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Vector valued Fourier hyperfunctions

By

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Introduction

In this paper, we study H-valued Fourier hyperfunctions. Here H is a complex Hilbert space which is not necessarily separable. We realize H-valued Fourier hyperfunctions as elements of the dual space of the space of all rapidly decreasing H-valued real analytic functions or as "boundary values" of slowly increasing H-valued holomorphic functions and then show that they are the twofold realization of the same H-valued Fourier hyperfunctions. When we realize H-valued Fourier hyperfunctions using H-valued analytic functionals, our treatment is more general than other works in the point that test functions are vector valued. This idea is also used in Brüning-Nagamachi [2], which I knew after submission of the present paper.

Next, we define the Fourier transformation of H-valued Fourier hyperfunctions and show that the space of H-valued Fourier hyperfunctions on the entire space is stable under the Fourier transformation. Further we prove the Paley-Wiener theorem for H-valued Fourier hyperfunctions.

Until now, many mathematicians have studied (vector valued) Fourier hyperfunctions: Sato [22], Kawai [13], [14], Ito-Nagamachi [8], [9], Junker [10], [11], Ito [5], Kaneko [12], Saburi [21], Nagamachi [20], Ito [6], [7]. Sato first introduced the notion of Fourier hyperfunctions in case of one variable and Kawai completed the theory of Fourier hyperfunctions. Ito-Nagamachi studied vector valued Fourier hyperfunctions with values in a separable Hilbert space. Junker and Ito studied vector valued Fourier hyperfunctions with values in a Fréchet space. Kaneko improved the theory of Fourier hyperfunctions. Saburi studied modified Fourier hyperfunctions. Nagamachi studied vector valued Fourier hyperfunctions of mixed type. Ito [6] studied 6 types of Fourier hyperfunctions and their vector valued versions by the algebro-analytic method. Developing Junker's method, Ito [7] constructed the unified theory of (vector valued) Sato-Fourier hyperfunctions by the duality method.

In this paper, we study the theory of Fourier hyperfunctions with values in a general Hilbert space which is not necessarily separable, by the duality method and also by the algebro-analytic method, and prove their equivalence and mutual independence of these two methods. In case of the duality method, it is characteristic that test functions are also vector valued. Thereby, we can use the vector valued version of the method of L_2 estimates for $\bar{\partial}$ operator in Hörmander [3], [4] and prove fundamental

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theorems of this paper. The proof of the equivalence of two realizations of hyperfunctions (or generalized functions) can be found in several works as Schapira [24], Komatsu [17] and etc.. But self-awakening description of this point in one paper is first appeared in this paper.

Here I wish to express my hearty thanks to Professor Hirai and the referee for their invaluable advices. They are very useful for improving the several points of the argument. Without them, we could not have obtained this completed new version of this paper.

Chapter 1. Preparations from the theory of (vector valued) functions of several complex variables

1.1. The Oka-Cartan-Kawai Theorem B. In this paper, we always assume that H is a complex Hilbert space which is not necessarily separable.

In this section, we prove the Oka-Cartan-Kawai Theorem B for the sheaf ${}^{H}\tilde{\mathcal{O}}$ of slowly increasing H-valued holomorphic functions and some of their consequences.

Here we remember the definition of the radial compactification $\boldsymbol{D}^n$ of the n-dimensional real Euclidean space $\boldsymbol{R}^n$ following Kawai [14], Definition 1.1.1.

Definition 1.1.1 (Kawai). We denote by $\boldsymbol{D}^n$ the radial compactification $\boldsymbol{R}^n \sqcup S_\infty^{n-1}$ which denotes the disjoint union of $\boldsymbol{R}^n$ and the $(n-1)$-dimensional sphere S_∞^{n-1} at infinity. When x is a vector in $\boldsymbol{R}^n \setminus \{0\}$, we denote by $x\infty$ the point in S_∞^{n-1} whose representative is x in the identification of S_∞^{n-1} with $(\boldsymbol{R}^n \setminus \{0\})/\boldsymbol{R}^+$. Here $\boldsymbol{R}^+$ denotes the set of all positive real numbers. Each element in $\boldsymbol{R}^+$ is considered as a multiplcation operator on $\boldsymbol{R}^n \setminus \{0\}$. The space $\boldsymbol{D}^n$ is endowed with the following natural topology. Namely, (i) if a point x of $\boldsymbol{D}^n$ belongs to $\boldsymbol{R}^n$, a fundamental system of neighborhoods of x is given by the family of all open spheres in $\boldsymbol{R}^n$ including x. (ii) If a point x of $\boldsymbol{D}^n$ belongs to S_∞^{n-1}, a fundamental system of neighborhoods of $x(=y\infty)$ is given by the family $\{(C+a)\cup C_\infty;\ C_\infty \ni y\infty\}$. Here a runs through all points in $\boldsymbol{R}^n$ and C runs through all open cones in $\boldsymbol{R}^n$ with the vertex at the origin which contains $y \in \boldsymbol{R}^n \setminus \{0\}$ and C_∞ denotes the set $\{z\infty;\ z \in C\}$.

We denote by $\tilde{\boldsymbol{C}}^n$ the space $\boldsymbol{D}^n \times \sqrt{-1}\,\boldsymbol{R}^n$ endowed with the direct product topology. $\boldsymbol{D}^n$ and S_∞^{n-1} are identified with the subsets of $\tilde{\boldsymbol{C}}^n$ by the relations $\boldsymbol{D}^n \simeq \boldsymbol{D}^n \times \sqrt{-1}\,\{0\} \hookrightarrow \tilde{\boldsymbol{C}}^n$ and $S_\infty^{n-1} \simeq S_\infty^{n-1} \times \sqrt{-1}\,\{0\} \hookrightarrow \tilde{\boldsymbol{C}}^n$. For a subset E of $\tilde{\boldsymbol{C}}^n$, we denote by $\mathrm{int}(E)$ its interior and by E^a its closure in $\tilde{\boldsymbol{C}}^n$.

In this paper, we denote by $(\cdot, \cdot)$ the inner product of H and by ${}^{H}\|\cdot\|$ the norm of H.

Let U be a measurable set in $\boldsymbol{C}^n$. A measurable H-valued function $f(z)$ on U is said to be square integrable if the integral $\int_U {}^{H}\|f(z)\|^2 d\lambda$ converges, where $d\lambda$ denotes the Lebesgue measure on $\boldsymbol{C}^n$. Let $L_2(U;\ H)$ be the space of all square integrable H-valued functions on U. A measurable H-valued function $f(z)$ on U is said to be locally square integrable if, for every relatively compact subset ω of U, $f(z)|_\omega$ belongs to $L_2(\omega;\ H)$. Let $L_{2,\mathrm{loc}}(U;\ H)$ be the space of all locally square integrable H-valued functions on U.

If $\{e_\alpha\}_{\alpha\in A}$ is an orthonormal basis of H, each $f(z)\in L_2(U;H)$ can be expanded by the formula

$$f(z)=\sum_{\alpha\in A} f_\alpha(z)e_\alpha, \qquad f_\alpha(z)=(f(z), e_\alpha),$$

where every $f_\alpha(z)$ is square integrable on U and $f_\alpha(z)=0$ except for at most countable numbers of $\alpha\in A$. Then we have the formula

$$\int_U {}^H\|f(z)\|^2 d\lambda=\sum_{\alpha\in A}\int_U |f_\alpha(z)|^2 d\lambda.$$

Here we remember the notion of H-valued holomorphic functions. Let $\boldsymbol{C}^n$ be the n-dimensional complex Euclidean space and Ω an open subset of $\boldsymbol{C}^n$. An H-valued smooth function $f(z)$ on Ω is said to be holomorphic if it satisfies the Cauchy-Riemann equation $\bar{\partial} f=0$ on Ω. We denote by $\mathcal{O}(\Omega;H)$ the space of H-valued holomorphic functions on Ω. We define the sheaf ${}^H\mathcal{O}$ of H-valued holomorphic functions over $\boldsymbol{C}^n$ to be the sheaf $\{\mathcal{O}(\Omega;H);\ \Omega$ is an open set in $\boldsymbol{C}^n\}$. We put $\mathcal{O}={}^C\mathcal{O}$.

If $\{e_\alpha\}_{\alpha\in A}$ is an orthonormal basis of H, each $f(z)\in\mathcal{O}(\Omega;H)$ can be expanded by the formula

$$f(z)=\sum_{\alpha\in A} f_\alpha(z)e_\alpha, \qquad f_\alpha(z)=(f(z), e_\alpha),$$

where every $f_\alpha(z)$ is holomorphic on Ω and $f_\alpha(z)=0$ except for at most countable numbers of $\alpha\in A$. As to the notion of an orthonormal basis of a Hilbert space H, we refer Bourbaki [1], § 2-3, Chapter 5. The norm ${}^H\|f(z)\|$ of $f(z)$ can be calculated as follows:

$${}^H\|f(z)\|=(\sum_{\alpha\in A}|f_\alpha(z)|^2)^{1/2}.$$

For every $f(z)\in\mathcal{O}(\Omega;H)$, $\sup\{{}^H\|f(z)\|;\ z\in K\}<\infty$ holds for every compact subset of Ω. If we define a seminorm ${}^H\|f\|_K$ of $\mathcal{O}(\Omega;H)$ by the relation ${}^H\|f\|_K=\sup\{{}^H\|f(z)\|;\ z\in K\}$, $\mathcal{O}(\Omega;H)$ becomes a Fréchet space with respect to the topology defined by the family of seminorms $\{{}^H\|f\|_K;\ K$ is a compact set in $\Omega\}$.

Definition 1.1.2 (The sheaf ${}^H\tilde{\mathcal{O}}$ of slowly increasing H-valued holomorphic functions). We define the sheaf ${}^H\tilde{\mathcal{O}}$ over $\tilde{\boldsymbol{C}}^n$ to be the sheaf $\{\tilde{\mathcal{O}}(\Omega;H);\ \Omega$ is an opn set in $\tilde{\boldsymbol{C}}^n\}$, where the section module $\tilde{\mathcal{O}}(\Omega;H)$ on an open set Ω in $\tilde{\boldsymbol{C}}^n$ is the space of all H-valued holomorphic functions $f(z)$ on $\Omega\cap\boldsymbol{C}^n$ such that, for any positive number ε and for any compact set K in Ω, the estimate $\sup\{{}^H\|f(z)\|e(-\varepsilon|z|);\ z\in K\cap\boldsymbol{C}^n\}<\infty$ holds. Here $e(t)$ denotes the function $e^t=\exp(t)$ of $t\in\boldsymbol{C}$ and we put $|z|=(|z_1|^2+\cdots+|z_n|^2)^{1/2}$. We put $\tilde{\mathcal{O}}={}^C\tilde{\mathcal{O}}$.

If we define a seminorm ${}^H\|f\|_{K,\varepsilon}$ of $\tilde{\mathcal{O}}(\Omega;H)$ by the relation ${}^H\|f\|_{K,\varepsilon}=\sup\{{}^H\|f(z)\| e(-\varepsilon|z|);\ z\in K\cap\boldsymbol{C}^n\}$, $\tilde{\mathcal{O}}(\Omega;H)$ becomes an FS*-space with respect to the topology defined by the family of seminorms $\{{}^H\|f\|_{K,\varepsilon};\ K$ is a compact set in Ω and ε is a positive number$\}$. As to the notion of FS*-spaces, we refer Komatsu [15].

Definition 1.1.3 (The sheaf ${}^H\mathcal{Q}$ of rapidly decreasing H-valued holomorphic functions). We define the sheaf ${}^H\mathcal{Q}$ over $\tilde{\boldsymbol{C}}^n$ to be the sheaf $\{\mathcal{Q}(\Omega;H);\ \Omega$ is an open set in $\tilde{C}^n\}$, where the section module $\mathcal{Q}(\Omega;H)$ on an open set Ω in $\tilde{C}^n$ is the space of all H-valued holomorphic functions $f(z)$ on $\Omega\cap C^n$ such that, for any compact set K in Ω, there exists some positive constant δ so that the estimate $\sup\{{}^H\|f(z)\|e(\delta|z|);\ z\in K\cap C^n\}<\infty$ holds. We put $\mathcal{Q}=\mathcal{Q}^C$.

Definition 1.1.4 (Definition of the space $\mathcal{O}_b^\eta(U;H)$). Let U be an open set in $\tilde{C}^n$. For $\eta\in\boldsymbol{R}$, the Banach space $\mathcal{O}_b^\eta(U;H)$ is defined to be the space

$$\mathcal{O}_b^\eta(U;H)=\{f\in C(U^a\cap C^n;H);\ f|_{U\cap C^n}\in\mathcal{O}(U\cap C^n;H),$$
$$\sup\{{}^H\|f(z)\|e(-\eta|z|);\ z\in U^a\cap C^n\}<\infty\}.$$

Let K be a compact set in $\tilde{C}^n$. Let $\mathcal{Q}(K;H)$ be the space of all rapidly decreasing H-valued holomorphic functions on a certain neighborhood of K.

Let $\{U_m\}_{m\geq 1}$ be a fundamental system of neighborhoods of K such that $U_{m+1}\subset\subset U_m$ holds. Here $U_{m+1}\subset\subset U_m$ means that U_{m+1} has a compact neighborhood in U_m with respect to the topology of $\tilde{C}^n$. Then we have the isomorphism $\mathcal{Q}(K;H)\cong\lim\limits_{m}\text{ind}\ \mathcal{O}_b^{-1/m}(U_m;H)$.

Then $\mathcal{Q}(K;H)$ becomes a DFS*-space. As to the notion of DFS*-spaces, we refer Komatsu [15].

Let Ω be an open set in $\tilde{C}^n$ and $\{K_m\}_{m\geq 1}$ be an exhausting family of compact subsets of Ω such that $K_1\subset K_2\subset\cdots\subset K_m\subset\cdots\subset\Omega$ and $\cup_m K_m=\Omega$ hold. Then we have an isomorphism

$$\mathcal{Q}(\Omega;H)\cong\lim_{m}\text{proj}\ \mathcal{Q}(K_m;H).$$

Then $\mathcal{Q}(\Omega;H)$ becomes an FS*-space with respect to the projective limit topology by the following.

Lemma A. *Let* $X=\lim\limits_{m}\text{ind}\,X_m$ *and* $Y=\lim\limits_{m}\text{ind}\,Y_m$ *be inductive limits of sequences of locally convex spaces* $\{X_m\}$ *and* $\{Y_m\}$, *respectively. Assume that, for every* m, r_m: $X_m\to Y_m$ *be a (weakly) compact linear mapping. Put* $r=\lim\limits_{m}\text{ind}\,r_m$. *Then* r *is a (weakly) compact linear mapping.*

It is easy to see that ${}^H\tilde{\mathcal{O}}|_{C^n}={}^H\mathcal{Q}|_{C^n}={}^H\mathcal{O}$ holds.

Next, constructing the soft resolution of the sheaf ${}^H\tilde{\mathcal{O}}$ and using it, we will prove the Oka-Cartan-Kawai Theorem B for the sheaf ${}^H\tilde{\mathcal{O}}$.

First we mention the definition of the sheaf ${}^H\tilde{L}={}^H\tilde{L}_{2,\text{loc}}$ of slowly increasing H-valued locally square integrable functions over $\tilde{C}^n$.

Definition 1.1.5. We define the sheaf ${}^H\tilde{L}$ over $\tilde{C}^n$ to be the sheaf $\{\tilde{L}(\Omega;H);\ \Omega$ is an open set in $\tilde{C}^n\}$, where the section module $\tilde{L}(\Omega;H)$ on an open set Ω in $\tilde{C}^n$ is the space of all $f\in L_{2,\text{loc}}(\Omega\cap C^n;H)$ such that, for every positive number ε and for every relatively compact open subset ω of Ω, $e(-\varepsilon\|z\|)f(z)|_\omega\in L_2(\omega\cap C^n;H)$ holds. Here $\|z\|$ denotes a C^∞-plurisubharmonic function as a modification of $|z|_1=\sum_{j=1}^n|z_j|$ by a certain mollifier.

Then ${}^{H}\tilde{L}$ constitutes a soft sheaf and, for an open set Ω in $\tilde{C}^n$, $\tilde{L}(\Omega; H)$ becomes an FS*-space.

If $\mathcal{F}$ is the sheaf of functions of some class over $\tilde{C}^n$ and Ω is an open set in $\tilde{C}^n$, let $\mathcal{F}^{p,q}(\Omega)$ be the space of all differential forms of type (p, q) with coefficients in the section module $\mathcal{F}(\Omega)$, $(p, q \geqq 0)$. Let $\mathcal{F}^{p,q}$ denote the sheaf $\{\mathcal{F}^{p,q}(\Omega); \Omega$ is an open set in $\tilde{C}^n\}$.

Definition 1.1.6 (the sheaf ${}^{H}\tilde{\mathcal{I}}^{p,q}$). We define the sheaf ${}^{H}\tilde{\mathcal{I}}^{p,q}$ over $\tilde{C}^n$ to be the sheaf $\{\tilde{\mathcal{I}}^{p,q}(\Omega; H); \Omega$ is an open set in $\tilde{C}^n\}$, where the section module $\tilde{\mathcal{I}}^{p,q}(\Omega; H)$ on an open set Ω in $\tilde{C}^n$ is the space of all $f \in \tilde{L}^{p,q}(\Omega; H)$ such that $\bar{\partial} f \in \tilde{L}^{p,q+1}(\Omega; H)$. Here $\bar{\partial} f$ is defined in the distribution sense. Especially we put ${}^{H}\tilde{\mathcal{I}} = {}^{H}\tilde{\mathcal{I}}^{0,0}$.

Then ${}^{H}\tilde{\mathcal{I}}^{p,q}$ constitutes a soft sheaf. Equipped with a graph topology with respect to the $\bar{\partial}$ oprator, $\tilde{\mathcal{I}}^{p,q}(\Omega; H)$ becomes an FS*-space for an open set Ω in $\tilde{C}^n$.

Definition 1.1.7. An open set V in $\tilde{C}^n$ is said to be an $\tilde{\mathcal{O}}$-pseudoconvex open set if it satisfies the conditions:

(1) $\sup\{|\mathrm{Im}\, z|; z \in V \cap C^n\} < \infty$, where we put $\mathrm{Im}\, z = (\mathrm{Im}\, z_1, \cdots, \mathrm{Im}\, z_n)$.

(2) There exists a strictly plurisubharmonic C^∞-function $\varphi(z)$ on $V \cap C^n$ having the following two properties:

(i) The closure of $V_t = \{z \in V \cap C^n; \varphi(z) < t\}$ in $\tilde{C}^n$ is a compact subset of V for every $t \in R$.

(ii) $\varphi(z)$ is bounded on $L \cap C^n$ for every compact subset L of V.

As typical examples of $\tilde{\mathcal{O}}$-pseudoconvex open sets, we have the following: $D^n \times \sqrt{-1}\{y \in R^n; |y| < \varepsilon\}$, $\mathrm{int}(\{z = x + \sqrt{-1}\, y \in C^n; |x| > b, |y| < \varepsilon\}^a)$, $(b > 0, \varepsilon > 0)$. Then the families $\{D^n \times \sqrt{-1}\{y \in R^n; |y| < \varepsilon\}; \varepsilon$ is a positive number$\}$, $\{\mathrm{int}(\{z = x + \sqrt{-1}\, y \in C^n; |x| > b, |y| < \varepsilon\}^a; b$ and ε are positive numbers$\}$ and $\{\mathrm{int}\{z = x + \sqrt{-1}\, y \in C^n; |x| > b, |y| < r + \varepsilon\}^a; b$ and ε are positive numbers$\}$ are fundamental systems of $\tilde{\mathcal{O}}$-pseudoconvex open neighborhoods of D^n, S_∞^{n-1} and $S_\infty^{n-1} \times \sqrt{-1}\{y \in R^n; |y| \leqq r\}$ in $\tilde{C}^n$ respectively, where r is some nonnegative constant.

For $e = (1, 0, \cdots, 0) \in S^{n-1}$, $b \in R^n$ and $\varepsilon > 0$, we put $C_{e,\varepsilon} = \{x \in R^n; x_2^2 + \cdots + x_n^2 < \varepsilon^2 (x_1 - \varepsilon^{-1})^2, x_1 > 0\}$ and $B_{b,\varepsilon} = \{y \in R^n; |y - b| < \varepsilon\}$. Then $\Omega(e\infty + \sqrt{-1}\, b; \varepsilon) = \mathrm{int}((C_{e,\varepsilon})^a) \times \sqrt{-1}\, B_{b,\varepsilon}$ is an $\tilde{\mathcal{O}}$-pseudoconvex open set in $\tilde{C}^n$. Then the family $\{\Omega(e\infty + \sqrt{-1}\, b; \varepsilon); \varepsilon > 0\}$ is a fundamental system of $\tilde{\mathcal{O}}$-pseudoconvex open neighborhoods of the point $e\infty + \sqrt{-1}\, b$.

Further, consider $a \in S^{n-1}$ and $b \in R^n$. Then there exists a rotation $T \in SO(n)$ such that $a = Te$. If we put $\Omega(a\infty + \sqrt{-1}\, b; \varepsilon) = \mathrm{int}((TC_{e,\varepsilon})^a) \times \sqrt{-1}\, B_{b,\varepsilon}$ for an arbitrary point $a\infty + \sqrt{-1}\, b \in S_\infty^{n-1} \times \sqrt{-1}\, R^n$, then the family $\{\Omega(a\infty + \sqrt{-1}\, b; \varepsilon); \varepsilon > 0\}$ is a fundamental system of $\tilde{\mathcal{O}}$-pseudoconvex open neighborhoods of the point $a\infty + \sqrt{-1}\, b$.

Theorem 1.1.8 (Hörmander-Kaneko). *Put* $U = \mathrm{int}(\{z \in C^n; |\mathrm{Im}\, z| < 1 + |\mathrm{Re}\, z|/\sqrt{3}\}^a)$ *and let* Ω *be an arbitrary* $\tilde{\mathcal{O}}$*-pseudoconvex open set included in* U. *Then, for every* $f \in \tilde{\mathcal{I}}^{p,q+1}(\Omega; H)$ *such that* $\bar{\partial} f = 0$, *there exists a solution* $u \in \tilde{\mathcal{I}}^{p,q}(\Omega; H)$ *so that* $\bar{\partial} u = f$ *holds. Here* p *and* q *are nonnegative integers.*

Proof. This is an H-valued version of Kaneko [12], Theorem 8.6.6 (p. 412).

We can prove this in a similar way as the proof of Kaneko [12], Theorem 8.6.6 (p. 412) using the following lemma 1.1.9 which is an H-valued version of Hörmander [3], Theorem 2.2.1′ (p. 105). (Q. E. D.)

Lemma 1.1.9. *Let Ω be a pseudoconvex open set in $\boldsymbol{C}^n$, let φ be plurisubharmonic in Ω and let e^{χ} where $\chi \in C(\Omega)$ be a lower bound for the plurisubharmonicity of φ. For every $f \in L_{2,\mathrm{loc}}^{p,q}(\Omega; H)$, $q>0$, such that $\bar{\partial} f=0$ and*

$$\int_{\Omega}{}^{H}\|f\|^2 e^{-(\varphi+\chi)} d\lambda<\infty,$$

one can then find a form $u \in L_2^{p,q-1}(\Omega, \varphi; H)$ such that $\bar{\partial} u=f$ and

$$q\int_{\Omega}{}^{H}\|u\|^2 e^{-\varphi} d\lambda \leqq \int_{\Omega}{}^{H}\|f\|^2 e^{-(\varphi+\chi)} d\lambda.$$

Here $L_2(\Omega, \varphi; H)$ denotes the space of all H-valued functions in Ω which are square integrable with respect to the density $e^{-\varphi}$.

Proof. If $\{e_\alpha\}_{\alpha\in A}$ is an orthonormal basis and $f(z) \in L_{2,\mathrm{loc}}^{p,q}(\Omega; H)$ satisfies the assumptions in the Lemma, $f(z)$ can be expanded by the formula

$$f(z)=\sum_{\alpha\in A} f_\alpha(z) e_\alpha, \qquad f_\alpha(z)=(f(z), e_\alpha), \qquad \bar{\partial} f_\alpha(z)=0$$

and

$$\int_{\Omega}{}^{H}\|f(z)\|^2 e^{-(\varphi+\chi)} d\lambda=\sum_{\alpha\in A}\int_{\Omega}|f_\alpha(z)|^2 e^{-(\varphi+\chi)} d\lambda$$

holds. Here $f_\alpha(z)=0$ except at most countable number of $\alpha \in A$. Now, applying Hörmander [3], Theorem 2.2.1′ (p. 105) to every $f_\alpha(z)$, we can find a form $u_\alpha(z) \in L_2^{p,q-1}(\Omega, \varphi)=L_2^{p,q-1}(\Omega, \varphi; \boldsymbol{C})$ such that $\bar{\partial} u_\alpha=f_\alpha$ and

$$q\int_{\Omega}|u_\alpha(z)|^2 e^{-\varphi} d\lambda \leqq \int_{\Omega}|f_\alpha(z)|^2 e^{-(\varphi+\chi)} d\lambda.$$

If we put $u=\sum_{\alpha\in A} u_\alpha(z) e_\alpha$, then we have $\bar{\partial} u=f$ and

$$q\int_{\Omega}{}^{H}\|u\|^2 e^{-\varphi} d\lambda \leqq \int_{\Omega}{}^{H}\|f\|^2 e^{-(\varphi+\chi)} d\lambda$$

holds. (Q. E. D.)

Theorem 1.1.10 (Dolbeault-Grothendieck resolution). *The sequence of sheaves over $\tilde{\boldsymbol{C}}^n$*

$$0 \longrightarrow {}^{H}\tilde{\mathcal{O}}^p \longrightarrow {}^{H}\mathcal{I}^{p,0} \xrightarrow{\bar{\partial}} {}^{H}\mathcal{I}^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} {}^{H}\mathcal{I}^{p,n} \longrightarrow 0$$

is exact. Here p is a nonnegative integer.

Proof. We have only to prove the exactness of the sequence

$$0 \longrightarrow {}^{H}\tilde{\mathcal{O}}_x^p \longrightarrow {}^{H}\mathcal{I}_x^{p,0} \xrightarrow{\bar{\partial}} \mathcal{I}_x^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{I}_x^{p,n} \longrightarrow 0$$

at every point $x \in \tilde{\boldsymbol{C}}^n$.

Let U be as in Theorem 1.1.8. At first we will prove the exactness of the above sequence at every point $x \in U$. The exactness of the sequence

$$0 \longrightarrow {}^{H}\tilde{\mathcal{O}}^{p}_{x} \longrightarrow {}^{H}\tilde{\mathcal{L}}^{p,0}_{x} \xrightarrow{\bar{\partial}} {}^{H}\tilde{\mathcal{L}}^{p,1}_{x}$$

follows from the ellipticity of $\bar{\partial}$ and the fact that, for H-valued holomorphic functions, we can interchange sup-norm and L_2-norm locally.

As to the exactness of the sequence

$${}^{H}\tilde{\mathcal{L}}^{p,0}_{x} \xrightarrow{\bar{\partial}} {}^{H}\tilde{\mathcal{L}}^{p,1}_{x} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} {}^{H}\tilde{\mathcal{L}}^{p,n}_{x} \longrightarrow 0,$$

we have the conclusion by virtue of Theorem 1.1.8 because each point in U has a fundamental system of $\tilde{\mathcal{O}}$-pseudoconvex open neighborhoods.

As to the point x outside of U, we can show the exactness by translation.

(Q. E. D.)

Corollary 1.1.11. *For every open set Ω in $\tilde{\boldsymbol{C}}^n$, we have the isomorphism*

$$H^q(\Omega, {}^{H}\tilde{\mathcal{O}}^p) \cong \{f \in \tilde{\mathcal{L}}^{p,q}(\Omega; H); \bar{\partial} f = 0\} / \{\bar{\partial} g; g \in \tilde{\mathcal{L}}^{p,q-1}(\Omega; H)\},$$

$$(p \geqq 0, q \geqq 1).$$

Using this Corollary, we can prove the Oka-Cartan-Kawai Theorem B.

Theorem 1.1.12. *Let U be as in Theorem 1.1.8 and Ω an arbitrary $\tilde{\mathcal{O}}$-pseudoconvex open set in $\tilde{\boldsymbol{C}}^n$ such that some translation of $\Omega \cap \boldsymbol{C}^n$ is included in U. Then we have $H^q(\Omega, {}^{H}\tilde{\mathcal{O}}^p) = 0$, $(p \geqq 0, q \geqq 1)$.*

Proof. This follows from Theorem 1.1.8 and Corollary 1.1.11. (Q. E. D.)

Now we define the sheaf ${}^{H}\underset{\sim}{L} = {}^{H}\underset{\sim}{L}_{2,\mathrm{loc}}$ of rapidly decreasing H-valued locally square integrable functions.

Definition 1.1.13. We define the sheaf ${}^{H}\underset{\sim}{L}$ over $\tilde{\boldsymbol{C}}^n$ to be the sheaf $\{\underset{\sim}{L}(\Omega; H); \Omega$ is an open set in $\tilde{\boldsymbol{C}}^n\}$, where the section module $\underset{\sim}{L}(\Omega; H)$ on an open set Ω in $\tilde{\boldsymbol{C}}^n$ is the space

$$\underset{\sim}{L}(\Omega; H) = \{f \in L_{2,\mathrm{loc}}(\Omega \cap \boldsymbol{C}^n; H); \text{ for every } \omega \subset\subset \Omega, \text{ there exists some } \delta > 0 \text{ such that } e(\delta\|z\|) f(z)|_{\omega \cap \boldsymbol{C}^n} \in L_2(\omega \cap \boldsymbol{C}^n; H)\}.$$

Then the sheaf ${}^{H}\underset{\sim}{L}$ constitutes a soft sheaf. For a compact set K in $\tilde{\boldsymbol{C}}^n$, ${}^{H}\underset{\sim}{L}(K) = \underset{\sim}{L}(K; H)$ is defined to be the space $\varinjlim_{K \subset U} \underset{\sim}{L}(U; H)$ where U runs through all open neighborhoods of K in $\tilde{\boldsymbol{C}}^n$. $\underset{\sim}{L}(K; H)$ is endowed with the following topology. Let $\{U_m\}$ be a fundamental system of neighborhoods of K such that $U_{m+1} \subset\subset U_m$ holds. Here $U_{m+1} \subset\subset U_m$ means that $(U_{m+1})^a$ is a compact subset of U_m. Let $L_{2,b}^{-1/m}(U_m; H)$ be the space of all $f \in L_{2,\mathrm{loc}}(U_m \cap \boldsymbol{C}^n; H)$ such that

$$\int_{U_m \cap \boldsymbol{C}^n} {}^{H}\|f\|^2 e(|z|/m)\, d\lambda < \infty$$

holds. Then the family $\{L_{2,b}^{-1/m}(U_m; H)\}$ constitutes a weakly compact inductive sequence with respect to the restriction mappings and we have the isomorphism

$$\underset{\sim}{L}(K;\ H)=\lim_{m}\operatorname{ind} L_{2,b}^{-1/m}(U_m;\ H).$$

By this topology, $\underset{\sim}{L}(K;\ H)$ becomes a DFS*-space.

Then $\underset{\sim}{L}(\Omega;\ H)$ becomes an FS*-space with respect to the projective limit topology:

$$\underset{\sim}{L}(\Omega;\ H)=\lim_{\Omega\supset K}\operatorname{proj} \underset{\sim}{L}(K;\ H),$$

where K runs through all compact subsets of Ω (see the above Lemma A).

Definition 1.1.14 (the sheaf of ${}^H\underset{\sim}{\mathcal{L}}^{p,q}$). We define the sheaf ${}^H\underset{\sim}{\mathcal{L}}^{p,q}$ over $\tilde{\boldsymbol{C}}^n$ to be the sheaf $\{\underset{\sim}{\mathcal{L}}^{p,q}(\Omega;\ H);\ \Omega$ is an open set in $\tilde{\boldsymbol{C}}^n\}$, where the section module $\underset{\sim}{\mathcal{L}}^{p,q}(\Omega;\ H)$ on an open set Ω in $\tilde{\boldsymbol{C}}^n$ is the space

$$\underset{\sim}{\mathcal{L}}^{p,q}(\Omega;\ H)=\{f\in \underset{\sim}{L}^{p,q}(\Omega;\ H);\ \bar{\partial}f\in \underset{\sim}{L}^{p,q+1}(\Omega;\ H)\}.$$

We put ${}^H\underset{\sim}{\mathcal{L}}={}^H\underset{\sim}{\mathcal{L}}^{0,0}$.

Then ${}^H\underset{\sim}{\mathcal{L}}^{p,q}$ constitutes a soft sheaf. Equipped with a graph topology with respect to the $\bar{\partial}$-operator, ${}^H\underset{\sim}{\mathcal{L}}^{p,q}(K)$ becomes a DFS*-space for a compact set K in $\tilde{\boldsymbol{C}}^n$. Then we have the following.

Theorem 1.1.15 (Hörmander-Kaneko). *Put* $U=\operatorname{int}(\{z\in \boldsymbol{C}^n;\ |\operatorname{Im} z|<1+|\operatorname{Re} z|/\sqrt{3},\ |\operatorname{Im} z|^2<1/2+|\operatorname{Re} z|^2\}^a)$. *Let Ω be an arbitrary $\tilde{\mathcal{O}}$-pseudoconvex open set included in U. Let f be an element in $\underset{\sim}{\mathcal{L}}^{p,q+1}(\Omega;\ H)$ such that $\bar{\partial}f=0$. Then, for any open set $\omega\subset\subset\Omega$, there exists a solution $u\in \underset{\sim}{\mathcal{L}}^{p,q}(\omega;\ H)$ of the equation $\bar{\partial}u=f$ on $\omega\cap \boldsymbol{C}^n$. Here p and q are nonnegative integers.*

Proof. Let $f\in \underset{\sim}{\mathcal{L}}^{p,q}(\Omega;\ H)$ such that $\bar{\partial}f=0$. Next, $\varphi(z)$ being a plurisubharmonic function $\Omega\cap \boldsymbol{C}^n$ which satisfies the conditions in Definition 1.1.6. For a $t\in \boldsymbol{R}$, we put $\Omega_t=\operatorname{int}(\{z\in\Omega\cap \boldsymbol{C}^n;\ \varphi(z)<t\}^a)$. Then, for any open set $\omega\subset\subset\Omega$, there exists a $t\in \boldsymbol{R}$ such that $\omega\subset\subset\Omega_t$. Then there exists a $\delta>0$ so that $h_\delta f|_{\Omega_t\cap C^n}\in \tilde{\mathcal{L}}^{p,q+1}(\Omega_t;\ H)$ holds, where we put $h_\delta(z)=e(\delta\sqrt{z^2+1})$ and $z^2=z_1^2+\cdots+z_n^2$. Since h_δ is holomorphic on $U\cap \boldsymbol{C}^n$ and f is $\bar{\partial}$-closed on $\Omega\cap \boldsymbol{C}^n$. Then, by virtue of Theorem 1.1.7, we have $v\in \tilde{\mathcal{L}}^{p,q}(\Omega_t;\ H)$ so that $\bar{\partial}v=h_\delta f$ holds on $\Omega_t\cap \boldsymbol{C}^n$. Put $u=(v/h_\delta)|_\omega$. Then we have $u\in \underset{\sim}{\mathcal{L}}^{p,q}(\omega;\ H)$ and $\bar{\partial}u=f$ on $\omega\cap \boldsymbol{C}^n$. (Q.E.D.)

Theorem 1.1.16 (Dolbeault-Grothendieck resolution). *The seqnence of sheaves over* $\tilde{\boldsymbol{C}}^n$

$$0\longrightarrow {}^H\underset{\sim}{\mathcal{O}}^p\longrightarrow {}^H\underset{\sim}{\mathcal{L}}^{p,0}\xrightarrow{\bar{\partial}} {}^H\underset{\sim}{\mathcal{L}}^{p,1}\xrightarrow{\bar{\partial}}\cdots\xrightarrow{\bar{\partial}} {}^H\underset{\sim}{\mathcal{L}}^{p,n}\longrightarrow 0$$

is exact, $(p\geqq 0)$.

Proof. Since the exactness of the above sequence is equivalent to the local solvability of the $\bar{\partial}$ equation, we have the conclusion similarly to Theorem 1.1.10 by virtue of Theorem 1.1.15 and the ellipticity of $\bar{\partial}$-operator at the term ${}^H\underset{\sim}{\mathcal{L}}^{p,0}$. (Q.E.D.)

Corollary 1.1.17. *For an open set Ω in $\tilde{\boldsymbol{C}}^n$, we have the isomorphism*

$$H^q(\Omega, {}^H\mathcal{O}^p) \cong \{f \in \mathcal{L}^{p,q}(\Omega; H); \bar{\partial} f = 0\} / \{\bar{\partial} g; g \in \mathcal{L}^{p,q+1}(\Omega; H)\},$$

$(p \geqq 0, q \geqq 1)$.

Then we have the following.

Theorem 1.1.18. *Let V be an open set in $\tilde{\boldsymbol{C}}^n$ such that some translation of $V \cap \boldsymbol{C}^n$ can be contained in U of Theorem* 1.1.14. *Assume that K is a compact subset of $\tilde{\boldsymbol{C}}^n$ included in V and has a fundamental system of neighborhoods composed of $\tilde{\mathcal{O}}$-pseudoconvex open sets. Then we have $H^q(K, {}^H\mathcal{O}^p)=0$, $(p \geqq 0, q \geqq 1)$.*

Proof. Every open set in $\tilde{\boldsymbol{C}}^n$ is paracompact, we have

$$H^q(K, {}^H\mathcal{O}^p) = \lim_{K \subset \Omega} \operatorname{ind} H^q(\Omega, {}^H\mathcal{O}^p).$$

Here Ω runs through all open neighborhoods of K. Since K has a fundamental system of neighborhoods composed of $\tilde{\mathcal{O}}$-pseudoconvex open sets, we have the conclusion by virtue of Theorem 1.1.15 and Corollary 1.1.17. (Q. E. D.)

At last we have the following.

Theorem 1.1.19 (Malgrange). *For every open subset S of $\boldsymbol{D}^n$, we have $H^q(S, {}^H\tilde{\mathcal{A}}^p)=0$, $(p \geqq 0, q \geqq 1)$. Here we put ${}^H\tilde{\mathcal{A}} = {}^H\tilde{\mathcal{O}}|_{D^n}$.*

Proof. By virtue of the Grauert Theorem in Kawai [14], Theorem 2.1.6 (p. 473), S has a fundamental system of neighborhoods $\{\Omega_m\}$ composed of $\tilde{\mathcal{O}}$-pseudoconvex open sets. Thus we have

$$H^q(S, {}^H\tilde{\mathcal{A}}^p) = \lim_{m} \operatorname{ind} H^q(\Omega_m, {}^H\tilde{\mathcal{O}}^p) = 0$$

by virtue of Theorem 1.1.12. (Q. E. D.)

Theorem 1.1.20 (Malgrange). *For every compact subset K of $\boldsymbol{D}^n$, we have $H^q(K, {}^H\mathcal{A}^p)=0$, $(p \geqq 0, q \geqq 1)$. Here we put ${}^H\mathcal{A} = {}^H\mathcal{O}|_{D^n}$.*

This is a corollary of Theorem 1.1.18.

1.2. Approximation Theorem. In this section, we prove an approximation theorem of Runge's type for ${}^H\mathcal{A}$.

Theorem 1.2.1. *For a compact set K in $\boldsymbol{D}^n$, $\mathcal{A}(\boldsymbol{D}^n; H)$ is dense in $\mathcal{A}(K; H)$.*

Proof. By virtue of the fact mentioned after Definition 1.1.4, we have the isomorphism $\mathcal{A}(K; H) \cong \lim_{m} \operatorname{ind} \mathcal{O}_b^{-1/m}(U_m; H)$ in the notation used there. We have to prove that $\mathcal{A}(\boldsymbol{D}^n; H)$ is dense in every $\mathcal{O}_b^{-1/m}(U_m; H)$. Now, let $\{e_\alpha\}_{\alpha \in A}$ be a complete orthonormal basis of H. Then every $f \in \mathcal{O}_b^{-1/m}(U_m; H)$ can be expanded as follows:

$$f = \sum_{\alpha \in A} f_\alpha e_\alpha, \qquad f_\alpha = (f, e_\alpha).$$

Then $f_\alpha \in \mathcal{O}_b^{-1/m}(U_m)=\mathcal{O}_b^{-1/m}(U_m;\ \boldsymbol{C})$ holds. Then we have only to apply Theorem 2.2.1 of Kawai [14] (p. 474) to every f_α. (Q.E.D.)

Chapter 2. The realization of H-valued Fourier hyperfunctions using H-valued Fourier analytic functionals

2.1. H-valued Fourier analytic functinals. In this section, we introduce the notion of H-valued Fourier analytic functionals as the dual object of rapidly decreasing H-valued holomorphic or real analytic functions.

Definition 2.1.1. For an open set Ω in $\tilde{\boldsymbol{C}}^n$, we consider the dual space $\mathcal{Q}(\Omega;\ H)'$ of the space $\mathcal{Q}(\Omega;\ H)$ and $\mathcal{Q}(\Omega;\ H)'$ is endowed with the topology of uniform convergence on every bounded set in $\mathcal{Q}(\Omega;\ H)$. An element u of $\mathcal{Q}(\Omega;\ H)'$ is said to be an H-valued Fourier analytic functional on Ω. We say that $u\in\mathcal{Q}(\Omega;\ H)'$ is carried by a compact subset K of Ω if u can be extended to $\mathcal{Q}(K;\ H)$. Then we call K a carrier of u. We also say that $u\in\mathcal{Q}(\Omega;\ H)'$ is carried by an open subset ω of Ω if u is carried by some compact subset of ω. Then ω is said to be a carrier of u. Similarly we define the spaces $\mathcal{Q}(K;\ H)'$, $\mathcal{A}(K;\ H)'$ and $\mathcal{A}(\Omega;\ H)'$ for a compact set K in $\tilde{\boldsymbol{C}}^n$ or in $\boldsymbol{D}^n$ and an open set Ω in $\boldsymbol{D}^n$ respectively. We define the notion of carriers in these cases similarly to the case of $\mathcal{Q}(\Omega;\ H)'$. We put $\mathcal{Q}(\emptyset;\ H)'=\mathcal{A}(\emptyset;\ H)'=0$.

Here we note that $\mathcal{Q}(K;\ H)'$ and $\mathcal{A}(K;\ H)'$ become FS*-spaces.

Proposition 2.1.2. *Let Ω be an open set in $\tilde{\boldsymbol{C}}^n$. Suppose that a compact set K in Ω has the Runge property such as $\mathcal{Q}(\Omega;\ H)$ is dense in $\mathcal{Q}(K;\ H)$. Let $u\in\mathcal{Q}(\Omega;\ H)'$. Then u is carried by K if and only if u is carried by all open neighborhood of K.*

Let $\{\Omega_j\}_{j=1}^{\infty}$ be a fundamental system of Runge open neighborhoods of K in Ω. Then Proposition 2.1.2 means that the isomorphism

$$(\lim_{\overrightarrow{j}} \operatorname{ind} \mathcal{Q}(\Omega_j;\ H))' \cong \mathcal{Q}(K;\ H)'$$

holds. An element of $\mathcal{A}(\boldsymbol{D}^n;\ H)'$ is said to be an H-valued real Fourier analytic functional or an H-valued analytic functional with carrier in $\boldsymbol{D}^n$.

Theorem 2.1.3. *For every at most countable family $\{K_i\}_{i\in I}$ of compact subsets of $\boldsymbol{D}^n$, then $\bigcap_{i\in I}\mathcal{A}(K_i;\ H)'=\mathcal{A}(\bigcap_{i\in I}K_i;\ H)'$ holds.*

Proof. If $I=\{1, 2\}$, we can conclude that $\mathcal{A}(K_1;\ H)'\cap\mathcal{A}(K_2;\ H)'=\mathcal{A}(K_1\cap K_2;\ H)'$. In fact, by virtue of Theorem 1.1.20, we have an exact sequence

$$0 \longrightarrow \mathcal{A}(K_1\cup K_2;\ H) \xrightarrow{\rho} \mathcal{A}(K_1;\ H)\oplus\mathcal{A}(K_2;\ H) \xrightarrow{\tau} \mathcal{A}(K_1\cap K_2;\ H)$$
$$\longrightarrow H^1(K_1\cup K_2;\ {}^H\mathcal{A})=0,$$

where $\rho(f)=(f|_{K_1}, -f|_{K_2})$ and $\tau((f_1, f_2))=f_1|_{K_1\cap K_2}+f_2|_{K_1\cap K_2}$. Since ρ and τ are continuous, we have the dual exact sequence

$$0 \longleftarrow \mathcal{A}(K_1\cup K_2;\ H)' \xleftarrow{\rho'} \mathcal{A}(K_1;\ H)'\oplus\mathcal{A}(K_2;\ H)' \xleftarrow{\tau'} \mathcal{A}(K_1\cap K_2;\ H)' \longleftarrow 0,$$

by virtue of Komatsu [15], Theorem 19 (p. 381). We note that $\mathcal{A}(K_1\cap K_2;\ H)'\subset \mathcal{A}(K_1;\ H)'\cap\mathcal{A}(K_2;\ H)'$. On the other hand, from the above exact sequence, we have $\{(u, u);\ u\in\mathcal{A}(K_1;\ H)'\cap\mathcal{A}(K_2;\ H)'\}\subset\mathrm{Ker}\,\rho'=\mathrm{Im}\,\tau'=\{(u, u);\ u\in\mathcal{A}(K_1\cap K_2;\ H)'\}$. Hence we have the above conclusion.

Assume $I=\boldsymbol{N}$. Then, by finite induction, we have, for every $n\geqq 1$,

$$\bigcap_{i=1}^{n}\mathcal{A}(K_i;\ H)'=\mathcal{A}\Big(\bigcap_{i=1}^{n}K_i;\ H\Big)'.$$

Then, by the natural inclusion map, the family $\{\mathcal{A}(\bigcap_{i=1}^{n}K_i;\ H)'\}$ becomes a projective family. Thus we have

$$\bigcap_{i\in I}\mathcal{A}(K_i;\ H)'=\lim_{n}\mathrm{proj}\bigcap_{i=1}^{n}\mathcal{A}(K_i;\ H)'$$

$$=\lim_{n}\mathrm{proj}\,\mathcal{A}\Big(\bigcap_{i=1}^{n}K_i;\ H\Big)'=\mathcal{A}\Big(\bigcap_{i\in I}K_i;\ H\Big)'. \quad \text{(Q. E. D.)}$$

Theorem 2.1.4. *Let $u\in\mathcal{A}(\boldsymbol{D}^n;\ H)'$, $u\neq 0$. Then there exists the smallest compact set in $\boldsymbol{D}^n$, which is a carrier of u, among the carriers of u. We call it the support of u and denote it by* supp(u).

Proof. By virtue of Theorem 2.1.3, we can prove the theorem by way of Zorn's Lemma. (Q. E. D.)

We remark that, for u, u_1 and $u_2\in\mathcal{A}(\boldsymbol{D}^n;\ H)'$, we have

$$\mathrm{supp}(u_1+u_2)\subset\mathrm{supp}(u_1)\cup\mathrm{supp}(u_2),$$

$$\mathrm{supp}(\lambda u)\subset\mathrm{supp}(u), \qquad \text{for } \lambda\in\boldsymbol{C}.$$

Theorem 2.1.5. *Let $K=\bigcup_{i=1}^{p}K_i$ be the union of compact sets K_i in $\boldsymbol{D}^n$. Let $u\in\mathcal{A}(K;\ H)'$. Then there exist $u_i\in\mathcal{A}(K_i;\ H)'$, $(i=1, 2, \cdots, p)$, such that $u=\sum_{i=1}^{p}u_i$.*

Proof. We note first that the mapping

$$\mathcal{A}(K;\ H)\longrightarrow\bigoplus_{i=1}^{p}\mathcal{A}(K_i;\ H),$$

$$f\longmapsto(f|K_i)_{1\leqq i\leqq p}$$

is continuous, injective and of closed range. By this mapping, we can identify $\mathcal{A}(K;\ H)$ with a closed subspace of $\bigoplus_{i=1}^{p}\mathcal{A}(K_i;\ H)$. Since $\mathcal{A}(K;\ H)$ and $\bigoplus_{i=1}^{p}\mathcal{A}(K_i;\ H)$ are DFS*-spaces, we have the following surjection from the Serre-Komatsu duality theorem

$$\prod_{i=1}^{p}\mathcal{A}(K_i;\ H)'\longrightarrow\mathcal{A}(K;\ H)',$$

$$(u_i)_{1\leqq i\leqq p}\longmapsto\sum_{i=1}^{p}u_i. \quad \text{(Q. E. D.)}$$

Let Ω be an open set in $\boldsymbol{D}^n$ and K a compact subset of Ω. Then the envelope $\tilde{K}$ of K in Ω is defined to be the closure of the union of K and the relatively compact (in Ω) connected components of $\Omega \backslash K$.

Theorem 2.1.6. *Let Ω be an open set in $\boldsymbol{D}^n$, and K_1 and K_2 two compact subsets of Ω with $K_1 \subset K_2$. Further assume that $K_i = \tilde{K}_i$ holds for $i=1, 2$. Then $\mathcal{A}((\Omega \backslash K_2)^a)'$ is dense in $\mathcal{A}((\Omega \backslash K_1)^a)'$.*

Proof. It is sufficient to see that the natural restriction mapping $\mathcal{A}((\Omega \backslash K_1)^a)$ into $\mathcal{A}((\Omega \backslash K_2)^a)$ is injective. But this is evident under the assumption of this theorem. (Q. E. D.)

2.2. The sheaf $^H\mathfrak{R}$ of H-valued Fourier hyperfunctions. In this section we construct the sheaf $^H\mathfrak{R}$ of H-valued Fourier hyperfunctions. At first we have the following.

Theorem 2.2.1. *The following are valid for the H-valued Fourier analytic functionals with carriers in $\boldsymbol{D}^n$.*

(1) *If K_1 and K_2 are two compact subsets of $\boldsymbol{D}^n$ with $K_1 \subset K_2$, there exists a continuous injection i_{K_1, K_2}: $\mathcal{A}(K_1; H)' \to \mathcal{A}(K_2; H)'$.*

(2) *If K_1 and K_2 are as in (1) and each connected component of K_2 meets K_1, then i_{K_1, K_2} has a dense image.*

(3) *If K_1 and K_2 are two compact subsets of $\boldsymbol{D}^n$ and we put $K = K_1 \cup K_2$, then, for every $u \in \mathcal{A}(K; H)'$, there exist $u_1 \in \mathcal{A}(K_1; H)'$ and $u_2 \in \mathcal{A}(K_2; H)'$ so that $u = u_1 + u_2$ holds, where u_1 and u_2 are considered as elements of $\mathcal{A}(K; H)'$ by virtue of (1).*

(4) *For every at most countable family $\{K_i\}_{i \in I}$ of compact subsets of $\boldsymbol{D}^n$, $\bigcap_{i \in I} \mathcal{A}(K_i; H)' = \mathcal{A}(\bigcap_{i \in I} K_i; H)'$ holds.*

Proof. (1) The natural restriction mapping i_{K_1, K_2}' of $\mathcal{A}(K_2; H)$ into $\mathcal{A}(K_1; H)$ is continuous and has a dense range. Hence its dual i_{K_1, K_2} is a continuous injection $\mathcal{A}(K_1; H)' \to \mathcal{A}(K_2; H)'$.

(2) It is evident from the identity theorem that i_{K_1, K_2}: $\mathcal{A}(K_1; H)' \to \mathcal{A}(K_2; H)'$ has a dense image.

(3) See Theorem 2.1.5.

(4) See Theorem 2.1.3. (Q. E. D.)

Theorem 2.2.2. *There exists one and only one flabby sheaf $^H\mathfrak{R}$ over $\boldsymbol{D}^n$ so that, for every compact subset K of $\boldsymbol{D}^n$, $\Gamma_K(\boldsymbol{D}^n, {}^H\mathfrak{R}) = \mathcal{A}(K; H)'$ holds. The section module $\mathfrak{R}(\Omega; H)$ of the sheaf $^H\mathfrak{R}$ on an open set Ω in $\boldsymbol{D}^n$ is defined to be the space*

$$\mathfrak{R}(\Omega; H) = \mathcal{A}(\Omega^a; H)' / \mathcal{A}(\partial\Omega; H)'.$$

Proof. It follows from Theorem 2.2.1 and Ito [7], Theorem 1.2.1. (Q. E. D.)

Here, for completeness, we mention the Schapira-Junker Theorem (Ito [7], Theorem 1.2.1).

Theorem 2.2.3 (Schapira-Junker). *Let X be a σ-compact locally compact topological* *space satisfying the second axiom of countability. We assume that, for every compact subset K of X, there exists a Fréchet space F_K such that the following conditions are fulfilled;*

(1) *For two compact subsets K_1 and K_2 of X with $K_1 \subset K_2$, there exists a continuous injection i_{K_1,K_2}: $F_{K_1} \to F_{K_2}$.*

(2) *If K_1 and K_2 are two compact subsets of X with $K_1 \subset K_2$ and each connected component of K_2 meets K_1, then i_{K_1,K_2} has a dense image.*

(3) *If K_1 and K_2 are two compact subsets of X and $K = K_1 \cup K_2$ holds, then, for every $u \in F_K$, there exist $u_1 \in F_{K_1}$ and $u_2 \in F_{K_2}$ so that $u = u_1 + u_2$ holds, where u_1 and u_2 are considered as elements of F_K by virtue of* (1).

(4) *For every at most countable family $\{K_i; i \in I\}$ of compact subsets of X, $\bigcap_{i \in I} F_{K_i} = F_K$ holds, where $K = \bigcap_{i \in I} K_i$.*

(5) *$F_{\emptyset} = 0$ holds.*

Then there exists one and only one flabby sheaf $\mathcal{F}$ over X so that, for every compact subset K of X, $\Gamma_K(X, \mathcal{F}) = F_K$ holds.

Definition 2.2.4. The sheaf ${}^H\mathfrak{R}$ is said to be the sheaf of H-valued Fourier hyperfunctions over $\boldsymbol{D}^n$ and a section f of ${}^H\mathfrak{R}$ on an open set Ω in $\boldsymbol{D}^n$ is said to be an H-valued Fourier hyperfunction on Ω.

At last we note that H-valued Fourier analytic functionals with carrier in a compact set K in $\tilde{\boldsymbol{C}}^n$ can be considered as Fourier analytic linear mappings on K. As to the notion of Fourier analytic linear mappings, we refer Ito [7].

Theorem 2.2.5. *Let K be a compact set in $\tilde{\boldsymbol{C}}^n$. Then we have the isomorphism*

$$\mathcal{Q}(K;\ H)' \cong L(\mathcal{Q}(K);\ H).$$

Proof. It goes in a similar way to Ito-Nagamachi [9], section 6, p. 21. (Q.E.D.)

Chapter 3. The realization as boundary values of slowly increasing H-valued holomorphic functions

3.1. The Malgrange Theorem. In this section we prove the Malgrange Theorem.

Theorem 3.1.1. *Let $\tilde{\Omega}$ be an open set in $\tilde{\boldsymbol{C}}^n$ for which $H^n(\tilde{\Omega}, {}^H\tilde{\mathcal{O}}) = 0$ holds and Ω an arbitrary open set contained in $\tilde{\Omega}$. Then we have $H^n(\Omega, {}^H\tilde{\mathcal{O}}) = 0$.*

Proof. By virtue of Corollary 1.1.11, we have only to prove the exactness of the sequence

$$\tilde{\mathcal{I}}^{0,n-1}(\Omega;\ H) \xrightarrow{\bar{\partial}} \tilde{\mathcal{I}}^{0,n}(\Omega;\ H) \longrightarrow 0$$

in the notations of Theorem 1.1.10. This is equivalent to proving the exactness of the sequence

$$\tilde{L}^{0,n-1}(\Omega;\ H) \xrightarrow{\bar{\partial}} \tilde{L}^{0,n}(\Omega;\ H) \longrightarrow 0.$$

By virtue of the Serrer-Komatsu duality theorem (Theorem 19 in Komatsu [15], p. 381) for FS*-spaces, it suffices to show the injectiveness and the closedness of the range of $-\bar{\partial}=(\bar{\partial})'$ in the dual sequence

$$(3.1.1)\qquad \underset{\sim}{L}_c^{0,1}(\Omega;\ H) \xleftarrow{-\bar{\partial}} \underset{\sim}{L}_c^{0,0}(\Omega;\ H) \longleftarrow 0.$$

Here $\underset{\sim}{L}_c^{p,q}(\Omega;\ H)$ denotes the space of sections with compact support of $\underset{\sim}{L}^{p,q}$ on Ω. Since $-\bar{\partial}$ is elliptic, its injectivity is an immediate consequence of the unique continuation property. Now we will prove the closedness of its range. This is surely true if Ω is replaced by the open set $\tilde{\Omega}$ in the assumption of this theorem because then $H^n(\tilde{\Omega},\ {}^H\tilde{\mathcal{O}})=0$ holds. Thus the problem reduces to the estimation of support of the solution $u\in\underset{\sim}{L}_c^{0,0}(\tilde{\Omega};\ H)$ of the system $-\bar{\partial}u=f$ for an $f\in[\mathrm{Im}(-\bar{\partial}^{\Omega})]^a$. Then there exists a sequence $\{u_\alpha\}_{\alpha\in A}\subset\underset{\sim}{L}_c^{0,0}(\Omega;\ H)$ and $-\bar{\partial}u_\alpha\to f$ in $\underset{\sim}{L}_c^{0,1}(\Omega;\ H)$. Then the convergence takes place in $\underset{\sim}{L}_c^{0,1}(\tilde{\Omega};\ H)$. Hence by the closed range property of

$$(3.1.2)\qquad \underset{\sim}{L}_c^{0,1}(\tilde{\Omega};\ H) \xleftarrow{-\bar{\partial}} \underset{\sim}{L}_c^{0,0}(\tilde{\Omega};\ H) \longleftarrow 0,$$

we can find $u\in\underset{\sim}{L}_c^{0,0}(\tilde{\Omega};\ H)$ such that $-\bar{\partial}u=f$. We note that u is holomorphic on $(\tilde{\Omega}\backslash\mathrm{supp}(f))\cap\boldsymbol{C}^n$ and that $\mathrm{supp}(f)\subset\subset\Omega$. Hence if we show that $u=0$ on $(\tilde{\Omega}\backslash\Omega)\cap\boldsymbol{C}^n$, we find that $\mathrm{supp}(u)\subset\subset\Omega$ from the uniqueness of analytic continuation. By the closedness of the range of $-\bar{\partial}$ in (3.1.2), we can apply the homomorphism theorem and find some sequence $v_\alpha,\ v\in\underset{\sim}{L}_c^{0,0}(\tilde{\Omega};\ H)$ such that $-\bar{\partial}v_\alpha=-\bar{\partial}u_\alpha$, $-\bar{\partial}v=-\bar{\partial}u$ and $v_\alpha\to v$ in $\underset{\sim}{L}_c^{0,0}(\tilde{\Omega};\ H)$ (cf. Köthe [18], §33, 4(2), p. 18). Since $-\bar{\partial}$ is an injective operator of $\underset{\sim}{L}_c^{0,0}(\tilde{\Omega};\ H)$ into $\underset{\sim}{L}_c^{0,1}(\tilde{\Omega};\ H)$, we must have $v_\alpha=u_\alpha$, $v=u$. Hence $u_\alpha\to u$ in $\underset{\sim}{L}_c^{0,0}(\tilde{\Omega};\ H)$. Since $\mathrm{supp}(u_\alpha)\subset\Omega$, we find $\mathrm{supp}(u)\subset\Omega^a$. (Q.E.D.)

Note. The author owe the above proof of this theorem to Kaneko's kind advices.

Corollary. Flabby $\dim{}^H\tilde{\mathcal{O}}\leqq n$.

3.2. The Serre Duality Theorem. In this section we prove a Serre Duality Theorem.

Theorem 3.2.1. *Let Ω be an open set in $\tilde{\boldsymbol{C}}^n$ such that* $\dim H^p(\Omega,\ {}^H\tilde{\mathcal{O}})<\infty(p\geqq 1)$ *holds. Then we have the isomorphism* $[H^p(\Omega,\ {}^H\tilde{\mathcal{O}})]'\cong H_c^{n-p}(\Omega,\ {}^H\underset{\sim}{\mathcal{O}})$, $(0\leqq p\leqq n)$.

Proof. By virtue of Corollaries 1.1.11 and 1.1.17, cohomology groups $H^p(\Omega,\ {}^H\tilde{\mathcal{O}})$ and $H_c^{n-p}(\Omega,\ {}^H\underset{\sim}{\mathcal{O}})$ are cohomology groups respectively of the complexes

$$(3.2.1)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\mathcal{E}}^{0,0}(\Omega;\ H) & \xrightarrow{\bar{\partial}} & \tilde{\mathcal{E}}^{0,1}(\Omega;\ H) & \xrightarrow{\bar{\partial}}\cdots\xrightarrow{\bar{\partial}} & \tilde{\mathcal{E}}^{0,n}(\Omega;\ H) & \longrightarrow & 0 \\ & & \updownarrow & & \updownarrow & & \updownarrow & & \\ 0 & \longleftarrow & \underset{\sim}{\mathcal{L}}_c^{0,n}(\Omega;\ H) & \xleftarrow{-\bar{\partial}} & \underset{\sim}{\mathcal{L}}_c^{0,n-1}(\Omega;\ H) & \xleftarrow{-\bar{\partial}}\cdots\xleftarrow{-\bar{\partial}} & \underset{\sim}{\mathcal{L}}_c^{0,0}(\Omega;\ H) & \longleftarrow & 0. \end{array}$$

Here the upper complex is composed of FS*-spaces and the lower complex is composed of DFS*-spaces. The ranges of operators $\bar{\partial}$ in the upper complex are all closed by virtue of Schwartz' Lemma (Theorem 20 in Komatsu [15]). Hence the ranges of operators $-\bar{\partial}=(\bar{\partial})'$ in the lower complex are also all closed and we have the isomorphism

$$[H^p(\Omega, {}^H\tilde{\mathcal{O}})]' \cong H_c^{n-p}(\Omega, {}^H\underset{\sim}{Q})$$

by virtue of the Serre-Komatsu duality theorem (cf. Komatsu [15]). (Q. E. D.)

Remark. The conclusion of the theorem works for any open set Ω for which every $\bar{\partial}$ operator in the diagram (3.2.1) is of closed range.

3.3. The Martineau-Harvey Theorem. In this section we prove the Martineau-Harvey Theorem.

Theorem 3.3.1 (the Martineau-Harvey Theorem). *Let K be a compact set in $\tilde{\boldsymbol{C}}^n$ and assume the following:*

(i) $H^p(K, {}^H\underset{\sim}{Q})=0$, $(p\geqq 1)$.

(ii) *Ω is an open neighborhood of K such that $H^p(\Omega, {}^H\tilde{\mathcal{O}})=0$, $(p\geqq 1)$ holds.*

Then

(1) $H_K^p(\Omega, {}^H\tilde{\mathcal{O}})=0$, $(p\neq n)$.

(2) *If $n\geqq 2$, we have algebraic isomorphisms*

$$H_K^n(\Omega, {}^H\tilde{\mathcal{O}})\cong H^{n-1}(\Omega\backslash K, {}^H\tilde{\mathcal{O}})\cong \underset{\sim}{Q}(K;\ H)',$$

(3) *If $n=1$, we have topological isomorphisms*

$$H_K^1(\Omega, {}^H\tilde{\mathcal{O}})\cong\tilde{\mathcal{O}}(\Omega\backslash K;\ H)/\tilde{\mathcal{O}}(\Omega;\ H)\cong\underset{\sim}{Q}(K;\ H)'.$$

Remark. If a compact set K has a fundamental system of $\tilde{\mathcal{O}}$-pseudoconvex open neighborhoods, it satisfies the assumptions in Theorem 3.3.1.

Proof. It goes in a similar way to Kawai [14]. From a general theory of relative cohomology groups (cf. Komatsu [16], Theorem II. 3.2), we have

$$\begin{aligned}0 &\longrightarrow H_K^0(\Omega, {}^H\tilde{\mathcal{O}}) \longrightarrow H^0(\Omega, {}^H\tilde{\mathcal{O}}) \longrightarrow H^0(\Omega\backslash K, {}^H\tilde{\mathcal{O}})\\ &\longrightarrow H_K^1(\Omega, {}^H\tilde{\mathcal{O}}) \longrightarrow H^1(\Omega, {}^H\tilde{\mathcal{O}}) \longrightarrow H^1(\Omega\backslash K, {}^H\tilde{\mathcal{O}}) \longrightarrow \cdots\\ &\longrightarrow H_K^n(\Omega, {}^H\tilde{\mathcal{O}}) \longrightarrow H^n(\Omega, {}^H\tilde{\mathcal{O}}) \longrightarrow H^n(\Omega\backslash K, {}^H\tilde{\mathcal{O}}) \longrightarrow \cdots.\end{aligned}$$

Then we have $H^p(\Omega, {}^H\tilde{\mathcal{O}})=0$, $(p\geqq 1)$ by the assumption and $H_K^0(\Omega, {}^H\tilde{\mathcal{O}})=0$ by the unique continuation theorem. Hence we have an exact sequence and algebraic isomorphisms

$$0\longrightarrow\tilde{\mathcal{O}}(\Omega;\ H)\longrightarrow\tilde{\mathcal{O}}(\Omega\backslash K;\ H)\longrightarrow H_K^1(\Omega, {}^H\tilde{\mathcal{O}})\longrightarrow 0,$$

$$H_K^p(\Omega, {}^H\tilde{\mathcal{O}})\cong H^{p-1}(\Omega\backslash K, {}^H\tilde{\mathcal{O}}), \qquad (p\geqq 2).$$

We also have the long exact sequence of cohomology groups with compact support (cf. Komatsu [16], Theorem II.3.15):

$$\begin{aligned}0 &\longrightarrow H_c^0(\Omega\backslash K, {}^H\underset{\sim}{Q}) \longrightarrow H_c^0(\Omega, {}^H\underset{\sim}{Q}) \longrightarrow H^0(K, {}^H\underset{\sim}{Q})\\ &\longrightarrow H_c^1(\Omega\backslash K, {}^H\underset{\sim}{Q}) \longrightarrow H_c^1(\Omega, {}^H\underset{\sim}{Q}) \longrightarrow H^1(K, {}^H\underset{\sim}{Q}) \longrightarrow \cdots\\ &\longrightarrow H_c^p(\Omega\backslash K, {}^H\underset{\sim}{Q}) \longrightarrow H_c^p(\Omega, {}^H\underset{\sim}{Q}) \longrightarrow H^p(K, {}^H\underset{\sim}{Q}) \longrightarrow \cdots.\end{aligned}$$

Here $H^p(K, {}^H\mathcal{O})=0$, $(p\geqq 1)$ by the assumption on K. From Theorem 3.2.1 and the fact $H^p(\Omega, {}^H\tilde{\mathcal{O}})=0$, $(p\geqq 1)$, we also have $H^p_c(\Omega, {}^H\mathcal{O})=0$, $(p\neq n)$. Therefore we obtain an exact sequence and topological isomorphisms:

When $n=1$,

$$0 \longrightarrow \mathcal{O}(K;\ H) \longrightarrow H^1_c(\Omega\setminus K, {}^H\mathcal{O}) \longrightarrow H^1_c(\Omega, {}^H\mathcal{O}) \longrightarrow 0,$$

when $n\geqq 2$,

$$H^1_c(\Omega\setminus K, {}^H\mathcal{O})\cong\mathcal{O}(K;\ H),$$

$$H^p_c(\Omega\setminus K, {}^H\mathcal{O})\cong H^p_c(\Omega, {}^H\mathcal{O})=0, \qquad (p\neq 1, n),$$

$$H^n_c(\Omega\setminus K, {}^H\mathcal{O})\cong H^n_c(\Omega, {}^H\mathcal{O}).$$

Now we consider the following dual complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{\mathcal{E}}^{0,0}(\Omega\setminus K;\ H) & \xrightarrow{\bar\partial_0} & \tilde{\mathcal{E}}^{0,1}(\Omega\setminus K;\ H) & \xrightarrow{\bar\partial_1} \cdots \longrightarrow & (*) \\ & & \updownarrow & & \updownarrow & & \\ 0 & \longleftarrow & \underset{\sim}{\mathcal{L}}^{0,n}_c(\Omega\setminus K;\ H) & \xleftarrow{-\bar\partial_{n-1}} & \underset{\sim}{\mathcal{L}}^{0,n-1}_c(\Omega\setminus K;\ H) & \xleftarrow{-\bar\partial_{n-2}} \cdots \longleftarrow & (**) \end{array}$$

$$\begin{array}{ccccccc} (*) & \xrightarrow{\bar\partial_{n-2}} & \tilde{\mathcal{E}}^{0,n-1}(\Omega\setminus K;\ H) & \xrightarrow{\bar\partial_{n-1}} & \tilde{\mathcal{E}}^{0,n}(\Omega\setminus K;\ H) & \longrightarrow & 0 \\ & & \updownarrow & & \updownarrow & & \\ (**) & \xleftarrow{-\bar\partial_1} & \underset{\sim}{\mathcal{L}}^{0,1}_c(\Omega\setminus K;\ H) & \xleftarrow{-\bar\partial_0} & \underset{\sim}{\mathcal{L}}^{0,0}_c(\Omega\setminus K;\ H) & \longleftarrow & 0. \end{array}$$

Then, since $H^p_c(\Omega\setminus K, {}^H\mathcal{O})=0$, $(p\neq 1, n)$, the range of $-\bar\partial_j=(\bar\partial_{n-j-1})'$ is closed for $j\neq 0$, $n-1$. However $\bar\partial_{n-1}$ is of closed range by the Malgrange Theorem. Hence, by the Serre-Komatsu duality theorem, $-\bar\partial$ is of closed range.

In order to prove the closedness of the range of $-\bar\partial_{n-1}$, we consider the following diagram:

$$\begin{array}{ccccc} 0 & \longleftarrow & \underset{\sim}{\mathcal{L}}^{0,n}_c(\Omega\setminus K;\ H) & \xleftarrow{-\bar\partial^{\Omega\setminus K}_{n-1}} & \underset{\sim}{\mathcal{L}}^{0,n-1}_c(\Omega\setminus K;\ H) \\ & & i\downarrow & & \downarrow i \\ 0 & \longleftarrow & \underset{\sim}{\mathcal{L}}^{0,n}_c(\Omega;\ H) & \xleftarrow{-\bar\partial^{\Omega}_{n-1}} & \underset{\sim}{\mathcal{L}}^{0,n-1}_c(\Omega;\ H), \end{array}$$

where the map i is the natural injection.

We conclude that $\bar\partial^{\Omega}_0$ is of closed range because $H^1(\Omega, {}^H\tilde{\mathcal{O}})=0$. Thus $-\bar\partial^{\Omega}_{n-1}$ is of closed range by the Serre-Komatsu duality theorem. Therefore $\mathrm{Im}\,(-\bar\partial^{\Omega\setminus K}_{n-1})=i^{-1}$ $(\mathrm{Im}(-\bar\partial^{\Omega}_{n-1}))$ is closed by the continuity of the map i. Therefore all $-\bar\partial^{\Omega\setminus K}_j$ are of closed range. Hence, by the Serre-Komatsu duality theorem, we have the isomorphisms

$$[H^p(\Omega\setminus K, {}^H\tilde{\mathcal{O}})]'\cong H^{n-p}_c(\Omega\setminus K, {}^H\mathcal{O}), \qquad (0\leqq p\leqq n).$$

If $n=1$, by the Serre duality theorem, we have the dual complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\mathcal{O}}(\Omega\,;\,H) & \longrightarrow & \tilde{\mathcal{O}}(\Omega\backslash K\,;\,H) & \longrightarrow & H^1_K(\Omega,\,{}^H\tilde{\mathcal{O}}) & \longrightarrow & 0 \\ & & \updownarrow & & \updownarrow & & & & \\ 0 & \longleftarrow & H^1_c(\Omega,\,{}^H\mathcal{Q}) & \longleftarrow & H^1_c(\Omega\backslash K,\,{}^H\mathcal{Q}) & \longleftarrow & \mathcal{Q}(K\,;\,H) & \longleftarrow & 0. \end{array}$$

Therefore we have topological isomorphisms

$$\begin{aligned}[][H^1_K(\Omega,\,{}^H\tilde{\mathcal{O}})]' &\cong [\mathrm{Coker}(\tilde{\mathcal{O}}(\Omega\,;\,H) \longrightarrow \tilde{\mathcal{O}}(\Omega\backslash K\,;\,H))]' \\ &\cong \mathrm{Ker}(H^1_c(\Omega\backslash K,\,{}^H\mathcal{Q}) \longrightarrow H^1_c(\Omega,\,{}^H\mathcal{Q})) \\ &\cong \mathcal{Q}(K\,;\,H). \end{aligned}$$

Thus we have topological isomorphisms

$$H^1_K(\Omega,\,{}^H\tilde{\mathcal{O}}) \cong \tilde{\mathcal{O}}(\Omega\backslash K\,;\,H)/\tilde{\mathcal{O}}(\Omega\,;\,H) \cong \mathcal{Q}(K\,;\,H)'.$$

This proves (3).

If $n \geqq 2$, since FS*- or DFS*-spaces are reflexive and we have

$$\tilde{\mathcal{O}}(\Omega\backslash K\,;\,H)' \cong H^n_c(\Omega\backslash K,\,{}^H\mathcal{Q}) \cong H^n_c(\Omega,\,{}^H\mathcal{Q}) \cong \tilde{\mathcal{O}}(\Omega\,;\,H)',$$

we have the isomorphism

$$\tilde{\mathcal{O}}(\Omega\,;\,H) \cong \tilde{\mathcal{O}}(\Omega\backslash K\,;\,H).$$

Thus we have

$$H^1_K(\Omega,\,{}^H\tilde{\mathcal{O}}) \cong \tilde{\mathcal{O}}(\Omega\backslash K\,;\,H)/\tilde{\mathcal{O}}(\Omega\,;\,H) = 0.$$

Further, for $p \geqq 2$, $p \neq n$, we have

$$\begin{aligned} 0 = [H^{n-p+1}_c(\Omega,\,{}^H\mathcal{Q})]' &\cong [H^{n-p+1}_c(\Omega\backslash K,\,{}^H\mathcal{Q})]' \\ &\cong H^{p-1}(\Omega\backslash K,\,{}^H\tilde{\mathcal{O}}) \cong H^p_K(\Omega,\,{}^H\tilde{\mathcal{O}}). \end{aligned}$$

Thus we have

$$H^p_K(\Omega,\,{}^H\tilde{\mathcal{O}}) = 0, \qquad (p \neq n).$$

This proves (1).

In the case $p = n$, we have algebraic isomorphisms

$$\begin{aligned} H^n_K(\Omega,\,{}^H\tilde{\mathcal{O}}) &\cong H^{n-1}(\Omega\backslash K,\,{}^H\tilde{\mathcal{O}}) \cong [H^1_c(\Omega\backslash K,\,{}^H\mathcal{Q})]' \\ &\cong \mathcal{Q}(K\,;\,H)'. \end{aligned}$$

This proves (2). (Q.E.D.)

Now we realize H-valued Fourier analytic functionals with certain compact carrier as relative cohomology classes with coefficients in ${}^H\tilde{\mathcal{O}}$.

Let $K_1, K_2, \cdots, K_n$ be subsets of $\boldsymbol{C}$ such that $K_1^a, K_2^a, \cdots, K_n^a$ are compact sets in $\tilde{\boldsymbol{C}}$. Then $K = (K_1 \times K_2 \times \cdots \times K_n)^a$ is a compact set in $\tilde{\boldsymbol{C}}^n$.

Let K be a compact set in $\tilde{\boldsymbol{C}}^n$ of the form $K = (K_1 \times \cdots \times K_n)^a$ as above. Then K has a fundamental system of $\tilde{\mathcal{O}}$-pseudoconvex open neighborhoods. Then we have

$$H^p(K,\,{}^H\mathcal{Q}) = 0, \qquad (p > 0).$$

By virtue of the Martineau-Harvey Theorem, there exists the algebraic isomorphism

$$\mathcal{Q}(K;\ H)'\cong H_K^n(\tilde{\boldsymbol{C}}^n,\ {}^H\tilde{\mathcal{O}}).$$

Let Ω be an $\tilde{\mathcal{O}}$-pseudoconvex open neighborhood of K. Then

$$\Omega_j=\Omega\cap\{z\in\tilde{\boldsymbol{C}}^n;\ z_j\notin K_j^a\}$$

is also an $\tilde{\mathcal{O}}$-pseudoconvex open set for $j=1, 2, \cdots, n$. Then, putting $\Omega_0=\Omega$, $\mathcal{V}=\{\Omega_j\}_{j=0}^n$ and $\mathcal{V}'=\{\Omega_j\}_{j=1}^n$ form ${}^H\tilde{\mathcal{O}}$-acyclic coverings of Ω and $\Omega\backslash K$. Namely,

$$H^q(\Omega_{j_0}\cap\Omega_{j_1}\cap\cdots\cap\Omega_{j_p},\ {}^H\tilde{\mathcal{O}})=0,$$

$$(q>0,\ \{j_0, j_1, \cdots, j_p\}\subset\{0, 1, 2, \cdots, n\},\ \ p=0, 1, 2, \cdots, n)$$

hold. Thus, by Leray's Theorem (Theorem II.3.29 in Komatsu [16]), we obtain the algebraic isomorphisms

$$H_K^n(\tilde{\boldsymbol{C}}^n,\ {}^H\tilde{\mathcal{O}})\cong H_K^n(\Omega,\ {}^H\tilde{\mathcal{O}})\cong H^n(\mathcal{V},\ \mathcal{V}',\ {}^H\tilde{\mathcal{O}}).$$

Since the covering $\mathcal{V}$ is composed of only $n+1$ open sets $\Omega_j (j=0, 1, \cdots, n)$, we easily obtain the isomorphisms

$$Z^n(\mathcal{V},\ \mathcal{V}',\ {}^H\tilde{\mathcal{O}})\cong C^n(\mathcal{V},\ \mathcal{V}',\ {}^H\tilde{\mathcal{O}})\cong\tilde{\mathcal{O}}(\Omega\sharp K;\ H),$$

$$C^{n-1}(\mathcal{V},\ \mathcal{V}',\ {}^H\tilde{\mathcal{O}})\cong\bigoplus_{j=1}^n\tilde{\mathcal{O}}(\Omega\sharp_j K;\ H),$$

where we put

$$\Omega\sharp K=\bigcap_{j=1}^n\Omega_j,\qquad \Omega\sharp_j K=\bigcap_{i\neq j}\Omega_i.$$

Now $\sum_j\tilde{\mathcal{O}}(\Omega\sharp_j K;\ H)$ denotes the image in $\tilde{\mathcal{O}}(\Omega\sharp K;\ H)$ of $\bigoplus_{j=1}^n\tilde{\mathcal{O}}(\Omega\sharp_j K;\ H)$ by the coboundary operator

$$\delta:\ (f_j)_{j=1}^n\longmapsto\sum_{j=1}^n(-1)^j f_j',$$

f_j' being the restriction of f_j to $\Omega\sharp K$. Hence we have

$$\delta C^{n-1}(\mathcal{V},\ \mathcal{V}',\ {}^H\tilde{\mathcal{O}})=\sum_{j=1}^n\tilde{\mathcal{O}}(\Omega\sharp_j K;\ H)|_{\Omega\sharp K}.$$

Thus we have the algebraic isomorphisms

$$\begin{aligned}\mathcal{Q}(K;\ H)'&\cong H_K^n(\Omega,\ {}^H\tilde{\mathcal{O}})\cong H^n(\mathcal{V},\ \mathcal{V}',\ {}^H\tilde{\mathcal{O}})\\&\cong Z^n(\mathcal{V},\ \mathcal{V}',\ {}^H\tilde{\mathcal{O}})/\delta C^{n-1}(\mathcal{V},\ \mathcal{V}',\ {}^H\tilde{\mathcal{O}})\\&\cong\tilde{\mathcal{O}}(\Omega\sharp K;\ H)/\sum_j\tilde{\mathcal{O}}(\Omega\sharp_j K;\ H).\end{aligned}$$

Thus we have the following.

Theorem 3.3.2. *We use notations as above. Then we have the algebraic isomorphisms*

$$\begin{aligned}\mathcal{Q}(K;\ H)'&\cong H_K^n(\Omega,\ {}^H\tilde{\mathcal{O}})\cong H^n(\mathcal{V},\ \mathcal{V}',\ {}^H\tilde{\mathcal{O}})\\&\cong\tilde{\mathcal{O}}(\Omega\sharp K;\ H)/\sum_{j=1}^n\tilde{\mathcal{O}}(\Omega\sharp_j K;\ H).\end{aligned}$$

For $g(z)\in\mathcal{Q}(K;\ H)$, we define its complex conjugate $\overline{g(z)}=\bar{g}(z)$ so that the following conditions are satisfied:

(1) If $g\in\mathcal{Q}(K;\ H)$, $\bar{\bar{g}}=g$.

(2) If f, $g\in\mathcal{Q}(K;\ H)$, $\overline{(f+g)}=\bar{f}+\bar{g}$,

(3) If $g\in\mathcal{Q}(K;\ H)$ and $\alpha\in\boldsymbol{C}$, $\overline{(\alpha g)}=\bar{\alpha}\bar{g}$.

We define the complex conjugate of any H-valued function similarly. Then we have the following.

Theorem 3.3.3. *We use the same notations as in Theorem* 3.3.2. *If we define the canonical mapping*

$$b:\ \tilde{\mathcal{O}}(\Omega\sharp K;\ H)\longrightarrow\mathcal{Q}(K;\ H)'$$

as follows. Let $f\in\tilde{\mathcal{O}}(\Omega\sharp K;\ H)$ and $g\in\mathcal{Q}(K;\ H)$. Let $\omega=\mathrm{int}((\omega_1\times\cdots\times\omega_n)^a)\subset\Omega$ be an open neighborhood of K with $K_j\subset\omega_j\subset\boldsymbol{C}(j=1, 2, \cdots, n)$ and $g\in\mathcal{Q}(\tilde{\omega};\ H)$ where $\tilde{\omega}$ is an open neighborhood of ω^a with $\tilde{\omega}\subset\Omega$. Let $L_j(j=1, 2, \cdots, n)$ be open sets in $\boldsymbol{C}$ with regular boundary such that $K_j\subset L_j\subset\omega_j$ and let Γ_j be the boundary of L_j and oriented in the positive sense. Then we define $b(f)\in\mathcal{Q}(K;\ H)'$ by the formula

$$\langle b(f),\ g\rangle=(-1)^n\int_{\Gamma_1}\cdots\int_{\Gamma_n}(f(z),\ \overline{g(z)})dz_1\cdots dz_n.$$

Then b is the surjective homomorphism of $\tilde{\mathcal{O}}(\Omega\sharp K;\ H)$ onto $\mathcal{Q}(K;\ H)'$ whose kernel is $\sum_j\tilde{\mathcal{O}}(\Omega\sharp_j K;\ H)$.

Remark 1. The canonical mapping b defines the isomorphism

$$H_K^n(\Omega,\ {}^H\tilde{\mathcal{O}})\cong\mathcal{Q}(K;\ H)'.$$

Thereby we can realize H-valued Fourier analytic functionals as boundary values of slowly increasing H-valued holomorphic functions.

Remark 2. If we define the FS*-space topology of the space $\tilde{\mathcal{O}}(\Omega\sharp K;\ H)/\sum_j\tilde{\mathcal{O}}(\Omega\sharp_j K;\ H)$ by the canonical way, then we have the topological isomorphism

$$\tilde{\mathcal{O}}(\Omega\sharp K;\ H)/\sum_j\tilde{\mathcal{O}}(\Omega\sharp_j K;\ H)\cong\mathcal{Q}(K;\ H)'.$$

Proof. (i) At first we note that the integral

$$(-1)^n\int_{\Gamma_1}\cdots\int_{\Gamma_n}(f(z),\ \overline{g(z)})dz_1\cdots dz_n$$

does not depend on the chosen curves $\Gamma_1, \cdots, \Gamma_n$ and defines a continuous linear mapping b from $\tilde{\mathcal{O}}(\Omega\sharp K;\ H)$ into $\mathcal{Q}(K;\ H)'$.

(ii) Now we prove the surjectivity of the mapping b. Let $u\in\tilde{\mathcal{O}}(K;\ H)'$ and put

$$\tilde{u}(z)=(2\pi i)^{-n}\langle u_\xi,\ (\xi-z)^{-1}e(-(\xi-z)^2)\rangle,$$
$$(z\in(\Omega\sharp K)\cap\boldsymbol{C}^n),$$

where we put

$$(\xi-z)^{-1}e(-(\xi-z)^2)=\prod_{j=1}^n(\xi_j-z_j)^{-1}e(-(\xi_j-z_j)^2).$$

Then

$$\langle b(\tilde{u}), g\rangle$$
$$=(-2\pi i)^{-n}\int_{\Gamma_1}\cdots\int_{\Gamma_n}(\langle u_\xi, (\xi-z)^{-1}e(-(\xi-z)^2)\rangle, \overline{g(\bar{z})})dz$$
$$=\langle u_\xi, (2\pi i)^{-n}\int_{\Gamma_1}\cdots\int_{\Gamma_n}g(z)(z-\xi)^{-1}e(-(z-\xi)^2)dz\rangle$$
$$=\langle u, g\rangle.$$

Thus we have $b(\tilde{u})=u$. Hence b is a surjection.

(iii) At last we prove $\mathrm{Ker}(b)=\sum_j\tilde{\mathcal{O}}(\Omega\sharp_j K; H)$. For the sake of simplicity, we prove this fact in the case $n=1$.

If $f\in\tilde{\mathcal{O}}(\Omega; H)$, $b(f)=0$ is evident. Conversely assume that $f\in\tilde{\mathcal{O}}(\Omega\backslash K; H)$ satisfies $b(f)=0$. Let $\{L_j\}$ be an increasing exhausting sequence of compact sets in Ω including K whose boundaries $\partial L_j\cap\boldsymbol{C}=\Gamma_j$ are regular curves oriented in the positive sense. Then, if we put

$$F_j(z)=\int_{\Gamma_j}f(\zeta)(2\pi i(\zeta-z))^{-1}e(-(\zeta-z)^2)d\zeta,$$

$F_j(z)$ belongs to $\tilde{\mathcal{O}}(\mathrm{int}(L_j); H)$ and $F_{j+1}|_{\mathrm{int}(L_j)}=F_j(z)$ holds ($j=1, 2, \cdots$). Hence we can define $F(z)\in\tilde{\mathcal{O}}(\Omega; H)$ so that $F(z)|_{\mathrm{int}(L_j)}=F_j(z)$ holds ($j=1, 2, \cdots$). Then $F(z)|_{\Omega\backslash K}=f(z)$ holds. In fact, if we let $z\in\Omega\backslash K$, there exists some L_j so that z is in the interior of L_j. Then we can choose the open set L in this Theorem so that $L\subset\mathrm{int}(L_j)$ and z is in the outside of L^a. Then, since $f(z)$ is holomorphic in the region enclosed by $\Gamma_j-\Gamma$ with $\Gamma=\partial L\cap\boldsymbol{C}$, we have

$$f(z)=\int_{\Gamma_j-\Gamma}f(\zeta)(2\pi i(\zeta-z))^{-1}e(-(\zeta-z)^2)d\zeta.$$

Hence we have

$$F(z)=F_j(z)=f(z)+\int_{\Gamma}f(\zeta)(2\pi i(\zeta-z))^{-1}e(-(\zeta-z)^2)d\zeta.$$

Then, since z is in the outside of L^a, $(2\pi i(\zeta-z))^{-1}e(-(\zeta-z)^2)$ belongs to $\underset{\sim}{\mathcal{O}}(K; H)$ as a function of ζ. Thus, we have, by the assumption,

$$\int_{\Gamma}f(\zeta)(2\pi i(\zeta-z))^{-1}e(-(\zeta-z)^2)d\zeta$$
$$=\langle b(f), (2\pi i(\zeta-z))^{-1}e(-(\zeta-z)^2)\rangle=0.$$

Hence, $F(z)|_{\Omega\backslash K}=f(z)$ holds. (Q.E.D.)

3.4. The Sato Theorem. In this section we prove the pure-codimensionality of $\boldsymbol{D}^n$ with respect to ${}^H\tilde{\mathcal{O}}$. Then we realize H-valued Fourier hyperfunctions as "boundary values" of slowly increasing H-valued holomorphic functions or as relative cohomology classes of slowly increasing H-valued holomorphic functions.

Theorem 3.4.1 (The Sato Theorem). *Let Ω be an open set in $\boldsymbol{D}^n$ and V an open set in $\tilde{\boldsymbol{C}}^n$ which contains Ω. Then we have the following:*

(1) *The relative cohomology groups* $H^p_\Omega(V, {}^H\tilde{\mathcal{O}})$ *are zero for* $p \neq n$.

(2) *The presheaf over* $\boldsymbol{D}^n$, $\Omega \mapsto H^n_\Omega(V, {}^H\tilde{\mathcal{O}})$ *is a flabby sheaf.*

(3) *This sheaf defined in* (2) *is isomorphic to the sheaf* ${}^H\mathcal{R}$ *of H-valued Fourier hyperfunctions in Theorem* 2.2.2.

Remark. The sheaf defined in (2) is denoted by $\mathcal{H}^n_{\boldsymbol{D}^n}({}^H\tilde{\mathcal{O}}) = \mathrm{Dist}^n(\boldsymbol{D}^n, {}^H\tilde{\mathcal{O}})$, where the notation in the right hand side is due to Sato [23], p. 465.

Proof. (1) It goes in a similar way to Kawai [14], (p. 482).

(2) It follows from (1), flabby dim ${}^H\tilde{\mathcal{O}} \leq n$ and Theorem II.3.24 in Komatsu [16].

(3) Consider the following exact sequence of relative cohomology groups

$$\begin{aligned} 0 &\longrightarrow H^0_{\partial\Omega}(V, {}^H\tilde{\mathcal{O}}) \longrightarrow H^0_{\Omega^a}(V, {}^H\tilde{\mathcal{O}}) \longrightarrow H^0_\Omega(V, {}^H\tilde{\mathcal{O}}) \\ &\longrightarrow H^1_{\partial\Omega}(V, {}^H\tilde{\mathcal{O}}) \longrightarrow \cdots \longrightarrow H^{n-1}_\Omega(V, {}^H\tilde{\mathcal{O}}) \\ &\longrightarrow H^n_{\partial\Omega}(V, {}^H\tilde{\mathcal{O}}) \longrightarrow H^n_{\Omega^a}(V, {}^H\tilde{\mathcal{O}}) \longrightarrow H^n_\Omega(V, {}^H\tilde{\mathcal{O}}) \longrightarrow 0. \end{aligned}$$

Here Ω^a denotes the closure of Ω and an open set V is taken so that $V \supset \Omega^a$. Then, by (1) and the Martineau-Harvey Theorem, we have $H^{n-1}_\Omega(V, {}^H\tilde{\mathcal{O}}) = 0$. Thus we have the exact sequence

$$0 \longrightarrow H^n_{\partial\Omega}(V, {}^H\tilde{\mathcal{O}}) \longrightarrow H^n_{\Omega^a}(V, {}^H\tilde{\mathcal{O}}) \longrightarrow H^n_\Omega(V, {}^H\tilde{\mathcal{O}}) \longrightarrow 0.$$

Since, by the Martineau-Harvey Theorem, we have isomorphisms

$$H^n_{\partial\Omega}(V, {}^H\tilde{\mathcal{O}}) \cong \tilde{\mathcal{A}}(\partial\Omega\,;\, H)', \qquad H^n_{\Omega^a}(V, {}^H\tilde{\mathcal{O}}) \cong \tilde{\mathcal{A}}(\Omega^a\,;\, H)',$$

we obtain the isomorphism

$$H^n_\Omega(V, {}^H\tilde{\mathcal{O}}) \cong \tilde{\mathcal{A}}(\Omega^a\,;\, H)'/\tilde{\mathcal{A}}(\partial\Omega\,;\, H)' = \mathcal{R}(\Omega\,;\, H).$$

Thus the sheaf $\Omega \mapsto H^n_\Omega(V, {}^H\tilde{\mathcal{O}})$ is isomorphic to the sheaf ${}^H\mathcal{R}$ of H-valued Fourier hyperfunctions over $\boldsymbol{D}^n$. (Q. E. D.)

Corollary 3.4.2. *Let* Ω *be an arbitrary open set in* $\boldsymbol{D}^n$ *and* V *an* $\tilde{\mathcal{O}}$*-pseudoconvex open neighborhood of* Ω *such that* Ω *is a closed subset of* V. *Then*

(1) *If* $n \geq 2$, *we have the algebraic isomorphism*

$$H^n_\Omega(V, {}^H\tilde{\mathcal{O}}) \cong H^{n-1}(V \backslash \Omega, {}^H\tilde{\mathcal{O}}).$$

(2) *If* $n = 1$, *we have the algebraic isomorphism*

$$H^1_\Omega(V, {}^H\tilde{\mathcal{O}}) \cong \tilde{\mathcal{O}}(V \backslash \Omega\,;\, H)/\tilde{\mathcal{O}}(V\,;\, H).$$

Proof. It follows from the long exact sequence of relative cohomology groups

$$\begin{aligned} 0 &\longrightarrow H^0_\Omega(V, {}^H\tilde{\mathcal{O}}) \longrightarrow H^0(V, {}^H\tilde{\mathcal{O}}) \longrightarrow H^0(V \backslash \Omega, {}^H\tilde{\mathcal{O}}) \\ &\longrightarrow H^1_\Omega(V, {}^H\tilde{\mathcal{O}}) \longrightarrow \cdots \longrightarrow H^{n-1}(V \backslash \Omega, {}^H\tilde{\mathcal{O}}) \\ &\longrightarrow H^n_\Omega(V, {}^H\tilde{\mathcal{O}}) \longrightarrow H^n(V, {}^H\tilde{\mathcal{O}}) \longrightarrow H^n(V \backslash \Omega, {}^H\tilde{\mathcal{O}}) \longrightarrow \cdots. \quad \text{(Q. E. D.)} \end{aligned}$$

Now assume $n \geq 2$. Let Ω be an open set in $\boldsymbol{D}^n$. Then there exists an $\tilde{\mathcal{O}}$-pseudoconvex open neighborhood V of Ω such that $V\cap D^n=\Omega$ (cf. Kawai [14], Theorem 2.1.6). We put $V_0=V$ and $V_j=V\backslash\{z\in V;\ \mathrm{Im}\, z_j=0\}$, $(j=1, 2, \cdots, n)$. Then $\mathcal{V}=\{V_j\}_{j=0}^n$ and $\mathcal{V}'=\{V_j\}_{j=1}^n$ cover V and $V\backslash\Omega$ respectively. Since V_j and their intersections are also $\tilde{\mathcal{O}}$-pseudoconvex open sets, the covering $(\mathcal{V}, \mathcal{V}')$ satisfies the conditions of Leray's Theorem (cf. Komatsu [16], p. 98).

Thus, by a similar way to Theorem 3.3.2, we have the following.

Theorem 3.4.3. *We use notations as above. Then we have the isomorphisms*

$$H_\Omega^n(V, {}^H\tilde{\mathcal{O}})\cong H^n(\mathcal{V}, \mathcal{V}', {}^H\tilde{\mathcal{O}})$$
$$\cong\tilde{\mathcal{O}}(V\sharp\Omega;\ H)/\sum_{j=1}^n\tilde{\mathcal{O}}(V\sharp_j\Omega;\ H),$$

where we put

$$V\sharp\Omega=\bigcap_{j=1}^n V_j, \qquad V\sharp_j\Omega=\bigcap_{i\neq j} V_i.$$

Chapter 4. Fourier transformation of H-valued Fourier hyperfunctions

4.1. Definition. In this section we introduce the notion of the Fourier transformation of H-valued Fourier hyperfunctions on D^n.

Proposition 4.1.1. *If we define $\mathcal{F}\varphi$ by the formula*

$$(\mathcal{F}\varphi)(\xi)=\int_{R^n}e(\sqrt{-1}(x\cdot\xi))\varphi(x)dx$$

for $\varphi\in\mathcal{A}(D^n;\ H)$, where $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$, then $\mathcal{F}$ gives a topological isomorphism of $\mathcal{A}(D^n;\ H)$ onto itself.

Proof. It is evident that $\mathcal{F}$ is an algebraic isomorphism. We have only to prove the continuity of $\mathcal{F}$ because the situation for $\mathcal{F}^{-1}$ is similar.

Now we prove the continuity of $\mathcal{F}$. Put $U_m=D^n\times\sqrt{-1}\{y\in R^n;\ |y|<1/m\}$, $(m=1, 2, 3, \cdots)$. Then we have the isomorphism

$$\mathcal{A}(D^n;\ H)\cong\lim_{\substack{\longrightarrow \\ m}}\mathrm{ind}\,\mathcal{O}_b^{-1/m}(U_m;\ H).$$

By this definition of the topology of $\mathcal{A}(D^n;\ H)$, we have only to show that, for every $m(m\geqq1)$, there exists some $m'(m<m')$ such that $\mathcal{F}:\ \mathcal{O}_b^{-1/m}(U_m;\ H)\to\mathcal{O}_b^{-1/m'}(U_{m'};\ H)$ is continuous.

For an arbitrary $m(m\geqq1)$ and an arbitrary $\varphi\in\mathcal{O}_b^{-1/m}(U_m;\ H)$, we have

$$\mathcal{F}\varphi(\xi)=\int_{R^n}e(\sqrt{-1}(x\cdot\xi))\varphi(x)dx.$$

For $m'>m$, we can easily see that

$$\mathcal{F}\varphi(\zeta)=\int_{R^n}e(\sqrt{-1}(x\cdot\zeta))\varphi(x)dx, \ (\zeta=\xi+\sqrt{-1}\eta)$$

can be defined and $(\mathcal{F}\varphi)(\zeta)\in\mathcal{O}(U_{m'}\cap C^n;\ H)$. By the Cauchy integral theorem, we have

$$(\mathcal{F}\varphi)(\zeta)=\int_{R^n}e(\sqrt{-1}(z\cdot\zeta))\varphi(z)dx,$$

$$(\zeta=\xi+\sqrt{-1}\,\eta,\quad z=x+\sqrt{-1}\,y,\quad |\eta|<1/m',\quad |y|<1/m).$$

Then if, for every $\zeta=\xi+\sqrt{-1}\,\eta$, we put $y=\varepsilon\xi/|\xi|$, $(\varepsilon=(m+m')/(2mm'))$, we have the estimate

$$e(|\zeta|/m')^{H}\|\mathcal{F}\varphi(\zeta)\|$$
$$\leqq C_{m,m'}\sup\{^{H}\|\varphi(z)\|e(|z|/m);\ z\in U_m\},\quad (\zeta\in U_{m'}),$$

where $C_{m,m'}$ is some constant depending on m and m'. This completes the proof.

(Q. E. D.)

Definition 4.1.2. Let T be an element in ${}^{H}\mathcal{R}(\boldsymbol{D}^n)\cong\mathcal{A}(\boldsymbol{D}^n;\ H)'$. Then we define $\mathcal{F}^*T$ by the formula

$$\langle\mathcal{F}^*T,\ \varphi\rangle=\langle T,\ \mathcal{F}\varphi\rangle,\qquad \text{for}\quad \varphi\in\mathcal{A}(\boldsymbol{D}^n;\ H).$$

We also define $\bar{\mathcal{F}}^*T$ by the formula

$$\langle\bar{\mathcal{F}}^*T,\ \varphi\rangle=\langle T,\ \bar{\mathcal{F}}\varphi\rangle,\qquad \text{for}\quad \varphi\in\mathcal{A}(\boldsymbol{D}^n;\ H),$$

where

$$(\bar{\mathcal{F}}\varphi)(\xi)=(2\pi)^{-n}\int_{\boldsymbol{R}^n}e(-\sqrt{-1}(x\cdot\xi))\varphi(x)dx.$$

$\mathcal{F}$ and $\bar{\mathcal{F}}$ are topological isomorphisms of $\mathcal{A}(\boldsymbol{D}^n;\ H)$ onto itself.

4.2. The Paley-Wiener Theorem. If we define $V_0=\tilde{\boldsymbol{C}}^n$ and $V_j=\boldsymbol{D}^n\times\sqrt{-1}\{y\in\boldsymbol{R}^n;\ y_j\neq 0\}$, $(j=1, 2, \cdots, n)$, and if we put $\mathcal{V}=\{V_j\}_{j=0}^{n}$ and $\mathcal{V}'=\{V_j\}_{j=1}^{n}$, then $(\mathcal{V},\ \mathcal{V}')$ is an relative covering of $(\tilde{\boldsymbol{C}}^n,\ \tilde{\boldsymbol{C}}^n\backslash\boldsymbol{D}^n)$. Then we can define the n-th cohomology group $H^n(\mathcal{V},\ \mathcal{V}',\ {}^{H}\tilde{\mathcal{O}})$ of the relative covering $(\mathcal{V},\ \mathcal{V}')$ as usual.

Then we have the isomorphisms

$$H^n(\mathcal{V},\ \mathcal{V}',\ {}^{H}\tilde{\mathcal{O}})\cong Z^n(\mathcal{V},\ \mathcal{V}',\ {}^{H}\tilde{\mathcal{O}})/\delta C^{n-1}(\mathcal{V},\ \mathcal{V}',\ {}^{H}\tilde{\mathcal{O}})$$
$$\cong\tilde{\mathcal{O}}(V_1\cap\cdots\cap V_n;\ H)/\sum_{j=1}^{m}\tilde{\mathcal{O}}(\bigcap_{i\neq j}V_i;\ H).$$

We denote $H^n(\mathcal{V},\ \mathcal{V}',\ {}^{H}\tilde{\mathcal{O}})$ by H^n for simplicity.

For an n-tuple $\sigma=(\sigma_1, \cdots, \sigma_n)$ of $+1$ or -1, we denote by Γ_σ the quadrant in $\boldsymbol{R}^n$ such as $\{x\in\boldsymbol{R}^n;\ \sigma_j x_j>0,\ (j=1, 2, \cdots, n)\}$. Then an element $\varphi\in\tilde{\mathcal{O}}(V_1\cap\cdots\cap V_n;\ H)$ is a 2^n-tuple of slowly increasing H-valued functions $\varphi=(\varphi_\sigma)_{\sigma\in\mathcal{P}}(\varphi_\sigma\in\tilde{\mathcal{O}}(\boldsymbol{D}^n\times\sqrt{-1}\,\Gamma_\sigma;\ H))$, where $\mathcal{P}$ denotes the set of all n-tuples $\sigma=(\sigma_1, \cdots, \sigma_n)$ of $+1$ or -1. We denote by $[\varphi]$ the cohomology class $\varphi+\sum_j\tilde{\mathcal{O}}(\cap_{i\neq j}V_i;\ H)\in H^n$. Then, for an element $[\varphi]$ in H^n, we define an element $b[\varphi]\in\mathcal{A}(\boldsymbol{D}^n;\ H)'$ by

(4.2.1)
$$\langle b[\varphi],\ f\rangle=\sum_{\sigma\in\mathcal{P}}(-1)^n\,\mathrm{sign}(\sigma)\int_{\boldsymbol{R}^n}(\varphi_\sigma(x+\sqrt{-1}\,\varepsilon_\sigma),\ \bar{f}(x+\sqrt{-1}\,\varepsilon_\sigma))dx$$
$$\text{for}\quad f\in\mathcal{A}(\boldsymbol{D}^n;\ H),$$

where we put $\mathrm{sign}(\sigma)=\prod_j\sigma_j$, ε_σ is in Γ_σ and $|\varepsilon_\sigma|$ is sufficiently small. We note that the above integral does not depend on a choice of $\varepsilon_\sigma\in\Gamma_\sigma$ as far as $|\varepsilon_\sigma|$ is sufficiently small by Cauchy's integral theorem. It follows also from Cauchy's integral theorem that (4.2.1) does not depend on a choice of a representative φ of a cohomology class $[\varphi]\in H^n$. Hence $b[\varphi]$ is well defined. We call b the boundary value operator of H^n into $\mathcal{A}(\boldsymbol{D}^n; H)'$. Note that b is continuous and linear. Further we have the isomorphism

$$b: H^n\cong\mathcal{A}(\boldsymbol{D}^n; H)'=\mathfrak{R}(\boldsymbol{D}^n; H).$$

For a cone Γ in $\boldsymbol{R}^n$ with vertex at the origin, we put $\Gamma^0=\{x\in\boldsymbol{R}^n; x\cdot\xi\geqq 0$ for every $\xi\in\Gamma\}$. Then we have the following.

Theorem 4.2.1 (the Paley-Wiener Theorem). *Let Γ be a closed strictly convex cone in $\boldsymbol{R}^n$ and K its closure in $\boldsymbol{D}^n$.*

For the sake of simplicity we assume that the vertex of the cone Γ be at the origin and $K\subset\subset[\{x_1\geqq-\varepsilon\}]^a$ holds for every $\varepsilon>0$ in the topology of $\boldsymbol{D}^n$. Let $T\in\mathfrak{R}(\boldsymbol{D}^n; H)$. Then $T\in\mathcal{A}(K; H)'$ if and only if $\langle T, e(\sqrt{-1}(z\cdot\zeta))\rangle\in\tilde{\mathcal{O}}(\boldsymbol{D}^n\times\sqrt{-1}\operatorname{int}(\Gamma^0); H)$ holds. Here we put

$$z\cdot\zeta=z_1\zeta_1+\cdots+z_n\zeta_n,\quad e(\sqrt{-1}(z\cdot\zeta))=\exp(\sqrt{-1}(z\cdot\zeta)).$$

This is an H-valued version of Kawai [14], Theorems 3.3.1 and 3.3.2 (p. 485).

Theorem 4.2.2. *Let K be the closure of Γ_σ in $\boldsymbol{D}^n$ for $\sigma=(\sigma_1, \sigma_2, \cdots, \sigma_n)$ with $\sigma_i=+1$ or -1, $(i=1, 2, \cdots, n)$. Then the following sequence is exact:*

$$\bigoplus'_{\sigma,\tau}\mathcal{A}(K_\sigma\cap K_\tau; H)'\xrightarrow{\alpha}\bigoplus_\sigma\mathcal{A}(K_\sigma; H)'\xrightarrow{\beta}\mathcal{A}(\boldsymbol{D}^n; H)'\longrightarrow 0,$$

where we put

$$\bigoplus'_{\sigma,\tau}\mathcal{A}(K_\sigma\cap K_\tau; H)'=\{(\mu_{\sigma\tau}); \mu_{\sigma\tau}\in\mathcal{A}(K_\sigma\cap K_\tau; H)',\quad \mu_{\sigma\tau}+\mu_{\tau\sigma}=0\},$$

and

$$\alpha: (\mu_{\sigma\tau})_{\sigma,\tau}\longrightarrow(\textstyle\sum_\tau\mu_{\sigma\tau})_\sigma,$$

$$\beta: (\mu_\sigma)_\sigma\longrightarrow\textstyle\sum_\sigma\mu_\sigma.$$

Proof. It follows from the flabbiness of the sheaf ${}^H\mathfrak{R}$ and Theorem 8.4.3 in Morimoto [19], (p. 210) and Martineau-Harvey Theorem. (Q.E.D.)

Definition 4.2.3. For $T\in\mathfrak{R}(\boldsymbol{D}^n; H)$, we decompose T as in Theorem 4.2.2:

$$T=\textstyle\sum_{\sigma\in\mathcal{P}}T_\sigma,\qquad T_\sigma\in\mathcal{A}(K_\sigma; H)'.$$

Putting

$$F_\sigma(\zeta)=(-1)^n\operatorname{sign}(\sigma)\langle T_{\sigma,z}, e(\sqrt{-1}(z\cdot\zeta))\rangle,$$
$$\text{for}\quad \zeta\in\boldsymbol{R}^n\times\sqrt{-1}\operatorname{int}(\Gamma^0_\sigma),$$

we define $\mathcal{F}_sT$ by the formula $\mathcal{F}_sT=[(F_\sigma)_{\sigma\in\mathcal{P}}]\in H^n$.

Then $\mathcal{F}_sT$ is well-defined. In fact, by the Paley-Wiener Theorem, we have $F_\sigma(\zeta)\in\tilde{\mathcal{O}}(\boldsymbol{D}^n\times\sqrt{-1}\operatorname{int}(\Gamma^0_\sigma); H)$. Here we note that $\operatorname{int}(\Gamma^0_\sigma)=\Gamma_\sigma$, $(\sigma\in\mathcal{P})$ holds. The class $[(F_\sigma)_{\sigma\in\mathcal{P}}]\in H^n$ is independent of the decomposition of T in Definition 4.2.3. Indeed, the ambiguity of the decomposition in Definition 4.2.3 comes from the element $(S_{\sigma\tau})\in \oplus'_{\sigma,\tau}\mathcal{A}(K_\sigma\cap K_\tau;\, H)'$ with $S_{\sigma\tau}+S_{\tau\sigma}=0$ by Theorem 4.2.2. Then we have $\langle S_{\sigma\tau,z},\, e(\sqrt{-1}(z\cdot\zeta))\rangle \in \tilde{\mathcal{O}}(\boldsymbol{D}^n\times\sqrt{-1}(\Gamma_\sigma\cap\Gamma_\tau)^0)$, $(\sigma\neq\tau)$. Hence $[(-1)^n \operatorname{sign}(\sigma)\langle\sum_\tau S_{\sigma\tau,z},\, e(\sqrt{-1}(z\cdot\zeta))\rangle]_{\sigma\in\mathcal{P}}$ belongs to $\sum_j\tilde{\mathcal{O}}(\cap_{i\neq j}V_i;\, H)$. Hence the ambiguity of the decomposition of T does not affect $\mathcal{F}_sT$. Thus $\mathcal{F}_sT$ is well-defined.

We call this $\mathcal{F}_s$ the Fourier-Carleman-Leray-Sato transformation.

Let $T\in\mathcal{R}(\boldsymbol{D}^n;\, H)$. We put $\mathcal{F}^*T=S=\sum_{\sigma\in\mathcal{P}}S_\sigma$, $S_\sigma\in\mathcal{A}(K_\sigma;\, H)'$. Then, for $f\in \mathcal{A}(\boldsymbol{D}^n;\, H)$, we have

$$\langle b\mathcal{F}_s(\mathcal{F}^*T),\, f\rangle=\sum_{\sigma\in\mathcal{P}}\int_{\boldsymbol{R}^n+\sqrt{-1}\varepsilon_\sigma}(\langle S_{\sigma,z},\, e(\sqrt{-1}(z\cdot\zeta))\rangle,\, \overline{f(\zeta)})d\zeta$$

$$=\langle \sum_{\sigma\in\mathcal{P}}S_{\sigma,z},\, \int_{\boldsymbol{R}^n}e(\sqrt{-1}(z\cdot\zeta))f(\zeta)d\zeta\rangle$$

$$=\langle S,\, \mathcal{F}f\rangle=\langle\mathcal{F}^*T,\, \mathcal{F}f\rangle=\langle T,\, f\rangle.$$

Thus we have the following.

Theorem 4.2.4. *We have* $b\mathcal{F}_s=\mathcal{F}^*$.

We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}(\boldsymbol{D}^n;\, H)' & \xrightarrow{\mathcal{F}_s} & H^n \\ \mathcal{F}^*\uparrow & \swarrow b & \\ \mathcal{A}(\boldsymbol{D}^n;\, H)' & & \end{array}$$

Since b is an isomorphism by Theorem 3.4.3 and the Martineau-Harvey Theorem, $\mathcal{F}_s$ is also an isomorphism. Thus we have the following.

Theorem 4.2.5. *In the above notations,* $\mathcal{F}_s$ *is an isomorphism of* $\mathcal{R}(\boldsymbol{D}^n;\, H)$ *onto* H^n.

Thus we have defined the Fourier transform of an element of $\mathcal{R}(\boldsymbol{D}^n;\, H)$ via "boundary values" of slowly increasing H-valued holomorphic functions in tublar domains.

We put $b\mathcal{F}_s=\mathcal{F}^*=\mathcal{F}$ and call it the Fourier transformation.

Definition 4.2.6. Let $T\in\mathcal{A}(\boldsymbol{R}^n;\, H)'=\mathcal{A}(\boldsymbol{R}^n;\, H)'$, whose support is a compact set K in $\boldsymbol{R}^n$. Then we define the Fourier-Borel transform $\hat{T}(\xi)$ of T by the formula

$$\hat{T}(\xi)=\langle T_x,\, e(\sqrt{-1}(x\cdot\xi))\rangle.$$

Then we have the following.

Proposition 4.2.7. *Let* $T\in\mathcal{A}(\boldsymbol{R}^n;\, H)'$ *whose support is a compact set* K *in* $\boldsymbol{R}^n$. *Then* $\hat{T}(\xi)\in\tilde{\mathcal{A}}(\boldsymbol{D}^n;\, H)$ *and* $\hat{T}(\xi)$ *can be extended to* $\hat{T}(\zeta)=\langle T_x,\, e(\sqrt{-1}(x\cdot\zeta))\rangle\in\mathcal{O}(\boldsymbol{C}^n;\, H)$ *so that, for any* $\varepsilon>0$, *there exists a constant* $C_\varepsilon>0$ *such that* ${}^H\|\hat{T}(\zeta)\|\leqq C_\varepsilon e(I_K(\eta)+\varepsilon|\zeta|)$ *holds.*

Proof. This can be proved directly. (Q. E. D.)

Corollary. *Let T be as in Proposition* 4.2.8. *Then we have* $\mathcal{F}T=b\hat{T}$.

Proof. We have, for any $\varphi\in\mathcal{A}(\boldsymbol{D}^n;H)$,

$$\langle\mathcal{F}T,\varphi\rangle=\langle T_x,\int_{\boldsymbol{R}^n}e(\sqrt{-1}(x\cdot\xi))\varphi(\xi)d\xi\rangle$$
$$=\int_{\boldsymbol{R}^n}(\langle T_x,e(\sqrt{-1}(x\cdot\xi))\rangle,\overline{\varphi(\xi)})d\xi$$
$$=\langle b\hat{T},\varphi\rangle.$$

This completes the proof. (Q.E.D.)

Theorem 4.2.8 (the Paley-Wiener Theorem). *For a* $T\in\mathcal{A}(\boldsymbol{R}^n;H)'$, *the following are equivalent:*

(1) *The support of T is contained in a convex compact set K in* $\boldsymbol{R}^n$.

(2) *The Fourier-Borel transform* $\hat{T}(\zeta)\in\mathcal{O}(\boldsymbol{C}^n;H)$ *of T satisfies the following estimate:*

for any $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ such that

$$^{H}\|\hat{T}(\zeta)\|\leqq C_\varepsilon(I_K(\eta)+\varepsilon|\zeta|),$$

where we put

$$I_K(\eta)=\sup\{-x\cdot\eta;\ x\in K\}.$$

Proof. The fact that (1) implies (2) is evident.

Now we prove that (2) implies (1). Let K^0 be the polar set of K:

$$K^0=\{\eta\in\boldsymbol{R}^n;\ |x\cdot\eta|\leqq1\ (x\in K)\}.$$

Then K^0 is a convex balanced subset of $\boldsymbol{R}^n$. Then, by the assumptions, $\hat{T}(\zeta)\in\tilde{\mathcal{O}}(\boldsymbol{D}^n\times\sqrt{-1}\,\mathrm{int}(K^0);H)$ holds. Since K is a convex compact set, we can represent K as $K=\cap_g K_g$, where K_g is the image of K_σ, ($\sigma=(1,1,\cdots,1)$), by some regular inhomogeneous linear transformatin g. Since K is a compact set K in $\boldsymbol{R}^n$, $K=\cap_g\Gamma_g$, where $\Gamma_g=K_g\cap\boldsymbol{R}^n$. Since $K^0=(\cap_g\Gamma_g)^0\supset\cup_g\Gamma_g^0$ holds, we have

$$\hat{T}(\zeta)\in\tilde{\mathcal{O}}(\boldsymbol{D}^n\times\sqrt{-1}\,\mathrm{int}(K^0);H)$$
$$\subset\cap_g\tilde{\mathcal{O}}(\boldsymbol{D}^n\times\sqrt{-1}(\Gamma_g^0);H).$$

Hence, we have $\hat{T}(\zeta)\in\tilde{\mathcal{O}}(\boldsymbol{D}^n\times\sqrt{-1}\,\mathrm{int}(\Gamma_g^0);H)$ for every g. Hence we have $T\in\mathcal{A}(K_g;H)'$ by the general version of Theorem 4.2.2. Since this holds for every g, we have

$$T\in\cap_g\mathcal{A}(K_g;H)'=\mathcal{A}(\cap_g K_g;H)'$$
$$=\mathcal{A}(K;H)'=\mathcal{A}(K;H)'.$$

This completes the proof. (Q.E.D.)

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