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Bernstein-Sato's polynomial for several analytic functions

By

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Introduction

Let X be a complex manifold, $\mathscr{D}_X$ (resp. $\mathscr{O}_X$) the sheaf of differential operators (resp. holomorphic functions), $x_0 \in X$, $f_1, \ldots, f_l \in \mathscr{O}_{X,x_0}$, $\zeta_1, \ldots, \zeta_l$ independent complex variables, and $\mathscr{D}_X[\zeta_1, \ldots, \zeta_l] = \mathscr{D}_X \otimes_{\mathbf{C}} \mathbf{C}[\zeta_1, \ldots, \zeta_l]$. The purpose of this paper is to prove the following theorem.

Theorem. (1) *For any $\mu = (\mu_1, \ldots, \mu_l) \in \mathbf{N}^l$, there exist a differential operator $P_\mu(\zeta) \in \mathscr{D}_{X,x_0}[\zeta_1, \ldots, \zeta_l]$ and a non-zero polynomial $b_\mu(\zeta) \in \mathbf{C}[\zeta_1, \ldots, \zeta_l]$ such that*

$$P_\mu(\zeta) f_1^{\zeta_1+\mu_1} \cdots f_l^{\zeta_l+\mu_l} = b_\mu(\zeta) f_1^{\zeta_1} \cdots f_l^{\zeta_l}.$$

(2) *Moreover, we can take $b_\mu(\zeta)$ so that*

$$b_\mu(\zeta) = \prod_i (\alpha_{i1}\zeta_1 + \cdots + \alpha_{il}\zeta_l + a_i),$$

where $\alpha_{ij} \in \mathbf{N}$, $\mathrm{GCD}(\alpha_{i1}, \ldots, \alpha_{il}) = 1$ and $a_i \in \mathbf{Q}_{>0}$ for any i.

Historically, the above theorem goes back to a conjecture of M. Sato based on his theory of prehomogeneous vector spaces (see [8]). When $l = 1$, (1) is proved by I. N. Bernstein [1] for $X = \mathbf{C}^n$ and a polynomial f, and by J. E. Björk in general, and (2) is proved by M. Kashiwara [3]. For a general l, the whole of the assertion except the part concerning a_i is proved by C. Sabbah [7]. Thus what is new in the above theorem is that we can take a_i to be positive rational numbers. Our proof here is based on the results and techniques given in [3], [4], [6] and [7].

In order to make the exposition easier to read, we exclusively consider a connected smooth affine variety X over $\mathbf{C}$ and the ring $D = D_X$ of (global sections of) algebraic differential operators, except the final step of the proof. Thus what we shall prove in this paper is an algebraic version of the above theorem. But the necessary modification to get the above theorem will be obvious.

The author learnt from Prof. C. Sabbah that he has also got the same result (unpublished). The author would like to express his thanks to Prof. C. Sabbah for his kind communication.

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Convention. The field of complex (resp. rational) numbers is denoted by $\mathbf{C}$ (resp. $\mathbf{Q}$). The rational integer ring is denoted by $\mathbf{Z}$, and $\mathbf{N}=\{0, 1, 2,\dots\}$.

Except for $f=(f_1,\dots,f_l)$, we denote an l-tuple by a lowercase Greek letter, and its i-th component by the same letter with the suffix i. Concerning the l-tuples, we use the following notations. For an l-tuple τ of variables, $\partial_\tau=(\partial_{\tau_1},\dots)$. $\left(\text{For a complex variable, say } u, \text{ sometimes we write } \partial_u \text{ for } \frac{\partial}{\partial u}.\right)$ For l-tuples α and β, $\alpha^{-1}=(\alpha_1^{-1},\dots)$, $\alpha\beta=\alpha\cdot\beta=(\alpha_1\beta_1,\dots)$ and $(\alpha|\beta)=\sum\alpha_i\beta_i$, e.g., $\partial_\tau\tau=(\partial_{\tau_1}\tau_1,\dots)$. For $\gamma\in\mathbf{N}^l$ (or $\mathbf{Z}^l$ in some cases), $\alpha^\gamma=\prod\alpha_i^{\gamma_i}$ and $u^\gamma=(u^{\gamma_1},\dots)$, where α is an l-tuple and u is a single variable etc. For $\alpha, \beta\in\mathbf{Z}^l$, $\alpha\geq\beta$ means that $\alpha_i\geq\beta_i$ for any i. We denote the standard basis of $\mathbf{C}^l$ by $\varepsilon(i)$ $(1\leq i\leq l)$.

1. Preliminaries

1.1. Let X be a connected smooth affine variety over $\mathbf{C}$, and D_X the ring of algebraic differential operators. Let $\mathbf{C}[s, t]$ be the $\mathbf{C}$-algebra with the defining relation $ts=(s+1)t$, and $\mathscr{M}$ the abelian category of pairs (M, i), where M is a finitely generated left D-module and i is a $\mathbf{C}$-algebra homomorphism $\mathbf{C}[s, t]\to\mathrm{End}_{\mathbf{C}}(M)$ such that $i(t)\in\mathrm{End}_D(M)$. (A morphism in $\mathscr{M}$ is a D-homomorphism which commutes with the $\mathbf{C}[s, t]$-module structures.) We write s and t for $i(s)$ and $i(t)$, and we write $M\in\mathscr{M}$ instead $(M, i)\in\mathscr{M}$. For $M\in\mathscr{M}$, let $b_M(x)=b(x, M)$ be the monic generator of the ideal of polynomials $b(x)\in\mathbf{C}[x]$ such that $b(s)=0$ as a $\mathbf{C}$-endomorphism of M/tM (not necessarily compatible with the D-module structure). Here we admit b_M to be zero.

A finitely generated D-module M is called *holonomic* (resp. *subholonomic*) if the dimension of the characteristic variety is $\dim X$ (resp. $\leq\dim X+1$).

Lemma 1.2. *Let $N, N'\in\mathscr{M}$, where N' is a quotient of N. Then $b_{N'}$ divides b_N.*

The proof is easy and omitted.

Lemma 1.3. *Let $N, N', N''\in\mathscr{M}$, where $N''\subset N'$ and N is a quotient of N''. Assume that N is subholonomic and $t\colon N\to N$ is injective. Then $b(x, N)$ divides $\prod_{i=0}^k b(x+i, N')$ for a sufficiently large k.*

Proof. Let $K=\ker(N''\to N)$. then $b_{N'/K}$ divides $b_{N'}$, and $N=N''/K\subset N'/K$. Hence we may assume from the beginning that $N=N''\subset N'$.

The increasing sequence $\ker(t^j|N')$ $(j=1, 2,\dots)$ becomes stationary for sufficiently large j. (Note that D is a left noetherian ring.) Let $\ker(t^p|N')=\ker(t^{p+1}|N')=\cdots=:L$. Then $L\in\mathscr{M}$, $L\cap N=0$ and $N\subset N'/L$. Since $b_{N'/L}$ divides $b_{N'}$, we may assume from the beginning that $t\colon N'\to N'$ is injective. Then $b(s, N')=t\cdot r$ with some $r\in\mathrm{End}_{\mathbf{C}}(N')$.

Put $t^{-j}N=\{u\in N'|t^ju\in N\}$. The increasing sequence $t^{-j}N$ $(j=1, 2,\dots)$ becomes stationary for sufficiently large j. Let $t^{-q}N=t^{-q-1}N=\cdots=:N_0$. As is easily seen, $N_0\in\mathscr{M}$. Especially $b(s, N')N_0\subset N_0$. Hence $t^q\cdot trN_0\subset N$, i.e.,

$rN_0 \subset t^{-q-1}N = N_0$. Thus r induces an endomorphism, say r_0, of N_0, and we have $b(s, N') = tr_0$ in $\mathrm{End}_{\mathbf{C}}(N_0)$, which implies that b_{N_0} divides $b_{N'}$. Hence we may assume that $N' = N_0$.

Let C be an irreducible subvariety of the cotangent bundle of X. For a finitely generated D-module M, denote its multiplicity along C by $m(C, M)$. Since $N \subset N_0$, $t^q N_0 \subset N$ and $t\colon N_0 \to N_0$ is injective, we have

$$m(C, N) \leq m(C, N_0) = m(C, t^q N_0) \leq m(C, N).$$

Thus the characteristic cycle of N_0 coincides with that of N. Since N is subholonomic, N_0/N is holonomic. (Cf. the proof of [3, Corollary (3.2)].) Similar argument shows that N/tN is also holonomic. Hence by [3, Corollary (5.14)], $b(x, N)$ divides $\prod_{i=0}^{k} b(x+i, N_0)$ for a sufficiently large k. (In [3], s is assumed to be commutative with the D-module structure. But this assumption in not used there.) Thus we have completed the proof.

1.4. Let X_0 be a simply connected open subset of $\bigcap_{i=1}^{l} f_i^{-1}(\mathbf{C}^\times)$, $X_0 \times \mathbf{C}^l \ni (x, \zeta) \to f^\zeta = \prod_i f_i(x)^{\zeta_i}$ a single-valued branch, $\mathbf{C}[\zeta] = \mathbf{C}[\zeta_1, \ldots, \zeta_l]$ and $D_X[\zeta] = D_X \otimes_{\mathbf{C}} \mathbf{C}[\zeta]$. Thus we get a $D_X[\zeta]$-module $D_X[\zeta]f^\zeta$. Let $\delta(u)$ be the standard generator of $D_{\mathbf{C}}/D_{\mathbf{C}}u$, $\tau = (\tau_1, \ldots, \tau_l)$ an l-tuple of independent complex variables, and $\delta(\tau - f(x)) = \prod_i \delta(\tau_i - f_i(x))$. Thus we get a $D_X[-\partial_\tau \tau]$ module $D_X[-\partial_\tau \tau]\delta(\tau - f(x))$.

Lemma 1.5. *By the correspondence* $\zeta \leftrightarrow -\partial_\tau \tau$ *and* $f^\zeta \leftrightarrow \delta(\tau - f)$, $D_X[\zeta]f^\zeta \simeq D_X[-\partial_\tau \tau]\delta(\tau - f)$.

Proof. Let $\zeta = (\zeta_1, \ldots, \zeta_l)$, $\tau = (\tau_1, \ldots, \tau_l)$, and $\mathbf{C}[\zeta, \tau, \tau^{-1}]$ be the $\mathbf{C}$-algebra defined by the relations $\tau_i \tau_i^{-1} = \tau_i^{-1}\tau_i = 1$, $[\zeta_i, \zeta_j] = [\tau_i, \tau_j] = 0$ and $\tau_i \zeta_j = (\zeta_j + \delta_{ij})\tau_i$, where δ_{ij} is the Kronecker delta. By the correspondence $\zeta \leftrightarrow -\partial_\tau \tau$, $\mathbf{C}[\zeta, \tau, \tau^{-1}] \simeq \mathbf{C}[-\partial_\tau \tau, \tau, \tau^{-1}] = D_{\mathbf{C}^l}[\tau^{-1}]$. Hence $D_X[\zeta, \tau, \tau^{-1}] := D_X \otimes_{\mathbf{C}} \mathbf{C}[\zeta, \tau, \tau^{-1}] \simeq D_{X \times \mathbf{C}^l}[\tau^{-1}]$. The annihilator of $\delta(\tau - f)$ in $D_{X \times \mathbf{C}^l}[\tau^{-1}]$ is the left ideal generated by

$$\tau_i - f_i (1 \leq i \leq l), \quad \text{and} \quad v + \sum_{i=1}^{l} v(f_i)\frac{\partial}{\partial \tau_i},$$

where v is any vector field on X. These elements correspond to

$$(1.5.1) \qquad \tau_i - f_i, \quad \text{and} \quad v - \sum_{i=1}^{l} v(f_i)\zeta_i \tau_i^{-1} \in D_X[\zeta, \tau, \tau^{-1}]$$

by the above isomorphism. Define the $D[\zeta, \tau, \tau^{-1}]$-module structure of $D_X[\zeta, f^{-1}]f^\zeta$ by

$$\zeta_i\colon P(\zeta)f^\zeta \longrightarrow \zeta_i P(\zeta)f^\zeta \quad \text{and} \quad \tau_i\colon P(\zeta)f^\zeta \longrightarrow P(\zeta + \varepsilon(i))f^{\zeta + \varepsilon(i)}.$$

We can show that the elements (1.5.1) generate the annihilator of f^ζ in $D_X[\zeta, \tau, \tau^{-1}]$. Hence $\delta(\tau - f) \to f^\zeta$ gives an isomorphsim $D_{X \times \mathbf{C}^l}[\tau^{-1}]\delta(\tau - f) \to D_X[\zeta, \tau, \tau^{-1}]f^\zeta = D_X[\zeta, f^{-1}]f^\zeta$. Since the image of $D_X[-\partial_\tau \tau]\delta(\tau - f)$ is $D_X[\zeta]f^\zeta$, we get the desired isomorphism.

2. Proof (first step)

2.0. In this section, we follow the argument of [6] and [7].

2.1. Let A be the set of $\alpha \in \mathbf{N}^l$ such that $\mathrm{GCD}(\alpha_1, \ldots, \alpha_l) = 1$. We fix an element $\alpha \in A$ throughout this section. Put $E = \mathbf{C}^l$, $M = D_{X \times E}\delta(\tau - f)$,

$$D_E^\mu = \sum_{\substack{\beta, \gamma \in \mathbf{N}^l \\ \beta - \gamma \geq \mu}} \mathbf{C}\tau^\beta \partial_\tau^\gamma, \quad D_E^{\alpha \geq j} = \sum_{\substack{\mu \in \mathbf{Z}^l \\ (\alpha|\mu) \geq j}} D_E^\mu, \quad D_E(u) = \sum_{j \in \mathbf{Z}} D_E^{\alpha \geq j} u^{-j},$$

$$D_{X \times E}^\mu = D_X \otimes_{\mathbf{C}} D_E^\mu, \quad D_{X \times E}^{\alpha \geq j} = D_X \otimes_{\mathbf{C}} D_E^{\alpha \geq j}, \quad D_{X \times E}(u) = D_X \otimes_{\mathbf{C}} D_E(u),$$

$$M^\mu = D_{X \times E}^\mu \delta(\tau - f), \quad M^{\alpha \geq j} = D_{X \times E}^{\alpha \geq j}\delta(\tau - f), \quad M(u) = D_{X \times E}(u)\delta(\tau - f),$$

where u is a new complex variable. We consider $D_E(u)$ as a subring of $D_E \otimes_{\mathbf{C}} \mathbf{C}(u)$, and $M(u)$ as a submodule of $(D_{X \times E} \otimes_{\mathbf{C}} \mathbf{C}(u))\delta(\tau - f)$. Let E' be a copy of E. By the transformation $B\colon E' \times \mathbf{C} \to E \times \mathbf{C}$ defined by $(\tau, u) = B(\tau', u) = (u^\alpha \tau', u)$, $D_E(u)$ is identified with $D_{E'}[u] = D_{E'} \otimes_{\mathbf{C}} \mathbf{C}[u]$. Hence $M(u)$ has a $D_{X \times E'}[u]$-module structure. Let u' be a new complex variable, and put $N = N_f = M(u) \otimes_{\mathbf{C}[u]} \mathbf{C}[u, \partial_{u'}]\delta(u - u')$. Then N is a left module over $D_{X \times E'}[u] \otimes_{\mathbf{C}} \mathbf{C}[u', \partial_{u'}] = D_{X \times E' \times \mathbf{C}}[u]$.

Lemma 2.2. (1) *N is a subholonomic $D_{X \times E' \times \mathbf{C}}$-module.*
(2) *N/uN is a holonomic $D_{X \times E' \times \mathbf{C}}$-module.*

Proof. Since $M(u) = D_{X \times E}(u)\delta(\tau - f) = D_{X \times E'}[u]\delta(u^\alpha \tau' - f)$,

$$N = D_{X \times E' \times \mathbf{C}}\delta(u^\alpha \tau' - f(x))\delta(u - u').$$

Hence there exists a maximum subholonomic submodule N_1 of N [3, Theorem (2.10)]. Since $g := \delta(u^\alpha \tau' - f)\delta(u - u')$ satisfies

$$\left(\frac{\partial}{\partial x_i} + \sum_{j=1}^{l} \frac{\partial f_j}{\partial x_i} u'^{-\alpha} \frac{\partial}{\partial \tau_j'}\right) g = 0 \quad \text{and} \quad (\tau_j' - u'^{-\alpha_j} f_j)g = 0,$$

$N[u'^{-1}]$ is a subholonomic $D_{X \times E' \times \mathbf{C}}[u'^{-1}]$-module. (Here $(x_1, \ldots)$ is a local coordinate system of X.) Hence $(N/N_1)[u'^{-1}] = 0$. Especially, $u'^k g = u^k g \in N_1$ for a sufficiently large k. Since $u\colon N \to N$ is an injective $D_{X \times E' \times \mathbf{C}}$-endomorhpism, $N \simeq u^k N = D_{X \times E' \times \mathbf{C}}(u^k g) \subset N_1$. Hence N is subholonomic. Hence N/uN is holonomic.

2.3. Let u be the $D_{X \times E'}$-endomorphism of $M(u)$ defined by the multiplication by u. Define a D_X-endomorphism $C = C_M$ of $M(u)$ by

$$M(u) \supset M^{\alpha \geq j} u^{-j} \ni m \cdot u^{-j} \xrightarrow{C} ((\alpha| - \partial_\tau \tau) + j)m \cdot u^{-j}.$$

Lemma 2.4. (1) $uC = (C+1)u$.

(2) *For* $P \in D_X$, $\beta, \gamma \in \mathbf{N}^l$ *and* $k \in \mathbf{N}$,

$$C(P\tau'^{\beta}\partial_{\tau'}^{\gamma}\delta(u^{\alpha}\tau' - f)\cdot u^k) = P\tau'^{\beta}\partial_{\tau'}^{\gamma}((\alpha| - \partial_{\tau'}\tau') - k)\delta(u^{\alpha}\tau' - f)\cdot u^k.$$

Especially, C *is a* $D_{X\times E'}$-endomorphism of $M(u)$.

These assertions can be proved by a direct calculation.

2.5. Let us "extend" the endomorphism C of $M(u)$ to N. As a $\mathbf{C}$-vector space, $L := \mathbf{C}[u, \partial_{u'}]\delta(u - u') \simeq \mathbf{C}[u, \partial_{u'}]$. Let $C(u^p\partial_{u'}^q) = -pu^p\partial_{u'}^q$, and consider it as an endomorphism of $\mathbf{C}[u, \partial_{u'}]\delta(u-u')$, which we shall denote by $C = C_L$. Define an endomorphism $C = C_N$ of N by $C_N = C_M \otimes 1 + 1 \otimes C_L$.

Lemma 2.6. *For any* $P \in \mathbf{C}[u', \partial_{u'}]$, $[C_L, P]L \subset uL$.

Proof. We may assume that $P = u'^i\partial_{u'}^j$. Then

$$[C, P]u^p\partial_{u'}^q\delta(u-u') \in uL \qquad \text{if } p > 0, \text{ and}$$

$$[C, P]\partial_{u'}^q\delta(u-u') \in C(L) = uL.$$

Lemma 2.7. (1) $C = C_N$ *is a well-defined* D_X*-endomorphism of* N.

(2) $uC = (C+1)u$.

(3) C *induces a* $D_{X\times E'\times\mathbf{C}}$*-endomorphism of* N/uN.

Proof. (1) In order to prove the well-definedness, it suffices to show that

$$Cum \otimes n + um \otimes Cn = Cm \otimes un + m \otimes Cun \tag{2.7.1}$$

for $m \in M(u)$ and $n \in \mathbf{C}[u, \partial_{u'}]\delta(u-u')$. Since we have $uC = (C+1)u$ as endomorphisms of $M(u)$ and $\mathbf{C}[u, \partial_{u'}]\delta(u-u')$, both members of (2.7.1) are equal to

$$-um \otimes n + Cm \otimes un + um \otimes Cn.$$

(2) is now obvious. It remains to prove that the endomorphism C of N/uN commutes with the $D_{\mathbf{C}}$-module structure. For $P \in D_{\mathbf{C}}$, $m \in D_{X\times E'}\delta(u^{\alpha}\tau' - f)$ $(\subset M(u))$ and $n \in \mathbf{C}[u, \partial_{u'}]\delta(u-u') = \mathbf{C}[u', \partial_{u'}]\delta(u-u')$, we have

$$\begin{aligned} &CP(m\otimes n) - PC(m\otimes n) \\ &= (Cm \otimes Pn + m \otimes CPn) - (Cm \otimes Pn + m \otimes PCn) \\ &= m \otimes [C, P]n \in uN \end{aligned}$$

by (2.6).

Lemma 2.8 ([4, Theorem 4.8] and [5, §5]). *The endomorphism algebra of a holonomic D-module is a finite dimensional* $\mathbf{C}$*-vector space.*

2.9. By (2.2, (2)) and (2.8), the totality of $D_{X\times E'\times\mathbf{C}}$-endomorphisms of N/uN forms a finite dimensional $\mathbf{C}$-vector space. Since C is a $D_{X\times E'\times\mathbf{C}}$-endomorphism of N/uN by (2.7, (3)), there is a non-zero polynomial $b(x) \in \mathbf{C}[x]$ such that $b(C) = 0$ as an endomorphism of N/uN.

Lemma 2.10. *For a polynomial $b(x) \in \mathbf{C}[x]$, the following conditions are equivalent.*

(1) $b(C) = 0$ *as an endomorphism of* N/uN.

(2) $b(C) = 0$ *as an endomorphism of* $M(0) := M(u)/uM(u)$.

(3) $b((\alpha| - \partial_\tau \tau) + j)M^{\alpha \geq j} \subset M^{\alpha \geq j+1}$ *for any* $j \in \mathbf{Z}$.

(4) $b((\alpha| - \partial_\tau \tau))M^{\alpha \geq 0} \subset M^{\alpha \geq 1}$.

(5) $b((\alpha| - \partial_\tau \tau))\delta(\tau - f) \in M^{\alpha \geq 1}$.

(6) $b((\alpha| - \partial_\tau \tau))\delta(\tau - f) \in uD_{X \times E}(u)\delta(\tau - f)$.

(7) $b((a| - \partial_{\tau'} \tau'))\delta(u^\alpha \tau' - f) \in uD_{X \times E'}[u]\delta(u^\alpha \tau' - f)$.

(8) $b((\alpha| - \partial_{\tau'} \tau'))\delta(u^\alpha \tau' - f)\delta(u - u') \in uD_{X \times E' \times \mathbf{C}}\delta(u^\alpha \tau' - f)\delta(u - u')$.

Proof. Note that the restriction of C to $M(0) \otimes_{\mathbf{C}} \mathbf{C}\delta(u - u')(\subset M(0) \otimes_{\mathbf{C}} \mathbf{C}[\partial_{u'}]\delta(u - u') = N/uN)$ can be naturally identified with $C|M(0)$. Since C commutes with $\mathbf{C}[\partial_{u'}]$, $b(C)|(N/uN) \equiv 0$ if and only if $b(C)|M(0) \otimes \mathbf{C}\delta(u - u') \equiv 0$, i.e., $b(C)|M(0) \equiv 0$. Thus we get (1) $\Longleftrightarrow$ (2). Since

$$M(0) = M(u)/uM(u) = \sum_{j \in \mathbf{Z}} \frac{M^{\alpha \geq j}}{M^{\alpha \geq j+1}} \cdot u^{-j},$$

we get (2) $\Leftrightarrow$ (3). The implications (1) $\Leftrightarrow$ (8), (3) $\Rightarrow$ (4) $\Rightarrow$ (5), and (6) $\Leftrightarrow$ (7) are trivial. For $P_{\beta,\gamma} \in D_X$, we have

$$b((\alpha| - \partial_\tau \tau) + j) \sum_{\substack{\beta, \gamma \in \mathbf{N}^l \\ (\alpha|\beta - \gamma) \geq j}} P_{\beta\gamma} \tau^\beta \partial_\tau^\gamma \delta(\tau - f)$$

$$= \sum_{(\alpha|\beta - \gamma) \geq j} P_{\beta\gamma} \tau^\beta \partial_\tau^\gamma b((\alpha| - \partial_\tau \tau) + j - (\alpha|\beta - \gamma))\delta(\tau - f)$$

$$\equiv \sum_{(\alpha|\beta - \gamma) = j} P_{\beta\gamma} \tau^\beta \partial_\tau^\gamma b((\alpha| - \partial_\tau \tau))\delta(\tau - f) \bmod M^{\alpha \geq j+1}.$$

Hence (3) $\Leftarrow$ (5). Since

$$uD_{X \times E}(u)\delta(\tau - f) = \sum_{j \in \mathbf{Z}} M^{\alpha \geq j} u^{-j+1},$$

and $b((\alpha| - \partial_\tau \tau))\delta(\tau - f)$ is free from u, we get (5) $\Longleftrightarrow$ (6).

2.11. For $\nu \in \mathbf{Z}^l$, put

$$\bar{M}^\nu := \bigcap_{\alpha \in A} M^{\alpha \geq (\alpha|\nu)} = \bigcap_{\alpha \in A} \Big(\sum_{\substack{\mu \in \mathbf{Z}^l \\ (\alpha|\mu) \geq (\alpha|\nu)}} M^\mu \Big).$$

(See (2.1) for A.)

Lemma 2.12 [6, 2.2]. (1) *There are finite number of* $\alpha(i)$*'s in* A *such that* $\bar{M}^{\mu}=\bigcap_i M^{\alpha(i)\geq(\alpha(i)|\mu)}$ *for any* $\mu\in\mathbf{Z}^l$.

(2) *There exists* $\kappa\in\mathbf{N}^l$ *such that* $M^{\mu}\subset\bar{M}^{\mu}\subset M^{\mu-\kappa}$ *for any* $\mu\in\mathbf{Z}^l$.

2.13. Let $b_{\alpha}(x)=b(\alpha,x)$ be the monic polynomial of the minimal degree satisfying the equivalent conditions of (2.10). (See (2.1) for α.) For a $\mu\in\mathbf{N}^l$, we have

$$\prod_{0\leq j<(\alpha(i)|\mu)} b(\alpha(i),(\alpha(i)|-\partial_\tau\tau)+j)M^{\alpha(i)\geq 0}\subset M^{\alpha(i)\geq(\alpha(i)|\mu)},$$

$$\prod_{\substack{i\\0\leq j<(\alpha(i)|\mu)}} b(\alpha(i),(\alpha(i)|-\partial_\tau\tau)+j)\bar{M}^{0}\subset\bar{M}^{\mu},\text{ and}$$

$$\prod_{\substack{i\\0\leq j<(\alpha(i)|\mu+\kappa)}} b(\alpha(i),(\alpha(i)|-\partial_\tau\tau)+j)M^{0}\subset M^{\mu}.$$

By the last inclusion relation and by the isomorphism given in (1.5), we get the functional equation

$$\prod_{\substack{i\\0\leq j<(\alpha(i)|\mu+\kappa)}} b(\alpha(i),(\alpha(i)|\zeta)+j)\cdot f^{\zeta}=P_{\mu}(\zeta)f^{\zeta+\mu}$$

with some $P_{\mu}(\zeta)\in D_X[\zeta]$. (Note that M^0 contains $\delta(\tau-f)$, which corresponds to f^{ζ} by the isomorphism of (1.5).)

Thus it remains to prove that the zeros of the minimal polynomial $b_{\alpha}(x)$ of $C\in\mathrm{End}_D(N/uN)$ are negative rational numbers for each $\alpha\in A$.

3. Proof (second step)

3.0. Here we consider the case where $\bigcup_i f_i^{-1}(0)$ is normal crossing. For the sake of simplicity, we assume that $X=\mathbf{C}^n$ and f_i's are monomials of the coordinate functions.

Lemma 3.1. *Let* $\alpha,\xi(1),\dots,\xi(h)$ *be vectors in* $\mathbf{Q}^l$. *Put* $H=\{1,2,\dots,h\}$, $\langle I\rangle:=\sum_{i\in I}\mathbf{Q}_{\geq 0}\xi(i)$ *for* $I\subset H$, $\bar{I}:=\{i\in H|\xi(i)\in\langle I\rangle\}$, $\mathscr{T}:=\{\bar{I}|I\subset H\}$, $\mathscr{I}:=\{I\in\mathscr{T}|\alpha\notin\langle I\rangle\}$, *and let* $i:\mathscr{I}\to H$ *be a mapping such that* $i(I)\notin I$ *for any* I. *Then* $\alpha\in\langle i(\mathscr{I})\rangle$.

Proof. Assume that $\alpha\notin\langle i(\mathscr{I})\rangle(=\langle\overline{i(\mathscr{I})}\rangle)$. Then $I_0:=\overline{i(\mathscr{I})}\in\mathscr{I}$. But by the assumption on i, $i(I_0)\notin I_0$, and hence $I_0\notin\mathscr{I}$.

Lemma 3.2. *The following conditions for* $I\subset H$ *are equivalent.*

(1) $\alpha\notin\langle I\rangle$.

(2) *There exists* $\mu \in \mathbf{Z}^l$ *such that* $(\alpha|\mu) > 0$ *and* $(\xi(i)|\mu) \le 0$ *for any* $i \in I$.

We omit the proof.

Lemma 3.3. *Keep the notations of* (3.1). *Let* $M = M(\alpha) = \{\mu \in \mathbf{Z}^l | (\alpha|\mu) > 0\}$, $I(\mu) = \{i \in H | (\xi(i)|\mu) \le 0\}$ *and* $i\colon M \to H$ *be a mapping such that* $i(\mu) \notin I(\mu)$ *for any* $\mu \in M$. *Then* $\alpha \in \langle i(M) \rangle$.

Proof. By (3.2), $\{I(\mu)|\mu \in M\} = \mathscr{I}$. For any $I \in \mathscr{I}$, take $\mu(I) \in M$ so that $I(\mu(I)) = I$, and put $i(I) = i(\mu(I))$. Then $i(I) = i(\mu(I)) \notin I(\mu(I)) = I$ and

$$\alpha \in \langle i(\mathscr{I}) \rangle = \langle i\mu(\mathscr{I}) \rangle \subset \langle i(M) \rangle.$$

Lemma 3.4. (1) *Keep the notations of the previous lemmas. Let* $K(i, \mu)$ $(i \in H, \mu \in \mathbf{Z}^l)$ *be finite subsets of* $\mathbf{Q}$, *and put*

$$b_\mu(\zeta) = b(\mu, \zeta) = \prod_{\substack{i \in H \\ a \in K(i,\mu)}} ((\xi(i)|\zeta) + a).$$

Assume that $K(i, \mu) = \phi$ *if* $(\xi(i)|\mu) \le 0$. *Then the ideal* $J(\alpha)$ *of* $\mathbf{C}[\zeta] = \mathbf{C}[\zeta_1, \dots, \zeta_l]$ *generated by* $\{b_\mu | \mu \in M(\alpha)\}$ *contains a polynomial of the form* $\prod_j((\alpha|\zeta) + a_j)$ *with* $a_j \in \mathbf{Q}$.

(2) *Assume further that* $K(i, \mu) \subset \mathbf{Q}_{\ge 0}$ *for any* i *and* μ. *Let* $H' = \{i \in H | 0 \in K(i, \mu)$ *for some* $\mu \in M(\alpha)\}$, *and assume also that* $\alpha \notin \langle H' \rangle$. *Then we can take the above polynomial so that* $a_j > 0$.

Proof. Let

$$M = M(\alpha) \ni \mu \longrightarrow (i(\mu), a(\mu)) \in H \times \mathbf{Q}$$

by any mapping such that $a(\mu) \in K(i(\mu), \mu)$. (Possibly such a mapping does not exist.) If $i(\mu) \in I(\mu)$ for some $\mu \in M$, then $(\xi(i(\mu))|\mu) \le 0$ and $K(i(\mu), \mu) = \phi$, which is absurd. Hence $i(\mu) \notin I(\mu)$ for any $\mu \in M$. By (3.3), $\alpha \in \langle i(M) \rangle$. Let $\alpha = \sum_{\mu \in M} c(\mu)\xi(i(\mu))$, where $c(\mu) \in \mathbf{Q}_{\ge 0}$ and $c(\mu) = 0$ for almost all μ's. Put $a = \sum_\mu c(\mu)a(\mu)$.

If the assumptions in (2) are satisfied, then $a > 0$. In fact, it suffices to show that $c(\mu) > 0$ and $a(\mu) > 0$ for some $\mu \in M$. Since $\alpha \notin \langle H' \rangle$, we have $c(\mu_0) > 0$ and $i(\mu_0) \notin H'$ for some $\mu_0 \in M$. Then $0 \notin K(i(\mu_0), \mu_0)$, and $a(\mu_0) \in K(i(\mu_0), \mu_0) \subset \mathbf{Q}_{>0}$.

Since (α, a) is a linear combination of $(\xi(i(\mu)), a(\mu))$ $(\mu \in M)$, we have

$$\bigcap_{\mu \in M} \{\zeta \in \mathbf{C}^l | (\xi(i(\mu))|\zeta) + a(\mu) = 0\} \subset \{\zeta \in \mathbf{C}^l | (\alpha|\zeta) + a = 0\}. \tag{3.4.1}$$

Since $\bigcap_{\mu \in M} b_\mu^{-1}(0)$ is a finite union of sets of the form of the left hand side of (3.4.1), we have

$$\bigcap_{\mu \in M} b_\mu^{-1}(0) \subset \bigcup_j \{\zeta \in \mathbf{C}^l | (\alpha|\zeta) + a_j = 0\}$$

with finite number of rational numbers a_j. Moreover, under the assumptions of (2), we may assume that $a_j > 0$. By the Hilbert's Nullstellensatz, the ideal $J(\alpha)$ contains a polynomial $\prod_j((\alpha|\zeta) + a_j)^k$ for a sufficiently large integer k. Thus we have completed the proof.

Lemma 3.5. *Let $x = (x_1, \dots, x_n) \in \mathbf{C}^n = X$, $\beta(i) \in \mathbf{N}^l$ $(1 \le i \le n)$, $f_j(x) = \prod_{i=1}^n x_i^{\beta(i)_j}$, $f = (f_1, \dots, f_l)$, and $\alpha \in \mathbf{N}^l \setminus \{0\}$. Then there exist $R(u) \in D_{X \times \mathbf{C}^l}[u]$ $(\subset D_{X \times \mathbf{C}^l} \otimes_{\mathbf{C}} \mathbf{C}(u))$ and a polynomial $b(s) \in \mathbf{C}[s]$ such that $b(s)$ is of the form $\prod_j(s + a_j)$ with $a_j \in \mathbf{Q}_{>0}$ and*

$$b((\alpha| - \partial_{\tau'}\tau'))\delta(u^\alpha\tau' - f) = u \cdot R(u)\delta(u^\alpha\tau' - f).$$

Proof. In order to make the argument easier to understand, we formally introduce the Mellin transformation $\varphi(\tau') \to \int \varphi(\tau')\tau'^\zeta d\tau'$. Our transformation here is a formal one and every expression should be understood in the form before the transformation.

Let t be a single complex variable, and put

$$[t]^i = \begin{cases} t^i & (i \ge 0) \\ \left(\dfrac{\partial}{\partial t}\right)^{-i} & (i < 0), \end{cases}$$

$$P_\mu = \prod_{j=1}^l [\tau'_j]^{\mu_j}, \quad Q_\mu = \prod_{i=1}^n [x_i]^{-(\beta(i)|\mu)},$$

$$h = n + l,\ \xi(i) = \beta(i)\ (1 \le i \le n),\ \xi(n + i) = -\varepsilon(i)\ (1 \le i \le l),$$

$$H = \{1, 2, \dots, h\},$$

$$K(i, \mu) = \begin{cases} \{1, 2, \dots, (\beta(i)|\mu)\}, & \text{if } 1 \le i \le n,\ (\beta(i)|\mu) > 0, \\ \phi, & \text{if } 1 \le i \le n,\ (\beta(i)|\mu) \le 0, \end{cases}$$

$$K(n + i, \mu) = \begin{cases} \{0, 1, \dots, -\mu_i - 1\}, & \text{if } 1 \le i \le l,\ \mu_i < 0, \\ \phi, & \text{if } 1 \le i \le l,\ \mu_i \ge 0, \end{cases}$$

$$\operatorname{sgn}(t) = \begin{cases} 1 & (t \ge 0) \\ -1 & (t < 0) \end{cases}, \quad \operatorname{sgn}(\mu) = \prod_{j=1}^l \operatorname{sgn}(\mu_j)^{\mu_j},$$

and $P_\mu^* = \operatorname{sgn}(\mu)P_\mu$.

Then

$$P_\mu^*\tau'^\zeta = \prod_{\substack{n < i \le n+l \\ a \in K(i,\mu)}} ((\xi(i)|\zeta) + a) \cdot \tau'^{\zeta+\mu} =: c_\mu(\zeta)\tau'^{\zeta+\mu}, \text{ and}$$

$$Q_\mu f^{\zeta+\mu} = Q_\mu \prod_{i=1}^n x_i^{(\beta(i)|\zeta+\mu)}$$

$$= \prod_{\substack{1 \le i \le n \\ a \in K(i,\mu)}} ((\xi(i)|\zeta) + a) \cdot f^\zeta =: d_\mu(\zeta)f^\zeta.$$

Put

$$b_\mu(\zeta) := c_\mu(\zeta) d_\mu(\zeta) = \prod_{\substack{i\in H \\ a\in K(i,\mu)}} ((\xi(i)|\zeta) + a).$$

Then

$$\int (P_\mu Q_\mu \delta(\tau' - u^{-\alpha}f)) \cdot \tau'^\zeta d\tau' = Q_\mu \int \delta(\tau' - u^{-\alpha}f) \cdot P_\mu^* \tau'^\zeta d\tau'$$
$$= c_\mu(\zeta) Q_\mu (u^{-\alpha}f)^{\zeta+\mu} = b_\mu(\zeta)(u^{-\alpha}f)^\zeta \cdot u^{-(\alpha|\mu)}.$$

Note that $K(i, \mu) = \phi$ if $(\xi(i)|\mu) \leq 0$ and $K(i, \mu) \subset \mathbf{Q}_{\geq 0}$ for any μ. Let H' be as in (3.4, (2)). Then $\{\xi(i)|i\in H'\} \subset \{-\varepsilon(1), \ldots, -\varepsilon(l)\}$. Since $\alpha\in\mathbf{N}^l\setminus\{0\}$, $\alpha\notin\langle H'\rangle$. Thus we can apply (3.4), and we get polynomials $e_\mu(\zeta)$ $(\mu\in M(\alpha))$ such that $\sum_{\mu\in M(\alpha)} e_\mu(\zeta) b_\mu(\zeta) = b((\alpha|\zeta))$, where $b(s) = \prod_j (s + a_j)$ with $a_j\in\mathbf{Q}_{>0}$. Put

$$R = \sum_{\mu\in M(\alpha)} u^{(\alpha|\mu)-1} e_\mu(-\partial_{\tau'}\tau') P_\mu Q_\mu.$$

Then

$$\int (R\delta(\tau' - u^{-\alpha}f)) \cdot \tau'^\zeta d\tau'$$
$$= \sum_{\mu\in M(\alpha)} u^{(\alpha|\mu)-1} \int e_\mu(-\partial_{\tau'}\tau') P_\mu Q_\mu \delta(\tau' - u^{-\alpha}f) \cdot \tau'^\zeta d\tau'$$
$$= \sum_{\mu\in M(\alpha)} u^{(\alpha|\mu)-1} \int P_\mu Q_\mu \delta(\tau' - u^{-\alpha}f) \cdot e_\mu(\tau'\partial_{\tau'}) \tau'^\zeta d\tau'$$
$$= \sum_{\mu\in M(\alpha)} u^{(\alpha|\mu)-1} e_\mu(\zeta) b_\mu(\zeta) (u^{-\alpha}f)^\zeta \cdot u^{-(\alpha|\mu)}$$
$$= u^{-1} \cdot b((\alpha|\zeta))(u^{-\alpha}f)^\zeta$$
$$= u^{-1} \int b((\alpha| -\partial_{\tau'}\tau'))\delta(\tau' - u^{-\alpha}f) \cdot \tau'^\zeta d\tau'.$$

Hence

$$R\delta(\tau' - u^{-\alpha}f) = u^{-1} b((\alpha| -\partial_{\tau'}\tau'))\delta(\tau' - u^{-\alpha}f),$$

which is equivalent to the desired equality. (In fact, we can identify $\delta(\tau' - u^{-\alpha}f)$ with $u^{\Sigma\alpha_i}\delta(u^\alpha\tau' - f)$.)

3.6. By (2.10, (7)) and (3.5), the zeros of the polynomials $b_\alpha(\alpha\in A)$ in (2.13) are negative rational numbers if $\bigcup_i f_i^{-1}(0)$ is normal crossing. (In this case, f_i's are monomials in local coordinate functions multiplied with invertible elements, say g_i. By the change of variables $\tau_i' = g_i\tau_i''$, the situation becomes the one considered in this section.)

4. Proof (final step)

4.0. Here we prove that the zeros of b_α in (2.13) are in general negative rational numbers. We shall prove this assertion by reducing to the normal crossing case by using the desingularization theorem of Hironaka [2]. Thus we can not stay within the affine varieties, and we need the sheaf theory. Localizing the D-modules considered in the previous sections, we get quasi-coherent sheaves, which we shall denote by the corresponding script letters, e.g., $\mathscr{D}_X = \mathscr{O}_X \otimes_{\mathbf{C}[X]} D_X$ etc. Since any variety is covered by affine open subsets, such sheaves can be defined even if X is not affine, and the argument of the previous sections works in general (with an obvious modification, if necessary).

4.1. Let X be as before, and $F: X' \to X$ a projective morphism such that $F: X' - \bigcup_i f_i'^{-1}(0) \xrightarrow{\sim} X - \bigcup_i f_i^{-1}(0)$, where $f_i' = f_i \circ F$, and that $\bigcup_i f_i'^{-1}(0)$ is normal crossing. Such a pair (X', F) always exists [2]. Put $n = \dim X = \dim X'$.

4.2. Let $g = \delta(u^\alpha \tau' - f(x))\delta(u - u')$, $g' = \delta(u^\alpha \tau' - f'(x'))\delta(u - u')$, $\mathscr{N} = \mathscr{N}_f = (\mathscr{D}_X \otimes_{\mathbf{C}} D_{E' \times \mathbf{C}})g$, $\mathscr{N}_{f'} = (\mathscr{D}_{X'} \otimes_{\mathbf{C}} D_{E' \times \mathbf{C}})g'$, $i: X' \to X' \times X$ be the mapping $x' \to (x', F(x'))$, and, $p': X' \times X \to X'$ and $p: X' \times X \to X$ the projections. For any variety Z, Ω_Z^j denotes the sheaf of regular j-forms on Z. Put

$$\mathscr{D}_{X' \times X \leftarrow X'} = i^{-1}(\mathscr{D}_{X' \times X} \otimes_{\mathscr{O}_{X' \times X}} (\Omega_{X' \times X}^{2n})^{-1}) \otimes_{i^{-1}\mathscr{O}_{X' \times X}} \Omega_{X'}^n.$$

For $\omega \in \Omega_X^n$ and $\omega' \in \Omega_{X'}^n$, put

$$1_{X \leftarrow X'} = F^*\omega \otimes (1 \otimes (\omega' \wedge \omega)^{-1} \otimes \omega') \in p'^{-1}\Omega_{X'}^n \otimes_{p'^{-1}\mathscr{O}_{X'}} i_* \mathscr{D}_{X' \times X \leftarrow X'}.$$

Then, we can show that $1_{X \leftarrow X'}$ does not depend on ω or ω', and it defines a global section. Especially,

$$1_{X \leftarrow X'} \otimes g' \in p_*(p'^{-1}\Omega_{X'}^n \otimes_{p'^{-1}\mathscr{O}_{X'}} i_* \mathscr{D}_{X' \times X \leftarrow X'} \otimes_{\mathscr{D}_{X'}} \mathscr{N}_{f'}).$$

Let g'' be the image of $1_{X \leftarrow X'} \otimes g'$ by

$$\begin{aligned} & p_*(p'^{-1}\Omega_{X'}^n \otimes_{p'^{-1}\mathscr{O}_X} i_* \mathscr{D}_{X' \times X \leftarrow X'} \otimes_{\mathscr{D}_{X'}} \mathscr{N}_{f'}) \\ & \to H^n(p_*(p'^{-1}\Omega_{X'}^{\bullet} \otimes_{p'^{-1}\mathscr{O}_{X'}} i_* \mathscr{D}_{X' \times X \leftarrow X'} \otimes_{\mathscr{D}_{X'}} \mathscr{N}_{f'})) \\ & \to H^n(Rp_*(p'^{-1}\Omega_{X'}^{\bullet} \otimes_{p'^{-1}\mathscr{O}_{X'}} i_* \mathscr{D}_{X' \times X \leftarrow X'} \otimes_{\mathscr{D}_{X'}} \mathscr{N}_{f'})) \\ & = H^0\left(\int_F \mathscr{N}_{f'}\right) =: \mathscr{N}'. \end{aligned}$$

Here $p'^{-1}\Omega_{X'}^{\bullet} \otimes$ is the relative de Rham complex. We can define endomorphisms C and u of $\mathscr{N}_{f'}$ in the same way as in §2. Since C is a $\mathscr{D}_{X'}$-endomorphism and u is a $\mathscr{D}_{X'} \otimes_{\mathbf{C}} D_{E' \times \mathbf{C}}$-endomorphsim, C induces a $\mathscr{D}_X$-endomorphism of $\mathscr{N}'$ and u induces a $\mathscr{D}_X \otimes_{\mathbf{C}} D_{E' \times \mathbf{C}}$-endomorphism. Put

$$\mathscr{N}'' = \sum_{j \geq 0} (\mathscr{D}_X \otimes_{\mathbf{C}} D_{E' \times \mathbf{C}}) C^j g'' \ (\subset \mathscr{N}').$$

Since $\mathcal{N}'$ is coherent over $\mathcal{D}_X \otimes_{\mathbf{C}} D_{E' \times \mathbf{C}}$ and $\mathcal{N}''$ is a union of an increasing sequence of coherent $\mathcal{D}_X \otimes_{\mathbf{C}} D_{E' \times \mathbf{C}}$-submodules of $\mathcal{N}'$, $\mathcal{N}''$ is also coherent over $\mathcal{D}_X \otimes D_{E' \times \mathbf{C}}$.

Lemma 4.3. *$\mathcal{N}''$ is stable by the endomorphisms C and u of $\mathcal{N}'$.*

Proof. For $P \in \mathcal{D}_{X'} \otimes_{\mathbf{C}} D_{E'} \otimes_{\mathbf{C}} \mathbf{C}[\partial_{u'}]$, we have

$$C(Pu^i \delta(u^\alpha \tau' - f')\delta(u - u')) = P((\alpha| - \partial_{\tau'}\tau') - i)u^i \delta(u^\alpha \tau' - f')\delta(u - u').$$

From this equality, we can show that $[C, P] = 0$, $uC = (C + 1)u$ and $u'C = Cu' + u$ as $\mathcal{D}_{X'}$-endomorphisms of $\mathcal{N}_{f'}$. These relations also hold in $\mathrm{End}(\mathcal{N}')$. Especially, $u'Cg'' = (Cu' + u)g'' = (Cu' + u')g''$ and $C^i u'g'' = u'(C - 1)^i g''$. For $P \in \mathcal{D}_X \otimes_{\mathbf{C}} D_{E'} \otimes_{\mathbf{C}} \mathbf{C}[\partial_{u'}]$, we have $CPu'^i C^j g'' = PC(C + i)^j u'^i g'' = Pu'^i (C - i)C^j g''$, and hence $\mathcal{N}''$ is C-stable. For $Q \in \mathcal{D}_X \otimes_{\mathbf{C}} D_{E' \times \mathbf{C}}$,

(4.3.1) $$uQC^j g'' = Q(C + 1)^j ug'' = Q(C + 1)^j u'g'' = Qu'C^j g'',$$

and hence $\mathcal{N}''$ is u-stable.

4.4. Define a surjective homomorphism $\Phi: \mathcal{N}'' \to \mathcal{N}$ by

$$\Phi(\sum_{j \geq 0} P_j C^j g'') = \sum_{j \geq 0} P_j C^j g$$

for $P_j \in \mathcal{D}_X \otimes_{\mathbf{C}} D_{E' \times \mathbf{C}}$.

Lemma 4.5. *Φ is well-defined and $\Phi u = u\Phi$.*

Proof. Put $X_1 = X - \bigcup_i f_i^{-1}(0)$. Then $(\mathcal{N}'', g'')$ can be identified with $(\mathcal{N}_f, g)$ on X_1. Hence, if $P = \sum P_j C^j$ annihilates g'', then it also annihilates $g = \delta(u^\alpha \tau' - f(x))\delta(u - u')$ on X_1. Since $\mathcal{N}_f \otimes_{\mathbf{C}} \mathbf{C}(u)$ is a simple $\mathcal{D}_X \otimes_{\mathbf{C}} D_{E' \times \mathbf{C}} \otimes_{\mathbf{C}} \mathbf{C}(u)$-module, P annihilates g everywhere on X. Hence Φ is well-defined. By (4.3.1) and by the formula obtained by replacing g'' with g,

$$\Phi(u \sum P_j C^j g'') = \Phi(\sum P_j u' C^j g'') = \sum P_j u' C^j g = u \sum P_j C^j g.$$

Hence $\Phi u = u\Phi$.

4.6. Let $\Gamma(X, \mathcal{N}') = N'$ and $\Gamma(X, \mathcal{N}'') = N''$. By $s \to C$ and $t \to u$, $\mathcal{N}_f$, $\mathcal{N}_{f'}$, $\mathcal{N}'$ and $\mathcal{N}''$ have $\mathbf{C}[s, t]$-module structures. (See (1.1) for $\mathbf{C}[s, t]$.) Since u commutes with the respective $\mathcal{D}$-module structures and since $\mathcal{N} = \mathcal{N}_f$, $\mathcal{N}'$ and $\mathcal{N}''$ are coherent over $\mathcal{D}_X \otimes_{\mathbf{C}} D_{E' \times \mathbf{C}}$, $N, N', N'' \in \mathcal{M}$. (Cf. (1.1).) Since $\mathcal{N}''$ and $\mathcal{N}$ are quasi-coherent over $\mathcal{O}_X$, the surjective homomorphism $\Phi: \mathcal{N}'' \to \mathcal{N}$ induces a surjective morphism $N'' \to N$ in $\mathcal{M}$. By (2.10, (1)) and (3.6), the zeros of $b_{N'}$ are negative rational numbers. Hence, by (1.3), the zeros of b_N are negative rational numbers. Thus we have completed the proof.

Remark 4.7. Let $\mathcal{B}$ be the ideal of $\mathbf{C}[\zeta]$ consisting of b_μ's as in (1) of our Theorem. It seems that $\mathcal{B}$ is a principal ideal of $\mathbf{C}[\zeta]$, but the author can not prove. In our subsequent paper, we shall show that $\mathcal{B}$ is a principal ideal in a certain special case.

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