioconvective flow. Here "bioconvection" is a convection caused by the concentration of upward swiming microorganisms in culture fluid. To describe this phenomena, a fluid dynamical model was presented by Levandowsky et al. [5] and Moribe [9] independently. The model consists of the equations for the motion of the culture fluid assumed to be viscous and incompressible and for the concentration of microorganisms. Both papers [5, 9] discuss underlying biological and physical idea leading to the equations, and give some qualitative descriptions based on intuitive arguments. To the best of our knowledge, formal mathematical analysis has never been carried out. So we treat this model in this paper and give some results.

After a brief description of the fluid dynamical model in Section 2, we show in Section 3 that, for an arbitrarily given $\alpha > 0$, there is a solution of the stationary problem with total concentration equal to α. Section 4 deals with the pointwise positivity of the concentration obtained in Section 3. The following sections (5 to 7) treat the nonstationary problem. We formulate in Section 5 the decay problem for the equations governing the disturbances from the stationary solution whose total concentration is equal to that of the initial data, and define a global weak solution for this problem. Then we show that, if the stationary solution is small enough, there is a global weak soluiton. In Section 6 we prove that the above weak solution becomes regular after some instant, by transforming the equations into an evolution equation in some Hilbert space. The solvability of this evolution equation is proved by the method developed in [2]. Using the results in Section 6, we show in Section 7 the uniform decay of the weak solution obtained in Section 5. The arguments employed in Section 5–7 are similar to those in [7]. In the final Appendix we show the self-adjointness of the operator introduced in Section 6.

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2. Fluid dynamical model

Let Ω be a bounded domain in R^3 with smooth boundary $\partial\Omega$. Let $c(x, t)$ denote the concentration of microorganisms at point $x = (x_1, x_2, x_3) \in \Omega$ at time $t \geq 0$, and let $u = \{u_j(x, t)\}_{j=1}^3$ and $p = p(x, t)$ denote the velocity and pressure of the culture fluid at $x \in \Omega$ at t. u, p and c are governed by a system of the equations:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u, \nabla)u + \nabla p = -g(1 + \gamma c)\chi + f, \qquad x \in \Omega,\ t > 0, \tag{2.1}$$

$$div\, u = 0, \qquad x \in \Omega,\ t > 0, \tag{2.2}$$

$$\frac{\partial c}{\partial t} - \theta \Delta c + (u, \nabla)c + U\frac{\partial c}{\partial x_3} = 0, \qquad x \in \Omega,\ t > 0. \tag{2.3}$$

Here $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ and $\Delta = \sum_{j=1}^3 \left(\frac{\partial}{\partial x_j}\right)^2$. $f = \{f_j(x)\}_{j=1}^3$ is the given external force. For simplicity f is assumed to be independent of t. g is the accelation of gravity, χ is the unit vector in the vertical direction, i. e., $\chi = {}^t(0, 0, 1)$. ν is the kinematic viscosity of the culture fluid, and the constant θ is the diffusion rate of microorganisms. The positive constant U denotes the mean speed of upward swimming of microorganisms. The positive constant γ is given by

$$\gamma = \frac{\rho_0}{\rho_m} - 1$$

where ρ_0 and ρ_m are the density of an individual organism and the culture fluid respectively. For the derivation of (2.1)–(2.3), see [5] and [9].

Put $c = \kappa(g\gamma)^{-1} m$ where $\kappa > 0$ is a constant specified later, and put $p = q - gx_3$. Then, (2.1)–(2.3) become

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u, \nabla)u + \nabla q = -\kappa m \chi + f, \qquad x \in \Omega,\ t > 0, \tag{2.4}$$

$$div\, u = 0. \qquad x \in \Omega,\ t \geq 0, \tag{2.5}$$

$$\frac{\partial m}{\partial t} - \theta \Delta m + (u, \nabla)m + U\frac{\partial m}{\partial x_3} = 0, \qquad x \in \Omega,\ t > 0. \tag{2.6}$$

We supplement (2.4)–(2.6) with the following initial and boundary conditions:

$$u(x, 0) = u_0(x, 0), \quad m(x, 0) = m_0(x), \qquad x \in \Omega, \tag{2.7}$$

$$u(x, t) = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{2.8}$$

$$\theta\frac{\partial m}{\partial n} - Un_3(x)m = 0, \qquad x \in \partial\Omega,\ t > 0. \tag{2.9}$$

Here $n(x) = \{n_j(x)\}_{j=1}^3$ is the unit outward normal at point $x \in \partial\Omega$, and $\frac{\partial}{\partial n}$ is the normal derivative on $\partial\Omega$.

Remark. (2.3) is the conservation equation

$$\left(\frac{d}{dt}\right)c + div\, J = 0, \qquad x \in \Omega,\ t > 0,$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + (u, \nabla)$ is the derivative along the fluid particle, and J is the flux of microorganisms given by $J = -\theta\nabla c + Uc\chi$. (2.9) states the no-flux condition at each point $x \in \partial\Omega$.

3. Stationary problem

We first introduce some function spaces used in this paper. $H^m(\Omega)$ denotes the Sobolev space of real valued functions on Ω which are in $L^2(\Omega)$ together with their weak derivatives of order less than or equal to m. By $C_{0,\sigma}^\infty(\Omega)$ we denote the space of all smooth solenoidal vector fields with compact support in Ω. Let V and H be the completion of $C_{0,\sigma}^\infty(\Omega)$ in $(H^1(\Omega))^3$ and $(L^2(\Omega))^3$ respectively. Let X denote the closed subspace of $L^2(\Omega)$ consisting of functions orthogonal to the constants, and set $B = H^1(\Omega) \cap X$. For $v \in V$ and $\phi \in B$ we have

$$|v| \leq C_\Omega |\nabla v|, \tag{3.1}$$

$$|\phi| \leq C_\Omega |\nabla \phi|, \tag{3.2}$$

where $|\cdot|$ denotes the usual $L^2(\Omega)$ norm and C_Ω is independent of v and ϕ. ((3.1) is Poincaré inequality. (3.2) is due to [10, Th. 3.6.5].) For $(v, \phi), (w, \psi) \in V \times H^1(\Omega)$ we set

$$[(v, \phi), (w, \psi)] = \nu(\nabla v, \nabla w) + \theta(\nabla \phi, \nabla \psi) \tag{3.3}$$

where $(\cdot,\cdot)$ is the usual $L^2(\Omega)$ inner product. Thanks to (3.1)–(3.2), this bilinear form is actually a scalar product on $V \times B$. The norm on $V \times B$ corresponding to (3.3) is denoted by $\|\cdot\|$. In what follows we write

$$b_0(u, v, w) = ((u, \nabla)v, w) = \int_\Omega u_j\left(\frac{\partial v_k}{\partial x_j}\right)w_k dx,$$

$$b_1(u, \phi, \psi) = ((u, \nabla)\phi, \psi) = \int_\Omega u_j\left(\frac{\partial \phi}{\partial x_j}\right)\psi dx,$$

where u, v and $w \in V$, and $\phi, \psi \in H^1(\Omega)$. Here and hereafter we use summation convention, i.e., sum over repeated indices. By the Hölder inequality and the Sobolev imbedding theorem, the tri-linear form b_0 makes sense and is estimated as

$$|b_0(u, v, w)| \leq |u|_{L^4}|\nabla v\|w|_{L^4} \leq C_0|\nabla u\|\nabla v\|\nabla w|. \tag{3.4}$$

See [12, Chap. II, Sect. 1]. Similarly, b_1 can be defined for $u \in V$ and $\phi, \psi \in H^1(\Omega)$. Further, if $\psi \in B$, then by (3.2) b_1 can be estimated as

$$|b_1(u, \phi, \psi)| \leq |u|_{L^4} |\nabla \phi| |\psi|_{L^4} \leq C_0' |\nabla u| |\phi| |\nabla \psi|. \tag{3.5}$$

(We may assume that $C_0 = C_0'$ by choosing larger one if necessary.) Note that, since $div\ u = 0$ in Ω and $u = 0$ on $\partial\Omega$ if $u \in V$, integration by parts gives

$$b_0(u, v, w) = -b_0(u, w, v), \qquad b_1(u, \phi, \psi) = -b_1(u, \psi, \phi). \tag{3.6}$$

The problem we consider in this section is the following: For an arbitrarily given $\alpha > 0$, find (u, m) such that

$$\int_\Omega m dx = \alpha, \tag{3.7}$$

$$-\nu \Delta u + (u, \nabla) u + \nabla q = -\kappa m \chi + f \quad \text{in } \Omega, \tag{3.8}$$

$$div\, u = 0 \quad \text{in } \Omega, \tag{3.9}$$

$$-\theta \Delta m + (u, \nabla) m + U \frac{\partial m}{\partial x_3} = 0 \quad \text{in } \Omega, \tag{3.10}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{3.11}$$

$$\theta \frac{\partial m}{\partial n} - U n_3 m = 0 \quad \text{on } \partial\Omega. \tag{3.12}$$

In what follows we set $\kappa = \left(\frac{2\nu U}{C_\Omega^3}\right)^{\frac{1}{2}}$. Throughout this paper we assume

$$\frac{U}{\theta} < (2C_\Omega)^{-1}. \tag{3.13}$$

The main result of this section is the following theorem.

Theorem 3.1. *Let U and θ be as above. Let $f \in H$. Then, there are $u_\alpha \in (H^2(\Omega))^3 \cap V$, $m_\alpha \in H^2(\Omega)$ and $p_\alpha \in H^1(\Omega)$ satisfying* (3.7)–(3.12).

We prove this in several steps. We seek m_α in the form $m_\alpha = \tilde{m} + E$, where $E(x) = C_\alpha \exp\left(\frac{U}{\theta} x_3\right)$. The constant C_α is chosen so that $\int_\Omega E(x) dx = \alpha$. A direct calculation shows

$$-\theta \Delta E + U \frac{\partial E}{\partial x_3} = 0 \quad \text{in } \Omega, \qquad \theta \frac{\partial E}{\partial x_3} - U n_3 E = 0 \quad \text{on } \partial\Omega. \tag{3.14}$$

Hence, the problem for u, q and $\tilde{m}$ becomes

$$-\nu \Delta u + (u, \nabla) u + \nabla (q + \kappa \theta U^{-1} E) = -\kappa \tilde{m} \chi + f \quad \text{in } \Omega, \tag{3.15}$$

$$div\, u = 0 \quad \text{in } \Omega, \tag{3.16}$$

(3.17) $$-\theta\Delta\tilde{m}+(u,\nabla)(\tilde{m}+E)+U\frac{\partial\tilde{m}}{\partial x_3}=0 \quad \text{in } \Omega,$$

(3.18) $$u=0 \quad \text{on } \partial\Omega,$$

(3.19) $$\theta\frac{\partial\tilde{m}}{\partial n}-Un_3\tilde{m}=0 \quad \text{on } \partial\Omega,$$

(3.20) $$\int_\Omega \tilde{m}(x)dx=0.$$

Definition 3.2. For $f\in H$, we call an element $(u,\tilde{m})\in V\times B$ a weak solution of (3.15)–(3.20), if and only if the following identity is satisfied:

(3.21) $$[(u,\tilde{m}),(v,\phi)]-b_0(u,v,u)-b_1(u,\phi,\tilde{m}+E)+U\left(\tilde{m},\frac{\partial\phi}{\partial x_3}\right)+\kappa(\tilde{m},\chi\cdot v)=0$$

for any $(v,\phi)\in V\times B$. ($\chi\cdot v$ is the third component v_3 of $v\in V$.)

Proposition 3.3. *For each $f\in H$, there is a weak solution $(u,\tilde{m})$ of* (3.15)–(3.20).

Proof. Let $(w,\psi)\in V\times B$. By (3.1–2), (3.4–5) and Schwarz's inequality, we have

$$\begin{aligned}&|b_0(w,v,w)+b_1(w,\phi,\psi+E)-U\left(\psi,\frac{\partial\phi}{\partial x_3}\right)-\kappa(\psi,\chi\cdot v)|\\&\le C_0|\nabla w|^2|\nabla v|+C_0|\nabla w\|\nabla\phi\|\nabla\psi|+C|w\|\nabla\phi|+U|\psi\|\nabla\phi|+\kappa|\psi\|v|\\&\le C(|\nabla w|^2+|\nabla\psi\|\nabla w|+|w|+|\psi|)\|(v,\phi)\|\end{aligned}$$

for any $(v,\phi)\in V\times B$. Hence, by Riesz's theorem, there exists an element $A(w,\psi)$ in $V\times B$ such that

$$\begin{aligned}&[A(w,\psi),(v,\phi)]\\&\quad=b_0(w,v,w)+b_1(w,\phi,\psi+E)-U\left(\psi,\frac{\partial\phi}{\partial x_3}\right)-\kappa(\psi,\chi\cdot v)\end{aligned}$$

for any $(v,\phi)\in V\times B$. Since we can regard f as the linear form $(v,\phi)\in V\times B\to(f,v)$, by Riesz' theorem we can choose an element $F\in V\times B$ such that $[F,(v,\phi)]=(f,v)$. Employing the nonlinear operator A and the element F, we can rewrite (3.21) as

$$[(u,\tilde{m})-A(u,\tilde{m})-F,(v,\phi)]=0 \qquad \text{for any } (v,\phi).$$

Therefore, our problem is reduced to find

(3.22) $$(u,\tilde{m})-A(u,\tilde{m})-F=0.$$

We can prove in just the same way as in [4, page 97] that the mapping

$$(v, \phi) \in V \times B \to A(v, \phi) + F \in V \times B$$

is completely continuous in $V \times B$. We next show that the norms of all possible solution $(u^\sigma, m^\sigma) \in V \times B$ of the equation

$$(u^\sigma, m^\sigma) - \sigma\{A(u^\sigma, m^\sigma) + F\} = 0 \qquad \text{for } 0 < \sigma \leq 1 \tag{3.23}$$

are uniformly bounded. This can be done as follows: Taking the scalar product in $V \times B$ of (3.23) with $(u^\sigma, m^\sigma + E - \alpha) \in V \times B$, we obtain

$$\begin{aligned}&[(u^\sigma, m^\sigma), (u^\sigma, m^\sigma)] + \theta(\nabla m^\sigma, \nabla E)\\ &= \sigma\left\{U\left(m^\sigma, \frac{\partial(m^\sigma + E)}{\partial x_3}\right) - \kappa(m^\sigma, \chi \cdot u^\sigma) + (f, u^\sigma)\right\}.\end{aligned} \tag{3.24}$$

Here we have used the fact that

$$b_0(u, v, v) = 0, \ b_1(u, \phi, \phi) = 0 \qquad \text{for } u, v \in V \text{ and } \phi \in B,$$

which follows from (3.6). By (3.1–2) and the fact that $0 < \sigma \leq 1$, one can deduce from (3.24) that

$$\begin{aligned}&v|\nabla u^\sigma|^2 + \theta|\nabla m^\sigma|^2\\ &\leq UC_\Omega|\nabla m^\sigma|^2 + (UC_\Omega + \theta)|\nabla E\,\|\nabla m^\sigma|\\ &\quad + \kappa C_\Omega^2|\nabla m^\sigma\|\nabla u^\sigma| + C_\Omega|f\,\|\nabla u^\sigma|\end{aligned}$$

Put $\kappa = \left(\dfrac{2vU}{C_\Omega^3}\right)^{\frac{1}{2}}$ in this inequality. By Schwarz's inequality, we can deduce

$$\begin{aligned}&\frac{v}{2}|\nabla u^\sigma|^2 + (\theta - 2UC_\Omega)|\nabla m^\sigma|^2\\ &\leq (UC_\Omega + \theta)|\nabla E\,\|\nabla m^\sigma| + C_\Omega|f\,\|\nabla u^\sigma|.\end{aligned}$$

Noting (3.13), from this inequality we can obtain the uniform boundedness of the norms of $(u^\sigma, m^\sigma)(0 < \sigma \leq 1)$ in $V \times B$. The proof of Proposition 3.3 is completed if we apply the Leray-Schauder principle (see [4, Chap. 1, Sect. 3]).

Proof of Theorem 3.1. Let $(u, \tilde{m})$ be the weak solution of (3.15)–(3.20) obtained in Proposition 3.3. Set $u_\alpha = u$ and $m_\alpha = \tilde{m} + E$, where E is the function introduced before Definition 2.2. Putting $\phi \equiv 0$ in (3.21), we see that u_α satisfies

$$v(\nabla u_\alpha, \nabla v) + b_0(u_\alpha, u_\alpha, v) - \kappa(m_\alpha, \chi \cdot v) = 0 \qquad \text{for any } v \in V.$$

Here we have used (3.6) and the fact that $(E, \chi \cdot v) = \dfrac{\theta}{U}(\nabla E, v)$. Then, by the regularity result given in [4, Chap. 5, Sect. 5], we see that $u_\alpha \in (H^2(\Omega))^3$. We next put $v \equiv 0$ in (3.21). Then,

$$\theta(\nabla m_\alpha, \nabla \phi) + b_1(u_\alpha, m_\alpha, \phi) + U\left(\frac{\partial m_\alpha}{\partial x_3}, \phi\right) - U\int_{\partial\Omega} m_\alpha \phi n_3 dS = 0$$

for any $\phi \in H^1(\Omega)$, which states that m_α is a generalized solution of (3.10) with (3.12) where u is replaced by u_α. Applying the regularity theorem in [8, Chap. 3, Sect. 12], we can show that $m_\alpha \in H^2(\Omega)$ and satisfies (3.12). For the existence of $p_\alpha \in H^1(\Omega)$ such that

$$\nabla p_\alpha = \nu \Delta u_\alpha - (u_\alpha, \nabla) u_\alpha - \kappa m_\alpha \chi + f,$$

see [12, Chap. I] or [4, Chap. 2].

4. Positivity of concentration

In this section we prove

Theorem 4.1. *Let* (u_α, m_α) *be the solution of* (3.7)–(3.12) *given in Theorem* 3.1. *Then,* $m_\alpha(x) > 0$ *for any* $x \in \Omega$.

To prove this we need to consider an auxiliary linear problem:

$$\frac{\partial h}{\partial t} - \theta \Delta h + (u_\alpha, \nabla) h + U \frac{\partial h}{\partial x_3} = 0, \qquad (x, t) \in \Omega \times (0, \infty), \tag{4.1}$$

$$\theta \frac{\partial h}{\partial n} - U n_3 h = 0, \qquad (x, t) \in \partial\Omega \times (0, \infty), \tag{4.2}$$

$$h(x, 0) = E(x), \qquad x \in \Omega, \tag{4.3}$$

where $E = C_\alpha \exp\left(\frac{U}{\theta} x_3\right)$ with $\int_\Omega E\, dx = \alpha$. For the existence of $h \in C^1((0, \infty); L^2(\Omega)) \cap C([0, \infty); H^2(\Omega))$ satisfying (4.1)–(4.3), see [1, Part 2] or [8, Chap. 5].

Lemma 4.2. *For any* $t \geq 0$, $\int_\Omega h(x, t) dx = \alpha$.

Proof. Differentiating $\int_\Omega h(x, t) dx$ in t, and making use of (4.1), we have

$$\left(\frac{d}{dt}\right) \int_\Omega h(x, t) dx = \int_\Omega \left(\theta \Delta h - U \frac{\partial h}{\partial x_3}\right) dx - \int_\Omega (u_\alpha, \nabla) h dx.$$

By the divergence theorem and (4.2), it holds that the first term in the right hand side vanishes. Since $div\, u_\alpha = 0$ and $u_\alpha = 0$ on $\partial\Omega$, integration by parts implies that the second term also vanishes. Hence the conclusion holds.

Lemma 4.3. *For any* $(x, t) \in \Omega \times [0, \infty)$, $h(x, t) > 0$.

Proof. We first note that $h(x, 0) > 0$. Suppose this lemma is false. Then there is a $y \in \bar{\Omega}$ and $T > 0$ such that $h(y, T) = 0$ and $h(x, t) > 0$ for $(x, t) \in \bar{\Omega} \times [0, T)$. Then, by the maximum principle [11, pp. 174–175], y is on the boundary $\partial\Omega$ and $\frac{\partial h}{\partial n}(y, T) < 0$. This is impossible since $\frac{\partial h}{\partial n}(y, T) = 0$ by (4.2), which shows the lemma.

Proof of Theorem 4.1. Note that $h(x, t) - m_\alpha(x) \in B \cap H^2(\Omega)$ for $t > 0$ by Lemma 4.2 and satisfies

$$\frac{\partial(h - m_\alpha)}{\partial t} - \theta\Delta(h - m_\alpha) \tag{4.4}$$

$$+ (u_\alpha, \nabla)(h - m_\alpha) + U\frac{\partial(h - m_\alpha)}{\partial x_3} = 0 \quad \text{in } \Omega \times (0, \infty),$$

$$\theta\frac{\partial(h - m_\alpha)}{\partial n} - Un_3(h - m_\alpha) = 0 \quad \text{on } \partial\Omega \times [0, \infty). \tag{4.5}$$

Since $b_1(u_\alpha, h - m_\alpha, h - m_\alpha) = 0$ by (3.6), the inner product of (4.4) with $h - m_\alpha$ becomes as follows

$$\left(\frac{d}{dt}\right)|h - m_\alpha|^2 - 2(\theta\Delta(h - m_\alpha), h - m_\alpha) + 2\left(U\frac{\partial(h - m_\alpha)}{\partial x_3}, h - m_\alpha\right) = 0.$$

Integrating by parts and using (4.5), we obtain

$$\left(\frac{d}{dt}\right)|h - m_\alpha|^2 + 2\theta|\nabla(h - m_\alpha)|^2 - 2U\left(h - m_\alpha, \frac{\partial(h - m_\alpha)}{\partial x_3}\right) = 0.$$

Then, by Schwarz's inequality and (3.2),

$$\left(\frac{d}{dt}\right)|h - m_\alpha|^2 + C|h - m_\alpha|^2 \leq 0,$$

where $C \equiv 2C_\Omega^{-2}(\theta - UC_\Omega) > 0$ by the assumption (3.13). From this one easily deduces that $h - m_\alpha$ tends to zero in $L^2(\Omega)$ as $t \to \infty$. Since $h(x, t) > 0$ for $(x, t) \in \Omega \times (0, \infty)$ by Lemma 4.3, we see that $m_\alpha(x) \geq 0$ for $x \in \Omega$. Suppose $m_\alpha(y) = 0$ for some $y \in \bar{\Omega}$, then y must be on $\partial\Omega$ and $\left(\frac{\partial m_\alpha}{\partial n}\right)(y) < 0$ by the maximum principle [11, pp. 65–66]. On the other hand, from (3.12) $\left(\frac{\partial m_\alpha}{\partial n}\right)(y) = \left(\frac{U}{\theta}\right)n_3(y)m_\alpha(y) = 0$, which leads to a contradiction. Thus we have proved Theorem 4.1.

5. Reduction to decay problem

Let us consider the initial boundary value problem (2.4)–(2.9) with the initial value of concentration m_0 satisfying $\int_\Omega m_0 dx = \alpha(> 0)$. Let u_α, p_α and m_α be the solution of the stationary problem (3.7)–(3.12) obtained in Theorem 3.1. By the same argument as in Lemma 4.2, we can show that, if there is a smooth solution of

(2.4)–(2.9), its concentration m satisfies $\int_{\Omega} m(x, t)dx = \alpha$ for any $t \geq 0$. From this observation we are led to the following problem: Set $v = u - u_\alpha$ and $\mu = m - m_\alpha$, then consider the equations governing the disturbances from (u_α, m_α),

$$\frac{\partial v}{\partial t} - \nu \Delta v + (u_\alpha, \nabla)v + (v, \nabla)u_\alpha + (v, \nabla)v + \nabla(q - p_\alpha) = -\kappa\mu\chi \quad \text{in } \Omega \times (0, T), \tag{5.1}$$

$$div\, v = 0 \quad \text{in } \Omega \times [0, T), \tag{5.2}$$

$$\frac{\partial \mu}{\partial t} - \theta\Delta\mu + (u_\alpha, \nabla)\mu + (v, \nabla)m_\alpha + (v, \nabla)\mu + U\frac{\partial \mu}{\partial x_3} = 0 \quad \text{in } \Omega \times (0, T), \tag{5.3}$$

$$v = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{5.4}$$

$$\theta\frac{\partial \mu}{\partial n} - Un_3\mu = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{5.5}$$

$$v(x, 0) = a(x), \quad \mu(x, 0) = b(x), \quad x \in \Omega, \tag{5.6}$$

where $(a, b) \in H \times X$.

Definition 5.1. We call (v, μ) a weak solution of (5.1)–(5.6) if (v, μ) satisfies the following conditions:

i) $v \in L^2(0, \infty; V) \cap L^\infty(0, \infty; H)$, $\mu \in L^2(0, \infty; B) \cap L^\infty(0, \infty; X)$,

ii) the identity

$$\begin{aligned} &-\int_0^\infty \{(v(t), w'(t)) + (\mu(t), \phi'(t))\}dt + \int_0^\infty [(v(t), \mu(t)), (w(t), \phi(t))]dt \\ &+ \int_0^\infty \Big\{ b_0(u_\alpha, v(\tau), w(\tau)) + b_0(v(\tau), u_\alpha, w(\tau)) + b_0(v(\tau), v(\tau), w(\tau)) \\ &+ b_1(u_\alpha, \mu(\tau), \phi(\tau)) + b_1(v(\tau), m_\alpha, \phi(\tau)) + b_1(v(\tau), \mu(\tau), \phi(\tau)) \\ &+ \kappa(\mu(\tau, \chi\cdot w(\tau)) - U\left(\mu(\tau), \frac{\partial \phi(\tau)}{\partial x_3}\right)\Big\}d\tau = (w(0), a) + (\phi(0), b) \end{aligned} \tag{5.7}$$

holds for any $(w, \phi) \in L^2(0, \infty; V \times B) \cap H^1(0, \infty; H \times X)$,

iii) the energy inequality

$$\begin{aligned} &|v(t)|^2 + |\mu(t)|^2 + 2\int_0^t \|(v(\tau), \mu(\tau))\|^2 d\tau \\ &+ 2\int_0^t \Big\{ b_0(v(\tau), u_\alpha, v(\tau)) + b_1(v(\tau), m_\alpha, \mu(\tau)) + \kappa(\mu(\tau), \chi\cdot v(\tau)) \end{aligned} \tag{5.8}$$

$$- U\left(\mu(\tau), \frac{\partial \mu(\tau)}{\partial x_3}\right)\Big\}d\tau \le |a|^2 + |b|^2,$$

holds for almost every $t \ge 0$.

Theorem 5.2. *Let* (u_α, m_α) *be as above. If* $|\nabla u_\alpha|$ *and* $|\nabla m_\alpha|$ *are so small that*

$$|\nabla m_\alpha| < 2\frac{\theta - 2UC_\Omega}{C_0}, \quad 2|\nabla u_\alpha| + |\nabla m_\alpha| < \frac{\nu}{C_0}, \tag{5.9}$$

where C_0 *is the constant in* (3.4–5), *then, for an arbitrarily given* $(a, b) \in H \times X$, *there is a weak solution* (v, μ) *of* (5.1)–(5.6).

This theorem is proved by the usual Galerkin approximation. We give only an outline of the proof:

(i) Take a complete orthonormal basis $\{(w_j, \phi_j)\}_{j=1}^{\infty}$ in $H \times X$ such that $w_j \in C_{0,\sigma}^{\infty}(\Omega)$ and $\phi_j \in C^\infty(\bar{\Omega}) \cap X$ satisfying $\theta \frac{\partial \phi_j}{\partial n} - Un_3\phi_j = 0$ on $\partial\Omega, j = 1, 2, \ldots$. (For the existence of such a basis, see Appendix.) We take as an l-th approximation the solution $(v_l, \mu_l) = \sum_{j=1}^{l} h_{jl}(t)(w_j, \phi_j)$ for the system of ordinary differential equations

$$\left(\frac{d}{dt}\right)\{(v_l, w_j) + (\mu_l, \phi_j)\} \tag{5.10}$$

$$= - [(v_l, \mu_l), (w_j, \phi_j)] - b_0(u_\alpha, v_l, w_j) - b_0(v_l, u_\alpha, w_j)$$

$$- b_0(v_l, v_l, w_j) - b_1(u_\alpha, \mu_l, \phi_j) - b_1(v_l, m_\alpha, \phi_j) - b_1(v_l, \mu_l, \phi_j)$$

$$- \kappa(\mu_l, \chi \cdot w_j) + U\left(\mu_l, \frac{\partial \phi_j}{\partial x_3}\right), \qquad j = 1, \ldots l,$$

with initial conditions

$$h_{jl}(0) = (a, w_j) + (b, \phi_j), \qquad j = 1, \ldots, l.$$

Multiplying (5.10) by $h_{jl}(t)$ and summing in j, we obtain

$$\left(\frac{d}{dt}\right)\{|v_l|^2 + |\mu_l|^2\} + 2\nu|\nabla v_l|^2 + 2\theta|\nabla \mu_l|^2$$

$$\le 2\left\{- b_0(v_l, u_\alpha, v_l) - b_1(v_l, m_\alpha, \mu_l) - \kappa(\mu_l, \chi \cdot v_l) + U\left(\mu_l, \frac{\partial \mu_l}{\partial x_3}\right)\right\}$$

$$\le 2\{C_0|\nabla u_\alpha\|\nabla v_l|^2 + C_0|\nabla m_\alpha\|\nabla v_l\|\mu_l| + \kappa C_\Omega^2|\nabla \mu_l\|\nabla v_l| + UC_\Omega|\nabla \mu|^2\}$$

by (3.4–5), Schwarz's inequality and (3.1–2). By the definition of κ, (3.13) and (5.9), one easily deduces

$$\frac{d}{dt}\{|v_l|^2 + |\mu_l|^2\} + C\,\|(v_l, \mu_l)\|^2 \le 0,$$

where $C = \min\left\{1 - \frac{C_0}{\nu}(2|\nabla u_\alpha| + |\nabla m_\alpha|),\ 2 - \frac{1}{\theta}(4UC_\Omega + C_0|\nabla m_\alpha|)\right\} > 0$. Integrating this yields

$$|u_l(t)|^2 + |\mu_l(t)|^2 + C\int_0^t \|(v_l(s), \mu_l(s))\|^2\, ds \le |a|^2 + |b|^2, \qquad l = 1, 2, 3, \dots, \tag{5.11}$$

as long as $(v_l(t), \mu_l(t))$ is defined. From this, we see that each $(v_l(t), \mu_l(t))$ is defined for all $t \ge 0$ and belongs to $L^2(0, \infty; V \times B) \cap L^\infty(0, \infty; H \times X)$. Furthermore, $\{(v_l, \mu_l)\}_{l=1}^\infty$ forms a bounded sequence in $L^2(0, \infty; V \times B) \cap L^\infty(0, \infty; H \times X)$.

(ii) Applying the argument in [12, Chap. III, Sect.3], we can choose a subsequence $\{(v_{l_h}, \mu_{l_h})\}_{h=1}^\infty$ and an element $(v, \mu) \in L^2_{loc}(0, \infty; V \times B) \cap L^\infty(0, \infty; H \times X)$ such that, for an arbitrarily fixed $T > 0$,

$(v_{l_h}, \mu_{l_h}) \to (v, \mu)$ in the weak topology of $L^2(0, T; V \times B)$ and in the weak-star topology of $L^\infty(0, T; H \times X)$;
$(v_{l_h}, \mu_{l_h}) \to (v, \mu)$ in $L^2(0, T; H \times X)$.

Also, by the argument in [12, Chap. III, Sect. 3, Remark 3.2], it follows that (v, μ) satisfies the energy inequality (5.8). Finally, by letting $h \to \infty$ in (5.11) with l replaced by l_h, we see that $(v, \mu) \in L^2(0, \infty; V \times B)$. This element (v, μ) is our desired weak solution.

6. Regularity of weak solutions of decay problem

In this section we transform (5.1)–(5.6) into an abstract initial value problem in $H \times X$. Let P_0 denote the orthogonal projection: $(L^2(\Omega))^3 \to H$. Let P_1 denote the orthogonal projection: $L^2(\Omega) \to X$. $A_0 \equiv P_0(-\nu\Delta)$ denotes the Stokes operator with $D(A_0) = (H^2(\Omega))^3 \cap V$ (see [2, 4, 7]). A_1 is *the Friedrichs extension* of the symmetric operator $P_1(-\theta\Delta)$ defined for $\phi \in X \cap H^2(\Omega)$ satisfying $\theta\frac{\partial\phi}{\partial n} - Un_3\phi = 0$ on $\partial\Omega$. As shown in **Appendix**, A_1 is the positive self-adjoint operator with $D(A_1) = \left\{\phi \in X \cap H^2(\Omega);\ \theta\frac{\partial\phi}{\partial n} - Un_3\phi = 0 \text{ on } \partial\Omega\right\}$. From the definition of A_1, it follows that $D(A_1^{\frac{1}{2}}) = B$ and

$$(\theta - 2UC_\Omega)^{\frac{1}{2}}|\nabla u| \le |A_1^{\frac{1}{2}}u| \le (\theta + 2UC_\Omega)^{\frac{1}{2}}|\nabla u| \qquad \text{for } u \in B. \tag{6.1}$$

For $u, v \in V$ and $\phi \in B$ we put

$$B_0(u, v) = -P_0(u, \nabla)v, \quad B_1(u, \phi) = -P_1(u, \nabla)\phi. \tag{6.2}$$

Applying P_0 and P_1 to (5.1) and (5.3) respectively, we obtain

$$\frac{dv}{dt} + A_0 v - B_0(u_\alpha, v) - B_0(v, u_\alpha) - B_0(v, v) + \kappa P_0\mu\chi = 0, \qquad t > 0, \tag{6.3}$$

$$\frac{d\mu}{dt} + A_1\mu - B_1(u_\alpha, \mu) - B_1(v, m_\alpha) - B_1(v, \mu) + UP_1\partial_3\mu = 0, \qquad t > 0, \tag{6.4}$$

where $\partial_3 = \dfrac{\partial}{\partial x_3}$.

In view of the spectral representation for A_0 and A_1, we have

Lemma 6.1. *Let* $\alpha \in (0, e)$. *Then*

$$|A_1^\alpha e^{-tA_1}| \le t^{-\alpha} \quad \text{for} \quad t>0, \tag{6.5}$$

$$|A_0^\alpha e^{-tA_0}| \le t^{-\alpha} \quad \text{for} \quad t>0. \tag{6.6}$$

Here and hereafter we use $|\cdot|$ to denote the operator norm in H and X. For the proof of this lemma, see [2, Section 2, III].

Lemma 6.2. *For* $v \in V$ *and* $\phi \in B$, *we have*

i) $|B_0(u_\alpha, v)| \le M_1 |A_0^{\frac{1}{2}} v|, \quad |B_0(v, u_\alpha)| \le M_1 |A_0^{\frac{1}{2}} v|$;

ii) $|B_1(u_\alpha, \phi)| \le M_1 |A_1^{\frac{1}{2}} v|, \quad |B_1(v, m_\alpha)| \le M_1 |A_0^{\frac{1}{2}} v|$;

iii) $|\kappa P_0 \phi \chi| \le M_1 |A_1^{\frac{1}{2}} \phi|, \quad |UP_1\partial_3\phi| \le M_1 |A_1^{\frac{1}{2}} \phi|$;

where M_1 *is independent of* v and ϕ.

These estimates can be easily proved by using the Sobolev imbedding theorem and (3.1–2). So we omit the proof.

Lemma 6.3. *We have*

i) $|B_0(u, v)| \le M_2 |A_0^{\frac{1}{2}} u\| A_0^{\frac{3}{4}} v|$ for $u \in D(A_0^{\frac{1}{2}})$ and $v \in D(A_0^{\frac{3}{4}})$,

ii) $|A_0^{-\frac{1}{4}} B_0(u, v)| \le M_2 |A_0^{\frac{1}{2}} u\| A_0^{\frac{1}{2}} v|$ for $u, v \in D(A_0^{\frac{1}{2}})$,

iii) $|B_1(u, \phi)| \le M_2 |A_0^{\frac{1}{2}} u\| A_1^{\frac{3}{4}} \phi|$ for $u \in D(A_0^{\frac{1}{2}})$ and $\phi \in D(A_1^{\frac{3}{4}})$,

iv) $|A_1^{-\frac{1}{4}} B_1(u, \phi)| \le M_2 |A_0^{\frac{1}{2}} u\| A_1^{\frac{1}{2}} \phi|$ for $u \in D(A_0^{\frac{1}{2}})$ and $\phi \in D(A_1^{\frac{1}{2}})$,

where M_2 *is independent of* u, v *and* ϕ.

Proof. *i)* and *ii)* are well known ([2, 3]) while *iii)* and *iv)* can be proved by the same arguments of those of [3, Lemmas 2.1–2.2] where we replace the Stokes operator by the positive operator A_1.

The main result of this section is the following theorem.

Theorem 6.4. *Let the assumptions in Theorem* 5.2 *hold. Let* (v, μ) *be a weak solution of* (5.1)–(5.6) *obtained in Theorem* 5.2. *Then, there is a* $t_0 > 0$ *such that* (v, μ) *belongs to* $C^1((t_0, \infty); H \times X) \cap C((t_0, \infty); D(A_0) \times (A_1))$ *and satisfies* (6.3)–(6.4) *for* $t > t_0$.

Since the method in proving this theorem is essentially due to [7, Sections 2, 3], we only review an outline of the proof. We first rewrite (6.3)–(6.4) into the integral form

$$(6.7)\qquad v(t+t_0)=e^{-tA_0}v(t_0)+\int_0^t e^{-(t-s)A_0}\{B_0(u_\alpha, v(s+t_0))+B_0(v(s+t_0),u_\alpha)$$
$$-\kappa P_0\mu(s+t_0)\chi+B_0(v(s+t_0),v(s+t_0))\}ds,$$

$$(6.8)\qquad \mu(t+t_0)=e^{-tA_1}\mu(t_0)+\int_0^t e^{-(t-s)A_1}\{B_1(u_\alpha, \mu(s+t_0))+B_1(v(s+t_0),m_\alpha)$$
$$-UP_1\partial_3\mu(s+t_0)+B_1(v(s+t_0),\mu(s+t_0))\}ds,$$

then consider the iteration scheme:

$$v_0(t+t_0)=e^{-tA_0}v(t_0),\quad \mu_0(t+t_0)=e^{-tA_1}\mu(t_0),$$
$$v_{j+1}(t+t_0)=v_0(t+t_0)$$
$$+\int_0^t e^{-(t-s)A_0}\{B_0(u_\alpha, v_j(s+t_0))+B_0(v_j(s+t_0),u_\alpha)$$
$$-\kappa P_0\mu_j(s+t_0)\chi+B_0(u_j(s+t_0),v_j(s+t_0))\}ds,$$
$$\mu_{j+1}(t+t_0)=\mu_0(t+t_0)$$
$$+\int_0^t e^{-(t-s)A_1}\{B_1(v_\alpha,\mu_j(s+t_0))+B_1(v_j(s+t_0),m_\alpha)$$
$$-UP_1\partial_3\mu_j(s+t_0)+B_1(v_j(s+t_0),\mu_j(s+t_0))\}ds,$$
$$j=0,1,2,\dots.$$

Let $T>0$ be a constant specified later. Put $k_0=\max\{|A_0^{\frac{1}{2}}v(t_0)|, |A_1^{\frac{1}{2}}\mu(t_0)|\}$, and define the sequences $\{K_{\gamma,j}\}_{j=0}^{\infty},\left(\gamma=\frac{1}{2},\frac{3}{4}\right)$ inductively by

$$K_{\gamma,0}=k_0,$$

$$K_{\gamma,j+1}=K_{\gamma,0}+3(1-\gamma)^{-1}M_1T^{\frac{1}{2}}K_{\frac{1}{2},j}+T^{\frac{1}{4}}B\left(1-\gamma,\frac{3}{4}\right)M_2K_{\frac{1}{2},j}K_{\frac{3}{4},j},$$
$$j=0,1,2,\dots,$$

where $B(p,q)$ is the beta function. Then, using Lemmas 6.1–3, we can estimate each step of the above scheme as

$$|A_0^{\gamma}v_j(t+t_0)|\le K_{\gamma,j}t^{\frac{1}{2}-\gamma},\quad |A_1^{\gamma}\mu_j(t+t_0)|\le K_{\gamma,j}t^{\frac{1}{2}-\gamma},$$
$$\text{for}\quad 0<t\le T,\qquad j=0,1,2,\dots,\quad \gamma=\frac{1}{2},\frac{3}{4}.$$

If we set $K_j=\max\{K_{\frac{1}{2},j}, K_{\frac{3}{4},j}\}$ $(j=1,2,\dots)$, then we have

$$k_{j+1} \le K_0 + 12M_1 T^{\frac{1}{2}} k_j + T^{\frac{1}{4}} \beta M_2 k_j^2 \qquad (j = 0, 1, 2, \ldots),$$

where $\beta = B\left(\frac{1}{3}, \frac{3}{4}\right)$. Take $T > 0$ so that $12M_1 T^{\frac{1}{2}} < 1$, and assume that

$$K_0 = \max\{|A_0^{\frac{1}{2}} v(t_0)|, |A_1^{\frac{1}{2}} \mu(t_0)|\} < K^* \tag{6.9}$$

where $K^* \equiv (1 - 12M_1 T^{\frac{1}{2}})^2 (4 T^{\frac{1}{4}} \beta M_2)^{-1}$. Employing the argument in [2] or [7, Sect. 3], we can show that $\{(v_j, \mu_j)\}_{j=1}^{\infty}$ converges to a solution of (6.7)–(6.8) on $(t_0, t_0 + T]$, which satisfies (6.3)-(6.4) there. Then, applying the uniqueness theorem in [4, Chap. 6, Sect. 2] to our case with some modification, we obtain

Proposition 6.5. *Let T and K^* be as above. Let (v, μ) be a weak solution given in Theorem* 5.2. *If there is a $t_0 > 0$ such that $(v(t_0), \mu(t_0)) \in V \times B$ and satisfies* (6.9), *then (v, μ) belongs to $C^1((t_0, T_0 + T]; H \times X) \cap C((t_0, t_0 + T]; D(A_0) \times D(A_1))$ and satisfies* (6.3)-(6.4) *on $(t_0, t_0 + T]$.*

To complete the proof of Theorem 6.4, we need

Lemma 6.6. *Let $\lambda \ge 0$. Let (w, ψ) belong to $L^2(0, \infty; H \times X)$. Then*

$$|A_0^{\frac{1}{2}} \int_0^t e^{-(t-s)(\lambda + A_0)} w(s) ds| \le \frac{1}{\sqrt{2}} \left(\int_0^t |w(s)|^2 ds \right)^{\frac{1}{2}},$$

$$|A_1^{\frac{1}{2}} \int_0^t e^{-(t-s)(\lambda + A_1)} \psi(s) ds| \le \frac{1}{\sqrt{2}} \left(\int_0^t |\psi(s)|^2 ds \right)^{\frac{1}{2}},$$

for any $t \ge 0$.

For the proof, see [7, Lemma 4].

Proof of Theorem 6.4. Let T and K^* be as above. Put $\lambda = \left(4M_2 \Gamma\left(\frac{1}{4}\right) K^*\right)^4$. We proceed as in [7, Lemma 19]. Since $(v, \mu) \in L^2(0, \infty; V \times B)$ there is a $t_0 > 0$ such that

$$|A_0^{\frac{1}{2}} v(t_0)| < \frac{K^*}{4}, \quad |A_1^{\frac{1}{2}} \mu(t_0)| < \frac{K^*}{4}, \tag{6.10}$$

$$\int_{t_0}^{\infty} |A_0^{\frac{1}{2}} v(s)|^2 ds < C^*, \quad \int_{t_0}^{\infty} |A_1^{\frac{1}{2}} \mu(s)|^2 ds < C^*, \tag{6.11}$$

where $C^* \equiv \min\{K^{*2}/(32\lambda), K^{*2}/72M_1^2\}$. Let δ^* be the least upper bound of δ such that $(v(t), \mu(t))$ belongs to $C^1((t_0, t_0 + \delta); H \times X) \cap C((t_0, t_0 + \delta; D(A_0) \times D(A_1))$ and satisfies

$$|A_0^{\frac{1}{2}} v(t)| < K^*, \quad |A_1^{\frac{1}{2}} \mu(t)| < K^* \tag{6.12}$$

on $[t_0, t_0 + \delta)$. By Proposition 6.5, δ^* is positive. Suppose that δ^* is finite.

From (6.3) one can deduce

$$v(t+t_0)=e^{-t(\lambda+A_0)}v(t_0)$$
$$+\int_0^t e^{-(t-s)(\lambda+A_0)}\{\lambda v(s+t_0)+B_0(u_\alpha,\ v(s+t_0))+B_0(v(s+t_0),u_\alpha)$$
$$-\kappa P_0\mu(s+t_0)\chi+B_0(v(s+t_0),v(s+t_0))\}ds$$

for $t\in(t_0,t_0+\delta^*)$. Applying $A_0^{\frac{1}{2}}$ to both sides, we estimate $|A_0^{\frac{1}{2}}v(t+t_0)|$ by using Lemmas 6.1–3 and 6.6. Then, by (6.10–12) and the definitions of λ and C^*, we have $|A_0^{\frac{1}{2}}v(t_0+\delta^*)|<K^*$. Similarly, we have $|A_1^{\frac{1}{2}}\mu(t_0+\delta^*)|<K^*$. Then, Proposition 6.5 implies that there is a $\delta'>\delta^*$ such that $(v(t),\ \mu(t))\in C^1((t_0,t_0+\delta');\ H\times X))\cap C((t_0,t_0+\delta');\ D(A_0)\times D(A_1))$ and (6.12) holds on $[t_0,t_0+\delta')$. This contradicts to the definition of δ^*. Hence, $\delta^*=\infty$, and the assertion of Theorem 6.4 follows from Proposition 6.5.

Remark 6.7. In proving Theorem 6.4, we easily see that $(A_0^\gamma v(t),A_1^\gamma\mu(t))$ $\left(\gamma=\frac{1}{2},\frac{3}{4}\right)$ are uniformly bounded and Hölder continuous on $[t_0+1,\infty)$ with values in $H\times X$, and that $(B_0(v(t),v(t)),\ B_1(v(t),\mu(t))$ are uniformly Hölder continuous on $[t_0+1,\infty)$ with values in $H\times X$.

7. Decay of weak solutions

Finally, under the same assumptions as in Theorem 5.2, we prove

Theorem 7.1. *Let* $(v,\ \mu)$ *be the weak solution given in Theorem* 5.2. *Then,* $\sup_{x\in\bar{\Omega}}|v(x,\ t)|$ *and* $\sup_{x\in\bar{\Omega}}|\mu(x,\ t)|$ *tend to zero as* $t\to\infty$.

First we have

Proposition 7.2. $|A_0^{\frac{1}{2}}v(t)|$ *and* $|A_1^{\frac{1}{2}}\mu(t)|$ *tend to zero as* $t\to\infty$.

For the proof, see [7, Lemma 22].

Since A_0 and A_1 are positive and self-adjoint in H and X respectively, we have

Lemma 7.3. *There is a constant* $\omega>0$ *such that*

$$|A_0e^{-tA_0}|\le t^{-1}e^{-\omega t},\qquad |A_1e^{-tA_1}|\le t^{-1}e^{-\omega t}\qquad \text{for}\quad t>0.$$

Proof of Theorem 7.1. Let t_0 be as in Theorem 6.4. Set $\zeta(s)\equiv e^{-(t-s)A_0}v(s)$. Using (6.3), we can deduce

$$v(t)=e^{-(t-s)A_0}v(s)+\int_s^t\zeta'(\tau)d\tau \tag{7.1}$$
$$=e^{-(t-s)A_0}v(s)+\int_s^t e^{-(t-s)A_0}\Psi(\tau)d\tau,\qquad \text{for}\quad t>s\ge t_0+1,$$

where

$$\Psi(s) \equiv B_0(u_\alpha, v(s)) + B_0(v(s), u_\alpha) - \kappa P_0 \mu(s)\chi + B_0(v(s), v(s)) \qquad (s > t_0).$$

As noted in Remark 6.7, $\Psi(s)$ is uniformly bounded and there is a $\gamma \in (0,1)$ such that

$$|\Psi(t) - \Psi(s)| \leq C|t - s|^\gamma \qquad \text{for} \quad t, s \geq t_0 + 1. \tag{7.2}$$

Also note that $|\Psi(t)| \to 0$ as $t \to \infty$ by Proposition 7.2 and Remark 6.7. Differentiate (7.1) in t, then, after some calculation, we obtain

$$\frac{dv(t)}{dt} = -A_0 e^{-(t-s)A_0} v(s) + e^{-(t-s)A_0} \Psi(t)$$
$$- \int_s^t A_0 e^{-(t-\tau)A_0} (\Psi(\tau) - \Psi(t)) d\tau.$$

From the boundedness of $\Psi(t)$ $(t \geq t_0 + 1)$ and (7.2), it holds that

$$|\Psi(t) - \Psi(\tau)| = (|\Psi(t) - \Psi(\tau)|^{\frac{1}{2}})^2 \leq C(s)|t - \tau|^{\frac{\gamma}{2}} \qquad (t, \tau \geq s \geq t_0 + 1)$$

Where $C(s) = o(1)$ as $s \to \infty$. Using this and Lemma 7.3, we have

$$\left| \int_s^t A_0 e^{-(t-\tau)A_0} (\Psi(\tau) - \Psi(t)) d\tau \right| \leq C(s) \int_s^t e^{-\omega(t-\tau)} (t - \tau)^{-1+\frac{\gamma}{2}} d\tau$$
$$\leq C_1(s) \quad \text{for} \quad t \geq s \geq t_0 + 1$$

where $C(s) = o(1)$ as $s \to \infty$. By Lemma 7.3 and the fact that $|\Psi(t)| \to 0$ as $t \to \infty$, we see that, for fixed $s \geq t_0 + 1$,

$$|-A_0 e^{-(t-s)A_0} v(s) + e^{-(t-s)A_0} \Psi(t)| \to 0 \qquad \text{as} \quad t \to \infty.$$

Collecting these gives that $\left|\frac{dv(t)}{dt}\right| \to 0$ as $t \to \infty$. Since $(v(t), \mu(t))$ satisfies

$$A_0 v(t) = -\frac{dv(t)}{dt} + \Psi(t),$$

and since the right hand side of this tends to zero as stated above, $|A_0 v(t)|$ tends to zero as $t \to \infty$. In the same way as above, we can show that $|A_1 \mu(t)| \to 0$ as $t \to \infty$. The uniform decay of $(v(t), \mu(t))$ now follows from the Sobolev imbedding theorem: $D(A_0) \times D(A_1) \subset (H^2(\Omega))^3 \times H^2(\Omega) \subset (C(\bar{\Omega}))^4$ with continuous injection.

8. Appendix: Self-adjointness of $P_1(-\theta\Delta)$

Let $\phi \in D(A_1) \equiv \left\{\psi \in H^2(\Omega) \cap X : \theta \frac{\partial \psi}{\partial n} - Un_3\psi = 0 \text{ on } \partial\Omega \right\}$. By the divergence theorem and the boundary condition of ϕ on $\partial\Omega$,

$$(A_1\phi, \phi) = \theta|\nabla\phi|^2 - U\int_{\partial\Omega} n_3(x)\phi(x)^2 dS$$
$$= \theta|\nabla\phi|^2 - 2U\int_\Omega \phi(x)\left(\frac{\partial\phi}{\partial x_3}\right)(x)dx \geq (\theta - 2UC_\Omega)|\nabla\phi|^2.$$

From this, (3.2) and (3.13), the positivity of A_1 follows.

We next show that $D(A_1^*) = D(A_1)$. Let $\psi \in D(A_1^*)$. By the definition of A_1^*, there is an element $f \in X$ such that

(8.1) $$(A_1\phi, \psi) = (\phi, f) \qquad \text{for any } \phi \in D(A_1).$$

Then, in the sense of distribution, it holds that

$$\langle \phi, -\theta\Delta\psi \rangle = (\phi, f) \qquad \text{for any } \phi \in C_0^\infty(\Omega) \cap X.$$

Put $[\phi] = \int_\Omega \phi dx$ for $\phi \in C_0^\infty(\Omega)$ and take $\psi_0 \in C_0^\infty$ so that $[\psi_0] = 1$. Since $\phi - [\phi]\psi_0 \in C_0^\infty(\Omega) \cap X$,

$$\langle \phi - [\phi]\psi_0, -\theta\Delta\psi \rangle = (\phi - [\phi]\psi_0, f).$$

From this we obtain

(8.2) $$\langle \phi, -\theta\Delta\psi - f \rangle = \langle \psi_0, -\theta\Delta\psi - f \rangle \int_\Omega \phi dx \qquad \text{for any } \phi \in C_0^\infty(\Omega).$$

(8.2) means that $-\theta\Delta\psi - f = \langle \psi_0, -\theta\Delta\psi - f \rangle 1$ in the sense of distribution. Since $f \in L^2(\Omega)$ and the right hand side of this equality is a constant function, $-\Delta\psi$ belongs to $L^2(\Omega)$. Hence, $\langle \psi_0, -\theta\Delta\psi - f \rangle$ can be rewritten as

$$\langle \psi_0, -\theta\Delta\psi - f \rangle = (\psi_0, -\theta\Delta\psi - f).$$

In this expression, approximate in $L^2(\Omega)$ the constant function

$$|\Omega|^{-1} \equiv \left(\int_\Omega 1dx\right)^{-1}$$

by $\psi_0 \in C_0^\infty(\Omega)$ with $\int_\Omega \psi_0 dx = 1$. Then, we obtain

$$-\theta\Delta\psi - f = (|\Omega|^{-1}, -\theta\Delta\psi - f) = -|\Omega|^{-1}\int_\Omega \theta\Delta\psi dx \qquad \text{in } L^2(\Omega).$$

Note that $f \in X$. Thus, regarding the right hand side as a constant function, we have

(8.3) $$-\theta\Delta\psi = f - |\Omega|^{-1}\int_\Omega \theta\Delta\psi dx \qquad \text{in } L^2(\Omega).$$

Taking the inner product of (8.3) with $\phi \in D(A_1)$, we obtain

(8.4) $$(-\theta\Delta\psi, \phi) = (f, \phi)$$

since $\phi \in X$. Using Green's formula ([6, Chap. 2, Sect. 6]), we can rewrite the left hand side as

$$(8.5) \qquad (-\theta\Delta\psi, \phi) = (\psi, -\theta\Delta\phi) - \left\langle \theta\frac{\partial\psi}{\partial n}, \phi \right\rangle + \left\langle \psi, \theta\frac{\partial\phi}{\partial n} \right\rangle$$

where the first bracket $\langle\cdot,\cdot\rangle$ denotes the duality between $H^{-\frac{3}{2}}(\partial\Omega)$ and $H^{\frac{3}{2}}(\partial\Omega)$, and the second denotes the duality between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$. On the other hand, since $\psi \in D(A_1^*) \subset X$ and $\phi \in D(A_1)$,

$$(\psi, -\theta\Delta\phi) = (P_1\psi, -\theta\Delta\phi) = (\psi, A_1\phi) = (f, \phi)$$

by (8.1). Also, using the boundary condition of ϕ, we obtain from (8.4–5)

$$\left\langle \theta\frac{\partial\psi}{\partial n} - Un_3\psi, \phi \right\rangle = 0.$$

For an arbitrary $\zeta \in C^\infty(\partial\Omega)$, we can easily construct $\phi \in D(A_1)$ such that $\phi|_{\partial\Omega} = \zeta$. Therefore, $\theta\frac{\partial\psi}{\partial n} - Un_3\psi = 0$ on $\partial\Omega$. From this and (8.3), using the regularity result in [6, Chap. 2], we see that $\psi \in H^2(\Omega) \cap X$ and satisfies the boundary condition, which states that $\psi \in D(A_1)$.

Finally we give a remark for the basis $\{(w_j, \phi_j)\}_{j=1}^\infty$ employed in Section 5. Since A_1 is self-adjoint, its eigenvectors form a complete orthonormal sytem in X. Making use of this system and the eigenvectors of the Stokes operator A_0, we can construct the desired basis.

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