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Graded CM modules over graded normal CM domains

By

Yuji Yoshino

1. Introduction

Remarkable progress has been made in recent years for the classification of maximal Cohen-Macaulay modules (abbr. CM modules) over Henselian Cohen-Macaulay local rings, and the computation using Auslander-Reiten quivers is known to be the steadiest way to do this. For the detail, we recommend the reader to refer to [6] where most of these topics are briefly summarized.

In the present paper we are interested in graded CM modules. Although it may be possible to determine them for several examples by the same method as in Henselian cases, it would require some hard computation. We shall propose, in this paper, a new method to classify graded CM modules over a graded normal CM domain, particularly of dimension two.

Our starting point is that the graded ring R has the Demazure's description, that is,

$$R = \sum_{n \geq 0} \mathrm{H}^0(X, \mathcal{O}_X(nD))\, t^n,$$

where $X = \mathrm{Proj}(R)$ and D is a $\mathbf{Q}$-Weil divisor on X such that ℓD is an ample Cartier divisor for some $\ell > 0$. A theorem of Demazure [2] says that every graded normal domain has such a description. The next section is occupied by some auxiliary results concerning this description that will be used repeatedly in this paper.

Taking an integer ℓ as above, we can define an $\mathcal{O}_X$-order Λ as

$$\Lambda = \begin{pmatrix} \mathcal{O}_X & \mathcal{O}_X(D) & \cdots & \mathcal{O}_X((\ell-1)D) \\ \mathcal{O}_X(-D) & \mathcal{O}_X & \cdots & \mathcal{O}_X((\ell-2)D) \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{O}_X(-(\ell-1)D) & \cdots & \cdots & \mathcal{O}_X \end{pmatrix}.$$

Some of the properties of Λ will be shown in §§4 and 5. Furthermore we can show in (4.7) that, for $n \geq 2$, the category of graded n-th syzygy modules over R is equivalent to that of right Λ-modules which are n-th syzygies. The proof

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of this fact is main part of §§3 and 4.

Let $bdl(\Lambda)$ be the category of locally free $\mathcal{O}_X$-modules with right Λ-module structure. It will be seen in §6 that $bdl(\Lambda)$ is equivalent to the category of d-th syzygy modules over Λ, where $d = \dim(X)$. Combining this with the above result, we will show in (6.8) that, if X is a curve, the category of graded CM modules over R is equivalent to $bdl(\Lambda)$. Thus in the case X is a curve, all the graded CM R-modules are obtained as vector bundles over X with Λ-module structure. And analysing the order Λ, we will be able to see into the latter modules in detail as we will develop in §7. Roughly speaking, one of our main theorems (cf. Theorem (7.6)) says that any CM module over R, when X is a curve, is obtained as an 'extension' of a vector bundle over X by a representation of a certain quiver determined by D.

2. Preliminary; Demazure's description of graded normal domains

Throughout the paper $R = \sum_{n\geq 0} R_n$ is assumed to be a graded normal domain with $R_0 = k$ an algebraically closed field and we adopt a non-essential assumption that $g.c.d.\{n | R_n \neq 0\} = 1$. We denote by K the graded quotient field of R, i.e. the set of all the fractions with homogeneous denominators. Then the assumption is equivalent to saying that there is a homogeneous element t of degree one in K. Letting K_0 be the degree 0 part in K, we know that

$$K = K_0[t, t^{-1}] \text{ (Laurent polynomial ring).}$$

If we denote $X = \mathrm{Proj}(R)$, K_0 is the function field of the projective variety X. By Demazure [2] it is known that there is a Weil divisor D on X with rational coefficients, which is ample and $\mathbf{Q}$-Cartier such that the graded ring R can be written as follows:

Lemma (2.1) $R = R(X, D) = \sum_{n\geq 0} \mathrm{H}^0(X, \mathcal{O}_X(nD))\, t^n$.

Here, for a $\mathbf{Q}$-divisor $D = \sum_i (q_i/p_i) D_i$ $(p_i, q_i \in \mathbf{Z})$, we denote $[D] = \sum_i [q_i/p_i] D_i$ with the convention:

$$[r] = \max\{n \in \mathbf{Z} | n \leq r\} \qquad \text{for } r \in \mathbf{Q},$$

and set $\mathcal{O}_X(D) = \mathcal{O}_X([D])$. In what follows we always denote the constant sheaf whose stalks are K_0 by $\mathscr{K}$, and we regard the sheaf $\mathcal{O}_X(D)$ as a subsheaf of $\mathscr{K}$.

The main purpose of this paper is to show how we can describe the category of graded CM modules by the datum (X, D). Therefore throughout the paper (X, D) is the one given as above. That is, $X = \mathrm{Proj}(R)$, and D is a $\mathbf{Q}$-Weil divisor on X such that ℓD is an ample Cartier divisor for some integer $\ell > 0$. Moreover we take the following notation:

(2.2) $$D = \sum_x (q_x/p_x)[x],$$

where x runs through all the points on X of codimension one. We assume that p_x is always taken to be a positive integer, and that if $q_x \neq 0$, then p_x and q_x are coprime, while $p_x = 1$ if $q_x = 0$. We often write $r_x = q_x/p_x$. Note that $r_x = 0$ for almost all $x \in X$, so that $p_x = 1$ for those x.

Recall the following fact (cf. Watanabe [5]):

Lemma (2.3) *Let x be a point of X of codimension one and let $\mathfrak{p}$ be a corresponding graded prime ideal of R of height one. Note that the graded localization $R_{(\mathfrak{p})}$ of R at $\mathfrak{p}$ is defined by*

$$R_{(\mathfrak{p})} = \{b/a \mid b \in R,\ a \in R - \mathfrak{p},\ a \text{ is homogeneous}\}.$$

If $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$, then we have the equality:

$$R_{(\mathfrak{p})} = \sum_{n \in \mathbf{Z}} \mathfrak{m}_x^{-[nr_x]} t^n.$$

For the points whose codimensions are larger than one, we also have a similar result to the above.

Lemma (2.4) *Let y be an irreducible point of X, and let $\mathfrak{q}$ be the graded prime ideal corresponding to y. Denote by $\Omega(y)$ the set of all points of codimension one that are generalizations of y. Then,*

$$R_{(\mathfrak{q})} = \sum_{n \in \mathbf{Z}} \Big(\bigcap_{x \in \Omega(y)} \mathfrak{m}_x^{-[nr_x]} \Big) t^n$$

Proof. Note that $R_{(\mathfrak{q})} = \cap R_{(\mathfrak{p})}$ where $\mathfrak{p}$ runs through all graded primes of height one with $\mathfrak{p} \subset \mathfrak{q}$. Thus the lemma follows from (2.3).

Now we define an $\mathcal{O}_X$-subalgebra $\mathscr{A}$ of $\mathscr{K}[t, t^{-1}]$ by the following:

Definition (2.5)

$$\mathscr{A} = \sum_{n \in \mathbf{Z}} \mathcal{O}_X(nD) t^n \subset \mathscr{K}[t, t^{-1}]$$

Notice that $\mathscr{A}$ is a graded $\mathcal{O}_X$-algebra, since $[nD + mD] \geq [nD] + [mD]$. Clearly we have the following result as a corollary of Lemma (2.4).

Lemma (2.6) *With the same notation as in Lemma* (2.4), *we have $\mathscr{A}_y = R_{(\mathfrak{q})}$. In particular, if $R_{(\mathfrak{q})}$ is nonsingular, then there is an equality:*

$$\text{gl.dim}(\mathscr{A}_y) = \text{codim}(y, X),$$

where the left hand side means the global dimension of the category of graded $\mathscr{A}_y$-modules.

Proof. This is immediate from (2.4). In fact, since $\mathfrak{m}_x^{-m} = \mathcal{O}_X(m[x])_x$ for any $x \in X$ and $m \in \mathbf{Z}$, we have $\mathcal{O}_X(nD)_y = \bigcap_{x \in \Omega(y)} \mathcal{O}_X(nr_x[x])_x = \bigcap_{x \in \Omega(y)} \mathfrak{m}_x^{-[nr_x]}$ for any $n \in \mathbf{Z}$.

Remark (2.7) In general we denote by $\tilde{M}$ the quasi-coherent sheaf corresponding to a graded R-module M. Then Lemma (2.6) can be stated as:

$$\widetilde{R(n)} = \mathcal{O}_X(nD)\,t^n,$$

for any $n \in \mathbf{Z}$. In fact, the both sheaves are regarded as subsheaves of $\mathscr{K}$ and their stalks coincide at any points by (2.6). Note also that a graded R-homomorphism $f\colon M \to N$ of graded modules induces a morphism $\tilde{f}\colon \tilde{M} \to \tilde{N}$ of quasi-coherent sheaves.

3. Graded $\mathscr{A}$-modules

Definition (3.1) (3.1.1) We denote by $Gr(R)$ the category of graded R-modules with degree 0 homomorphisms, likewise by $Gr(\mathscr{A})$ the category of graded $\mathscr{A}$-modules with degree 0 homomorphisms.
(3.1.2) Define the functor $\Delta\colon Gr(R) \to Gr(\mathscr{A})$ by:

$$(*) \qquad \Delta(M) = \sum_{n \in \mathbf{Z}} \widetilde{M(n)}, \qquad \Delta(f) = \sum_{n \in \mathbf{Z}} \widetilde{f(n)},$$

for an object M in $Gr(R)$ and a morphism $f\colon M \to N$ in $Gr(R)$, where $f(n)\colon M(n) \to N(n)$ denotes the shift of degrees by n, hence $f(n)_m = f_{n+m}$.

It is easily checked that Δ is a well-defined functor.

Lemma (3.2) (3.2.1) $\Delta(R) = \mathscr{A}$.
(3.2.2) *Δ is an exact functor.*

Proof. The first assertion follows from (2.7) and the second is straightforward.

Definition (3.3) We define the functor $\Gamma\colon Gr(\mathscr{A}) \to Gr(R)$ as follows: For a graded $\mathscr{A}$-module $\mathscr{F} = \sum_{n \in \mathbf{Z}} \mathscr{F}_n$ and for a graded homomorphism $\psi\colon \mathscr{F} \to \mathscr{G}$,

$$(**) \qquad \Gamma(\mathscr{F}) = \sum_{n \in \mathbf{Z}} \mathrm{H}^0(X, \mathscr{F}_n), \quad \Gamma(\psi) = \mathrm{H}^0(X, \psi)\colon \Gamma(\mathscr{F}) \longrightarrow \Gamma(\mathscr{G}).$$

Note the following obvious fact:

Lemma (3.4) *Γ is a left exact functor.*

Definition (3.5) For any integer n, we denote by $gr^n(R)$ the full subcategory of $Gr(R)$ whose objects are n-th syzygies. Here we say that an R-module M is an n-th syzygy if there is an exact sequence in $Gr(R)$:

$$0 \longrightarrow M \longrightarrow F_{n-1} \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0,$$

where each F_i is a finite direct sum of the modules of the form $R(a)$ ($a \in \mathbf{Z}$). Similarly, $gr^n(\mathscr{A})$ is the full subcategory of $Gr(\mathscr{A})$ consisting of all n-th syzygies. Here an $\mathscr{A}$-module $\mathscr{F}$ is an n-th syzygy if and only if there is an exact sequence in $Gr(\mathscr{A})$:

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G}_{n-1} \longrightarrow \mathscr{G}_{n-2} \longrightarrow \cdots \longrightarrow \mathscr{G}_1 \longrightarrow \mathscr{G}_0,$$

where each $\mathscr{G}_i$ is a finite direct sum of modules of the form $\mathscr{A}(a)$ $(a \in \mathbf{Z})$.

The following lemma is easily shown from the exactness of the functor Δ and the left-exactness of Γ.

Lemma (3.6) (3.6.1) *For any integer n, Δ induces the functor*:

$$\Delta_n\colon gr^n(R) \longrightarrow gr^n(\mathscr{A}).$$

(3.6.2) *Γ induces the functor $\Gamma_2\colon gr^2(\mathscr{A}) \to gr^2(R)$.*

Of most importance is the following proposition.

Proposition (3.7) *If n is an integer ≥ 2, then $\Delta_n\colon gr^n(R) \to gr^n(\mathscr{A})$ is an equivalence of categories.*

Proof. First of all assume $n = 2$. In this case we show that Δ_2 has Γ_2 as an inverse. Note that, for any integers a, b,

(3.7.1) $$\begin{aligned}\mathrm{Hom}_{Gr(\mathscr{A})}(\mathscr{A}(a), \mathscr{A}(b)) &= \mathrm{H}^0(X, \mathscr{A}_{b-a}) = \mathrm{H}^0(X, \mathcal{O}_X((b-a)D))\,t^{b-a}\\ &= R_{b-a} = \mathrm{Hom}_{Gr(R)}(R(a), R(b)).\end{aligned}$$

By definition any object $\mathscr{F} \in gr^2(\mathscr{A})$ has a presentation:

$$0 \longrightarrow \mathscr{F} \longrightarrow \textstyle\sum_i \mathscr{A}(a_i) \xrightarrow{(f_{ij})} \sum_j \mathscr{A}(b_j),$$

where $f_{ij} \in R_{(b_j - a_i)}$ by the above. Applying the functor Γ to this, we have the exact sequence:

$$0 \longrightarrow \Gamma(\mathscr{F}) \longrightarrow \textstyle\sum_i R(a_i) \xrightarrow{(f_{ij})} \sum_j R(b_j)$$

The sequence is still exact when applying the functor Δ:

$$0 \longrightarrow \Delta(\Gamma(\mathscr{F})) \longrightarrow \textstyle\sum_i \mathscr{A}(a_i) \xrightarrow{(f_{ij})} \sum_j \mathscr{A}(b_j)$$

As a consequnce we obtain that $\mathscr{F} \simeq \Delta(\Gamma(\mathscr{F}))$. Similarly $M \simeq \Gamma(\Delta(M))$ for $M \in gr^2(R)$. Therefore Δ_2 yields a bijective correspondence between the objects of $gr^2(R)$ and $gr^2(\mathscr{A})$.

It remains to show that Δ_2 is fully faithfull. Equivalently we have to show that, for any M, $N \in gr^2(R)$, the following map induced by Δ_2 is bijective:

(3.7.2) $$\mathrm{Hom}_{gr^2(R)}(M, N) \longrightarrow \mathrm{Hom}_{gr^2(\mathscr{A})}(\Delta(M), \Delta(N)).$$

To do this, notice that any M in $gr^2(R)$ are reflexive modules. For a morphism $f\colon M \to N$ in $gr^2(R)$, take a free resolution of $M^* = \mathrm{Hom}_R(M, R)$ (resp. $N^* = \mathrm{Hom}_R(N, R)$) as follows:

$$F_1 \xrightarrow{\varphi} F_0 \longrightarrow M^* \longrightarrow 0. \qquad (\text{resp. } G_1 \xrightarrow{\psi} G_0 \longrightarrow N^* \longrightarrow 0.)$$

Then $f^*: N^* \to M^*$ induces a commutative diagram:

$$\begin{tikzcd} F_1 \arrow[r, "\varphi"] & F_0 \arrow[r] & M^* \arrow[r] & 0 \\ G_1 \arrow[u, "h_1"] \arrow[r, "\psi"] & G_0 \arrow[u, "h_0"] \arrow[r] & N^* \arrow[u, "f^*"] \arrow[r] & 0. \end{tikzcd}$$

Hence the following diagram is also commutative with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & F_0^* \arrow[r, "\varphi^*"] \arrow[d, "h_0^*"] & F_1^* \arrow[d, "h_1^*"] \\ 0 \arrow[r] & N \arrow[r] & G_0^* \arrow[r, "\psi^*"] & G_1^*, \end{tikzcd}$$

where φ^*, ψ^*, h_0^* and h_1^* can be regarded as matrices whose entries are homogeneous elements in R. Then a similar argument using (3.7.1) shows that

$$\Gamma\Delta(\varphi^*) = \varphi^*,\ \Gamma\Delta(\psi^*) = \psi^*,\ \Gamma\Delta(h_0^*) = h_0^* \text{ and } \Gamma\Delta(h_1^*) = h_1^*.$$

Hence we conclude that $\Gamma\Delta(f) = f$ and that the map in (3.7.2) is a monomorphism.

To show that this is also an epimorphism, it is enough to show that the map induced by Γ;

$$\mathrm{Hom}_{gr^2(\mathscr{A})}(\Delta(M), \Delta(N)) \longrightarrow \mathrm{Hom}_{gr^2(R)}(M, N)$$

is a monomorphism. Let $\theta: \Delta(M) \to \Delta(N)$ be a morphism in $gr(\mathscr{A})$ and assume that $\Gamma(\theta) = 0$. Write $\theta = \sum_{n \in \mathbf{Z}} \theta_n$, where $\theta_n: \widetilde{M(n)} \to \widetilde{N(n)}$. Taking a system of homogeneous parameters $\{r_1, r_2, \ldots, r_v\}$ in $R_+ = \sum_{n>0} R_n$, we consider a covering $\{D(r_i) = \mathrm{Spec}(R_{r_i})_0 \mid i = 1, 2, \ldots, v\}$ of X. Then θ is a collection of the maps;

$$\theta^{(i)}: M_{r_i} \longrightarrow N_{r_i} \qquad \text{on } D(r_i)\ (i = 1, 2, \ldots, v)$$

with $\theta^{(i)}|_{D(r_i) \cap D(r_j)} = \theta^{(j)}|_{D(r_i) \cap D(r_j)}$. By definition $\Gamma(\theta) = 0$ implies that each map induced by $\theta^{(i)}$:

$$M = \bigcap_{i=1}^{v} M_{r_i} \longrightarrow N = \bigcap_{i=1}^{v} N_{r_i}$$

is trivial. Hence, for any $x = y/r_i^m \in (M_{r_i})_n$ ($y \in M_{d_i m+n}$, $d_i = \deg(r_i)$), we see that $0 = \theta_{d_i m+n}^{(i)}(y) = r_i^m \theta_n^{(i)}(x)$, and thus $\theta_n^{(i)}(x) = 0$. Therefore it is concluded that $\theta^{(i)} = 0$ for any $i = 1, 2, \ldots, v$. Consequently $\theta = 0$, and the map in (3.7.2) is an isomorphism.

Hence we showed that $\Gamma_2 \cdot \Delta_2 \simeq 1_{gr^2(R)}$ and $\Delta_2 \cdot \Gamma_2 \simeq 1_{gr^2(\mathscr{A})}$.

Now consider the case $n > 2$. Since the functor Γ_2 is exact, its restriction onto the subcategory $gr^n(\mathscr{A})$ is also an exact functor. In particular, it induces the functor $\Gamma_n: gr^n(\mathscr{A}) \to gr^n(R)$. Since Γ_2 is the inverse of Δ_2, Γ_n is the inverse of Δ_n.

Remark (3.8) When $n \leq 1$, notice that $\Delta_n\colon gr^n(R) \to gr^n(\mathscr{A})$ is not necessarily an equivalence of categories.

4. The $\mathcal{O}_X$-order Λ

Notation (4.1) In the following we fix a positive integer ℓ with the property that ℓD is an integral Cartier divisor on X. Moreover we write $E = \ell D$.

Remark (4.2) Notice that $\mathcal{O}_X(-E)t^{-\ell}$ and $\mathcal{O}_X(E)t^{\ell}$ appear in $\mathscr{A}$ as graded pieces. Therefore for an object $\mathscr{F} = \sum_{n\in\mathbf{Z}} \mathscr{F}_n$ in $gr(\mathscr{A})$, we have equalities of $\mathcal{O}_X$-modules:

$$\mathscr{F}_{n+\ell} = \mathscr{F}_n \otimes_{\mathcal{O}_X} \mathcal{O}_X(E),$$

for any $n \in \mathbf{Z}$. Actually, since $\mathscr{F}$ is an $\mathscr{A}$-module, we have

$$\mathscr{F}_n \otimes \mathcal{O}_X(E)t^{\ell} = \mathscr{F}_n \mathscr{A}_{\ell} \subset \mathscr{F}_{n+\ell},\ \mathscr{F}_{n+\ell} \otimes \mathcal{O}_X(-E)t^{-\ell} = \mathscr{F}_{n+\ell}\mathscr{A}_{-\ell} \subset \mathscr{F}_n.$$

Noting that $\mathcal{O}_X(E)$ is an invertible sheaf, we show the above equality.

Definitions (4.3) (4.3.1) We define an $\mathcal{O}_X$-order Λ in the full matrix algebra $M_{\ell}(\mathscr{K})$ over $\mathscr{K}$ by

$$\Lambda = (\mathcal{O}_X((j-i)D))_{1\leq i,j\leq \ell}$$
$$= \begin{pmatrix} \mathcal{O}_X & O_X(D) & \cdots & \mathcal{O}_X((\ell-1)D) \\ \mathcal{O}_X(-D) & \mathcal{O}_X & \cdots & \mathcal{O}_X((\ell-2)D) \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{O}_X(-(\ell-1)D) & \cdots & \cdots & \mathcal{O}_X \end{pmatrix}.$$

Furthermore we denote by $mod(\Lambda)$ the category of coherent right Λ-modules and Λ-homomorphisms.

(4.3.2) The functor $\Phi\colon gr^1(\mathscr{A}) \to mod(\Lambda)$ is defined as follows:

For $\mathscr{F} = \sum_{n\in\mathbf{Z}} \mathscr{F}_n t^n$, we define:

$$\Phi(\mathscr{F}) = (\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_{\ell-1}) \subset \mathscr{K}^{\ell}.$$

And for a morphism $\psi = \sum_{n\in\mathbf{Z}} \psi_n\colon \mathscr{F} \to \mathscr{G}$, define $\Phi(\psi)\colon \Phi(\mathscr{F}) \to \Phi(\mathscr{G})$ by:

$$\Phi(\psi)(x_0, x_1, \ldots, x_{\ell-1}) = (\psi_0(x_0), \psi_1(x_1), \ldots, \psi_{\ell-1}(x_{\ell-1})).$$

It is an easy exercise to see that $\Phi(\mathscr{F})$ is a right Λ-module and $\Phi(\psi)$ is a Λ-homomorphism.

(4.3.3) The functor $\Psi\colon mod(\Lambda) \to Gr(\mathscr{A})$ is defined as follows:

For any $\mathscr{M} \in mod(\Lambda)$ and any $n \in \mathbf{Z}$, let $\mathscr{F}_n$ be the $\mathcal{O}_X$-module $\mathcal{O}_X(aE) \otimes_{\mathcal{O}_X} \mathscr{M}e_b$, where $n = a\ell + b$, $a, b \in \mathbf{Z}$, $0 \leq b < \ell$ and e_b is an idempotent matrix in Λ whose entries are 1 in $(b+1, b+1)$-position and 0 otherwise. Then define:

$$\Psi(\mathscr{M}) = \sum_{n\in\mathbf{Z}} \mathscr{F}_n t^n.$$

Similarly, for a Λ-homomorphism φ, define $\Psi(\varphi)$ to be $\mathcal{O}_X(aE)\otimes_{\mathcal{O}_X}\varphi\cdot e_b$ on $\mathcal{F}_n$. Note that, for any integer m, there is a natural map:

$$\mathcal{F}_n \otimes_{\mathcal{O}_X} \mathcal{O}_X(mD) \longrightarrow \mathcal{F}_{n+m},$$

for each $m\in\mathbf{Z}$. In fact, if $m = a'\ell + b'(a', b'\in\mathbf{Z}, 0\leq b' < \ell)$, then

$$\mathcal{O}_X(mD) = \begin{cases} \mathcal{O}_X(a'E)\otimes_{\mathcal{O}_X} e_b\Lambda e_{b+b'} & \text{if } b+b' < \ell, \\ \mathcal{O}_X((a'+1)E)\otimes_{\mathcal{O}_X} e_b\Lambda e_{b+b'-\ell} & \text{if } b+b' \geq \ell. \end{cases}$$

Thus if $b + b' < \ell$, then

$$\begin{aligned}\mathcal{F}_n \otimes_{\mathcal{O}_X} \mathcal{O}_X(mD) &= \mathcal{O}_X((a+a')E)\otimes_{\mathcal{O}_X} \mathcal{M}e_b\cdot e_b\Lambda e_{b+b'} \\ &\subset \mathcal{O}_X((a+a')E)\otimes_{\mathcal{O}_X}\mathcal{M}e_{b+b'} = \mathcal{F}_{n+m}.\end{aligned}$$

Likewise we have the same natural inclusion when $b + b' \geq \ell$. Therefore $\Psi(\mathcal{M})$ is a graded $\mathcal{A}$-module and $\Psi(\varphi)$ is an $\mathcal{A}$-homomorphism.

(4.3.4) For an integer $n\geq 1$, a right Λ-module $\mathcal{M}$ is called an n-th syzygy if there is an exact sequence of right Λ-modules:

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{P}_{n-1} \longrightarrow \mathcal{P}_{n-2} \longrightarrow \cdots \longrightarrow \mathcal{P}_0,$$

where each $\mathcal{P}_i$ is a finite direct sum of Λ-modules of the form $\mathcal{O}_X(-aE)\otimes_{\mathcal{O}_X} e_b\Lambda\,(0\leq b<\ell, a\in\mathbf{Z})$. We denote by $mod^n(\Lambda)$ the full subcategory of $mod(\Lambda)$ whose objects are n-th syzygies.

The following lemma is almost clear from the definition.

Lemma (4.4) (4.4.1) *Φ is an exact functor and $\Phi(\mathcal{A}(-n))\simeq\mathcal{O}_X(-aE)\otimes_{\mathcal{O}_X}e_b\Lambda$, where $n = a\ell + b$, a, $b\in\mathbf{Z}$, $0\leq b<\ell$.*

(4.4.2) *Ψ is an exact functor and there are equalities*:

$$\Psi(\mathcal{O}_X(-aE)\otimes_{\mathcal{O}_X} e_b\Lambda) = \mathcal{A}(-(a\ell+b)) \qquad (a\in\mathbf{Z},\ 0\leq b<\ell).$$

Proof. (4.4.1): The exactness of Φ is obvious from the definition. The isomorphism follows from the fact that, for each c $(0\leq c<\ell)$,

$$(\mathcal{A}(-n))_c \simeq \mathcal{O}_X(-(n-c)D) = \mathcal{O}_X(-aE)\otimes_{\mathcal{O}_X}\mathcal{O}_X(-(b-c)D) = (\mathcal{O}_X(-aE)\otimes e_b\Lambda)e_c.$$

(4.4.2): To show that Ψ is an exact functor, let $0\to\mathcal{M}'\to\mathcal{M}\to\mathcal{M}''\to 0$ be an exact sequence of right Λ-modules. Then $0\to\mathcal{M}'e_b\to\mathcal{M}e_b\to\mathcal{M}''e_b\to 0$ is an exact sequence of $\mathcal{O}_X$-modules for any b $(0\leq b<\ell)$, hence the following is also an exact sequence of $\mathcal{O}_X$-modules for any $a\in\mathbf{Z}$:

$$0 \longrightarrow \mathcal{O}_X(aE)\otimes\mathcal{M}'e_b \longrightarrow \mathcal{O}_X(aE)\otimes\mathcal{M}e_b \longrightarrow \mathcal{O}_X(aE)\otimes\mathcal{M}''e_b \longrightarrow 0.$$

Summing up this, we thus see that $0\to\Psi(\mathcal{M}')\to\Psi(\mathcal{M})\to\Psi(\mathcal{M}'')\to 0$ is an exact sequnce of $\mathcal{A}$-modules. The last equality follows from a direct computation as follows:

$$\begin{aligned}\Psi(\mathcal{O}_X(-aE)\otimes_{\mathcal{O}_X} e_b\Lambda) &= \sum_{n\in\mathbf{Z},\, n=a'\ell+b'} \{\mathcal{O}_X((a'-a)E)\otimes_{\mathcal{O}_X} e_b\Lambda e_{b'}\}\, t^n \\ &= \sum_n \{\mathcal{O}_X((a'-a)E)\otimes_{\mathcal{O}_X}\mathcal{O}_X((b'-b)D)\}\, t^n \\ &= \sum_n \mathcal{O}_X((n-a\ell-b)D)\, t^n \\ &= \mathcal{A}(-(a\ell+b)).\end{aligned}$$

It is deduced from this lemma that the functors Φ and Ψ preserve n-th syzygies. Hence we get the following:

Corollary (4.5) *For an integer $n \geq 1$, Φ and Ψ induce the following functors*:

$$gr^n(\mathcal{A}) \longrightarrow mod^n(\Lambda),$$
$$mod^n(\Lambda) \longrightarrow gr^n(\mathcal{A}).$$

In fact these functors give rise to an equivalence of the categories.

Proposition (4.6) *For an integer $n \geq 1$, Φ and Ψ induce an equivalence of categories*:

$$gr^n(\mathcal{A}) \simeq mod^n(\Lambda).$$

Proof. It is evident that we have only to prove the proposition in the case $n = 1$. Note from the definition that for $\mathcal{M} \in mod^1(\Lambda)$, there is an equality

$$\Phi\Psi(\mathcal{M}) \simeq (\mathcal{M}e_0, \mathcal{M}e_1, \dots, \mathcal{M}e_{\ell-1}) = \mathcal{M}$$

On the other hand, for $\mathcal{F} \in gr^1(\mathcal{A})$,

$$\begin{aligned}\Psi\Phi(\mathcal{F}) &= \sum_{n\in\mathbf{Z},\, n=a\ell+b} (\mathcal{O}_X(aE)\otimes_{\mathcal{O}_X}\mathcal{F}_b)\, t^n \\ &= \sum_{n\in\mathbf{Z},\, n=a\ell+b} \mathcal{F}_{a\ell+b}\, t^n \simeq \mathcal{F}.\end{aligned}$$

(See (4.2) for the second equality.) Thus they induce a bijection between the sets of objects in $gr^1(\mathcal{A})$ and $mod^1(\Lambda)$.

To complete the proof, let $\mathcal{F}, \mathcal{G} \in gr^1(\mathcal{A})$ and let $\mathcal{M} = \Phi(\mathcal{F})$, $\mathcal{N} = \Phi(\mathcal{G})$. We want to show that the map $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_\Lambda(\mathcal{M}, \mathcal{N})$ induced by Φ is an isomorphism. Let $\psi = \sum_n \psi_n$ be a morphism from $\mathcal{F}$ to $\mathcal{G}$ and assume that $\Phi(\psi) = 0$. Then, by definition, $\psi_n = 0\,(0 \leq n < \ell)$. On the other hand, we know by (4.2) that $\psi_{n+\ell} = \psi_n \otimes \mathcal{O}_X(E)$. Hence $\psi_n = 0$ for all $n \in \mathbf{Z}$, showing the injectivity. To show the surjectivity, let f be an arbitrary element in $\mathrm{Hom}_\Lambda(\mathcal{M}, \mathcal{N})$. Then it is easily checked that $\psi = \sum_{n\in\mathbf{Z},\, n=a\ell+b} \mathcal{O}_X(aE)\otimes_{\mathcal{O}_X} f_b$ is an $\mathcal{A}$-homomorphism from $\mathcal{F} = \Psi(\mathcal{M})$ to $\mathcal{G} = \Psi(\mathcal{N})$ and that $\Phi(\psi) = f$.

Combining Proposition (3.7) with (4.6) we obtain the theorem:

Theorem (4.7) *If n is an integer ≥ 2, then the category $gr^n(R)$ is equivalent with $mod^n(\Lambda)$.*

Remark (4.8) Let $y \in X$ be an irreducible point and let $\mathfrak{q}$ be the graded prime ideal of R corresponding to y. With the notation in Lemma (2.4), set

$$I_n = \bigcap_{x \in \Omega(y)} \mathfrak{m}_x^{-[nr_x]} \qquad (n \in \mathbf{Z}).$$

Then, the stalk of Λ at y is

$$\Lambda_y = (I_{j-i})_{0 \leq i,j < \ell}.$$

And similarly,

$$\mathscr{A}_y = \sum_{n \in \mathbf{Z}} I_n t^n$$

Since $I_{-\ell} = \mathcal{O}_X(-E)_y = \mathcal{O}_X(E)_y^{-1} = I_\ell^{-1}$, we note that $\mathscr{A}_y$ contains $I_\ell t^\ell$ and $I_\ell^{-1} t^{-\ell}$ as graded pieces. Now define the functor:

$$\Phi_y \colon gr^1(\mathscr{A}_y) \longrightarrow mod^1(\Lambda_y);\ \sum_i M_i t^i \longmapsto (M_0, M_1, \ldots, M_{\ell-1}).$$

Then completely the same argument as in the proof of Proposition (4.6) shows that Φ_y yields an equivalence of categories. Thus the next lemma follows from Corollary (2.6).

Lemma (4.9) *Under the same assumption as the above, if $R_{(\mathfrak{q})}$ is nonsingular, then there is an equality*:

$$\mathrm{gl.dim}(\Lambda_y) = \mathrm{codim}(y, X).$$

In particular, if $x \in X$ is a point of codimension one, then Λ_x is a hereditary order.

5. The ramification index of Λ

In Lemma (4.9) we have shown that Λ is hereditary in codimension one. More precisely one can determine the ramification indices of Λ at those points of codimension one.

Theorem (5.1) *Let $x \in X$ be a point of codimension one. Then the ramification index of Λ_x as an $\mathcal{O}_{X,x}$-order is equal to p_x.* (See (2.2) for the precise definition of p_x.)

Proof. For simplicity we write $\mathfrak{m} = \mathfrak{m}_x \subset A = \mathcal{O}_{X,x}$, $p = p_x$ and $q = q_x$. Then notice that one can write $\Lambda_x = \Lambda_x^0 \otimes \Lambda_x^1$, where

$$\Lambda_x^0 = \begin{pmatrix} A & \mathfrak{m}^{-[q/p]} & \cdots & \mathfrak{m}^{-[(p-1)q/p]} \\ \mathfrak{m}^{[q/p]} & A & \cdots & \mathfrak{m}^{-[(p-2)q/p]} \\ \cdots & \cdots & \cdots & \cdots \\ \mathfrak{m}^{[(p-1)q/p]} & \cdots & \cdots & A \end{pmatrix},$$

$$\Lambda_x^1 = \begin{pmatrix} A & \mathfrak{m}^{-q} & \cdots & \mathfrak{m}^{-\ell q/p} \\ \mathfrak{m}^{q} & A & \cdots & \mathfrak{m}^{-(\ell-p)q/p} \\ \cdots & \cdots & \cdots & \cdots \\ \mathfrak{m}^{\ell q/p} & \cdots & \cdots & A \end{pmatrix}.$$

Since Λ_x^1 is isomorphic to the matrix algebra $M_{\ell/p}(A)$, the ramification index of Λ_x is equal to that of Λ_x^0. And this is equal to the number of isomorphism classes of indecomposable projective right Λ_x^0-modules, since Λ_x^0 is also hereditary. See Reiner [4]. Now let $\{e_0, e_1, \ldots, e_{p-1}\}$ be a complete system of orthogonal idempotents in Λ_x^0. It is sufficient to prove that if $i \neq j$, then $e_i\Lambda_x^0$ is not isomorphic to $e_j\Lambda_x^0$ as a Λ_x^0-module. Suppose that $e_i\Lambda_x^0 \simeq e_j\Lambda_x^0$ for some $i<j$. Then there is an integer s $(0<|s|<|q|)$ satisfying

(5.1.1) $$\mathfrak{m}^s e_i\Lambda_x^0 = e_j\Lambda_x^0.$$

Note that

$$e_j\Lambda_x^0 = (\mathfrak{m}^{[jq/p]}, \ldots, \mathfrak{m}^{-[(p-j)q/p]}),$$

in which each $\mathfrak{m}^k$ appears $\rho_{q/p}(k)$-times where $\rho_{q/p}$ is defined in the next lemma. It is then easy to see that the equality in (5.1.1) contradicts the following fact (cf. Yoshino-Osa [7, Lemma 4]).

Lemma (5.2) *Let $r = q/p$ $(p, q \in \mathbf{Z})$ be a rational number with p, q coprime. We define the function $\rho_r \colon \mathbf{Z} \to \mathbf{N}$ as follows:*

$$\rho_r(n) = \#\{v \in \mathbf{Z} \mid [vr] = n\}$$

Then $|q|$ is the minimum period of the function ρ_r.

Proof. We may assume that both p and q are positive. It is obvious that if $s=q$, then $\rho_r(i+s) = \rho_r(i)$ for any $i \in \mathbf{Z}$. Let s be the minimal positive integer with this property. Then clearly s divides q. We have to show that $s=q$. Letting $r' = q/(p+q)$, we can see that $\rho_{r'}(i) = \rho_r(i) + 1$ for any $i \in \mathbf{Z}$. Hence we may assume that $0<p<q$. Then ρ_r takes either 1 or 0, hence

$$\begin{aligned}\sum_{i=0}^{q-1} \rho_r(i) &= \#\{i \in \mathbf{Z} \mid 0 \leq i < q,\ \rho_r(i) = 1\} \\ &= \#\{j \mid 0 \leq j < p\} = p.\end{aligned}$$

On the other hand the leftmost of this equation is equal to $(q/s)\sum_{i=0}^{s-1} \rho_r(i)$, since s is the period of ρ_r. Therefore q/s divides p and thus $q=s$.

6. Vector bundles with Λ-module structure

Notation (6.1) In the rest of this paper we denote $d = \dim(X)$ (≥ 1), so that $\dim(R) = d + 1$. And denote by $bdl(\Lambda)$ the full subcategory of $mod(\Lambda)$ consisting of all right Λ-modules that are locally free as $\mathcal{O}_X$-modules. Therefore the objects of $bdl(\Lambda)$ are vector bundles on X with structure of right Λ-modules.

First we note the following fact:

Lemma (6.2) *Suppose that R has only an isolated singularity and that X is nonsingular. Then an object $\mathcal{M} \in mod(\Lambda)$ belongs to $bdl(\Lambda)$ if and only if $\mathcal{M}_x$ is a projective Λ_x-module for any $x \in X$.*

Proof. Since Λ_x is a free $\mathcal{O}_{X,x}$-module, the 'if' part is trivial. To prove the 'only if' part, let $\mathcal{M}$ be a right Λ-module that is locally free as an $\mathcal{O}_X$-module and let $x \in X$. Since Λ_x is an $O_{X,x}$-order whose global dimension equals the Krull dimension of $\mathcal{O}_{X,x}$; see (4.9), we may apply [1; Theorem IV. 1.9] to conclude that $\mathcal{M}_x$ is a projective Λ_x-module.

This lemma proves the following:

Theorem (6.3) *Suppose that R has only an isolated singularity and that X is nonsingular. Then $bdl(\Lambda) \subset mod^t(\Lambda)$ for any integer $t \geq 1$.*

Proof. Note first that the set of left Λ-modules $\{\Lambda e_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(nE) | n, i \in \mathbf{Z}, 0 \leq i < \ell\}$ generates the category of coherent left Λ-modules. To prove this, let $\mathcal{N}$ be a coherent left Λ-module. Since E is an ample divisor on X, for any i $(0 \leq i < \ell)$, there is an epimorphism of $\mathcal{O}_X$-modules:

$$\mathcal{O}_X^{(m)} \longrightarrow e_i \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{O}_X(nE) = \mathcal{H}om_\Lambda(\Lambda e_i, \mathcal{N}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(nE),$$

with some integers m and n. This induces the epimorphism $\mathcal{O}_X(-nE)^{(m)} \to \mathcal{H}om_\Lambda(\Lambda e_i, \mathcal{N})$ and hence the Λ-homomorphism $f_i \colon (\Lambda e_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(-nE))^{(m)} \to \mathcal{N}$. Then it is easy to see that the sum of the f_i gives rise to an epimorphism:

$$\sum_{i=0}^{\ell-1} (\Lambda e_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(-nE))^{(m)} \longrightarrow \mathcal{N}.$$

Hence $\{\Lambda e_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(nE)\}$ generates the category of left Λ-modules.

Now we prove the theorem. Let $\mathcal{M}$ be an object in $bdl(\Lambda)$. Then, by the above, there is an exact sequence of left Λ-modules:

$$\sum_{i=0}^{\ell-1} \Lambda e_i \otimes_{\mathcal{O}_X} P_{t,i} \longrightarrow \cdots \longrightarrow \sum_{i=0}^{\ell-1} \Lambda e_i \otimes_{\mathcal{O}_X} P_{1,i} \longrightarrow \sum_{i=0}^{\ell-1} \Lambda e_i \otimes_{\mathcal{O}_X} P_{0,i}$$
$$\longrightarrow \mathcal{H}om_\Lambda(\mathcal{M}, \Lambda) \longrightarrow 0,$$

where $P_{i,j}$ are direct sums of $\mathcal{O}_X$-modules of the form $\mathcal{O}_X(nE)$ $(n \in \mathbf{Z})$. Note that for any $x \in X$, the natural map:

$$\mathscr{M}_x \longrightarrow \mathrm{Hom}_{\Lambda_x}(\mathrm{Hom}_{\Lambda_x}(\mathscr{M}_x, \Lambda_x), \Lambda_x)$$

is an isomorphism, since $\mathscr{M}_x$ is Λ_x-projective. Hence the natural map of sheaves

$$\mathscr{M} \longrightarrow \mathscr{H}om_\Lambda(\mathscr{H}om_\Lambda(\mathscr{M}, \Lambda), \Lambda)$$

is also an isomorphism. Therefore applying the functor $\mathscr{H}om_\Lambda(\ , \Lambda)$ to the above exact sequence, we have the following complex of right Λ-modules:

$$(6.3.1) \qquad 0 \longrightarrow \mathscr{M} \longrightarrow \sum_{i=0}^{\ell-1} P'_{0,i} \otimes_{\mathcal{O}_X} e_i\Lambda \longrightarrow \cdots \longrightarrow \sum_{i=0}^{\ell-1} P'_{t-1,i} \otimes_{\mathcal{O}_X} e_i\Lambda \longrightarrow \sum_{i=0}^{\ell-1} P'_{t,i} \otimes_{\mathcal{O}_X} e_i\Lambda$$

where $P'_i = \mathscr{H}om_{\mathcal{O}_X}(P_i, \mathcal{O}_X)$. Note from the same argument as in (6.2) that $\mathscr{H}om_\Lambda(\mathscr{M}, \Lambda)_x$ is a projective left Λ_x-module for any $x \in X$. Thus the homologies $\mathscr{E}xt^i_\Lambda(\mathscr{H}om_\Lambda(\mathscr{M}, \Lambda), \Lambda)$ $(1 \le i < t)$ of the above complex vanish, and hence the sequence (6.3.1) is exact. Therefore $\mathscr{M}$ is a t-th syzygy as a right Λ-module.

Corollary (6.4) *Under the same assumption as in* (6.3) *there is an equality*:

$$mod^d(\Lambda) = bdl(\Lambda).$$

Proof. Let $\mathscr{M} \in mod^d(\Lambda)$. Then for any $x \in X$, we see that $\mathscr{M}_x$ is a d-th syzygy as an Λ_x-module. Since Λ_x is a free $\mathcal{O}_{X,x}$-module, $\mathscr{M}_x$ is a d-th syzygy as an $\mathcal{O}_{X,x}$-module, hence it must be $\mathcal{O}_{X,x}$-free, for $\mathcal{O}_{X,x}$ is a regular local ring of dimension not more than d. Therefore $\mathscr{M} \in bdl(\Lambda)$.

This, together with (6.3) and (4.7), shows the following:

Corollary (6.5) (6.5.1) *If* $d = 1$, *then* $bdl(\Lambda) = mod^2(\Lambda) \simeq gr^2(R)$.
(6.5.2) *Suppose that R has only an isolated singularity and that X is nonsingular. If* $d \ge 2$, *then* $bdl(\Lambda) = mod^d(\Lambda) \simeq gr^d(R)$.

Proof. Since $mod^2(\Lambda) \subset mod^1(\Lambda)$, the first equality in (6.5.1) follows from (6.3) and (6.4).

Next we note the relationship of syzygy modules with CM modules. For this we settle the notation.

Notation (6.6) We denote by $\mathrm{gr}\mathfrak{C}(R)$ the full subcategory of $Gr(R)$ whose objects are graded CM modules.

It is known that $\mathrm{gr}\mathfrak{C}(R)$ is equal to the category of syzygies under a suitable assumption.

Lemma (6.7) *If R is a CM ring that is an isolated singularity, then there is an equality*:

$$\mathrm{gr}\mathfrak{C}(R) = gr^{d+1}(R).$$

This lemma is well-known, but we include here a brief proof of this for the convenience of the reader.

Proof. If $M \in gr^{d+1}(R)$, then $\operatorname{depth}(M) = d + 1$, since R is a CM ring. Hence $M \in \mathrm{gr}\mathfrak{C}(R)$. To show the converse, we prove the stronger statement:

(6.7.1) Let R be a graded CM ring and let t be an integer with $1 \le t \le d + 1$. If M is a finitely generated R-module with $\operatorname{depth}_R(M) \ge t$ such that $M_{(\mathfrak{p})}$ is $R_{(\mathfrak{p})}$-free for any relevant homogeneous prime ideal $\mathfrak{p}$ of R, then $M \in gr^t(R)$.

If this is true, then we are through, since every CM module over an isolated singularity satisfies the assumption in (6.7.1) with $t = d + 1$. We prove (6.7.1) by induction on t. Let $\{f_i \mid 1 \le i \le N\}$ be a set of homogeneous generators of $\operatorname{Hom}_R(M, R)$ and consider the map $f = (f_1, f_2, \ldots, f_N)\colon M \to R^{(N)}$. First of all we assume that $t = 1$. It is enough to show that $\operatorname{Ker}(f) = 0$. Suppose that $\operatorname{Ker}(f) \ne 0$ and take a homogeneous associated prime ideal $\mathfrak{p}$ of $\operatorname{Ker}(f)$. Note that $\mathfrak{p}$ is not an irrelevant prime ideal of R, since if so, then $\operatorname{depth}(M) = 0$ which contradicts the assumption. Therefore $M_{(\mathfrak{p})}$ is a free $R_{(\mathfrak{p})}$-module, hence the map $f_{(\mathfrak{p})}\colon M_{(\mathfrak{p})} \to R_{(\mathfrak{p})}^{(N)}$ is a split monomorphism, since $\operatorname{Hom}_{R_{(\mathfrak{p})}}(M_{(\mathfrak{p})}, R_{(\mathfrak{p})}) = (f_1, f_2, \ldots, f_N) R_{(\mathfrak{p})}$. In particular $\operatorname{Ker}(f)_{(\mathfrak{p})} = 0$ that is also a contradiction, since $\mathfrak{p}$ is an associated prime of $\operatorname{Ker}(f)$.

Now suppose that $t \ge 2$. As above we can show that f is an injective map, hence we may have an exact sequence of R-modules:

$$0 \longrightarrow M \xrightarrow{f} R^{(N)} \longrightarrow L \longrightarrow 0.$$

For any relevant homogeneous prime ideal $\mathfrak{p}$ of R, since $M_{(\mathfrak{p})}$ is a free $R_{(\mathfrak{p})}$-module and since $\operatorname{Hom}_{R_{(\mathfrak{p})}}(M_{(\mathfrak{p})}, R_{(\mathfrak{p})}) = (f_1, f_2, \ldots, f_N) R_{(\mathfrak{p})}$, the map $f_{(\mathfrak{p})}$ is a split monomorphism, hence $L_{(\mathfrak{p})}$ is also a free $R_{(\mathfrak{p})}$-module. On the other hand, we have $\operatorname{depth}(L) \ge t - 1$, since $\operatorname{depth}(M) \ge t$ and $\operatorname{depth}(R) = d + 1 \ge t$. Thus we can apply the induction hypothesis to L to obtain that L is a $(t - 1)$-th syzygy as an R-module and that M is a t-th syzygy.

Combining this with Corollary (6.5) we obtain:

Theorem (6.8) *Let R be a CM normal domain.*
(6.8.1) *If $d = 1$, then* $\mathrm{gr}\mathfrak{C}(R) \simeq bdl(\Lambda)$.
(6.8.2) *Suppose that R has only an isolated singularity and that X is nonsingular. If $d \ge 2$, then* $\mathrm{gr}\mathfrak{C}(R)$ *can be embedded into $bdl(\Lambda)$ as a full subcategory.*

This shows that in the case that X is a curve, the classification of graded CM modules over R can be reduced to the classification of objects in $bdl(\Lambda)$.

Remark (6.9) Let Λ be an $\mathcal{O}_X$-order in $M_\ell(\mathcal{K})$ and let $\mathcal{T}$ be an invertible (fractional) ideal of $\mathcal{O}_X$. Assume that Λ is divided into smaller orders as shown in the following:

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \cdots & \Lambda_{1r} \\ \Lambda_{21} & \Lambda_{22} & \cdots & \Lambda_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ \Lambda_{r1} & \Lambda_{r2} & \cdots & \Lambda_{rr} \end{pmatrix},$$

where each Λ_{ij} is a set consisting of $\lambda_i \times \lambda_j$-matrices on $\mathscr{K}$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ a partition of ℓ. Then we can define a new order Λ' as:

$$\Lambda' = \begin{pmatrix} \Lambda_{11} & \mathscr{T}\Lambda_{12} & \cdots & \mathscr{T}^{r-1}\Lambda_{1r} \\ \mathscr{T}^{-1}\Lambda_{21} & \Lambda_{22} & \cdots & \mathscr{T}^{r-2}\Lambda_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ \mathscr{T}^{1-r}\Lambda_{r1} & \mathscr{T}^{2-r}\Lambda_{r2} & \cdots & \Lambda_{rr} \end{pmatrix}.$$

In this case there is a natural equivalence of the categories:

$$bdl(\Lambda) \simeq bdl(\Lambda'). \tag{6.9.1}$$

Actually the equivalence is defined by sending the object $(\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_{\ell-1})$ in $bdl(\Lambda)$ to $(\mathscr{G}_0, \mathscr{G}_1, \ldots, \mathscr{G}_{\ell-1})$, where $\mathscr{G}_i = \mathscr{T}^{u(i)}\mathscr{F}_i$ with $u(i) = \max\{j \mid \lambda_1 + \cdots + \lambda_j \leq i\} \cup \{0\}$.

By (6.8) this equivalence can be applied to several examples to show an equivalence of a category $\mathrm{gr}\mathfrak{C}(R)$ with another. For the easiest example, let $X = \mathbf{P}^1$ be a projective line and consider two divisors; $D_1 = -\frac{1}{2}(0) + \frac{1}{2}(1) + \frac{1}{2}(\infty)$, $D_2 = \frac{1}{2}(0) + \frac{1}{2}(1) + \frac{1}{2}(\infty)$. We can take 2 as ℓ in either case, hence

$$\Lambda_1 = \begin{pmatrix} \mathcal{O}_X & \mathcal{O}_X((0)) \\ \mathcal{O}_X(-(1) - (\infty)) & \mathcal{O}_X \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} \mathcal{O}_X & \mathcal{O}_X \\ \mathcal{O}_X(-(0) - (1) - (\infty)) & \mathcal{O}_X \end{pmatrix}.$$

Thus $bdl(\Lambda_1) \simeq bdl(\Lambda_2)$ by (6.9.1). Then (6.8.1) implies that $\mathrm{gr}\mathfrak{C}(R_1)$ is equivalent to $\mathrm{gr}\mathfrak{C}(R_2)$. Note that $R_1 = R(\mathbf{P}^1, D_1) \simeq k[x, y, z]/(x^2y + y^3 + z^2)$ is a hypersurface having D_4-singularity at the vertex of the cone, while $R_2 = R(\mathbf{P}^1, D_2) = k[x, y, z, w]/(yw - x^4, x^4 + yz - x^2z, x^2z + x^2w - zw)$ is not even a Gorenstein ring.

7. The case that X is a curve

In what follows we assume that X has dimension one, hence it is a curve or equivalently R has Krull dimension two. We would like to analyse the category $bdl(\Lambda)$. Note that in this case Λ is hereditary at any closed points. (See Lemma (4.9).)

Now take a maximal order Γ of $M_\ell(\mathscr{K})$ that contains Λ. Since the natural map between the Brauer groups $Br(\mathcal{O}_X) \to Br(\mathscr{K})$ is a monomorphism, the Brauer class of Γ must be trivial, hence Γ is Morita equivalent to $\mathcal{O}_X$. That is, there is an equivalence of categories:

$$bdl(\Gamma) \simeq bdl(\mathcal{O}_X).$$

For a vector bundle $\mathscr{F} \in bdl(\mathcal{O}_X)$ (resp. a morphism f in $bdl(\mathcal{O}_X)$), we denote by $\mathscr{F}'$ (resp. f') the object (resp. the morphism) in $bdl(\Gamma)$ corresponding to $\mathscr{F}$ (resp. f).

Definition (7.1) Define the functor ϕ_Γ: $bdl(\Lambda) \to bdl(\mathcal{O}_X)$ by

$$\phi_\Gamma(\mathscr{G})' = \mathscr{G} \otimes_\Lambda \Gamma,$$

for an object $\mathscr{G}$ in $bdl(\Lambda)$. Since Γ is locally projective as a Λ-module, it follows that $\mathscr{G} \otimes_\Lambda \Gamma \in bdl(\Gamma)$, hence that ϕ_Γ is a well-defined functor. By the same reason ϕ_Γ is an exact functor. For a vector bundle $\mathscr{F} \in bdl(\mathcal{O}_X)$, denote by $\phi_\Gamma^{-1}(\mathscr{F})$ the full subcategory of $bdl(\Lambda)$ consisting of all the objects $\mathscr{G}$ with $\phi_\Gamma(\mathscr{G}) \simeq \mathscr{F}$.

From the fact that Λ_x is a hereditary order over a discrete valuation ring $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ for any closed point $x \in X$, we see that

$$\mathfrak{m}_x \Gamma_x \subset \Lambda_x \subset \Gamma_x.$$

Notice from Lemma (5.1) that $\Lambda_x = \Gamma_x$ when $p_x = 1$. (Recall that $p_x = 1$ if $q_x = 0$.) Now considering the ideal sheaf $\mathscr{T} = \mathcal{O}_X(-\sum_{p_x \neq 1}[x])$, we have

$$\mathscr{T}\Gamma \subset \Lambda \subset \Gamma,$$

in $M_\ell(\mathscr{K})$. This shows that for any object $\mathscr{G}$ in $\phi_\Gamma^{-1}(\mathscr{F})$, the following condition holds true:

$$\mathscr{T}\mathscr{F}' \subset \mathscr{G} \subset \mathscr{G} \otimes_\Lambda \Gamma = \mathscr{F}'. \tag{7.2}$$

Conversely, the object $\mathscr{G}$ satisfying (7.2) belongs to $\phi_\Gamma^{-1}(\mathscr{F})$. Denote $\bar{\Lambda} = \Lambda/\mathscr{T}\Gamma \subset \bar{\Gamma} = \Gamma/\mathscr{T}\Gamma$. And we define the category $mod(\bar{\Lambda}, \mathscr{F})$ as follows: The objects are right $\bar{\Lambda}$-submodules $\bar{\mathscr{G}}$ of $\mathscr{F}'/\mathscr{T}\mathscr{F}'$ with $\bar{\mathscr{G}} \cdot \bar{\Gamma} = \mathscr{F}'/\mathscr{T}\mathscr{F}'$. A morphism from $\bar{\mathscr{G}}_1$ to $\bar{\mathscr{G}}_2$ is a $\bar{\Lambda}$-homomorphism φ such that there is an endomorphism f of the vector bundle $\mathscr{F}$ with the commutative diagram:

$$\begin{array}{ccc}
\bar{\mathscr{G}}_1 \otimes_{\bar{\Lambda}} \bar{\Gamma} & \longrightarrow & \bar{\mathscr{G}}_1 \cdot \bar{\Gamma} = \mathscr{F}'/\mathscr{T}\mathscr{F}' \\
{\scriptstyle \varphi\otimes\bar{\Gamma}}\downarrow & & {\scriptstyle f'\otimes\bar{\Gamma}}\downarrow \\
\bar{\mathscr{G}}_2 \otimes_{\bar{\Lambda}} \bar{\Gamma} & \longrightarrow & \bar{\mathscr{G}}_2 \cdot \bar{\Gamma} = \mathscr{F}'/\mathscr{T}\mathscr{F}'
\end{array}$$

Naturally there is a functor

$$\alpha\colon \phi_\Gamma^{-1}(\mathscr{F}) \longrightarrow mod(\bar{\Lambda}, \mathscr{F})$$

defined by $\alpha(\mathscr{G}) = \mathscr{G}/\mathscr{T}\mathscr{F}'$ for any $\mathscr{G} \in \phi_\Gamma(\mathscr{F})$; see (7.2). Now we can show:

Lemma (7.3) *The functor α yields a bijective correspondence between the sets of isomorphism classes of objects in $\phi_\Gamma^{-1}(\mathscr{F})$ and of objects in $mod(\bar{\Lambda}, \mathscr{F})$.*

Proof. For $\bar{\mathscr{G}} \in mod(\bar{\Lambda}, \mathscr{F})$, we define a Λ-module $\beta(\bar{\mathscr{G}})$ by the pull-back diagram:

$$\begin{array}{ccc} \beta(\bar{\mathscr{G}}) & \longrightarrow & \mathscr{F}' \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_1} \\ \bar{\mathscr{G}} & \xrightarrow{\pi_2} & \bar{\mathscr{G}}\cdot\bar{\Gamma} \simeq \mathscr{F}'/\mathscr{T}\mathscr{F}', \end{array}$$

where π_1 and π_2 are natural morphisms. Note that $\beta(\bar{\mathscr{G}})$ is a Λ-submodule of $\mathscr{F}'$ and that it is a locally free $\mathcal{O}_X$-module, as X is a curve. Furthermore $\beta(\bar{\mathscr{G}}) \otimes_\Lambda \Gamma \simeq \beta(\bar{\mathscr{G}})\cdot\Gamma = \mathscr{F}'$, since $\beta(\bar{\mathscr{G}})$ is a projective Λ-module at any closed point by (6.2). Hence $\beta(\bar{\mathscr{G}}) \in \phi_\Gamma^{-1}(\mathscr{F})$. Clearly $\alpha\beta(\bar{\mathscr{G}}) \simeq \bar{\mathscr{G}}$. And by the definition of α, we see that $\beta\alpha(\mathscr{G}) \simeq \mathscr{G}$ for any $\mathscr{G} \in \phi_\Gamma^{-1}(\mathscr{F})$. Thus the lemma follows.

This lemma reduces the classification of objects in $\phi_\Gamma^{-1}(\mathscr{F})$ to that of $\bar{\Lambda}$-modules. We can describe the category $mod(\bar{\Lambda}, \mathscr{F})$ in more visible way. For this purpose we introduce the following notation.

Definition (7.4) Let $\mathscr{F}$ be a vector bundle on X. Define the category $rep(\mathscr{F}, D)$ as follows:

The objects of $rep(\mathscr{F}, D)$ are the sets of vector spaces:

$$\{\{V_{x,i} | x \in X,\ 0 \le i < p_x\} |\ \text{each } V_{x,i} \text{ is a } \kappa(x)\text{-vector space and}$$
$$V_{x,i} \subset V_{x,i-1}\,(0 < i \le p_x), \text{ and } \mathscr{F} \otimes_{\mathcal{O}_X} \kappa(x) = V_{x,0}\}$$

For two objects $\{V_{x,i}\}$ and $\{V'_{x,i}\}$, morphisms between them are the endomorphisms φ of $\mathscr{F}$ satisfying:

$$(\varphi \otimes \kappa(x))(V_{x,i}) \subset V'_{x,i} \qquad (x \in X,\ 0 \le i < p_x).$$

Note that $rep(\mathscr{F}, D)$ is determined by p_x and by $\mathscr{F}$, but independent of q_x.

Lemma (7.5) *There is an equivalence of categories*:

$$mod(\bar{\Lambda}, \mathscr{F}) \simeq rep(\mathscr{F}, D).$$

Proof. Note that $\mathcal{O}_X/\mathscr{T} \simeq \prod_{p_x \neq 1} \kappa(x)$ a finite product of fields. Writing a ring of upper triangular matrices of size $p_x \times p_x$ over $\kappa(x)$ as $T_{p_x}(\kappa(x))$, we see from Theorem (5.1) that $\bar{\Lambda}$ is Morita-eqivalent to the ring $T = \prod_{p_x \neq 1} T_{p_x}(\kappa(x))$. Hence there is an equivalence of categories: $mod(\bar{\Lambda}) \simeq mod(T)$. Notice that any T-module is a set $\{V_{x,i} | x \in X,\ 0 \le i < p_x\}$ where each $V_{x,i}$ is a $\kappa(x)$-vector space and $V_{x,i} \subset V_{x,i-1}$ $(0 < i \le p_x)$. Then under the above equivalence, the subcategory $mod(\bar{\Lambda}, \mathscr{F})$ maps onto $rep(\mathscr{F}, D)$.

Now we obtain the following theorem from (7.3) and (7.5).

Theorem (7.6) *There is a bijective correspondence between the sets of isomorphism classes of objects in $\phi_\Gamma^{-1}(\mathscr{F})$ and of objects in $rep(\mathscr{F}, D)$.*

We say that the category $\mathrm{gr}\mathfrak{C}(R)$ is of finite representation type if there are a finite number of graded CM modules $M_1, M_2, \ldots, M_n$ such that any graded CM module over R is isomorphic to a direct sum of the modules of the form

$M_i(m)$ $(i, m \in \mathbf{Z},\ 1 \leq i \leq n)$. Notice that if this is the case, the category $\mathrm{gr}\mathfrak{C}(R)$ has only a countably many classes of objects. As a corollary of (7.6), in the case that X is a curve, we obtain a necessary condition for R to be of finite representation type in terms of X and D.

Corollary (7.7) *Let $k = \mathbf{C}$ the field of complex numbers. Assume that $\mathrm{gr}\mathfrak{C}(R)$ is of finite representation type. Then $X = \mathbf{P}^1$, $P = \{x \mid p_x \neq 1\}$ contains at most three points and if P consists of three points then the following inequality holds:*

$$\sum_{x \in P} \frac{1}{p_x} > 1.$$

Proof. First we prove that under the assumption, X must be isomorphic to $\mathbf{P}^1$. For this, assume that X has genus greater than one. Then the category $bdl(\mathcal{O}_X)$ of vector bundles has uncountably many isomorphism classes. Note that, for any $\mathscr{F} \in bdl(\mathcal{O}_X)$, there is at least one object $\mathscr{G} \in \phi_\Gamma^{-1}(\mathscr{F})$ by (7.6) and that if $\mathscr{F}_1 \not\simeq \mathscr{F}_2$ in $bdl(\mathcal{O}_X)$, then $\mathscr{G}_1 \not\simeq \mathscr{G}_2$ in $bdl(\Lambda)$ for $\mathscr{G}_i \in \phi_\Gamma^{-1}(\mathscr{F}_i)$ $(i = 1, 2)$. Thus $bdl(\Lambda)$, hence $\mathrm{gr}\mathfrak{C}(R)$, contains uncountably many classes of objects. This contradicts that $\mathrm{gr}\mathfrak{C}(R)$ is of finite representation type. Therefore $X \simeq \mathbf{P}^1$.

Let $P = \{x_1, x_2, \ldots, x_N\}$ and let $p_i = p_{x_i}$ $(1 \leq i \leq N)$. We consider the branched quiver Q with N branches, each of which has length p_i $(1 \leq i \leq N)$. See Figure (7.7.1).

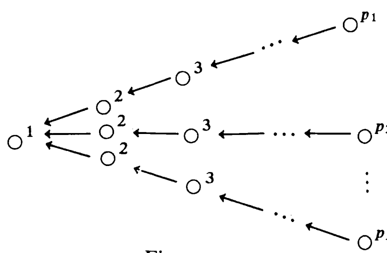

Figure (7.7.1)

Then it is obvious that the condition concerning p_i in (7.7) is equivalent to saying that the undirected graph $|Q|$ of Q is one of the Dynkin diagrams.

Now assume that $\mathrm{gr}\mathfrak{C}(R)$ is of finite representation type and that $X = \mathbf{P}^1$. Let $\mathscr{F} = \mathcal{O}_X^n$. Since $\mathrm{End}_{\mathcal{O}_X}(\mathscr{F}) = M_n(\mathbf{C})$ the complete matrix algebra and since $\mathscr{F} \otimes_{\mathcal{O}_X} \kappa(x) \simeq \mathbf{C}^n$, we can see that $rep(\mathscr{F}, D)$ is the category of representations of the quiver Q with $\mathbf{C}^n$ on the center of Q and with each arrow representing a monomorphism of $\mathbf{C}$-vector spaces. Hence if $|Q|$ is not a Dynkin diagram, then, by a theorem of Gabriel, $mod(\mathscr{F}, D)$ has uncountably many objects. (For example see [3; 8.5], where, for suitable n, uncountably many representations of this kind are constructed very concretely.) This contradicts that $\mathrm{gr}\mathfrak{C}(R)$ is of finite representation type, hence $|Q|$ is a Dynkin diagram.

Added in Proof:

In the last paragraph of the proof of Proposition (3.7), we claimed that Γ_2 is an exact functor and hence that it induces the functor Γ_n: $gr^n(\mathscr{A}) \to gr^n(R)$. But this is not correct. The assertion of (3.7) should be changed into the following:

Propositon (3.7). *Δ_2: $gr^2(R) \to gr^2(\mathscr{A})$ is an equivalence of categories. In general, Δ_n: $gr^n(R) \to gr^n(\mathscr{A})$ is a full embedding, when $n \geq 3$.*

The second claim of this is immediate form the bijectivity of the map in (3.7.2). According to this change, the statements of (4.7) and (6.5.2) should be:

(4.7). *There is an equivalence $gr^2(R) \simeq mod^2(\Lambda)$. And if $n \geq 3$, then $gr^n(R)$ can be fully embedded in $mod^n(\Lambda)$.*
(6.5.2). *Suppose that R has an isolated singularity and that X is nonsingular. If $d \geq 2$, then $bdl(\Lambda) = mod^d(\Lambda)$, in which $gr^d(R)$ can be fully embedded.*

After these alteration, the rest of the paper is valid as it is.

DEPARTMENT OF MATHEMATICS,
YOSHIDA COLLEGE,
KYOTO UNIVERSITY

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