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# On limit theorems related to a class of "winding-type" additive functionals of complex brownian motion

By

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## § 0. Introduction

Let $z(t)=x(t)+\sqrt{-1}y(t)$, $z(0)=0$, be a complex Brownian motion starting at the origin. Many works have been done on the limit theorems for additive functionals of $z(t)$. Well-known classical results are due to G. Kallianpur and H. Robbins ([5]) for occupation times and to F. Spitzer ([10]) for winding number of $z(t)$ around a given non-zero point. The former result has been extended by Y. Kasahara and S. Kotani ([6]): They obtained scaling limit processes for a class of additive functionals of $z(t)$ including occupation times of $z(t)$ in bounded sets. The latter result has been extended by J. Pitman and M. Yor ([7], [8], [9]) who obtained the joint limit distribution, as time tends to infinity, of a class of additive functionals of $z(t)$ including winding numbers around several non zero points. Main purpose of the present paper is to reproduce and extend some results of Pitman-Yor by the method of Kasahara-Kotani: In particular we discuss the convergence as *stochastic processes* of time scaled additive functionals belonging to a little more general class.

First, we describe briefly the main idea of Kasahara-Kotani. In order to study the limit process as $\lambda\to\infty$ of additive functionals $A^\lambda(t)$, $\lambda>0$, given in the form

$$A^\lambda(t)=\frac{1}{\lambda N(\lambda)}\int_0^{u(\lambda t)} f(z_s)dz_s$$

where $u(t)=e^{2t}-1$ and $N(\lambda)$ is some normalizing function, we set

$$Z(t)=\log(z(t)+1)$$

and introduce an increasing process

$$\tau^\lambda(t)=\frac{1}{\lambda}u^{-1}(\langle Z\rangle^{-1}(\lambda^2 t)).$$

(Generally, $\langle M\rangle(t)$ is the usual quadratic variation process of a conformal (local) martingale $M(t)$ and $g^{-1}(t)$ is the right continuous inverse function of a continuous increasing function $g(t)$.) Then, by the time substitution, we have

$$A^\lambda(\tau^\lambda(t))=\frac{1}{N(\lambda)}\int_0^t f(e^{\lambda\hat{Z}^\lambda(s)}-1)e^{\lambda\hat{Z}^\lambda(s)}d\hat{Z}^\lambda(s)$$

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where $\hat{Z}^{\lambda}(t)=(1/\lambda)Z(\langle Z\rangle^{-1}(\lambda^2 t))$. Note that $\hat{Z}^{\lambda}(t)$ is a complex Brownian motion for every $\lambda>0$. The limit process of $A^{\lambda}(t)$ can be found if we can obtain the limit process as $\lambda\to\infty$ of the joint continuous processes $\{A^{\lambda}(\tau^{\lambda}(t)),\ \hat{Z}^{\lambda}(t),\ \tau^{\lambda}(t)\}$. The limit process of $\{\hat{Z}^{\lambda}(t),\ \tau^{\lambda}(t)\}$ is given by $\{b(t),\ \mu(t)\}$ where $b(t)$ is a complex Brownian motion and $\mu(t)=\max_{0\leq s\leq t}\mathcal{R}e[b(s)]$ (*cf.* Lemma 3.1 of [6]). The study of convergence for the above joint processes is therefore reduced to that for

$$\frac{1}{N(\lambda)}\int_0^t f(e^{\lambda b(s)}-1)e^{\lambda b(s)}db(s)$$

as $\lambda\to\infty$. If we represent $b(t)$ as

$$b(t)=x(t)+\sqrt{-1}\int_0^t d\theta(s),$$

where $\theta(t)$ is a Brownian motion on the unit circle $\boldsymbol{T}=\boldsymbol{R}/2\pi\boldsymbol{Z}\cong[0,\ 2\pi]$ so that $(x(t),\ \theta(t))$ is a Brownian motion on the Riemannian manifold $\boldsymbol{R}\times\boldsymbol{T}$, then, in this study, the ergodic property of $\theta(t)$ plays an important role; indeed, it is a homogenization problem for $(x(t),\ \theta(t))$.

We would apply this method of Kasahara-Kotani to some problems discussed by Pitman-Yor, namely to the study of joint limit distribution, as $\lambda\to\infty$, of the processes $(A_{ij}{}^{\lambda})$ given by

$$A_{ij}{}^{\lambda}(t)=\frac{1}{\lambda N_{ij}(\lambda)}\int_0^{u(\lambda t)}\frac{f_j(z_s)}{z_s-a_i}dz_s, \tag{0.1}$$

where $a_1,\ \cdots,\ a_n$ are distinct points on $\boldsymbol{C}\setminus\{0\}$ and $f_j,\ j=1,\ \cdots,\ m$, are some Borel functions on $\boldsymbol{C}$. If $f_j\equiv 1$, then $\mathcal{I}m[A_{ij}{}^{\lambda}(t)]$ is a normalized algebraic total angle wound by $z(t)$ around $a_i$ up to the time $e^{2\lambda t}-1$. Writing

$$f_j(a_i-a_ie^{x+\sqrt{-1}\theta})=g_{ij}(x,\ \theta),\qquad (x,\ \theta)\in\boldsymbol{R}\times\boldsymbol{T},$$

Pitman-Yor discussed the case when $g_{ij}$ depend only on $\theta$. Here we consider a more general case by introducing a notion of functions regularly varying at point $a_i$ and also at the point at infinity. This class of functions was introduced by S. Watanabe in an unpublished note. In order to apply Kasahara-Kotani's method to this class of additive functionals, we need an ergodic theorem for Brownian motion $(x(t),\ \theta(t))$ on $\boldsymbol{R}\times\boldsymbol{T}$ which we establish in §1 by using the method of eigenfunction expansions.

Finally, we summarize the contents of this paper. In §1, we consider a class of diffusion processes on $\boldsymbol{R}^d\times M$ where $M$ is a compact Riemannian manifold and obtain an ergodic theorem for them. In §2, we apply the result of §1 to a homogenization problem for Brownian motion $(x(t),\ \theta(t))$ on $\boldsymbol{R}\times\boldsymbol{T}$ and thereby describe the limit process as $\lambda\to\infty$ of the joint processes

$$\left\{\frac{1}{N_i(\lambda)}\int_0^t f_i(a-ae^{\lambda z(s)})dz(s)\right\}_{1\leq i\leq m},$$

where $a\in\boldsymbol{C}\setminus\{0\}$, $z(t)=x(t)+\sqrt{-1}\int_0^t d\theta(s)$ so that $z(t)$ is a complex Brownian motion, and $f_i$ are taken from the class of regularly varying functions in the sense given by

Definition 2.1. Here, the asymptotic Knight's theorem of Pitman-Yor [9] for a class of conformal martingales also plays an important role. In § 3, we obtain the joint limit theorem for additive functionals of the form (0.1) by applying the results in § 2.

## § 1. An ergodic theorem for some class of diffusion processes on compact manifolds

Let $M$ be an $m$-dimensional compact (connected) $C^{\infty}$-Riemannian manifold without boundary and $(\Theta_t)_{t\geq 0}$ be a Brownian motion on $M$ (see Ikeda and Watanabe [4], Chapter 5, section 4). The generator of $(\Theta_t)$ is $(1/2)\Delta_M$, where $\Delta_M$ is the Laplace-Beltrami operator for $M$. Since $M$ is compact, $\Delta_M$ has pure point spectrum

$$0=\lambda_0>-\lambda_1\geqq-\lambda_2\geqq\cdots \tag{1.1}$$

and we denote the corresponding normalized eigenfunctions by $\{\varphi_n\}$. It is known that the transition density $q(t, \theta, \eta)$ of $(\Theta_t)$ has the following expansion:

$$q(t, \theta, \eta)=\sum_{n=0}^{\infty} e^{-\lambda_n t}\varphi_n(\theta)\varphi_n(\eta), \tag{1.2}$$

which converges uniformly in $(\theta, \eta)$ for every $t>0$ (see Chavel [1] p. 140).

Let $(X_t)_{t\geq 0}$ be an $\boldsymbol{R}^d$-valued diffusion process determined by the stochastic differential equation

$$dX_t=\sigma(X_t)dB_t+b(X_t)dt, \tag{1.3}$$

where $\sigma(x)$ and $b(x)$ are bounded and smooth, $\sigma(x)$ is uniformly non-degenerate and $(B_t)_{t\geq 0}$ is a $d$-dimensional Brownian motion.

We assume that $X$ and $\Theta$ are independent and $X_0=0$ and $\Theta_0=\theta_0(\theta_0\in M)$ throughout this section.

Our main result in this section is as follows:

**Theorem 1.1.** *Let $h$ be a Borel measurable function from $\boldsymbol{R}^d$ to $\boldsymbol{R}^1$ and $f$ be a Borel measurable function from $M$ to $\boldsymbol{R}^1$ satisfying the following conditions:*

(1) $\quad |h(x)|\leqq\text{const.}|x|^{\alpha} \quad$ *for every* $\quad x\in\boldsymbol{R}^d$

*for some $\alpha>-\min(2, d)$,*

(2) *$f$ is in $L^p(M)=L^p(M, d\theta)$ for some $p$ with $p\geqq 1$ and $p>m/(\alpha+2)$, where $d\theta$ is the volume element of $M$,*

(3) *$f$ is null charged i.e.*

$$\int_M f(\theta)d\theta=0.$$

*Then for every $T>0$, it holds that*

$$E_{(0,\theta_0)}\left[\sup_{0\leq t\leq T}\left|\int_0^t h(X_s)f(\Theta_{\lambda s})ds\right|\right]\longrightarrow 0$$

*as $\lambda\to\infty$.*

To prove our theorem, we prepare some estimates for $E_x|h(X_t)|$ and $E_\theta|f(\Theta_t)|$.

**Lemma 1.1.** *Suppose that* $h$: $\boldsymbol{R}^d \to \boldsymbol{R}^1$ *satisfies* $|h(x)| \leqq \text{const.}|x|^\alpha$ *for every* $x \in \boldsymbol{R}^d$, *where* $\alpha > -d$. *Then for every* $x \in \boldsymbol{R}^d$ *and* $t>0$,

$$E_x|h(X_t)| \leqq \text{const. } t^{\alpha/2} + \text{const.}|x|^\alpha 1_{(\alpha>0)}.$$

*Proof.* From the assumption of $X_t$, we have the following estimate for the transition density $p(t, x, y)$ of $X_t$:

$$p(t, x, y) \leqq \text{const. } t^{-d/2} \exp\left\{-\frac{\text{const.}|x-y|^2}{2t}\right\}. \tag{1.4}$$

(See Friedman [3], p. 141, Theorem 4.5.)

Then, from the assumption for $h(x)$,

$$\begin{aligned} E_x|h(X_t)| &\leqq \text{const. } t^{-d/2} \int_{R^d} \exp\left\{-\frac{\text{const.}|x-y|^2}{2t}\right\}|y|^\alpha dy \\ &= \text{const.} \int_{R^d} \exp\left(-\frac{\text{const.}|\xi|^2}{2}\right)|\sqrt{t}\,\xi + x|^\alpha d\xi. \end{aligned} \tag{1.5}$$

If $\alpha > 0$, the right hand side (RHS) of (1.5) is bounded by

$$\begin{aligned} &\text{const.} \int_{R^d} \exp\left(-\frac{\text{const.}|\xi|^2}{2}\right)(\sqrt{t}\,|\xi|)^\alpha d\xi \\ &\qquad + \text{const.} \int_{R^d} \exp\left(-\frac{\text{const.}|\xi|^2}{2}\right)|x|^\alpha d\xi \\ &= \text{const. } t^{\alpha/2} + \text{const. } |x|^\alpha. \end{aligned}$$

If $-d < \alpha \leqq 0$, the RHS of (1.5) is bounded by

$$\begin{aligned} &\text{const. } t^{\alpha/2} \int_{R^d} \exp\left(-\frac{\text{const.}|\xi|^2}{2}\right)|\xi + x/\sqrt{t}\,|^\alpha d\xi \\ &= \text{const. } t^{\alpha/2} \int_{|\xi + x/\sqrt{t}|<1} \exp\left(-\frac{\text{const.}|\xi|^2}{2}\right)|\xi + x/\sqrt{t}\,|^\alpha d\xi \\ &\qquad + \text{const. } t^{\alpha/2} \int_{|\xi + x/\sqrt{t}| \geqq 1} \exp\left(-\frac{\text{const.}|\xi|^2}{2}\right)|\xi + x/\sqrt{t}\,|^\alpha d\xi \\ &\leqq \text{const. } t^{\alpha/2} \int_{|\xi + x/\sqrt{t}|<1} |\xi + x/\sqrt{t}\,|^\alpha d\xi \\ &\qquad + \text{const. } t^{\alpha/2} \int_{|\xi + x/\sqrt{t}| \geqq 1} \exp\left(-\frac{\text{const.}|\xi|^2}{2}\right) d\xi \\ &\leqq \text{const. } t^{\alpha/2}. \quad \text{Q.E.D.} \end{aligned}$$

**Lemma 1.2.** *Suppose* $f \in L^p(M \to \boldsymbol{R}^1, d\theta)$ *with some* $p \geqq 1$. *Then for every* $\theta \in M$ *and* $t>0$,

$$E_\theta|f(\Theta_t)| \leqq (\text{const. } t^{-m/2p} + \text{const.})\|f\|_p.$$

*Proof.* For a moment, let $p>1$. First note that

$$E_\theta|f(\Theta_t)|=\int_M q(t,\ \theta,\ \eta)|f(\eta)|d\eta\leqq\left(\int_M q(t,\ \theta,\ \eta)^q d\eta\right)^{1/q}\|f\|_p,$$

where $q(t,\ \theta,\ \eta)$ is the transition density of $\Theta_t$ and $1/q+1/p=1$.

Since we have the uniform estimate

$$q(t,\ \theta,\ \eta)\leqq\text{const.}\ t^{-m/2}\qquad(t\downarrow 0)$$

(see Chavel [1], p. 154~155), setting $\delta>0$ small enough,

$$\left(\int_M q(t,\ \theta,\ \eta)^q d\eta\right)^{1/q}1_{(t<\delta)}\leqq\left(\text{const.}\int_M t^{-m(q-1)/2}q(t,\ \theta,\ \eta)d\eta\right)^{1/q}1_{(t<\delta)}$$

$$=\text{const.}\ t^{-m/2p}1_{(t<\delta)}.$$

On the other hand, from (1.2) we have

$$\begin{aligned}q(t,\ \theta,\ \eta)1_{(t>\delta)}&\leqq\left(\sum_{n=0}^{\infty}e^{-\lambda_n t}\varphi_n(\theta)^2\right)^{1/2}\left(\sum_{n=0}^{\infty}e^{-\lambda_n t}\varphi_n(\eta)^2\right)^{1/2}1_{(t>\delta)}\\&\leqq\left(\sum_{n=0}^{\infty}e^{-\lambda_n\delta}\varphi_n(\theta)^2\right)^{1/2}\left(\sum_{n=0}^{\infty}e^{-\lambda_n\delta}\varphi_n(\eta)^2\right)^{1/2}\\&=q(\delta,\ \theta,\ \theta)^{1/2}q(\delta,\ \eta,\ \eta)^{1/2}\\&\leqq\text{const.}.\end{aligned}$$

Hence

$$\left(\int_M q(t,\ \theta,\ \eta)^q d\eta\right)^{1/q}1_{(t>\delta)}\leqq\text{const.}\ V(M)^{1/q},$$

where $V(M)=\int_M d\theta$.

Therefore,

$$\begin{aligned}E_\theta|f(\Theta_t)|&\leqq(\text{const.}\ t^{-m/2p}1_{(t<\delta)}+\text{const.}\ V(M)^{1/q})\|f\|_p\\&\leqq(\text{const.}\ t^{-m/2p}+\text{const.})\|f\|_p.\end{aligned}$$

In the case that $p=1$, we can prove the lemma similarly by replacing $\left(\int_M q(t,\ \theta,\ \eta)^q d\eta\right)^{1/q}$ with $\sup_\eta q(t,\ \theta,\ \eta)$. Q.E.D.

*Proof of Theorem* 1.1. First we will prove the theorem in the case that $f=\varphi_n$ for some $n\geqq 1$. From now on we write the expectation $E_{(0,\theta_0)}$ simply by $E$.

Set

$$u_\lambda(x,\ \theta)=\int_0^\infty E_{(x,\theta)}(h(X_s)\varphi_n(\Theta_{\lambda s}))ds$$

and

$$M_t^\lambda=u_\lambda(X_t,\ \Theta_{\lambda t})+\int_0^t h(X_s)\varphi_n(\Theta_{\lambda s})ds.$$

In order to prove that

(1.6)
$$E\sup_{0\leq t\leq T}\left|\int_0^t h(X_s)\varphi_n(\Theta_{\lambda s})ds\right| \longrightarrow 0 \qquad (\lambda \longrightarrow \infty),$$

it is clearly sufficient to prove that

(1.7)
$$E\sup_{0\leq t\leq T}|u_\lambda(X_t, \Theta_{\lambda t})| \longrightarrow 0 \qquad (\lambda \longrightarrow \infty)$$

and

(1.8)
$$E\sup_{0\leq t\leq T}|M_t^\lambda| \longrightarrow 0 \qquad (\lambda \longrightarrow \infty).$$

The convergence (1.7) is proved as follows. By the orthonormality of $\{\varphi_k\}$ and (1.2), we see the following identity:

$$\begin{aligned} E_\theta(\varphi_n(\Theta_{\lambda s})) &= \int_M q(\lambda s, \theta, \eta)\varphi_n(\eta)d\eta \\ &= e^{-\lambda_n \lambda s}\varphi_n(\theta) \qquad \text{for every} \quad \theta\in M. \end{aligned}$$

Clearly $\varphi_n(\theta)$ is bounded since $\varphi_n$ is continuous and $M$ is compact and hence we have the basic estimate

(1.9)
$$E_\theta(\varphi_n(\Theta_{\lambda s}))\leqq \text{const.}\ e^{-\lambda_n \lambda s}.$$

By Lemma 1.1 and (1.9), we obtain the following estimate for $u_\lambda$:

(1.10)
$$\begin{aligned} |u_\lambda(x, \theta)| &\leqq \int_0^\infty E_x|h(X_s)|\,|E_\theta(\varphi_n(\Theta_{\lambda s}))|\,ds \\ &\leqq \text{const.}\int_0^\infty s^{\alpha/2}e^{-\lambda_n\lambda s}ds + \text{const.}\,|x|^\alpha 1_{(\alpha>0)}\int_0^\infty e^{-\lambda_n\lambda s}ds \\ &= \text{const.}\ \lambda^{-\alpha/2-1} + \text{const.}\,|x|^\alpha 1_{(\alpha>0)}\lambda^{-1}. \end{aligned}$$

Hence

$$E\sup_{0\leq t\leq T}|u_\lambda(X_t, \Theta_{\lambda t})| \leqq \text{const.}\,\lambda^{-\alpha/2-1} + \text{const.}\,\lambda^{-1}E\sup_{0\leq t\leq T}|X_t|^\alpha 1_{(\alpha>0)}.$$

Here

(1.11)
$$E\sup_{0\leq t\leq T}|X_t|^\alpha < +\infty$$

holds for $\alpha>0$. Indeed, if $\alpha>1$, we have by (1.3) and the martingale inequality that

$$\begin{aligned} E\sup_{0\leq t\leq T}|X_t|^\alpha &= E\sup_{0\leq t\leq T}\left|\int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds\right|^\alpha \\ &\leqq \text{const.}\,E\sup_{0\leq t\leq T}\left|\int_0^t \sigma(X_s)dB_s\right|^\alpha + \text{const.} \\ &\leqq \text{const.}\,E\left|\int_0^T \sigma(X_s)dB_s\right|^\alpha + \text{const.} \\ &\leqq \text{const.}\,E|X_T|^\alpha + \text{const.}, \end{aligned}$$

and the finiteness of $E|X_T|^\alpha$ follows from (1.4). It is easy to see that (1.11) is also valid for $0<\alpha\leqq 1$ since

$$E\sup_{0\le t\le T}|X_t|^\alpha \leqq \text{const.}(E\sup_{0\le t\le T}|X_t|^2)^{\alpha/2}$$

by Hölder's inequality. Thus (1.7) is proved.

We now show (1.8). Fixing $\lambda>0$ and setting $\mathcal{F}_t=\sigma\{(X_s,\ \Theta_{\lambda s});\ s\leqq t\}$, we can prove that $M_t^\lambda$ becomes an $(\mathcal{F}_t)$-martingale by a repeated use of Fubini's theorem. (Note by Lemma 1.1 and (1.9) that

$$E\left[\int_0^\infty |E_{(X_t,\Theta_{\lambda t})}(h(X_u)\varphi_n(\Theta_{\lambda u}))|\,du\right]$$

$$=E\left[\int_0^\infty |E_{X_t}(h(X_u))|\,|E_{\Theta_{\lambda t}}(\varphi_n(\Theta_{\lambda u}))|\,du\right]<+\infty\,.)$$

Then we have

$$E\sup_{0\le t\le T}|M_t^\lambda| \leqq (E\sup_{0\le t\le T}|M_t^\lambda|^2)^{1/2} \leqq \text{const.}(E|M_T^\lambda|^2)^{1/2}$$

$$\leqq\left(\text{const.}E|u_\lambda(X_T,\ \Theta_{\lambda T})|^2+\text{const.}E\left|\int_0^T h(X_s)\varphi_n(\Theta_{\lambda s})ds\right|^2\right)^{1/2}$$

by the martingale inequality. Hence it is only necessary to show that

$$I_1=E|u_\lambda(X_T,\ \Theta_{\lambda T})|^2 \longrightarrow 0 \qquad \text{as} \quad \lambda \longrightarrow \infty$$

and

$$I_2=E\left(\int_0^T h(X_s)\varphi_n(\Theta_{\lambda s})ds\right)^2 \longrightarrow 0 \qquad \text{as} \quad \lambda \longrightarrow \infty\,.$$

We can easily see that $I_1\to 0$ as $\lambda\to\infty$. Indeed, (1.10) implies that

$$I_1\leqq \text{const.}\lambda^{-\alpha-2}+\text{const.}\lambda^{-\alpha/2-2}E|X_T|^\alpha 1_{(\alpha>0)}+\text{const.}\lambda^{-2}E|X_T|^{2\alpha}1_{(\alpha>0)}$$

and the finiteness of $E|X_T|^\alpha 1_{(\alpha>0)}$ and $E|X_T|^{2\alpha}1_{(\alpha>0)}$ follows from (1.4).

Finally we shall prove that $I_2\to 0$ as $\lambda\to\infty$. By Lemma 1.1, (1.9) and Fubini's theorem, we have

$$I_2=2E\left[\int_0^T ds\int_0^s du\ h(X_s)h(X_u)\varphi_n(\Theta_{\lambda s})\varphi_n(\Theta_{\lambda u})\right]$$

$$=2\int_0^T ds\int_0^s du\ E\left[h(X_u)E_{X_u}(h(X_{s-u}))\right]E\left[\varphi_n(\Theta_{\lambda u}(E_{\Theta_{\lambda u}}(\varphi_n(\Theta_{\lambda(s-u)}))\right].$$

Since Lemma 1.1 implies that

$$|E[h(X_u)E_{X_u}(h(X_{s-u}))]|\leqq E[|h(X_u)|E_{X_u}(|h(X_{s-u})|)]$$

$$\leqq \text{const.}(s-u)^{\alpha/2}u^{\alpha/2}+u^\alpha 1_{(\alpha>0)}$$

and (1.9) implies that

$$|E[\varphi_n(\Theta_{\lambda u})E_{\Theta_{\lambda u}}(\varphi_n(\Theta_{\lambda(s-u)}))]|\leqq \text{const.}\,e^{-\lambda_n\lambda(s-u)},$$

$$I_2\leqq \text{const.}\int_0^T ds\int_0^s du\ e^{-\lambda_n\lambda(s-u)}(s-u)^{\alpha/2}u^{\alpha/2}$$

$$+\text{const.}\int_0^T ds\int_0^s du\ e^{-\lambda_n\lambda(s-u)}u^\alpha 1_{(\alpha>0)}$$

$$\leqq \text{const.}\, \lambda^{-\alpha/2-1} + \text{const.}\, \lambda^{-1} 1_{(\alpha>0)}$$

$$\longrightarrow 0 \qquad (\lambda \longrightarrow \infty).$$

Thus the proof of (1.6) is complete.

Next we will show Theorem 1.1 for general $f$ satisfying the conditions (II) and (III). Let $\mathcal{L}$ be the set of all linear combinations of finite number of $\varphi_1, \varphi_2, \cdots$. We know by (1.6) that Theorem 1.1 holds for $f \in \mathcal{L}$. Furthermore, by Lemma 1.1 and Lemma 1.2 we have that

$$E \sup_{0 \leqq t \leqq T} \left| \int_0^t h(X_s) f(\Theta_{\lambda s}) ds \right| \leqq \int_0^T (E|h(X_s)|)(E|f(\Theta_{\lambda s})|) ds$$

$$\leqq \left( \text{const.} \int_0^T s^{\alpha/2} (\lambda s)^{-m/2p} ds + \text{const.} \int_0^T s^{\alpha/2} ds \right) \|f\|_p$$

$$\leqq (\text{const.}\, \lambda^{-m/2p} + \text{const.}) \|f\|_p .$$

Therefore,

$$E \sup_{0 \leqq t \leqq T} \left| \int_0^t h(X_s) f(\Theta_{\lambda s}) ds \right| \leqq (o(1) + \text{const.}) \|f\|_p \qquad (\lambda \longrightarrow \infty).$$

To complete the proof we have only to note the following facts: Since $M$ is compact, any continuous function $f$ on $M$ satisfying the null charged condition (III) is uniformly approximated by functions of $\mathcal{L}$ (*cf.* Chavel [1], p. 139-140), and continuous functions are dense in $L^p(M)$. Q.E.D.

## § 2. Some limit theorem for additive functionals of a Brownian motion on the cylinder

In this section, we will prove some limit theorem (Theorem 2.1) for additive functionals of a Brownian motion on the cylinder $\boldsymbol{R} \times \boldsymbol{T}$, $\boldsymbol{T} = \boldsymbol{R}/2\pi \boldsymbol{Z} \cong [0, 2\pi]$, as an application of Theorem 1.1 in the previous section.

First of all we prepare some notations for conformal martingales. Let $z(t) = x(t) + \sqrt{-1} y(t)$ be a conformal martingale *i.e.* $\langle x \rangle(t) = \langle y \rangle(t)$ and $\langle x, y \rangle(t) = 0$. We denote these common processes $\langle x \rangle(t)$ and $\langle y \rangle(t)$ by $\langle z \rangle(t)$. Throughout this paper we always denote by $\langle z \rangle^{-1}(t)$ the process obtained by the right continuous inverse function of $t \to \langle z \rangle(t)$. If $\langle z \rangle(t) \to \infty (t \to \infty)$ a.s., then the time changed process $z(\langle z \rangle^{-1}(t))$ becomes a complex Brownian motion by the Knight theorem. We always denote this Brownian motion by $\hat{z}(t)$.

If $z_1(t) = x_1(t) + \sqrt{-1} y_1(t)$ and $z_2(t) = x_2(t) + \sqrt{-1} y_2(t)$ are conformal martingales, then we denote by $\langle z_1, z_2 \rangle(t)$ the matrix of quadratic variation processes $\begin{pmatrix} \langle x_1, x_2 \rangle(t) & \langle x_1, y_2 \rangle(t) \\ \langle y_1, x_2 \rangle(t) & \langle y_1, y_2 \rangle(t) \end{pmatrix}$. Note that

$$\langle z, z \rangle(t) = \langle z \rangle(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and}$$

$$\left\langle \int_0^{\cdot} \Phi_1(s) dz_s, \int_0^{\cdot} \Phi_2(s) dz_s \right\rangle(t) = \int_0^t \mathcal{R}e(\Phi_1 \Phi_2^*)(s) d\langle z \rangle_s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$+ \int_0^t \mathcal{I}m(\Phi_1 \Phi_2^*)(s) d\langle z \rangle_s \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(Here $\Phi^*$ represents the complex conjugate of $\Phi$.)

Let $(S, \mathscr{B}(S), \mu)$ be a measure space and set $\mathscr{F}=\{A\in\mathscr{B}(S);\ \mu(A)<+\infty\}$. A family of random variables $M=\{M(A);\ A\in\mathscr{F}\}$ is called a (real) Gaussian random measure on $S$ with mean 0 and variance measure $\mu$ if and only if $M$ is a Gaussian system such that $E[M(A)]=0$ and $E[M(A)M(B)]=\mu(A\cap B)$ hold for any $A, B\in\mathscr{F}$. Furthermore, a complex Gaussian random measure $M$ on $S$ with mean 0 and variance measure $\mu$ is by definition a family of complex random variables $M(A)$ which can be expressed in the form $M(A)=M_1(A)+\sqrt{-1}M_2(A)$ where $M_1$ and $M_2$ are mutually independent Gaussian random measures with mean 0 and the same variance measure $\mu$.

Throughout this section, we always denote $L^2(\boldsymbol{T}\to\boldsymbol{C}, d\theta/2\pi)$ by $L^2(0, 2\pi)$. Let us introduce a definition of regularly varying functions of a complex variable:

**Definition 2.1.** A function $f(z)$ defined on $0<|z-a|<R$ is called *regularly varying at* $a(\neq 0)$ *with order* $\rho(>-1/2)$ if there exist some slowly varying (at $\infty$) function $L(\lambda)$, $c(\theta)\in L^2(0, 2\pi)$ and $r>(\log|a/R|)\vee 0$, which have the following two properties:

1°) There exist some constants $\varepsilon\geqq 0$, $K>0$, and $\lambda_0>0$ such that $\varepsilon<\rho+1/2$ and

$$\int_0^{2\pi}d\theta\int_{-\infty}^{-r/\lambda}|(\lambda^{\rho}L(\lambda))^{-1}f(a-ae^{\lambda x+\sqrt{-1}\theta})|^2e^{-x^2/s}dx$$
$$\leqq K\cdot(s^{\rho-\varepsilon+1/2}1_{(0<s<1)}+s^{\rho+\varepsilon+1/2}1_{(s\geqq 1)})$$

for all $\lambda\geqq\lambda_0$ and $s>0$.

2°) For any $s>0$,

$$\int_0^{2\pi}d\theta\int_{-\infty}^{-r/\lambda}|(\lambda^{\rho}L(\lambda))^{-1}f(a-ae^{\lambda x+\sqrt{-1}\theta})-c(\theta)(-x)^{\rho}|^2e^{-x^2/s}dx$$
$$\longrightarrow 0 \quad \text{as} \quad \lambda\longrightarrow\infty\,.$$

For $a=0$, we substitute the condition $r>(\log|a/R|)\vee 0$ with the condition $r>(-\log R)\vee 0$ and $a-a\,e^{\lambda x+\sqrt{-1}\theta}$ with $e^{\lambda x+\sqrt{-1}\theta}$ in the above definition.

We call $N(\lambda)=\lambda^{\rho}L(\lambda)$ and $c(\theta)$ *the regular normalizing function of $f$ at $a$* and *the asymptotic angular component of $f$ at $a$*, respectively.

Furthermore, we call a function $f(z)$ defined on $|z|>R$ *regularly varying at* $\infty$ *with order* $\rho$ if $\tilde{f}(z)=f(1/z)$ is regularly varying at 0 with order $\rho$. The regular normalizing function of $f$ at $\infty$ and the asymptotic angular component of $f$ at $\infty$ are those of $\tilde{f}$ at 0, respectively.

**Remark 2.1.** The class of functions regularly varying both at $a$ and at $\infty$ defined above contains the original class of functions regularly varying at $a$ defined by Watanabe ([11]).

**Example 1.** For any given domain $D\subset\boldsymbol{C}$ such that $D$ or $D^c$ is bounded, the function $f(z)=1_D(z)$ is regularly varying at $a$ with order 0 for any $a\in\boldsymbol{C}\cup\{\infty\}\backslash\partial D$. The regular normalizing function of $f$ at $a$ is 1 and the asymptotic angular component of $f$ at $a$ is 1 if $a\in D$ and 0 if $a\notin D$. (Here we consider that $\infty\in D$ when $D^c$ is bounded.)

**Example 2.** Let $g(\theta)\in L^2(0, 2\pi)$ and let $h(x)$ be an ordinary regularly varying function at $\infty$ with exponent $\rho(<\infty)$ such that

$$\left|\frac{h(\lambda x)}{h(\lambda)}\right|\leqq K\cdot(x^{\rho-\varepsilon}1_{(|x|<1)}+x^{\rho+\varepsilon}1_{(|x|\geqq 1)})$$

for all $\lambda$, where $K>0$ and $\varepsilon\geqq 0$ are some constants satisfying $\varepsilon<\rho+1/2$. Then

$$f(z)=g\left(\arg\frac{z-a}{-a}\right)\cdot h\left(-\log\left|\frac{z-a}{-a}\right|\right)$$

is regularly varying at $a$ with order $\rho$. The regular normalizing function of $f$ at $a$ is $h(\lambda)$ and the asymptiotic angular component of $f$ at $a$ is $g(\theta)$.

When $f(z)$ is regularly varying at $\infty$, the asymptotic behavoir of $f(a-a\, e^{\lambda x+\sqrt{-1}\theta})1_{(x>0)}$ as $\lambda\to\infty$ for every $a\neq 0$ can be described using that of $f(e^{\lambda x+\sqrt{-1}\theta})1_{(x>0)}$:

**Proposition 2.1.** *Snppose that a function $f(z)$ defined on $|z|>R$ be regularly varying at $\infty$ with order $\rho$. Then for any $a\in \boldsymbol{C}\backslash\{0\}$, there exists $r'>\log(1+R/|a|)$ such that the following two properties hold:*

1°) *There exist some constants $\varepsilon\geqq 0$, $K>0$ and $\lambda_0>0$ such that $\varepsilon<\rho+1/2$ and*

$$\text{(2.1)}\qquad I_1:=\int_0^{2\pi}d\theta\int_{r'/\lambda}^{+\infty}\left|\frac{1}{N(\lambda)}f(a-a\, e^{\lambda x+\sqrt{-1}\theta})\right|^2 e^{-x^2/s}dx \leqq K\cdot(s^{\rho-\varepsilon+1/2}1_{(0<s<1)}+s^{\rho+\varepsilon+1/2}1_{(s\geqq 1)})$$

*for all $\lambda\geqq\lambda_0$ and $s>0$.*

2°) *For any $s>0$,*

$$\text{(2.2)}\qquad I_2:=\int_0^{2\pi}d\theta\int_{r'/\lambda}^{+\infty}\left|\frac{1}{N(\lambda)}f(a-a\, e^{\lambda x+\sqrt{-1}\theta})-c(-\theta-\arg(-a))\cdot x^{\rho}\right|^2 e^{-x^2/s}dx \longrightarrow 0 \quad \text{as } \lambda\longrightarrow\infty,$$

*where $N(\lambda)$ and $c(\theta)$ are the regular normalizing function of $f$ at $\infty$ and the asymptotic angular component of $f$ at $\infty$, respectively.*

*Proof.* By the assumptions, there exists some $r>(\log R)\vee 0$ which satisfies the following two properties:

1°) There exist some constants $\varepsilon\geqq 0$, $K>0$ and $\lambda_0>0$ such that $\varepsilon<\rho+1/2$ and

$$\text{(2.3)}\qquad \int_0^{2\pi}d\theta\int_{r/\lambda}^{+\infty}\left|\frac{1}{N(\lambda)}f(e^{\lambda x+\sqrt{-1}\theta})\right|^2 e^{-x^2/s}dx\leqq K\cdot(s^{\rho-\varepsilon+1/2}1_{(0<s<1)}+s^{\rho+\varepsilon+1/2}1_{(s\geqq 1)})$$

for all $\lambda\geqq\lambda_0$ and $s>0$.

2°) For any $s>0$,

$$\text{(2.4)}\qquad \int_0^{2\pi}d\theta\int_{r/\lambda}^{+\infty}\left|\frac{1}{N(\lambda)}f(e^{\lambda x+\sqrt{-1}\theta})-c(-\theta)\cdot x^{\rho}\right|^2 e^{-x^2/s}dx \longrightarrow 0 \quad \text{as } \lambda\longrightarrow\infty.$$

Now denoting $\max(r, \log(|a|^2+2|a|), \log(1+1/|a|))$ by $r$ again, we see that (2.3) and (2.4) clearly hold for this new $r$. Therefore we may assume that $r\geqq\log(|a|^2+$

$2|a|)$ and $r \geqq \log(1+1/|a|)$. Set $r'=\log(1+e^r/|a|)$. We have that $r'>\log(1+R/|a|)$ since $r>\log R$.

In order to change the variables of the integrals $I_1$ and $I_2$ above, we set

$$a-a\ e^{\lambda x'+\sqrt{-1}\theta'}=e^{\lambda x+\sqrt{-1}\theta}=z.$$

Then

$$x'=x-\frac{1}{\lambda}\log|aJ^\lambda|,$$

$$\theta'=\theta-\arg(-aJ^\lambda)$$

and

$$dx'\wedge d\theta'=(-2\sqrt{-1}\lambda|z-a|^2)^{-1}\cdot dz\wedge d\bar{z}=|J^\lambda|^2 dx\wedge d\theta,$$

where

$$J^\lambda(x,\ \theta)=\frac{z}{z-a}=(1-ae^{-\lambda x-\sqrt{-1}\theta})^{-1}.$$

Hence

$$I_1=\int_0^{2\pi}d\theta\int_R 1_{(\lambda x-\log|aJ^\lambda|>r')}\cdot|N(\lambda)^{-1}f(e^{\lambda x+\sqrt{-1}\theta})|^2$$
$$\times\exp\Big(-(x-\frac{1}{\lambda}\log|aJ^\lambda|)^2/s\Big)|J^\lambda|^2 dx.$$

Noting that $|J^\lambda|\geqq(1+|a|e^{-\lambda x})^{-1}$ and $r'=\log(1+e^r/|a|)$, we see that if $\lambda x-\log|aJ^\lambda|>r'$, then $\lambda x>r$. So we have

$$I_1\leqq\int_0^{2\pi}d\theta\int_{r/\lambda}^{+\infty}\left|\frac{1}{N(\lambda)}f(e^{\lambda x+\sqrt{-1}\theta})\right|\times\exp\Big(-(x-\frac{1}{\lambda}\log|aJ^\lambda|)^2/s\Big)|J^\lambda|^2 dx.$$

Moreover by the inequality $r\geqq\log(|a|^2+2|a|)$ it holds that

$$|J^\lambda|\leqq(1-|a|e^{-\lambda x})^{-1}<(1-|a|e^{-r})^{-1}<|a|^{-1}e^{r/2}.$$

This implies that

$$\frac{1}{\lambda}\log|aJ^\lambda|<\frac{x}{2} \tag{2.5}$$

for $x>r/\lambda$. Therefore,

$$I_1\leqq|a|^{-2}e^r\int_0^{2\pi}d\theta\int_{r/\lambda}^{+\infty}\left|\frac{1}{N(\lambda)}f(e^{\lambda x+\sqrt{-1}\theta})\right|^2 e^{-x^2/4s}dx$$

which proves (2.1) together with (2.3).

Similarly,

$$I_2\leqq|a|^{-2}e^r\int_0^{2\pi}d\theta\int_{r/\lambda}^{+\infty}\left|\frac{1}{N(\lambda)}f(e^{\lambda x+\sqrt{-1}\theta})\right. \tag{2.6}$$
$$\left.-c(-\theta+\arg J^\lambda)\cdot\Big(x-\frac{1}{\lambda}\log|aJ^\lambda|\Big)^\rho\right|^2 e^{-x^2/4s}dx.$$

On the other hand, by $r\geqq\log(1+1/|a|)$ it holds that

$$|J^{\lambda}| \geqq (1+|a|e^{-\lambda x})^{-1} > (1+|a|e^{-r})^{-1} > |a|^{-1}e^{-r}.$$

This implies that

$$\frac{1}{\lambda}\log|aJ^{\lambda}| > -x$$

for $x > r/\lambda$. Noting this and (2.5) we have the estimate

$$\left|x^{\rho}-\left(x-\frac{1}{\lambda}\log|aJ^{\lambda}|\right)^{\rho}\right|1_{(x>t/\lambda)} \leqq \text{const. } x^{\rho}1_{(x>0)}$$

for any $\rho > -1/2$. Hence we can easily prove that

$$(2.7) \qquad \int_0^{2\pi} d\theta \int_{r/\lambda}^{+\infty} \left|c(-\theta)x^{\rho}-c(-\theta)\left(x-\frac{1}{\lambda}\log|aJ^{\lambda}|\right)^{\rho}\right|^2 e^{-x^2/4s}dx$$
$$\longrightarrow 0 \quad \text{as} \quad \lambda \longrightarrow \infty$$

by Lebegue's convergence theorem and the fact that

$$J^{\lambda}(x, \theta) \longrightarrow 1 \qquad \text{as} \quad \lambda \longrightarrow \infty$$

uniformly in $\theta$ for any $x > 0$.

Since $\arg J^{\lambda} \to 0$ as $\lambda \to \infty$ uniformly in $\theta$ for any $x > 0$, we can also prove that

$$(2.8) \qquad \int_0^{2\pi} d\theta \int_{r/\lambda}^{+\infty} \left(x-\frac{1}{\lambda}\log|aJ^{\lambda}|\right)^{\rho}$$
$$\times |c(-\theta)-c(-\theta+\arg J^{\lambda})|^2 e^{-x^2/4s}dx \longrightarrow 0 \qquad \text{as} \quad \lambda \longrightarrow \infty$$

by Lebesgue's convergence theorem and the fact that

$$\int_0^{2\pi} |c(-\theta)-c(-\theta+\arg J^{\lambda})|^2 d\theta \longrightarrow 0 \qquad \text{as} \quad \lambda \longrightarrow \infty$$

for fixed $x > 0$.

Combining (2.6), (2.7), (2.8) and (2.4), we obtain (2.2). Q.E.D.

Let $(x_t, \theta_t)$ be a Brownian motion on the cylinder $\boldsymbol{R} \times \boldsymbol{T}$ satisfying $x_0 = 0$ and $\theta_0 = 0$ a.s. Clearly

$$z_t = x_t + \sqrt{-1}\int_0^t d\theta_s$$

becomes a complex Brownian motion. Our main theorem in this section is as follows:

**Theorem 2.1.** (1) *Suppose that the functions $f_1, \cdots, f_m$ defined on $0 < |z-a| < R$ are regularly varying at a with order $\rho_1, \cdots, \rho_m$, respectively. Denote the regular normalizing function of $f_i$ at a and the asymptotic angular component of $f_i$ at a by $N_i(\lambda)$ and $c_i(\theta)$, respectively for $i = 1, \cdots, m$. Then there exists some $r > (\log|a/R|) \vee 0$ and we have*

$$\left\{N_i(\lambda)^{-1}\int_0^{\cdot} f_i(a-ae^{\lambda z_s})1_{(\lambda x_s < -r)}dz_s,\right.$$

$$N_i(\lambda)^{-2}\int_0^{\cdot}|f_i(a-ae^{\lambda z_s})|^2 1_{(\lambda x_s<-r)}ds\Big\}_{1\leq i\leq m}$$

$$\longrightarrow\Big\{\overline{c_i}\int_0^{\cdot}(-x_s)^{\rho_i}1_{(x_s<0)}dz_s$$

$$+\int_0^{\cdot}\int_0^{2\pi}(c_i(\theta)-\overline{c_i})(-x_s)^{\rho_i}M(1_{(x_s<0)}ds,\ d\theta),$$

$$|\overline{c_i}|^2\int_0^{\cdot}(-x_s)^{2\rho_i}1_{(x_s<0)}ds\Big\}_{1\leq j\leq m}$$

*as* $\lambda\to\infty$ *in law, where* $\bar{c}=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}c(\theta)d\theta$ *and* $M$ *is a complex Gaussian random measure on* $[0,\infty)\times[0,2\pi]$ *with mean* 0 *and variance measure* $dt\cdot(d\theta/2\pi)$ *which is independent of* $z(t)$.

(2) *Suppose that the functions* $f_1,\cdots,f_m$ *defined on* $|z|>R$ *are regularly varying at* $\infty$ *with order* $\rho_1,\cdots,\rho_m$, *respectively. Denote the regular normalizing function of* $f_i$ *at* $\infty$ *and the asymptotic angular component of* $f_i$ *at* $\infty$ *by* $N_i(\lambda)$ *and* $c_i(\theta)$, *respectively for* $i=1,\cdots,m$. *Then, for every* $a\in C\setminus\{0\}$, *there exists some* $r>(\log(1+R/|a|))\vee 0$ *and we have*

$$\Big\{N_i(\lambda)^{-1}\int_0^{\cdot}f_i(a-ae^{\lambda z_s})1_{(\lambda x_s>r)}dz_s,$$

$$N_i(\lambda)^{-2}\int_0^{\cdot}|f_i(a-ae^{\lambda z_s})|^2 1_{(\lambda x_s>r)}ds\Big\}_{1\leq i\leq m}$$

$$\longrightarrow\Big\{\overline{c_i}\int_0^{\cdot}(x_s)^{\rho_i}1_{(x_s>0)}dz_s$$

$$+\int_0^{\cdot}\int_0^{2\pi}(c_i(\theta)-\overline{c_i})(x_s)^{\rho_i}M(1_{(x_s>0)}ds,\ d\theta),$$

$$|\overline{c_i}|^2\int_0^{\cdot}(x_s)^{2\rho_i}1_{(x_s>0)}ds\Big\}_{1\leq j\leq m}$$

*as* $\lambda\to\infty$ *in law, where* $\bar{c}=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}c(\theta)d\theta$ *and* $M$ *is a complex Gaussian random measure on* $[0,\infty)\times[0,2\pi]$ *with mean* 0 *and variance measure* $dt\cdot(d\theta/2\pi)$ *which is independent of* $z(t)$.

*Proof*. We will prove (1) only, because by Proposition 2.1, the proof of (2) proceeds similarly. (Note that

$$\Big\{\int_0^{\cdot}\int_0^{2\pi}(c_i(-\theta-\arg(-a))-\overline{c_i})(x_s)^{\rho_i}M(1_{(x_s>0)}ds,\ d\theta)\Big\}_{1\leq i\leq m}$$

is equivalent in law to

$$\Big\{\int_0^{\cdot}\int_0^{2\pi}(c_i(\theta)-\overline{c_i})(x_s)^{\rho_i}M(1_{(x_s>0)}ds,\ d\theta)\Big\}_{1\leq i\leq m}).$$

Let $\{e_0\equiv1,e_1,\cdots,e_p\}$ be some orthonormal system in $L^2(0,2\pi)$ such that

$$c_i(\theta)=\sum_{k=0}^{p}\alpha_i^{(k)}e_k(\theta), \qquad \alpha_i^{(k)}\in \boldsymbol{C} \qquad (k=0, \cdots, p)$$

for $i=1, \cdots, m$. Define

$$V_k^{\lambda}(t)=\int_0^t e_k(\lambda\theta_s)1_{(\lambda x_s<-r)}dz_s \qquad (k=0, \cdots, p) \tag{2.9}$$

for some $r>(\log|a/R|)\vee 0$. Then it holds that

$$\begin{aligned} I_\lambda=E\sup_{0\leq t\leq T}\Big|N_i(\lambda)^{-1}\int_0^t f_i(a-ae^{\lambda z_s})1_{(\lambda x_s<-r)}dz_s \\ -\sum_{k=0}^{p}\alpha_i^{(k)}\int_0^t(-x_s)^{\rho_i}dV_k^{\lambda}(s)\Big|^2 \longrightarrow 0 \qquad \text{as} \quad \lambda\longrightarrow\infty. \end{aligned} \tag{2.10}$$

The proof of (2.10) is as follows. Let $q(t, \theta, \eta)$ be the transition density of $\theta(t)$. Then

$$\begin{aligned} I_\lambda &= E\sup_{0\leq t\leq T}\Big|\int_0^t(N_i(\lambda)^{-1}f_i(a-ae^{\lambda z_s})-c_i(\lambda\theta_s)(-x_s)^{\rho_i})1_{(\lambda x_s<-r)}dz_s\Big|^2 \\ &\leqq \text{const.}\,E\int_0^T|N_i(\lambda)^{-1}f_i(a-ae^{\lambda z_s})-c_i(\lambda\theta_s)(-x_s)^{\rho_i}|^2 1_{(\lambda x_s<-r)}ds \\ &= \text{const.}\,E\int_0^T|N_i(\lambda)^{-1}f_i(a-ae^{\lambda x_s+\sqrt{-1}\theta(\lambda^2 s)})-c_i(\theta(\lambda^2 s))(-x_s)^{\rho_i}|^2 1_{(\lambda x_s<-r)}ds \\ &= \text{const.}\int_0^T ds\int_0^{2\pi}d\theta q(\lambda^2 s, 0, \theta)\int_{-r/\lambda}^{\infty}|N_i(\lambda)^{-1}f_i(a-ae^{\lambda x+\sqrt{-1}\theta}) \\ &\qquad -c_i(\theta)(-x)^{\rho_i}|^2\frac{1}{\sqrt{2\pi s}}e^{-x^2/2s}dx. \end{aligned}$$

Hence noting the inequality

$$q(s, \theta, \eta)\leqq \text{const.}\,s^{-1/2}+\text{const.}$$

which we have seen in the proof of Lemma 1.2, we have

$$\begin{aligned} I_\lambda &\leqq \text{const.}\int_0^T ds\,(\text{const.}\,\lambda^{-1}s^{-1}+\text{const.}\,s^{-1/2}) \\ &\times\int_0^{2\pi}d\theta\int_{-r/\lambda}^{\infty}|N_i(\lambda)^{-1}f_i(a-ae^{\lambda x-\sqrt{-1}\theta})-c_i(\theta)(-x)^{\rho_i}|^2 e^{-x^2/2s}dx. \end{aligned}$$

This last expression clearly tends to 0 as $\lambda\to 0$ for some $r>(\log|a/R|)\vee 0$ by the definition of regularly varying functions at $a$ and Lebesgue's convergence theorem.

Similarly we have

$$\begin{aligned} J_\lambda=E\sup_{0\leq t\leq T}\Big|N_i(\lambda)^{-2}\int_0^t|f_i(a-ae^{\lambda z_s})|^2 1_{(\lambda x_s<-r)}ds \\ -\overline{|c_i|^2}\int_0^t(-x_s)^{2\rho_i}d\langle V_0^{\lambda}\rangle_s\Big| \longrightarrow 0 \qquad \text{as} \quad \lambda\longrightarrow\infty. \end{aligned} \tag{2.11}$$

Actually,

$$J_\lambda = E\sup_{0\le t\le T}\left|\int_0^t (N_i(\lambda)^{-2}|f_i(a-ae^{\lambda z_s})|^2 - \overline{|c_i|^2}(-x_s)^{2\rho_i})1_{(\lambda x_s<-r)}ds\right|$$

$$= E\int_0^T |N_i(\lambda)^{-2}|f_i(a-ae^{\lambda z_s})|^2 - |c_i(\lambda\theta_s)|^2(-x_s)^{2\rho_i}|1_{(\lambda x_s<-r)}ds$$

$$+E\sup_{0\le t\le T}\left|\int_0^t (|c_i(\lambda\theta_s)|^2 - \overline{|c_i|^2})(-x_s)^{2\rho_i}1_{(\lambda x_s<-r)}ds\right|$$

$$:= J_\lambda^{(1)} + J_\lambda^{(2)}.$$

By Theorem 1.1, we have that $J_\lambda^{(2)}\to 0$ as $\lambda\to\infty$. As for $J_\lambda^{(1)}$,

$$J_\lambda^{(1)} \le \left(E\int_0^T |N_i(\lambda)^{-1}|f_i(a-ae^{\lambda z_s})| + |c_i(\lambda\theta_s)|(-x_s)^{\rho_i}|^2 1_{(\lambda x_s<-r)}ds\right)^{1/2}$$

$$\times\left(E\int_0^T |N_i(\lambda)^{-1}|f_i(a-ae^{\lambda z_s})| - |c_i(\lambda\theta_s)|(-x_s)^{\rho_i}|^2 1_{(\lambda x_s<-r)}ds\right)^{1/2}$$

by Schwartz' inequality. The first expectation in the last form is bounded by a constant by the definition of the regularly varying functions. The second expectation in the last form is bounded by the expectation

$$E\int_0^T |N_i(\lambda)^{-1}f_i(a-ae^{\lambda z_s}) - c_i(\lambda\theta_s)(-x_s)^{\rho_i}|^2 1_{(\lambda x_s<-r)}ds$$

which tends to 0 as $\lambda\to\infty$ as we have seen above in the proof of (2.10).

Therefore if we can prove that the joint processes

$$\left\{\int_0^\cdot (-x_s)^{\rho_i}dV_0^\lambda(s),\ \int_0^\cdot (-x_s)^{\rho_i}dV_k^\lambda(s),\right.$$

$$\left.\int_0^\cdot (-x_s)^{2\rho_i}d\langle V_0^\lambda\rangle_s,\ \int_0^\cdot (-x_s)^{2\rho_i}d\langle V_k^\lambda\rangle_s\right\}_{1\le i\le m}^{1\le k\le p}$$

converge to

$$\left\{\int_0^\cdot (-x_s)^{\rho_i}1_{(x_s<0)}dz_s,\ \int_0^\cdot\int_0^{2\pi} e_k(\theta)(-x_s)^{\rho_i}M(1_{(x_s<0)}ds,\ d\theta),\right.$$

$$\left.\int_0^\cdot (-x_s)^{2\rho_i}1_{(x_s<0)}ds,\ \int_0^\cdot (-x_s)^{2\rho_i}1_{(x_s<0)}ds\right\}_{1\le i\le m}^{1\le k\le p}$$

as $\lambda\to\infty$ in law, then we can finish the proof of our theorem. This follows at once from Lemma 2.3 and Lemma 2.4 below. Q.E.D.

Before stating these lemmas, we introduce the following two general lemmas which have been obtained in Watanabe [11].

**Lemma W1.** *Let $M_\lambda$ be a continuous conformal martingale for any $\lambda(1\le\lambda\le\infty)$ satisfying the following properties:*

(2.12) $$E(\langle M_\lambda\rangle(t))^2 \le K_1(t) \quad \text{for any } t>0 \text{ and } 1\le\lambda<\infty,$$

(2.13) $$E\left(\int_0^t |\Phi_\lambda(s)|^2 d\langle M_\lambda\rangle(s)\right)^2 \le K_2(t) \quad \text{for any } t>0 \text{ and } 1\le\lambda<\infty,$$

$$(2.14)\qquad \int_0^t |\Phi_\lambda(s)|^2 d\langle M_\lambda\rangle(s) \longrightarrow \infty \qquad \text{as } t\longrightarrow\infty \text{ a.s. for any } \lambda(1\leqq\lambda\leqq\infty),$$

*where $K_1(t)$ and $K_2(t)$ are some positive functions independent of $\lambda$, and $\Phi_\lambda(t)$ $(1\leqq\lambda\leqq\infty)$ are some $(\mathcal{F}_t^{M_\lambda})$-predictable real or complex valued processes.*

*If*

$$\left\{M_\lambda, \langle M_\lambda\rangle, \left\langle\int_0^\cdot \Phi_\lambda(s)dM_\lambda(s), M_\lambda\right\rangle, \int_0^\cdot |\Phi_\lambda(s)|^2 d\langle M_\lambda\rangle(s)\right\}$$

$$\longrightarrow \left\{M_\infty, \langle M_\infty\rangle, \left\langle\int_0^\cdot \Phi_\infty(s)dM_\infty(s), M_\infty\right\rangle, \int_0^\cdot |\Phi_\infty(s)|^2 d\langle M_\infty\rangle(s)\right\}$$

*as $\lambda\to\infty$ in law on $C([0,\infty)\to \boldsymbol{C}\times\boldsymbol{R}\times\boldsymbol{R}^4\times\boldsymbol{R})$, then*

$$\left\{M_\lambda, \int_0^\cdot \Phi_\lambda(s)dM_\lambda(s), \int_0^\cdot |\Phi_\lambda(s)|^2 d\langle M_\lambda\rangle(s)\right\}$$

$$\longrightarrow \left\{M_\infty, \int_0^\cdot \Phi_\infty(s)dM_\infty(s), \int_0^\cdot |\Phi_\infty(s)|^2 d\langle M_\infty\rangle(s)\right\}$$

*as $\lambda\to\infty$ in law on $C([0,\infty)\to\boldsymbol{C}^2\times\boldsymbol{R})$.*

*Proof.* We will prove the lemma assuming that $M_\lambda$ and $\Phi_\lambda$ are real valued, because the proof of the general case follows at once from this case. Set

$$N_\lambda(t)=\int_0^t \Phi_\lambda(s)dM_\lambda(s) \qquad (1\leqq\lambda\leqq\infty).$$

By the condition (2.14) and the Knight theorem, we see that $\hat{N}_\lambda$ $(1\leqq\lambda\leqq\infty)$ becomes a Brownian motion. Thus the laws induced by $N_\lambda=\hat{N}_\lambda(\langle N_\lambda\rangle)$ form a tight family, which implies that the family of laws induced by

$$\{M_\lambda, N_\lambda, \langle M_\lambda\rangle, \langle M_\lambda, N_\lambda\rangle, \langle N_\lambda\rangle\}$$

is tight. Hence we may choose one of the limit points of the above family which we may assume to be the law of

$$\{M_\infty, X, \langle M_\infty\rangle, \langle M_\infty, N_\infty\rangle, \langle N_\infty\rangle\},$$

where

$$N_\infty=\int_0^\cdot \Phi_\infty(s)dM_\infty(s)$$

and $X$ is some continuous process. Then we can conclude that $X=N_\infty$ as follows. We see from the condition (2.12) that both $\{M_\lambda^2(t)\}_{\lambda\geqq1}$ and $\{\langle M_\lambda\rangle(t)\}_{\lambda\geqq1}$ are uniformly integrable for any $t>0$. Similarly we see from the condition (2.13) that both $\{N_\lambda^2(t)\}_{\lambda\geqq1}$ and $\{\langle N_\lambda\rangle(t)\}_{\lambda\geqq1}$ are uniformly integrable for any $t>0$. Therefore $\{M_\lambda(t)N_\lambda(t)\}_{\lambda\geqq1}$ and $\{\langle M_\lambda, N_\lambda\rangle(t)\}_{\lambda\geqq1}$ are also uniformly integrable for any $t>0$. Consequently, we see that $M_\infty$ and $X$ are $(\mathcal{F}^{M_\infty, X})$-martingales and that

$$\langle X\rangle=\langle N_\infty\rangle=\int_0^\cdot |\Phi_\infty(s)|^2 d\langle M_\infty\rangle(s),$$

$$\langle X, M_\infty\rangle=\langle N_\infty, M_\infty\rangle=\int_0^\cdot \Phi_\infty(s)d\langle M_\infty\rangle(s)$$

from the Skorohod theorem realizing a sequence of random variables converging in law by an almost sure convergent sequence. From these we have

$$\begin{aligned}\langle X-N_\infty\rangle &= \langle X\rangle+\langle N_\infty\rangle-2\langle X, N_\infty\rangle \\ &= 2\int_0^{\cdot} |\Phi_\infty(s)|^2 d\langle M_\infty\rangle(s)-2\int_0^{\cdot} \Phi_\infty(s)d\langle X, M_\infty\rangle(s) \\ &= 2\int_0^{\cdot} |\Phi_\infty(s)|^2 d\langle M_\infty\rangle(s)-2\int_0^{\cdot} |\Phi_\infty(s)|^2 d\langle M_\infty\rangle(s) \\ &= 0 \quad \text{a.s.},\end{aligned}$$

which implies that $X=N_\infty$ a.s. Q.E.D.

**Lemma W2.** *Let $M_\lambda$ be a continuous conformal martingale such that* $\lim_{t\uparrow\infty}\langle M_\lambda\rangle(t)=\infty$ *a.s. for every $\lambda(1\leqq\lambda\leqq+\infty)$.*

*If*

$$\{M_\lambda, \langle M_\lambda\rangle\} \longrightarrow \{M_\infty, \langle M_\infty\rangle\} \qquad \text{as} \quad \lambda \longrightarrow \infty$$

*in law on $C([0, \infty)\to \boldsymbol{C}\times\boldsymbol{R})$, then*

$$\{M_\lambda, \langle M_\lambda\rangle, \hat{M}_\lambda\} \longrightarrow \{M_\infty, \langle M_\infty\rangle, \hat{M}_\infty\} \qquad \text{as} \quad \lambda \longrightarrow \infty$$

*in law on $C([0, \infty)\to \boldsymbol{C}\times\boldsymbol{R}\times\boldsymbol{C})$.*

*Proof.* Let $X(t)$ be a process such that

$$\{M_\lambda(t), \langle M_\lambda\rangle(t), \hat{M}_\lambda(t)\} \longrightarrow \{M_\infty(t), \langle M_\infty\rangle(t), X(t)\}$$

as $\lambda\to\infty$ in law and realize this sequence by an almost sure convergent sequence. Since $\hat{M}_\lambda(\langle M_\lambda\rangle(t))=M_\lambda(t)$, we have that $X(\langle M_\infty\rangle(t))=M_\infty(t)$. Hence $X(t)=M_\infty(\langle M_\infty\rangle^{-1}(t))=\hat{M}_\infty(t)$. Q.E.D.

Now we state our lemmas which are essential in our proof.

**Lemma 2.1.** *If $c\in L^1(0, 2\pi)$ and $\rho>-1$, then*

$$\begin{aligned}I_\lambda &= E \sup_{0\le t\le T}\left|\int_0^t c(\lambda\theta_s)(-x_s)^\rho 1_{(\lambda x_s<-r)}ds-\bar{c}\int_0^t(-x_s)^\rho 1_{(x_s<0)}ds\right| \\ &\longrightarrow 0 \qquad \text{as} \quad \lambda\longrightarrow\infty\end{aligned}$$

*for any $r\geqq 0$.*

*Proof.*

$$\begin{aligned}I_\lambda &\leqq \text{const.}\, E\int_0^T |c(\lambda\theta_s)(-x_s)^\rho(1_{(\lambda x_s<-r)}-1_{(x_s<0)})|\, ds \\ &\quad + E\sup_{0\le t\le T}\left|\int_0^t(c(\lambda\theta_s)-\bar{c})(-x_s)^\rho 1_{(x_s<0)}ds\right| \\ &:= I_\lambda^{(1)}+I_\lambda^{(2)}, \text{ say.}\end{aligned}$$

By Lemma 1.1 and Lemma 1.2 we have

$$E|c(\lambda\theta_s)(-x_s)^\rho(1_{(\lambda x_s<-r)}-1_{(x_s<0)})|\leqq E|2c(\lambda\theta_s)(-x_s)^\rho|$$
$$\leqq 2E|x_s|^\rho E|c(\lambda\theta_s)|=2E|x_s|^\rho E|c(\theta(\lambda^2 s))|$$
$$\leqq \text{const.}\|c\|_1 s^{\rho/2}(\lambda^{-1}s^{-1/2}+\text{const.})<+\infty.$$

Then we can see easily that

$$E|c(\lambda\theta_s)(-x_s)^\rho(1_{(\lambda x_s<-r)}-1_{(x_s<0)})|\longrightarrow 0 \quad \text{as} \quad \lambda\longrightarrow\infty$$

for any $s>0$ by Lebesgue's convergence theorem. Since

$$\int_0^T s^{\rho/2}(\lambda^{-1}s^{-1/2}+\text{const.})ds<+\infty,$$

we have that $I_\lambda^{(1)}\to 0$ as $\lambda\to\infty$ using Lebesgue's convergence theorem again.

On the other hand, it follows from Theorem 1.1 that $I_\lambda^{(2)}\to 0$ as $\lambda\to\infty$ since $\lambda\theta(t)$ has the same law as $\theta(\lambda^2 t)$. Q.E.D.

**Lemma 2.2.** *If $c(\not\equiv 0)\in L^1(0, 2\pi)$ and $\rho>-1$, then for any $\lambda(1\leqq\lambda<\infty)$ and any $r\geqq 0$,*

$$\int_0^t |c(\lambda\theta_s)|(-x_s)^\rho 1_{(\lambda x_s<-r)}ds\longrightarrow\infty \quad a.s. \quad as \quad t\longrightarrow\infty.$$

*Proof.* Fix $K>0$, $t>0$, and $1\leqq\lambda<\infty$. Then for any $\alpha>0$ we have

$$P\left[\int_0^{\alpha^2 t}|c(\lambda\theta_s)|(-x_s)^\rho 1_{(\lambda x_s<-r)}ds>K\right]$$
$$=P\left[\int_0^t|c(\lambda\theta(\alpha^2 s))|(-x(\alpha^2 s))^\rho 1_{(\lambda x(\alpha^2 s)<-r)}ds>K/\alpha^2\right]$$
$$=P\left[\int_0^t|c(\lambda\alpha\theta_s)|(-x_s)^\rho 1_{(\lambda\alpha x_s<-r)}ds>K/\alpha^{2+\rho}\right]$$

This, together with Lemma 2.1, gives an inequality

$$\liminf_{\alpha\to\infty} P\left[\int_0^{\alpha^2 t}|c(\lambda\theta_s)|(-x_s)^\rho 1_{(\lambda x_s<-r)}ds>K\right]\geqq P\left[\overline{|c|}\int_0^t(-x_s)^\rho 1_{(x_s<0)}ds>\varepsilon\right]$$

for any $\varepsilon>0$. The last expression obviously converges to 1 as $\varepsilon\to 0$ because $x_0=0$. Therefore, noting that the process involved is increasing in $t$, we obtain the lemma. Q.E.D.

**Lemma 2.3.** *Let $V_k{}^\lambda(t)$ $(k=0, \cdots, p)$ be as (2.9). Then*

$$\{V_0{}^\lambda, V_k{}^\lambda, \langle V_0{}^\lambda\rangle, \langle V_k{}^\lambda\rangle\}_{1\leq k\leq p}$$
$$\longrightarrow\left\{\int_0^\cdot 1_{(x_s<0)}dz_s, \int_0^\cdot\int_0^{2\pi}e_k(\theta)M(1_{(x_s<0)}ds, d\theta),\right.$$
$$\left.\int_0^\cdot 1_{(x_s<0)}ds, \int_0^\cdot 1_{(x_s<0)}ds\right\}_{1\leq k\leq p}$$

*as $\lambda\to\infty$ in law.*

*Proof.* First note by Lemma 2.1 that

$$\langle V_k^\lambda, V_l^\lambda\rangle(t)=\int_0^t \mathcal{R}e(e_k e_l^*)(\lambda\theta_s)1_{(\lambda x_s<-r)}ds\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$+\int_0^t \mathcal{I}m(e_k e_l^*)(\lambda\theta_s)1_{(\lambda x_s<-r)}ds\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$\longrightarrow \delta_{kl}\int_0^t 1_{(x_s<0)}ds\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{for} \quad k,\ l=0, \cdots, p$$

as $\lambda\to\infty$ on $C([0, \infty)\to \boldsymbol{R}^4)$ in probability for any $t>0$ and, also by Lemma 2.2 that

$$\langle V_k^\lambda\rangle(t) \longrightarrow \infty \quad \text{a.s.} \quad \text{as } t \longrightarrow \infty \qquad \text{for} \quad k=0, \cdots, p.$$

Fix $t>0$ and $\varepsilon>0$. Since the facts stated above imply that

$$P[\langle V_k^\lambda\rangle(n)<t] \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty,$$

$$P\left[\int_0^n 1_{(x_s<0)}ds<t\right] \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty$$

and

$$P[\langle V_k^\lambda\rangle(n)<t] \longrightarrow P\left[\int_0^n 1_{(x_s<0)}ds<t\right] \qquad \text{as} \quad \lambda \longrightarrow \infty$$

for any $n>0$, there exist $\lambda_0>0$ and $n_0>0$ such that

$$P[\langle V_k^\lambda\rangle^{-1}(t)>n_0]=P[\langle V_k^\lambda\rangle(n_0)<t]<\varepsilon$$

for all $\lambda\geqq\lambda_0$. Therefore there exists $\lambda_1>0$ such that

$$\begin{aligned}&P[|\langle V_k^\lambda, V_l^\lambda\rangle|(\langle V_k^\lambda\rangle^{-1}(t))>\varepsilon]\\ &\quad\leqq P[\langle V_k^\lambda\rangle^{-1}(t)>n_0]+P[\sup|\langle V_k^\lambda, V_l^\lambda\rangle|(t)>\varepsilon]\\ &\quad<2\varepsilon\end{aligned}$$

for all $\lambda\geqq\lambda_1$. Consequently we have

$$\langle V_k^\lambda, V_l^\lambda\rangle(\langle V_k^\lambda\rangle^{-1}(t)) \longrightarrow \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \qquad \text{if} \quad k\neq l \tag{2.15}$$

as $\lambda\to\infty$ in probability for any $t>0$, from which we obtain that $\{\widehat{V_0^\lambda}, \widehat{V_1^\lambda}, \cdots, \widehat{V_p^\lambda}\}$ converges in law to a $(p+1)$-dimensional complex Brownian motion as $\lambda\to\infty$ by the "asymptotic Knight's theorem" in Pitman and Yor [9] (p. 1008).

On the other hand, we easily see by Lemma W1 and Lemma W2 that the limit law of $\{V_0^\lambda, \langle V_0^\lambda\rangle, \widehat{V_0^\lambda}\}$ is that of

$$\left\{\int_0^\cdot 1_{(x_s<0)}dz_s,\ \int_0^\cdot 1_{(x_s<0)}dz,\ \widehat{\int_0^\cdot 1_{(x_s<0)}dz_s}\right\}.$$

Hence we can conclude that the limit law of $V_k^\lambda(t)$ $(k=1, \cdots, p)$ can be represented by the law of

$$\int_0^t\int_0^{2\pi} e_k(\theta)M(ds, d\theta) \qquad (k=1, \cdots, p).$$

Thus we have

$$\{V_0^\lambda,\ V_k^\lambda,\ \langle V_j^\lambda\rangle\}_{\substack{1\le k\le p\\ 0\le j\le p}} \longrightarrow \left\{\widehat{\int_0^\cdot 1_{(x_s<0)}dz_s},\ \int_0^\cdot\int_0^{2\pi} e_k(\theta)M(ds,\ d\theta),\ \int_0^t 1_{(x_s<0)}ds\right\}_{\substack{1\le k\le p\\ 0\le j\le p}}$$

as $\lambda\to\infty$ in law. This implies the assertion of the lemma. Q.E.D.

**Lemma 2.4.** *Let $V_k^\lambda(t)$ $(k=0,\cdots,p)$ be as (2.9). If $\rho>-1/2$, then*

$$\left\{V_0^\lambda,\ \int_0^\cdot(-x_s)^\rho dV_0^\lambda(s),\ \int_0^\cdot(-x_s)^{2\rho}d\langle V_0^\lambda\rangle_s\right\} \tag{2.16}$$

$$\longrightarrow\left\{\int_0^\cdot 1_{(x_s<0)}dz_s,\ \int_0^\cdot(-x_s)^\rho 1_{(x_s<0)}dz_s,\ \int_0^\cdot(-x_s)^{2\rho}1_{(x_s<0)}ds\right\}$$

*as $\lambda\to\infty$ in law and*

$$\left\{V_k^\lambda,\ \int_0^\cdot(-x_s)^\rho dV_k^\lambda(s),\ \int_0^\cdot(-x_s)^{2\rho}d\langle V_k^\lambda\rangle_s\right\} \tag{2.17}$$

$$\longrightarrow\left\{\int_0^\cdot\int_0^{2\pi}e_k(\theta)M(1_{(x_s<0)}ds,\ d\theta),\right.$$

$$\int_0^\cdot\int_0^{2\pi}e_k(\theta)(-x_s)^\rho M(1_{(x_s<0)}ds,\ d\theta),$$

$$\left.\int_0^\cdot(-x_s)^{2\rho}1_{(x_s<0)}ds\right\}$$

*for $k=1,\cdots,p$ as $\lambda\to\infty$ in law.*

*Proof.* Set

$$\begin{cases} V_0^\infty(t)=\int_0^t 1_{(x_s<0)}dz_s\\ V_k^\infty(t)=\int_0^t\int_0^{2\pi}e_k(\theta)M(1_{(x_s<0)}ds,\ d\theta).\end{cases}$$

By Lemma 2.1, we have

$$E\sup_{0\le t\le T}\left|\int_0^t(-x_s)^{2\rho}d\langle V_k^\lambda\rangle_s-\int_0^t(-x_s)^{2\rho}d\langle V_k^\infty\rangle_s\right|\longrightarrow 0\qquad\text{as}\quad\lambda\longrightarrow\infty$$

and

$$E\sup_{0\le t\le T}\left|\int_0^t(-x_s)^{\rho}d\langle V_k^\lambda\rangle_s-\int_0^t(-x_s)^{\rho}d\langle V_k^\infty\rangle_s\right|\longrightarrow 0\qquad\text{as}\quad\lambda\longrightarrow\infty$$

for $k=0,1,\cdots,p$. On the other hand, Lemma 2.3 implies that

$$\{V_k^\lambda,\ \langle V_k^\lambda\rangle\}\longrightarrow\{V_k^\infty,\ \langle V_k^\infty\rangle\}\qquad\text{as}\quad\lambda\longrightarrow\infty$$

in law for each $k$. Therefore,

$$\left\{V_k^\lambda,\ \langle V_k^\lambda\rangle,\ \left\langle\int_0^\cdot(-x_s)^\rho dV_k^\lambda(s),\ V_k^\lambda(s)\right\rangle,\ \left\langle\int_0^\cdot(-x_s)^\rho dV_k^\lambda(s)\right\rangle\right\}$$

$$=\left\{V_k^{\lambda},\ \langle V_k^{\lambda}\rangle,\ \int_0^{\cdot}(-x_s)^{\rho}d\langle V_k^{\lambda}\rangle_s\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \int_0^{\cdot}(-x_s)^{2\rho}d\langle V_k^{\lambda}\rangle_s\right\}$$

$$\longrightarrow\left\{V_k^{\infty},\ \langle V_k^{\infty}\rangle,\ \int_0^{\cdot}(-x_s)^{\rho}d\langle V_k^{\infty}\rangle_s\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \int_0^{\cdot}(-x_s)^{2\rho}d\langle V_k^{\infty}\rangle_s\right\}$$

as $\lambda\to\infty$ in law for each $k$.

Thus if we can prove that the above processes satisfy the conditions (2.12)-(2.14) in Lemma W1, then (2.16) and (2.17) follow from Lemma W1. It is easy to show that

$$E\langle V_k^{\lambda}\rangle(t)^2\leqq\text{const.}(t^{1/2}+\text{const.}\,t)^2,\qquad 1\leqq\lambda<\infty \tag{2.18}$$

for each $k$. Indeed,

$$E\langle V_k^{\lambda}\rangle(t)^2=2E\int_0^t ds\int_s^t|e_k(\lambda\theta_s)|^2|e_k(\lambda\theta_u)|^2 1_{(x_s<0)}1_{(x_u<0)}du$$

$$\leqq 2E\int_0^t ds\int_s^t|e_k(\lambda\theta_s)|^2|e_k(\lambda\theta_u)|^2du$$

$$=2E\int_0^t ds\int_s^t|e_k(\theta(\lambda^2 s))|^2|e_k(\theta(\lambda^2 u))|^2du$$

$$\leqq\text{const.}\int_0^t ds\int_s^t(\lambda^{-1}s^{-1/2}+\text{const.})\{\lambda^{-1}(u-s)^{-1/2}+\text{const.}\}du.$$

Here the last inequality follows from Lemma 1.2. Then we have (2.18).

We can also prove that

$$E\left|\int_0^t(-x_s)^{2\rho}d\langle V_k^{\lambda}\rangle_s\right|^2\leqq\text{const.}\,t^{2\rho}(t^{1/2}+\text{const.}\,t)^2,\qquad 1\leqq\lambda<\infty$$

for each $k$ by a similar argument as above using Lemma 1.1 and Lemma 1.2.

Further it has already been shown in Lemma 2.2 that

$$\int_0^t(-x_s)^{2\rho}d\langle V_k^{\lambda}\rangle_s\longrightarrow\infty\qquad(t\longrightarrow\infty)\qquad\text{a.s.},\quad 1\leqq\lambda<\infty$$

and

$$\int_0^t(-x_s)^{2\rho}d\langle V_k^{\infty}\rangle_s=\int_0^t(-x_s)^{2\rho}1_{(x_s<0)}ds\longrightarrow\infty\qquad(t\longrightarrow\infty)\quad\text{a.s.}$$

for each $k$. Consequently we have completed the proof of the lemma. Q.E.D.

## § 3. Application to a limit theorem for "winding-type" additive functionals

Throughout this section let $z(t)=x(t)+\sqrt{-1}y(t)$, $z(0)=0$, be a complex Brownian motion starting at the origin. Let $a_1, a_2, \cdots, a_n$ be given distinct points on $\boldsymbol{C}\setminus\{0\}$ and $a_\infty=\infty$. For $i=1,\cdots,n,\infty$, let $f_{i1}, f_{i2},\cdots,f_{im}$ be some regularly varying functions at $a_i$ with order $\rho_{i1},\rho_{i2},\cdots,\rho_{im}$, respectively. (See Definition 2.1.) We denote the regular normalizing function of $f_{ij}$ at $a_i$ by $N_{ij}(\lambda)$ and the asymptotic angular component of $f_{ij}$ at $a_i$ by $c_{ij}(\theta)$ for $i=1,\cdots,n,\infty$ and $j=1,\cdots,m$.

The main purpose of this section is to give the joint limit processes, as $\lambda\to\infty$, of the processes $\{A_{ij-}{}^{\lambda}, A_{ij+}{}^{\lambda}\}$ defined by

$$\tag{3.1} \begin{cases} A_{ij-}{}^{\lambda}(t)=\dfrac{1}{\lambda N_{ij}(\lambda)}\displaystyle\int_0^{u(\lambda t)}\frac{f_{ij}(z_s)}{z_s-a_i}1_{D(i-)}(z_s)dz_s \\ A_{ij+}{}^{\lambda}(t)=\dfrac{1}{\lambda N_{\infty j}(\lambda)}\displaystyle\int_0^{u(\lambda t)}\frac{f_{\infty j}(z_s)}{z_s-a_i}1_{D(i+)}(z_s)dz_s, \end{cases}$$

where $u(t)=e^{2t}-1$, $D(i-)$ is some bounded domain containing $a_i$ and $D(i+)$ is some domain such that $D(i+)^c$ is bounded and $a_i\notin\overline{D(i+)}$. As we shall see, a particular choice of $D(i-)$ and $D(i+)$ is immaterial in the limit theorem.

First, we introduce the notion of $K$-convergence for stochastic processes:

**Definition 3.1.** Let $D_1=D_1([0, \infty)\to\boldsymbol{R}^d)$ be the space of all $\boldsymbol{R}^d$-valued right continuous functions with left limits. A sequence of $D_1$-valued stochastic processes $\{X_n(t)\}$ is said to be $K$-convergent to $X_\infty(t)$ if there exist a sequence of $\boldsymbol{R}^d\times\boldsymbol{R}$-valued stochastic processes $\{(Y_n(t), \varphi_n(t))\}$ and $(Y_\infty(t), \varphi_\infty(t))$ such that

1°) $Y_n(t)$ $(1\leqq n\leqq\infty)$ and $\varphi_n(t)$ $(1\leqq n\leqq\infty)$ are all continuous stochastic processes,

2°) $\varphi_n(t)$ is non-decreasing a.s., $\varphi_n(0)=0$ and $\varphi_n(t)\to\infty$ as $t\to\infty$ a.s. for all $1\leqq n\leqq\infty$,

3°) $X_n(t)=Y_n(\varphi_n{}^{-1}(t))$ $(1\leqq n\leqq\infty)$,

4°) $\{(Y_n, \varphi_n)\}\to(Y_\infty, \varphi_\infty)$ as $n\to\infty$ in law on $C([0, \infty)\to\boldsymbol{R}^d\times\boldsymbol{R})$.

We remark that the main limit theorems by Kasahara and Kotani [6] are in the sense of $K$-convergence. If $\{X_n(t)\}$ is $K$-convergent to $X_\infty(t)$ as $n\to\infty$ and $X_\infty(t)$ is non-decreasing $w.p.$ 1, then $\{X_n(t)\}$ is weakly $M_1$-convergent to $X_\infty(t)$. Generally, $M_1$-convergence does not follow from $K$-convergence but, if $\{X_n(t)\}$ is $K$-convergent to $X_\infty(t)$ as $n\to\infty$ and $\varphi_\infty{}^{-1}$ has no fixed discontinuous point, then $\{X_n(t)\}$ converges to $X_\infty(t)$ as $n\to\infty$ *in the sense of finite dimensional distributions.* This fact is obviously derived from the following real variable proposition:

**Proposition 3.1.** *Let $\{y_n(t)\}$ and $\{\varphi_n(t)\}$ be sequences of continuous functions on $[0, \infty)$ such that $\varphi_n(t)$ is non-decreasing and $\varphi_n(t)\to\infty$ $(t\to\infty)$ $(n=1, 2, \cdots)$. Suppose $y_n(t)\to y(t)$ and $\varphi_n(t)\to\varphi(t)$ uniformly in $t$ on each compact sets as $n\to\infty$ and $\varphi(t)\to\infty$ $(t\to\infty)$.*

*If $y(t)$ is constant on $(\varphi^{-1}(t_0-), \varphi^{-1}(t_0))$ for some $t_0\in[0, \infty)$, then we have*

$$\tag{3.2} y_n(\varphi_n{}^{-1}(t_0))\longrightarrow y(\varphi^{-1}(t_0)) \qquad (n\longrightarrow\infty).$$

*Particularly, if $\varphi^{-1}(t_0-)=\varphi^{-1}(t_0)$ then we have* (3.2) *also.*

We omit the proof.

Next, in order to describe the joint limit processes, we introduce a particular system of $n$ complex Brownian motions and $n+1$ complex Gaussian random measures. As in the preceding section, we always denote by $\hat{M}(t)$ the time-changed process $M(\langle M\rangle^{-1}(t))$ for a conformal martingale $M(t)$.

Let $\zeta=(\zeta_1, \cdots, \zeta_n)$ be a $C^n$-valued continuous process which has the following properties:

(1) Each $\zeta_i=\xi_i+\sqrt{-1}\eta_i$ is a complex Brownian motion starting at the origin for $i=1, \cdots, n$.

(2) Setting

$$\begin{cases} \zeta_{i-}(t)=\int_0^t 1_{(\xi_i(s)<0)}d\zeta_i(s) \\ \zeta_{i+}(t)=\int_0^t 1_{(\xi_i(s)>0)}d\zeta_i(s), \end{cases}$$

the family $\{\widehat{\zeta_{1-}}, \cdots, \widehat{\zeta_{n-}}, \widehat{\zeta_{1+}}\}$ is mutually independent and $\widehat{\zeta_{1+}}=\widehat{\zeta_{2+}}=\cdots=\widehat{\zeta_{n+}}$.

An important fact is that a $C^n$-valued process with these properties exists uniquely in the sense of law. We will explain the structure of $\zeta$ in Remark 3.1 in the last part of this section.

Furthermore we take $n+1$ complex Gaussian random measures $M_1, \cdots, M_n, M_+$ with the following properties:

(3) Each $M_i$ is a complex Gaussian random measure on $[0, \infty)\times[0, 2\pi]$ with mean 0 and variance measure $dt\cdot d\theta/2\pi$ for $i=1, \cdots, n, +$.

(4) The family $\{\zeta, M_1, \cdots, M_n, M_+\}$ is mutually independent.

Now define, for $i=1, 2, \cdots, n$,

$$Z_i(t)=X_i(t)+\sqrt{-1}Y_i(t)=\int_0^t \frac{dz_s}{z_s-a_i},$$

$$\hat{Z}_i^\lambda(t)=\hat{X}_i^\lambda(t)+\sqrt{-1}\hat{Y}_i^\lambda(t)=\frac{1}{\lambda}Z_i(\langle Z_i\rangle^{-1}(\lambda^2 t))$$

and

$$\tau_i^\lambda(t)=\frac{1}{\lambda}u^{-1}(\langle Z_i\rangle^{-1}(\lambda^2 t))=\frac{1}{2\lambda}\log\left[\lambda^2|a_i|^2\int_0^t e^{2\lambda\hat{X}_i^\lambda(s)}ds+1\right]$$

Then our theorem can be stated as follows:

**Theorem 3.1.**

$$\{\hat{Z}_i^\lambda, \tau_i^\lambda, A_{ij-}^\lambda(\tau_i^\lambda), A_{ij+}^\lambda(\tau_i^\lambda)\}_{1\le i\le n}^{1\le j\le m} \longrightarrow \{\zeta_i, \mu_i, \mathcal{L}_{ij-}, \mathcal{L}_{ij+}\}_{1\le i\le n}^{1\le j\le m}$$

*as* $\lambda\to\infty$ *in law on* $C([0, \infty)\to \boldsymbol{C}^n\times\boldsymbol{R}^n\times\boldsymbol{C}^{mn}\times\boldsymbol{C}^{mn})$, *where*

$$\mu_i(t)=\max_{0\le s\le t}\xi_i(s),$$

(3.3) $$\mathcal{L}_{ij-}(t)=\overline{c_{ij}}\int_0^t(-\xi_i(s))^{\rho_{ij}}d\zeta_{i-}(s) +\int_0^t\int_0^{2\pi}(c_{ij}(\theta)-\overline{c_{ij}})(-\xi_i(s))^{\rho_{ij}}M_i(d\langle\zeta_{i-}\rangle(s), d\theta),$$

(3.4) $$\mathcal{L}_{ij+}(t)=\overline{c_{\infty j}}\int_0^t\xi_i(s)^{\rho_{\infty j}}d\zeta_{i+}(s) +\int_0^t\int_0^{2\pi}(c_{\infty j}(\theta)-\overline{c_{\infty j}})\xi_i(s)^{\rho_{\infty j}}M_+(d\langle\zeta_{i+}\rangle(s), d\theta)$$

*and* $\bar{c}=\frac{1}{2\pi}\int_0^{2\pi}c(\theta)d\theta$, *in general.*

As a corollary to Theorem 3.1, we can conclude the following:

**Theorem 3.2.**

$$\{\hat{Z}_i^\lambda,\ A_{ij-}^\lambda,\ A_{ij+}^\lambda\}_{1\le i\le n}^{1\le j\le m} \longrightarrow \{\zeta_i,\ \mathcal{L}_{ij-}(\mu_i^{-1}),\ \mathcal{L}_{ij+}(\mu_i^{-1})\}_{1\le i\le n}^{1\le j\le m}$$

*as* $\lambda\to\infty$ *in the sense of K-convergence.*

*Proof of Theorem* 3.1. The fact that

$$\{\hat{Z}_i^\lambda,\ \tau_i^\lambda\} \longrightarrow \{\zeta_i,\ \max_{0\le s\le\cdot}\xi_i(s)\} \qquad \text{as} \quad \lambda\longrightarrow\infty$$

in law on $C([0,\infty)\to C\times R)$ for each $i$ was obtained by Kasahara and Kotani ([6], Lemma 3.1).

The first important step in our proof is the following transformation:

$$\text{(3.5)} \qquad \frac{1}{\lambda}\int_0^{\langle z_i\rangle^{-1}(\lambda^2 t)}\frac{f(z_s)}{z_s-a_i}dz_s=\frac{1}{\lambda}\int_0^{\langle z_i\rangle^{-1}(\lambda^2 t)}f(a_i-a_ie^{Z_i(s)})dZ_i(s)$$
$$=\int_0^t f(a_i-a_ie^{\lambda\hat{Z}_i^\lambda(s)})d\hat{Z}_i^\lambda(s).$$

By this transformation, we have

$$\begin{cases} A_{ij-}^\lambda(\tau_i^\lambda(t))=\dfrac{1}{N_{ij}(\lambda)}\displaystyle\int_0^t(f_{ij}\cdot 1_{D(i-)})(a_i-a_ie^{\lambda\hat{Z}_i^\lambda(s)})d\hat{Z}_i^\lambda(s) \\ A_{ij+}^\lambda(\tau_i^\lambda(t))=\dfrac{1}{N_{\infty j}(\lambda)}\displaystyle\int_0^t(f_{\infty j}\cdot 1_{D(i+)})(a_i-a_ie^{\lambda\hat{Z}_i^\lambda(s)})d\hat{Z}_i^\lambda(s). \end{cases}$$

Fix sufficiently large $r>0$ and set

$$\begin{cases} F_{ij-}^\lambda(t)=\dfrac{1}{N_{ij}(\lambda)}\displaystyle\int_0^t f_{ij}(a_i-a_ie^{\lambda\hat{Z}_i^\lambda(s)})1_{(\lambda\hat{X}_i^\lambda(s)<-r)}d\hat{Z}_i^\lambda(s) \\ F_{ij+}^\lambda(t)=\dfrac{1}{N_{\infty j}(\lambda)}\displaystyle\int_0^t f_{\infty j}(a_i-a_ie^{\lambda\hat{Z}_i^\lambda(s)})1_{(\lambda\hat{X}_i^\lambda(s)>r)}d\hat{Z}_i^\lambda(s). \end{cases}$$

Since

$$\sup_{0\le\theta\le 2\pi}|1_{D(i-)}(a_i-a_ie^{\lambda x+\sqrt{-1}\theta})-1_{(\lambda x<-r)}|\longrightarrow 0$$

and

$$\sup_{0\le\theta\le 2\pi}|1_{D(i+)}(a_i-a_ie^{\lambda x+\sqrt{-1}\theta})-1_{(\lambda x>r)}|\longrightarrow 0$$

as $\lambda\to\infty$, we can easily deduce that

$$\text{(3.6)} \qquad E\sup_{0\le t\le T}|A_{ij\pm}^\lambda(\tau_i^\lambda(t))-F_{ij\pm}^\lambda(t)|^2\longrightarrow 0 \qquad \text{as} \quad \lambda\longrightarrow\infty$$

and

$$\text{(3.7)} \qquad E\sup_{0\le t\le T}|\langle A_{ij\pm}^\lambda(\tau_i^\lambda)\rangle_t-\langle F_{ij\pm}^\lambda\rangle_t|\longrightarrow 0 \qquad \text{as} \quad \lambda\longrightarrow\infty$$

by a similar argument as in the proof of Theorem 2.1.

Therefore the joint processes

$$\{\hat{Z}_i^\lambda,\ A_{ij-}^\lambda(\tau_i^\lambda),\ \langle A_{ij-}^\lambda(\tau_i^\lambda)\rangle,\ A_{ij+}^\lambda(\tau_i^\lambda),\ \langle A_{ij+}^\lambda(\tau_i^\lambda)\rangle\}_{1\le i\le n}^{1\le j\le m}$$

have the same limit law as the joint processes

$$\{\hat{Z}_i^\lambda,\ F_{ij-}^\lambda,\ \langle F_{ij-}^\lambda\rangle,\ F_{ij+}^\lambda,\ \langle F_{ij+}^\lambda\rangle\}_{1\le i\le n}^{1\le j\le m}.$$

We know by Theorem 2.1 that the joint limit processes as $\lambda\to\infty$ of $\{\hat{Z}_i^\lambda, F_{ij-}^\lambda, \langle F_{ij-}^\lambda\rangle\}_{1\le j\le m}$ and $\{\hat{Z}_i^\lambda, F_{ij+}^\lambda, \langle F_{ij+}^\lambda\rangle\}_{1\le j\le m}$ are $\{\zeta_i, \mathcal{L}_{ij-}, \langle\mathcal{L}_{ij-}\rangle\}_{1\le j\le m}$ and $\{\zeta_i, \mathcal{L}_{ij+}, \langle\mathcal{L}_{ij+}\rangle\}_{1\le j\le m}$ respectively for each $i$, where $\mathcal{L}_{ij-}$ and $\mathcal{L}_{ij+}$ are defined by (3.3) and (3.4). Then the laws of

$$\{\hat{Z}_i^\lambda,\ A_{ij-}^\lambda(\tau_i^\lambda),\ \langle A_{ij-}^\lambda(\tau_i^\lambda)\rangle,\ A_{ij+}^\lambda(\tau_i^\lambda),\ \langle A_{ij+}^\lambda(\tau_i^\lambda)\rangle\}_{1\le j\le m}^{1\le i\le n},$$

$\lambda>0$, form a tight family because each component converges in law. Further it is clear from the above argument that we may assume for any limit point of this family that it is the law of

$$\{\zeta_i,\ \mathcal{A}_{ij-},\ \langle\mathcal{A}_{ij-}\rangle,\ \ \mathcal{A}_{ij+},\ \langle\mathcal{A}_{ij+}\rangle\}_{1\le j\le m}^{1\le i\le n},$$

where $\zeta_1, \zeta_2, \cdots, \zeta_n$ are *some* complex Brownian motions,

$$\mathcal{A}_{ij-}(t)=\overline{c_{ij}}\int_0^t(-\xi_i(s))^{\rho_{ij}}d\zeta_{i-}(s)$$
$$+\int_0^t\int_0^{2\pi}(c_{ij}(\theta)-\overline{c_{ij}})(-\xi_i(s))^{\rho_{ij}}M_i(d\langle\zeta_{i-}\rangle(s),\ d\theta),$$

$$\mathcal{A}_{ij+}(t)=\overline{c_{\infty j}}\int_0^t\xi_i(s)^{\rho_{\infty j}}d\zeta_{i+}(s)$$
$$+\int_0^t\int_0^{2\pi}(c_{\infty j}(\theta)-\overline{c_{\infty j}})\xi_i(s)^{\rho_{\infty j}}\tilde{M}_i(d\langle\zeta_{i+}\rangle(s),\ d\theta)$$

and $M_1, M_2, \cdots, M_n, \tilde{M}_1, \tilde{M}_2, \cdots, \tilde{M}_n$ are *some* complex Gaussian random measures on $[0,\infty)\times[0,2\pi]$ with mean 0 and variance measure $dt\cdot d\theta/2\pi$. We fix these $\zeta_1, \cdots, \zeta_n, M_1, \cdots, M_n, \tilde{M}_1, \cdots, \tilde{M}_n$ below. It remains to prove the identity

$$\widehat{\zeta_{1+}}=\widehat{\zeta_{2+}}=\cdots=\widehat{\zeta_{n+}},\tag{3.8}$$

the identity

$$\tilde{M}_1=\tilde{M}_2=\cdots=\tilde{M}_n:=M_+\tag{3.9}$$

and the mutual independence of

$$\widehat{\zeta_{1-}},\ \widehat{\zeta_{2-}},\ \cdots,\ \widehat{\zeta_{n-}},\ \widehat{\zeta_{1+}},\ M_1,\ M_2,\ \cdots,\ M_n,\ M_+.\tag{3.10}$$

Firstly we prove the identity (3.8). As a consequence of (3.6) and (3.7), we may replace $D(i+)$ by $D(1+)\cap D(2+)\cap\cdots\cap D(n+)$. Therefore we may assume that

$$D(1+)=D(2+)=\cdots=D(n+):=D(\infty).$$

Set

$$W_{i+}{}^{\lambda}(t)=\frac{1}{\lambda}\int_0^{u(\lambda t)}\frac{1}{z_s-a_i}1_{D(\infty)}(z_s)dz_s\,.$$

This is the particular case of $A_{ij+}{}^{\lambda}(t)$. We remark that

$$\text{(3.11)} \qquad E\sup_{0\le t\le T}|W_{i+}{}^{\lambda}(\tau_1{}^{\lambda}(t))-W_{1+}{}^{\lambda}(\tau_1{}^{\lambda}(t))|^2\longrightarrow 0 \quad \text{as} \quad \lambda\longrightarrow\infty$$

and

$$\text{(3.12)} \qquad E\sup_{0\le t\le T}|\langle W_{i+}{}^{\lambda}(\tau_1{}^{\lambda})\rangle_t-\langle W_{1+}{}^{\lambda}(\tau_1{}^{\lambda})\rangle_t|\longrightarrow 0 \quad \text{as} \quad \lambda\longrightarrow\infty$$

for any $i$. To prove (3.11), note that

$$\begin{aligned} W_{i+}{}^{\lambda}(\tau_1{}^{\lambda}(t)) &= \frac{1}{\lambda}\int_0^{\langle z_1\rangle^{-1}(\lambda^2 t)}\frac{1}{z_s-a_i}1_{D(\infty)}(z_s)dz_s \\ &= \frac{1}{\lambda}\int_0^{\langle z_1\rangle^{-1}(\lambda^2 t)}\frac{1}{z_s-a_1}1_{D(\infty)}(z_s)\cdot\frac{z_s-a_1}{z_s-a_i}dz_s \\ &= \int_0^t 1_{D(\infty)}(a_1-a_1e^{\lambda\hat{Z}_1{}^{\lambda}(s)})R_i(\lambda\hat{X}_1{}^{\lambda},\lambda\hat{Y}_1{}^{\lambda})d\hat{Z}_1{}^{\lambda}(s), \end{aligned}$$

where

$$R_i(x,\theta)=-a_1e^{x+\sqrt{-1}\theta}/(a_1-a_i-a_1e^{x+\sqrt{-1}\theta}).$$

Hence

$$\begin{aligned} &E\sup_{0\le t\le T}|W_{i+}{}^{\lambda}(\tau_1{}^{\lambda}(t))-W_{1+}{}^{\lambda}(\tau_1{}^{\lambda}(t))|^2 \\ &\le \text{const.}\,E\int_0^T 1_{D(\infty)}(a_1-a_1e^{\lambda\hat{Z}_1{}^{\lambda}(s)})|R_i(\lambda\hat{X}_1{}^{\lambda},\lambda\hat{Y}_1{}^{\lambda})-1|^2ds\,. \end{aligned}$$

Since $1_{D(\infty)}(a_1-a_1e^{x\sqrt{-1}\theta})|R_i(x,\theta)|$ is bounded in $(x,\theta)\in \boldsymbol{R}\times\boldsymbol{T}$ and $\sup_{0\le\theta\le 2\pi}1_{D(\infty)}(a_1-a_1e^{\lambda x+\sqrt{-1}\theta})|R_i(\lambda x,\theta)-1|\to 0$ as $\lambda\to\infty$ for any $x\ne 0$, we can deduce the convergence (3.11). The proof of (3.12) can be given similarly.

Then the laws of $P_\lambda$, $\lambda>0$, of

$$\{\hat{Z}_i{}^{\lambda},\ W_{i+}{}^{\lambda}(\tau_i{}^{\lambda}),\ \langle W_{i+}{}^{\lambda}(\tau_i{}^{\lambda})\rangle,\ W_{i+}{}^{\lambda}(\tau_1{}^{\lambda}),\ \langle W_{i+}{}^{\lambda}(\tau_1{}^{\lambda})\rangle\}_{1\le i\le n}$$

form a tight family and we may assume one limit point $P_\infty$ of $\{P_\lambda\}$ to be the law of

$$\{\zeta_i,\ \zeta_{i+},\ \langle\zeta_{i+}\rangle,\ \zeta_{1+},\ \langle\zeta_{1+}\rangle\}_{1\le i\le n}\,.$$

Let $P_{\lambda_\nu}\to P_\infty$ for some subsequence and write $\lambda_\nu$ as $\lambda$ for the notational simplicity. We can prove that $\langle W_{i+}{}^{\lambda}(\tau_i{}^{\lambda})\rangle_t\to\infty$ and $\langle W_{i+}{}^{\lambda}(\tau_1{}^{\lambda})\rangle_t\to\infty$ as $t\to\infty$ by a similar argument as in the proof of Lemma 2.2, and hence we have, by Lemma W2, that

$$\begin{aligned} &\{\hat{Z}_i{}^{\lambda},\ \widehat{W_{i+}{}^{\lambda}(\tau_i{}^{\lambda})},\ W_i{}^{+\lambda}(\tau_i{}^{\lambda}),\ W_{i+}{}^{\lambda}(\tau_1{}^{\lambda}),\ \widehat{W_{i+}{}^{\lambda}(\tau_1{}^{\lambda})}\}_{1\le i\le n} \\ &\longrightarrow \{\zeta_i,\ \zeta_{i+},\ \hat{\zeta}_{i+},\ \zeta_{1+},\ \hat{\zeta}_{1+}\}_{1\le i\le n} \end{aligned}$$

as $\lambda\to\infty$ in law. We may assume by the Skorohod theorem that this convergence is uniform on each compact interval a.s. Then we see that $\hat{\zeta}_{i+}$ is identical to $\hat{\zeta}_{1+}$ for $i=1,\cdots,n$ because $\widehat{W_{i+}{}^{\lambda}(\tau_i{}^{\lambda})}=\widehat{W_{i+}{}^{\lambda}(\tau_1{}^{\lambda})}$. Thus the identity (3.8) is now proved.

Secondly, we prove the identity (3.9). We can prove similarly to (3.11) and (3.12) that

$$E\sup_{0\le t\le T}|A_{ij+}{}^{\lambda}(\tau_1{}^{\lambda}(t))-A_{1j+}{}^{\lambda}(\tau_1{}^{\lambda}(t))|^2 \longrightarrow 0 \qquad \text{as} \quad \lambda \longrightarrow \infty$$

and

$$E\sup_{0\le t\le T}|\langle A_{ij+}{}^{\lambda}(\tau_1{}^{\lambda})\rangle_t-\langle A_{1j+}{}^{\lambda}(\tau_1{}^{\lambda})\rangle_t| \longrightarrow 0 \qquad \text{as} \quad \lambda \longrightarrow \infty$$

for any $i$ and $j$. Let $P_\infty$ be one limit point of the tight family of the laws $P_\lambda$, $\lambda>0$, of

$$\{\hat{Z}_i{}^{\lambda},\ A_{ij+}{}^{\lambda}(\tau_i{}^{\lambda}),\ \langle A_{ij+}{}^{\lambda}(\tau_i{}^{\lambda})\rangle,\ A_{ij+}{}^{\lambda}(\tau_1{}^{\lambda}),\ \langle A_{ij+}{}^{\lambda}(\tau_1{}^{\lambda})\rangle\}_{1\le i\le n}^{1\le j\le m}.$$

We may assume the law of $P_\infty$ to be the law of

$$\{\zeta_i,\ \mathcal{A}_{ij+},\ \langle\mathcal{A}_{ij+}\rangle,\ \mathcal{A}_{1j+},\ \langle\mathcal{A}_{1j+}\rangle\}_{1\le i\le n}^{1\le j\le m}.$$

Let $P_{\lambda_\nu}\to P_\infty$ for some subsequence and write $\lambda_\nu$ simply as $\lambda$. Since we can prove that $\langle A_{ij+}(\tau_i{}^{\lambda})\rangle_t\to\infty$ and $\langle A_{ij+}(\tau_1{}^{\lambda})\rangle_t\to\infty$ as $\lambda\to\infty$ by a similar argument as in the proof of Lemma 2.2, and hence by Lemma W2 we have that

$$\begin{aligned}&\{\hat{Z}_i{}^{\lambda},\ A_{ij+}{}^{\lambda}(\tau_i{}^{\lambda}),\ \widehat{A_{ij+}{}^{\lambda}(\tau_i{}^{\lambda})},\ A_{ij+}{}^{\lambda}(\tau_1{}^{\lambda}),\ \widehat{A_{ij+}{}^{\lambda}(\tau_1{}^{\lambda})}\}_{1\le i\le n}^{1\le j\le m}\\ &\qquad\longrightarrow \{\zeta_i,\ \mathcal{A}_{ij+},\ \widehat{\mathcal{A}}_{ij+},\ \mathcal{A}_{1j+},\ \widehat{\mathcal{A}}_{1j+}\}_{1\le i\le n}^{1\le j\le m}\end{aligned}$$

as $\lambda\to\infty$ in law. We may assume by the Skorohod theorem that this convergence is uniform on each compact interval a.s. Then we have that

$$\widehat{\mathcal{A}}_{1j+}(t)=\widehat{\mathcal{A}}_{2j+}(t)=\cdots=\widehat{\mathcal{A}}_{nj+}(t) \qquad (j=1,\cdots,m) \tag{3.13}$$

because $\widehat{A_{ij+}{}^{\lambda}(\tau_i{}^{\lambda})}=\widehat{A_{ij+}{}^{\lambda}(\tau_1{}^{\lambda})}$.

Set

$$\mathcal{N}_{ij+}(t)=\mathcal{A}_{ij+}(\langle\zeta_{i+}\rangle^{-1}(t)).$$

The identity (3.13) implies that

$$\widehat{\mathcal{N}}_{1j+}(t)=\widehat{\mathcal{N}}_{2j+}(t)=\cdots=\widehat{\mathcal{N}}_{nj}(t) \qquad (j=1,\cdots,m). \tag{3.14}$$

On the other hand, note that

$$\begin{aligned}\mathcal{N}_{ij+}(t)&=\overline{c_{\infty j}}\int_0^t(\xi_i(\langle\zeta_{i+}\rangle^{-1}(s))\vee 0)^{\rho_{\infty j}}d\widehat{\zeta}_{i+}(s)\\ &\quad+\int_0^t\int_0^{2\pi}(c_{\infty j}(\theta)-\overline{c_{\infty j}})(\xi_i(\langle\zeta_{i+}\rangle^{-1}(s))\vee 0)^{\rho_{\infty j}}\tilde{M}_i(ds,d\theta)\end{aligned}$$

and

$$\langle\mathcal{N}_{ij+}\rangle(t)=\overline{|c_{\infty j}|^2}\int_0^t(\xi_i(\langle\zeta_{i+}\rangle^{-1}(s))\vee 0)^{2\rho_{\infty j}}ds.$$

Since $\xi_i(\langle\zeta_{i+}\rangle^{-1}(t))\vee 0$, $i=1,\cdots,n$, are the same reflecting Brownian motion by the identity (3.8) (See remark 3.1 below), we have that

$$\langle\mathcal{N}_{1j+}\rangle(t)=\langle\mathcal{N}_{2j+}\rangle(t)=\cdots=\langle\mathcal{N}_{nj+}\rangle(t) \qquad (j=1,\cdots,m). \tag{3.15}$$

Combining (3.14) and (3.15), we obtain the identity

$$\mathcal{N}_{1j+}(t)=\mathcal{N}_{2j+}(t)=\cdots=\mathcal{N}_{nj+}(t) \qquad (j=1,\cdots,m).$$

This clearly shows the identity (3.9).

Finally, we prove the mutual independence of (3.10). Let $\{e_0\equiv 1, e_1, \cdots, e_p\}$ be some orthonormal system in $L^2(0, 2\pi)$ such that

$$c_{ij}(\theta)=\sum_{k=0}^{p}\alpha_{ij}^{(k)}e_k(\theta), \qquad \alpha_{ij}^{(k)}\in C \quad (k=0,\cdots,p)$$

for $i=1,\cdots,n,\infty$ and $j=1,\cdots,m$. Set

$$\begin{cases} V_{ik-}^{\lambda}(t)=\int_0^t e_k(\lambda\hat{Y}_i^{\lambda}(s))1_{(\lambda\hat{X}_i^{\lambda}(s)<-r)}d\hat{Z}_i^{\lambda}(s) \\ V_{ik+}^{\lambda}(t)=\int_0^t e_k(\lambda\hat{Y}_i^{\lambda}(s))1_{(\lambda\hat{X}_i^{\lambda}(s)>r)}d\hat{Z}_i^{\lambda}(s). \end{cases}$$

By (2.10) and (3.6), we have

$$E\sup_{0\le t\le T}\left|A_{ij-}^{\lambda}(\tau_i^{\lambda}(t))-\sum_{k=0}^{p}\alpha_{ij}^{(k)}\int_0^t(-\hat{X}_i^{\lambda}(s))^{\rho ij}dV_{ik-}^{\lambda}(s)\right|^2 \longrightarrow 0 \quad \text{as} \quad \lambda\longrightarrow\infty$$

and

$$E\sup_{0\le t\le T}\left|A_{ij+}^{\lambda}(\tau_i^{\lambda}(t))-\sum_{k=0}^{p}\alpha_{\infty j}^{(k)}\int_0^t\hat{X}_i^{\lambda}(s)^{\rho\infty j}dV_{ik+}^{\lambda}(s)\right|^2 \longrightarrow 0 \quad \text{as} \quad \lambda\longrightarrow\infty.$$

Hence by Lemma 2.3, Lemma 2.4 and Lemma W2 we may assume the law of one limit point of the tight family of the laws of

$$\Big\{\hat{Z}_i^{\lambda},\ A_{ij-}^{\lambda}(\tau_i^{\lambda}),\ A_{ij+}^{\lambda}(\tau_i^{\lambda}),\ \int_0^{\cdot}(-\hat{X}_i^{\lambda}(s))^{\rho ij}dV_{ik-}^{\lambda}(s),\ V_{ik-}^{\lambda},\ \widehat{V_{ik-}^{\lambda}},$$

$$\int_0^{\cdot}(\hat{X}_i^{\lambda}(s))^{\rho\infty j}dV_{ik+}^{\lambda}(s),\ V_{ik+}^{\lambda},\ \widehat{V_{ik+}^{\lambda}}\Big\}_{1\le i\le n}^{1\le j\le m;\ 0\le k\le p},$$

$\lambda>0$, to be the law of

$$\Big\{\zeta_i,\ \mathcal{A}_{ij-},\ \mathcal{A}_{ij+},\ \int_0^{\cdot}(-\xi_i(s))^{\rho ij}d\mathcal{V}_{ik-}(s),\ \mathcal{V}_{ik-},\ \widehat{\mathcal{V}_{ik-}},$$

$$\int_0^{\cdot}\xi_i(s)^{\rho\infty j}d\mathcal{V}_{ik+}(s),\ \mathcal{V}_{ik+},\ \widehat{\mathcal{V}_{ik+}}\Big\}_{1\le i\le n}^{1\le j\le m;\ 0\le k\le p},$$

where

$$\begin{cases} \mathcal{V}_{i0-}(t)=\zeta_{i-}(t) \\ \mathcal{V}_{ik-}(t)=\int_0^t\int_0^{2\pi}e_k(\theta)M_i(d\langle\zeta_{i-}\rangle(s),d\theta) & (k=1,\cdots,p) \\ \mathcal{V}_{i0+}(t)=\zeta_{i+}(t) \\ \mathcal{V}_{ik+}(t)=\int_0^t\int_0^{2\pi}e_k(\theta)\tilde{M}_i(d\langle\zeta_{i+}\rangle(s),d\theta) & (k=1,\cdots,p). \end{cases}$$

Therefore if we can prove that

$$\{\widehat{\mathcal{V}_{1k-}},\ \widehat{\mathcal{V}_{2k-}},\ \cdots,\ \widehat{\mathcal{V}_{nk-}},\ \widehat{\mathcal{V}_{1k+}}\}_{0\le k\le p}$$

is an $(n+1)\cdot(p+1)$-dimensional Brownian motion, then the mutual independence of (3.10) follows at once.

To prove this, set

$$\begin{cases} G_{ik^-}{}^{\lambda}(t)=\dfrac{1}{\lambda}\displaystyle\int_0^{\lambda^2 t} e_k\Big(\arg\frac{z_s-a_i}{-a_i}\Big)1_{(\log|z_s-a_i|/-a_i|<-r)}\frac{dz_s}{z_s-a_i} \\ G_{ik^+}{}^{\lambda}(t)=\dfrac{1}{\lambda}\displaystyle\int_0^{\lambda^2 t} e_k\Big(\arg\frac{z_s-a_i}{-a_i}\Big)1_{(\log|z_s-a_i|/-a_i|>r)}\frac{dz_s}{z_s-a_i}\,. \end{cases}$$

By the transformation (3.5), we have

$$G_{ik\pm}{}^{\lambda}(\lambda^{-2}\langle Z_i\rangle^{-1}(\lambda^2 t))=V_{ik\pm}{}^{\lambda}(t)\,.$$

This implies that

$$\langle G_{ik^-}{}^{\lambda},\, G_{il^-}{}^{\lambda}\rangle(\langle G_{ik^-}{}^{\lambda}\rangle^{-1}(t))=\langle V_{ik^-}{}^{\lambda},\, V_{il^-}{}^{\lambda}\rangle(\langle V_{ik^-}{}^{\lambda}\rangle^{-1}(t))\,. \tag{3.16}$$

By (2.15), the right hand side of (3.16) converges to $\begin{pmatrix}0&0\\0&0\end{pmatrix}$ in probability as $\lambda\to\infty$ for any $t>0$ if $k\neq l$. On the other hand, since

$$1_{(\log|z_s-a_i|/-a_i|<-r)}\cdot 1_{(\log|z_s-a_j|/-a_j|<-r)}\equiv 0 \quad\text{if}\quad i\neq j$$

for sufficiently large $r$, we have that

$$\langle G_{ik^-}{}^{\lambda},\, G_{jl^-}{}^{\lambda}\rangle(t)\equiv\begin{pmatrix}0&0\\0&0\end{pmatrix} \quad\text{if}\quad i\neq j \tag{3.17}$$

for any $k$, $l$. Combining (3.16), (3.17) and the obvious relation

$$\langle G_{1k^+}{}^{\lambda},\, G_{il^-}{}^{\lambda}\rangle(t)\equiv\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

for any $i$, $k$, $l$, we can conclude by the asymptotic Knight's theorem in Pitman-Yor [9] that $\{\widehat{G_{1k^-}{}^{\lambda}},\cdots,\widehat{G_{nk^-}{}^{\lambda}},\widehat{G_{1k^+}{}^{\lambda}}\}_{0\le k\le p}$ converges in law to an $(n+1)\cdot(p+1)$-dimensional Brownian motion. Then noting that $\widehat{G_{ik\pm}{}^{\lambda}}(t)=\widehat{V_{ik\pm}{}^{\lambda}}(t)$, we arrive at the needed conclusion.

Now the proof is complete. Q.E.D.

**Remark 3.1.** (due to S. Watanabe)

The $\boldsymbol{C}^n$-valued process $\zeta=(\zeta_1,\cdots,\zeta_n)$ can be constructed as follows: We follow the notions and notations concerning Brownian excursions to [4], Chapter III, section 4.3. Take $n$ poisson point processes of Brownian negative excursions $p_1^-$, $p_2^-$, $\cdots$, $p_n^-$ (*i.e.* stationary Poisson point processes on $\mathcal{W}^-$ with the characteristic measure $\boldsymbol{n}^-$), a Poisson point process of Brownian positive excursion $p^+$ (*i.e.* a stationary Poisson point process on $\mathcal{W}^+$ with the characteristic measure $\boldsymbol{n}^+$) and $n+1$ one-dimensional Brownian motions $\beta_1,\beta_2,\cdots,\beta_n,\beta_+$ such that the family $(p_1^-,\cdots,p_n^-,p^+,\beta_1,\cdots,\beta_n,\beta_+)$ is mutually independent. The sum $p_i$ of $p_i^-$ and $p^+$ is a Poisson point process of Brownian excursions (*i.e.* a stationary Poisson point process on $\mathcal{W}=\mathcal{W}^-\cup\mathcal{W}^+$ with the characteristic measure $\boldsymbol{n}=\boldsymbol{n}^-+\boldsymbol{n}^+$) and we can construct a Brownian motion $\xi_i$ from

$p_i$ as in Chapter III, section 4.3 of [4], $i=1, \cdots, n$. Set

$$\eta_i(t)=\beta_i\Big(\int_0^t 1_{(\xi_i(s)<0)}ds\Big)+\beta_+\Big(\int_0^t 1_{(\xi_i(s)>0)}ds\Big)$$

and define finally

$$\zeta_i(t)=\xi_i(t)+\sqrt{-1}\eta_i(t), \qquad i=1, \cdots, n.$$

Then it is easy to see that $\{\zeta_1, \cdots, \zeta_n\}$ satisfies the conditions (1) and (2) above.

Conversely, suppose we are given a family $\{\zeta_1, \cdots, \zeta_n\}$ possessing the properties (1) and (2). Set

$$\begin{cases} \zeta_{i-}(t)=\int_0^t 1_{(\xi_i(s)<0)}d\zeta_i(s) & (i=1, \cdots, n) \\ \zeta_{i+}(t)=\int_0^t 1_{(\xi_i(s)>0)}d\zeta_i(s) & (i=1, \cdots, n) \end{cases}$$

and write

$$\begin{cases} \zeta_{i-}(t):=\xi_{i-}(t)+\sqrt{-1}\eta_{i-}(t) & (i=1, \cdots, n) \\ \zeta_{i+}(t):=\xi_{i+}(t)+\sqrt{-1}\eta_{i+}(t) & (i=1, \cdots, n) \end{cases} \tag{3.18}$$

and

$$\begin{cases} \widehat{\zeta_{i-}}(t):=\alpha_i(t)+\sqrt{-1}\beta_i(t) & (i=1, \cdots, n) \\ \widehat{\zeta_{1+}}(t)=\widehat{\zeta_{2+}}(t)=\cdots=\widehat{\zeta_{n+}}(t):=\alpha_+(t)+\sqrt{-1}\beta_+(t). \end{cases} \tag{3.19}$$

By the assumptions, $\alpha_1, \cdots, \alpha_n, \alpha_+, \beta_1, \cdots, \beta_n, \beta_+$ are mutually independent 1-dimensional Brownian motions. By Tanaka's formula, we have

$$\xi_i(t)\wedge 0=\xi_{i-}(t)-l_i(t) \qquad (i=1, \cdots, n) \tag{3.20}$$

and

$$\xi_i(t)\vee 0=\xi_{i+}(t)+l_i(t) \qquad (i=1, \cdots, n), \tag{3.21}$$

where $l_i(t)$ is the local time at 0 of one-dimensional Brownian motion $\xi_i(t)$. If we make a time change $t\mapsto\langle\xi_{i-}\rangle^{-1}(t)$ for (3.20) and $t\mapsto\langle\xi_{i+}\rangle^{-1}(t)$ for (3.21), then $\xi_i(\langle\xi_{i-}\rangle)^{-1}(t))\wedge 0$, $i=1, \cdots, n$, are mutually independent reflecting Brownian motions on $(-\infty, 0]$ and $\xi_i(\langle\xi_{i+}\rangle^{-1}(t))\vee 0$, $i=1, \cdots, n$, are the same reflecting Brownian motion on $[0, \infty)$. That is, from (3.20) and (3.21) we have $n+1$ equations

$$\begin{cases} r_i(t)=\alpha_i(t)-\phi_i(t) & (i=1, \cdots, n) \\ r_+(t)=\alpha_+(t)+\phi_+(t), \end{cases} \tag{3.22}$$

where $r_i(t)=\xi_i(\langle\xi_{i-}\rangle^{-1}(t))\wedge 0$, $i=1, \cdots, n$, $r_+(t)=\xi_i(\langle\xi_{i+}\rangle^{-1}(t))\vee 0$, $\phi_i(t)=l_i(\langle\xi_{i-}\rangle^{-1}(t))$, $i=1, \cdots, n$ and $\phi_+(t)=l_i(\langle\xi_{i+}\rangle^{-1}(t))$. These equations give the Skorohod decompositions of $r_i(t)$, $i=1, \cdots, n, +$; in particular,

$$\phi_i(t)=\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t 1_{(0<r_i(s)<\varepsilon)}ds \qquad (i=1, \cdots, n, +).$$

If $p^+$ is the Poisson point process of positive Brownian excursion corresponding to $r_+$

and $p_i^-$, $i=1, \cdots, n$, are the Poisson point processes of negative Brownian excursions corresponding to $r_i$, then $p_1^-, \cdots, p_n^-, p^+, \beta_1, \cdots, \beta_n, \beta_+$ are mutually independent. Thus we have recovered this independent family from $\{\zeta_i\}_{1 \le i \le n}$ and hence, the uniqueness in law of $\{\zeta_i\}_{1 \le i \le n}$ is now obvious.

Set

$$\mu_i(t) = \max_{0 \le s \le t} \xi_i(s) \qquad (i=1, \cdots, n)$$

and

$$\sigma_+(t) = (\max_{0 \le s \le \cdot} r_+(s))^{-1}(t) = \inf\{u\, ;\ r_+(u)=t\}.$$

Then we have the following:

$$l_i(\mu_i^{-1}(t)) = \phi_+(\sigma_+(t)) := e(t), \tag{3.23}$$

$$\begin{cases} \langle \xi_{i-} \rangle (l_i^{-1}(t)) = \phi_i^{-1}(t) \\ \langle \xi_{i+} \rangle (l_i^{-1}(t)) = \phi_+^{-1}(t) \end{cases} \tag{3.24}$$

and

$$\begin{cases} \langle \xi_{i-} \rangle (\mu_i^{-1}(t)) = \phi_i^{-1}(\phi_+(\sigma_+(t))) & (=\phi_i^{-1}(e(t))) \\ \langle \xi_{i+} \rangle (\mu_i^{-1}(t)) = \sigma_+(t) & (\neq \phi_+^{-1}(e(t))). \end{cases} \tag{3.25}$$

These properties are easily deduced by our way of construction of $\{\zeta_i(t)\}_{1 \le i \le n}$, *cf.* [4]. The structure of the process $t \mapsto e(t)$ is well known: It is the inverse of the Dwass's extremal process (*cf.* [2]), in particular, for fixed $t>0$ $e(t)$ has the exponential distribution with mean $t$.

Putting together (3.18), (3.19), (3.22), (3.24) and (3.25), and noting that $r_i(\phi_i^{-1}(t)) \equiv 0$ $(i=1, \cdots, n, +)$ and $r_+(\sigma_+(t)) \equiv t$, we can express $\zeta_{i\pm}(l_i^{-1}(t))$ and $\zeta_{i\pm}(\mu_i^{-1}(t))$ as follows:

$$\begin{cases} \zeta_{i-}(l_i^{-1}(t)) = t + \sqrt{-1}\, C_i(t) \\ \zeta_{i+}(l_i^{-1}(t)) = -t + \sqrt{-1}\, C_+(t) \end{cases} \tag{3.26}$$

and

$$\begin{cases} \zeta_{i-}(\mu_i^{-1}(t)) = e(t) + \sqrt{-1}\, C_i(e(t)) \\ \zeta_{i+}(\mu_i^{-1}(t)) = t - e(t) + \sqrt{-1}\, \beta_+(\sigma_+(t)), \end{cases} \tag{3.27}$$

where

$$C_i(t) = \beta_i(\phi_i^{-1}(t)) \qquad (i=1, \cdots, n, +).$$

Note that $C_1, \cdots, C_n, C_+$ are mutually independent Cauchy processes in (3.26). Note also that $C_1, \cdots, C_n, r_+, \beta_+$ are mutually independent in (3.27).

These processes appear as components of limit process of windings of $z(t)$: Theorem 3.2 implies that

$$\{W_{i-}^{\lambda}, W_{i+}^{\lambda}\}_{1 \le i \le n} \longrightarrow \{\zeta_{i-}(\mu_i^{-1}), \zeta_{i+}(\mu_i^{-1})\}_{1 \le i \le n}$$

as $\lambda \to \infty$ in the sense of $K$-convergence, where

$$W_{i\pm}^{\lambda}(t) = \frac{1}{\lambda} \int_0^{u(\lambda t)} \frac{1}{z_s - a_i} 1_{D(i\pm)}(z_s)\, dz_s \qquad (i=1, \cdots, n).$$

Taking $D(i+)=D(i-)^c$, the process $\mathcal{I}m[W_{i-}{}^{\lambda}(t)+W_{i+}{}^{\lambda}(t)]$ is a normalized algebraic total angle wound by $z(t)$ arocnd $a_i$ up to the time $u(\lambda t)=e^{2\lambda t}-1$. Then the imaginary parts of (3.27) clearly show that the primary description by Pitman and Yor ([7]) of the asymptotic joint distribution of windings of $z_t$.

In addition, using above analysis, we give an another description of the joint limit process of windings of $z_t$ below. Let $g(z)$ be a bounded function such that

$$\int_C |g(z)|\,|z|^{\varepsilon} m(dz)<\infty$$

for some $\varepsilon>0$, where $m(dz)$ denotes the Lebesgue integral. Set

$$\bar{g}=\frac{1}{2\pi}\int_C |g(z)|\,m(dz)$$

and

$$T^{\lambda}(t)=\frac{1}{\lambda}\int_0^{u(\lambda t)} g(z_s)ds\,.$$

Then, by Kasahara-Kotani's result (see [4]), we have

$$\{\hat{Z}_i{}^{\lambda},\ T_i{}^{\lambda}(\tau_i{}^{\lambda})\}_{1\leq i\leq n} \longrightarrow \{\zeta_i,\ 2\bar{g}l_i(\mu_i{}^{-1})\}_{1\leq i\leq n}=\{\zeta_i,\ 2\bar{g}e\}_{1\leq i\leq n} \tag{3.28}$$

as $\lambda\to\infty$ in the sense of $K$-convergence. Combining (3.28) and Theorem 3.1, we have

$$\{W_{i-}{}^{\lambda}((T_i{}^{\lambda})^{-1}),\ W_{i+}{}^{\lambda}((T_i{}^{\lambda})^{-1})\}_{1\leq i\leq n} \longrightarrow \{\zeta_{i-}(l_i{}^{-1}(\cdot/(2\bar{g}))),\ \zeta_{i+}(l_i{}^{-1}(\cdot/(2\bar{g})))\}_{1\leq i\leq n}$$

as $\lambda\to\infty$ in the sense of $K$-convegence if $g(z)>0$. By (3.26), we can express this last limit process as

$$\begin{cases} \zeta_{i-}(l_i{}^{-1}(t/(2\bar{g})))= t/(2\bar{g})+\sqrt{-1}C_i(t/(2\bar{g})) \\ \zeta_{i+}(l_i{}^{-1}(t/(2\bar{g})))=-t/(2\bar{g})+\sqrt{-1}C_+(t/(2\bar{g}))\,. \end{cases}$$

This is one of natural (symmetric) descriptions for the joint limit process of windings of $z(t)$ in the compact Riemannian surface $\boldsymbol{C}\cup\{\infty\}$.

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