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On a necessary condition for L^2 well-posedness of the Cauchy problem for some Schrödinger type equations with a potential term

By

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0. Introduction

In this paper we study a necessary condition in order that the Cauchy problem for Schrödinger type equations with a potential term

$$(0.1)\qquad \begin{cases} Lu(t,x)\equiv \dfrac{1}{i}\partial_t u + H_2(t,x,D_x)u + H_1(t,x,D_x)u = f(t,x) \\ \qquad\qquad \text{on } [0,T]\times R_x^n\ (t>0), \\ u(0,x)=u_0(x) \end{cases}$$

is L^2 well posed on $[0, T]$. Here, we suppose that the symbols $h_j(t, x, \xi)$ ($j=1, 2$) of pseudo-differential operators $H_j(t, x, D_x)$ are continuous functions on $[0, T]\times R^{2n}_{x,\xi}$ and C^∞ functions on $R^{2n}_{x,\xi}$ for each $t\in[0, T]$. Moreover, we assume that $h_2(t, x, \xi)$ is real valued and that

$$(0.2)\qquad \text{if } |\alpha+\beta|\geq j\,,\quad |\partial_\xi^\alpha \partial_x^\beta h_j(t,x,\xi)|\leq C_{\alpha,\beta}$$

holds for $j=1$ and 2, where α and β are multi-indices and $C_{\alpha,\beta}$ are constants independent of $(t, x, \xi)\in[0, T]\times R^{2n}_{x,\xi}$. Our result will be stated in Theorem 1.1.

In the preceding paper [6] we gave a sufficient condition under a weaker assumption on $h_1(t, x, \xi)$ than that in the present paper in order that the Cauchy problem (0.1) is L^2 well posed on $[0, T]$. Combining this result in [6] and Theorem 1.1 in the present paper, we can obtain a necessary and sufficient condition so that (0.1) is L^2 well posed on $[0, T]$, if we impose an additional assumption that $h_j(t, x, \xi)$ ($j=1, 2$) are independent of $t\in[0, T]$ (Theorem 1.2). We can see from this Theorem 1.2 that the invariance under the canonical transformations of L^2 well-posedness is valid in a sense (Corollary 1.3).

Some results on a necessary condition for L^2 well-posedness have been

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obtained. But, we have not had the informations at all of the case where equations have an unbounded potential term. The case where $h_2(t, x, \xi)=|\xi|^2$ and $h_1(t, x, \xi)=\sum_{j=1}^n b^j(x)\xi_j$ was treated in S. Mizohata [11], where $b^j(x)(j=1, 2, \cdots, n)$ are C^∞ functions. This result was generalized by the author [4] and [5] to the equations on the general Riemannian manifold. We must note that our assumption in the present paper on $h_2(t, x, \xi)$ is more general than in [4], [5] and [11], but one on $h_1(t, x, \xi)$ is more limited than in those. But, we want to emphasize that we can obtain a necessary and sufficient condition for L^2 well-posedness under our situation, not only a necessary condition. For it is difficult so far to obtain a necessary and sufficient condition under the situations in [4], [5] and [11]. See the introduction in [6] about results on a sufficient condition obtained already.

We shall state our results and examples in section 1. Section 2 will be devoted to the proof of Theorem 1.1. In section 3 we shall give another proof of the sufficient condition, limiting equations to ours. This result has been proven generally in [6]. But, we can prove it more easily than in [6], if we limit the equations to ours.

1. Results and examples

We shall use the same notations as in [6] through the present paper. Let $\mathcal{S}=\mathcal{S}(R^n)$ be the Schwartz space of rapidly decreasing functions on R^n. The Fourier transformation $\hat{u}(\xi)$ for $u(x)\in\mathcal{S}$ is defined by

$$\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx\,, \quad x\cdot\xi=x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n\,.$$

The symbol class $T^m=T^m(R^{2n})$ for a real m of pseudo-differential operators is defined by the set of all C^∞ functions $p(x, \xi)$ such that

$$|\partial_\xi^\alpha D_x^\beta p(x, \xi)|\le C_{\alpha,\beta}(1+|x|^2+|\xi|^2)^{m/2}$$

are valid for all multi-indices α and β with constants $C_{\alpha,\beta}$ independent of $(x, \xi)\in R^{2n}$, where $p_{(\beta)}^{(\alpha)}(x, \xi)=\partial_\xi^\alpha D_x^\beta p(x, \xi)$. The above constants $C_{\alpha,\beta}$ are different from the constants in (0.2). If there is no confusion, we shall often use the same symbol $C_{\alpha,\beta}$. Another symbol class $\mathcal{B}^{k,\infty}(R^{2n})$ $(k=0, 1, \cdots)$ is defined by the set of all C^∞ functions $p(x, \xi)$ suh that

$$\text{if} \quad |\alpha+\beta|\ge k\,, \quad |p_{(\beta)}^{(\alpha)}(x, \xi)|\le C_{\alpha,\beta}$$

are valid, where $C_{\alpha,\beta}$ are constants independent of $(x, \xi)\in R^{2n}$. We can easily see from [6] that $\mathcal{B}^{k,\infty}(R^{2n})$ is included in $T^k(R^{2n})$ $(k=0, 1, \cdots)$. The pseudo-differential operator $P=p(x, D_x)$ with a symbol $\sigma(P)(x, \xi)=p(x, \xi)\in T^m$ is defined by

$$Pu(x)=\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)đ\xi \qquad (đ\xi=(2\pi)^{-n}d\xi).$$

Let $\boldsymbol{B}$ be a Fréchet space. Then, we denote by $\mathcal{E}_t^0([0, T]; \boldsymbol{B})$ and by $L_t^1([0, T]; \boldsymbol{B})$ the space of all $\boldsymbol{B}$-valued continuous functions on $[0, T]$ and the space of all $\boldsymbol{B}$-valued L^1-functions on $[0, T]$ respectively. Through the present paper we adopt the following definition of L^2 well-posedness, which is weaker than that in [6], because we study a necessary condition in the present paper.

Definition 1.1. We say that the Caucy problem (0.1) is L^2 *well posed on* $[0, T]$, if for any $u_0(x)\in L^2$ and any $f(t,x)\in L_t^1([0, T]; L^2)$ there exists one and only one solution $u(t,x)$ of (0.1) in $\mathcal{E}_t^0([0, T]; L^2)$ in a distribution sense and the energy inequality

$$\|u(t,\cdot)\|\le C(T)\Big(\|u_0(\cdot)\|+\int_0^t\|f(\theta,\cdot)\|d\theta\Big)\ (0\le t\le T) \tag{1.1}$$

holds for a constant $C(T)\ge 1$. Here, $\|\cdot\|$ denotes the L^2 norm. Also, see Definition 1.1 in [6] for the meaning of the term "a distribution sense".

Let $(q,p)(t,s;y,\xi)=(q_1,\cdots,q_n,p_1,\cdots,p_n)(t,s;y,\xi)$ be the solution of the Hamilton canonical equations for $h_2(t,x,\xi)$ issuing from (y,ξ) at $t=s$, that is,

$$\frac{dq}{dt}=\frac{\partial h_2}{\partial\xi}(t,q,p),\quad \frac{dp}{dt}=-\frac{\partial h_2}{\partial x}(t,q,p),\quad (q,p)|_{t=s}=(y,\xi). \tag{1.2}$$

We know that the solution $(q,p)(t,s;y,\xi)$ exists on $[0, T]\times[0, T]\times R^{2n}_{y,\xi}$. See Proposition 3.1 in [7] or Lemma 2.1 in [6].

Theorem 1.1. *We assume* (0.2). *Then, if the Cauchy problem* (0.1) *is* L^2 *well posed on* $[0, T]$, *there must be a* $T_1(0<T_1\le T)$ *such that*

$$\sup_{0\le t\le T_1,(y,\xi)\in R^{2n}} Im\int_0^t h_1(\theta, q(\theta,0;y,\xi), p(\theta,0;y,\xi))d\theta<\infty \tag{1.3}$$

holds. Imc implies the imaginary part of c.

The proof of Theorem 1.1 will be given in section 2.

Remark 1.1. In this remark we will show that the inequality (1.3) is invariant under the general canonical transformations. Let Φ be a canonical transformation from $R^{2n}_{x',\xi'}$ onto $R^{2n}_{x,\xi}$. That is, $\Phi^*\sum_{j=1}^n dx_j\wedge d\xi_j=\sum_{j=1}^n dx'_j\wedge d\xi'_j$ is valid, where Φ^* denotes the pull back of differential forms. We set for $h_j(t,x,\xi)$ $(j=1,2)$ in Theorem 1.1

$$k_j(t,x',\xi')=h_j(t,\Phi(x',\xi'))\ (j=1,2) \tag{1.4}$$

and denote by $(q', p')(t, s; y', \xi')$ the solution of the Hamilton canonical equations for $k_2(t, x', \xi')$

$$(1.2)' \qquad \begin{cases} \dfrac{dq'}{dt} = \dfrac{\partial k_2}{\partial \xi'}(t, q', p'), \quad \dfrac{dp'}{dt} = -\dfrac{\partial k_2}{\partial x'}(t, q', p'), \\ (q', p')|_{t=s} = (y', \xi'). \end{cases}$$

Then, we know well that

$$(1.5) \qquad \begin{aligned} &\Phi(q'(t, s; y', \xi'), p'(t, s; y', \xi')) \\ &\quad = (q(t, s; y, \xi), p(t, s; y, \xi)) \qquad ((y, \xi) = \Phi(y', \xi')) \end{aligned}$$

is yielded from (1.4) with $j=2$ for any $(y', \xi') \in R^{2n}$ (see section 45 in [1]). So, we have from (1.4) with $j=1$

$$\begin{aligned} &k_1(t, q'(t, s; y', \xi'), p'(t, s; y', \xi')) \\ &\quad = h_1(t, q(t, s; y, \xi), p(t, s; y, \xi)) \qquad ((y, \xi) = \Phi(y', \xi')). \end{aligned}$$

Hence,

$$(1.6) \qquad \begin{aligned} &\int_0^t k_1(\theta, q'(\theta, 0; y', \xi'), p'(\theta, 0; y', \xi')) d\theta \\ &\quad = \int_0^t h_1(\theta, q(\theta, 0; y, \xi), p(\theta, 0; y, \xi)) d\theta \qquad ((y, \xi) = \Phi(y', \xi')) \end{aligned}$$

is valid for any $(y', \xi') \in R^{2n}$. Thus, (1.3) is invariant under the canonical transformations.

We can easily get the following theorem from the above Theorem 1.1 and Corollary 1.2 in [6].

Theorem 1.2. *We add in Theorem 1.1 an assumption that* $h_j(t, x, \xi)$ $(j=1, 2)$ *are independent of* $t \in [0, T]$. *Then, if and only if the Cauchy problem* (0.1) *is* L^2 *well posed on* $[0, T]$, *there exists a* $T_1 (0 < T_1 \leq T)$ *such that* (1.3) *is valid.*

Proof. We know from Theorem 1.1 that the condition (1.3) for a T_1 is necessary for (0.1) to be L^2 well posed on $[0, T]$. So, we have only to prove its sufficiency. We may express $h_j(t, x, \xi)$ $(j=1, 2)$ as $h_j(x, \xi)$. Since $h_2(x, \xi)$ is independent of t, we have

$$(q, p)(\theta, s; y, \xi) = (q, p)(\theta - s, 0; y, \xi)$$

and so,

$$\exp\left\{-i \int_s^t h_1(q(\theta, s; y, \xi), p(\theta, s; y, \xi)) d\theta\right\}$$

$$=\exp\left\{-i\int_0^{t-s} h_1(q(\theta, 0; y, \xi), p(\theta, 0; y, \xi))d\theta\right\} \quad (0\leq s\leq t\leq T).$$

Consequently,

$$\sup_{0\leq s\leq t\leq T_1,(y,\xi)\in R^{2n}} \exp\left\{\mathrm{Im}\int_s^t h_1(q(\theta, s; y, \xi), p(\theta, s; y, \xi))d\theta\right\}<\infty \tag{1.7}$$

follows from (1.3). Hence, we can see by Corollary 1.2 in [6] that the Cauchy problem (0.1) is L^2 well posed on $[0, T_1]$. So, we get the energy inequality (1.1) where we replace T by T_1. In the same way it follows that (0.1) is L^2 well posed on $[T_1, T_2]$ ($T_2=\min(2T_1, T)$), because the equation in (0.1) is independent of t. Let $u_1(t, x)\in \mathcal{E}_t^0([0, T_1]; L^2)$ be the solution of (0.1) and $u_2(t, x)\in \mathcal{E}_t^0([T_1, T_2]; L^2)$ the solution of the equation in (0.1) with initial data $u_2(T_1, x)=u_1(T_1, x)$. Then, we get

$$\begin{aligned}\|u_2(t, \cdot)\|&\leq C(T_1)\left(\|u_1(T_1, \cdot)\|+\int_{T_1}^t\|f(\theta, \cdot)\|d\theta\right)\\&\leq C(T_1)^2\left(\|u_0(\cdot)\|+\int_0^t\|f(\theta, \cdot)\|d\theta\right) \ (T_1\leq t\leq T_2),\end{aligned}$$

noting $C(T_1)\geq 1$. So, if we define the function $u(t, x)\in \mathcal{E}_t^0([0, T_2]; L^2)$ by

$$u(t, x)=\begin{cases} u_1(t, x) & (0\leq t\leq T_1)\\ u_2(t, x) & (T_1\leq t\leq T_2),\end{cases}$$

we can see that $u(t, x)$ is the solution of (0.1) and that (0.1) is L^2 well posed on $[0, T_2]$. We can complete the proof repeatedly. Q.E.D.

Remark 1.2. Suppose the same assumptions as in Theorem 1.2. Then, since the equation in (0.1) is independent of t, we can easily see that if (0.1) is L^2 well posed on $[0, T]$ in our sense, (0.1) is so in a sense of Definition 1.1 in [6]. So, the statement in Theorem 1.2 remains valid, even if we adopt the definition in [6] as that of L^2 well-posedness instead of ours.

The following theorem shows the invariance under the canonical transformations of L^2 well-posedness in a sense. We note that we considered only special canonical transformations in section 3 of [6].

Corollary 1.3. *Suppose the same assumptions as in Theorem 1.2 and let Φ be a canonical transformation from $R^{2n}_{x',\xi'}$ onto $R^{2n}_{x,\xi}$. We define $k_j(x', \xi')$ ($j=1, 2$) by (1.4) and consider the Cauchy poblem*

$$\begin{cases} L'v(t, x')\equiv\frac{1}{i}\partial_t v(t, x')+K_2(x', D_{x'})v+K_1(x', D_{x'})v\\ \qquad =g(t, x') \qquad on\ [0, T]\times R^n_{x'},\\ v(0, x')=v_0(x').\end{cases} \tag{1.8}$$

We assume that each $k_j(x', \xi')$ $(j=1, 2)$ *satisfies the same inequalities as* (0.2), *that is,*

$$\text{if } |\alpha+\beta|\geq j\,, \quad |k_{j(\beta)}^{(\alpha)}(x', \xi')|\leq C'_{\alpha,\beta}$$

with constants $C'_{\alpha,\beta}$ *independent of* $(x', \xi')\in R^{2n}$. *Then, if and only if* (0.1) *is* L^2 *well posed on* $[0, T]$, (1.8) *is* L^2 *well posed on* $[0, T]$.

Proof. Corollary 1.3 follows from Theorem 1.2 and (1.6) at once.
Q.E.D.

Example 1.1. Let $h_2(x, \xi)$ be a polynomial of degre 2 in only variables x and ξ with real coefficients satisfying $h_2(x, \xi)\geq 0$ on R^{2n}. We define $h_1(x, \xi)$ by

$$h_1(x, \xi)=c\{1+h_2(x, \xi)\}^{1/2}\,,$$

where c is a complex constant. We take these $h_j(x, \xi)$ $(j=1, 2)$ as $h_j(t, x, \xi)$ in (0.1). This example was stated in Example 1.4 in [6]. We can see from [6] that these $h_j(x, \xi)$ $(j=1, 2)$ satisfy the assumptions in Theorem 1.2. We note from the energy equality $\frac{d}{dt}h_2(q(t, s; y, \xi), p(t, s; y, \xi))=0$ that

$$\int_0^t h_1(q(\theta, 0; y, \xi), p(\theta, 0; y, \xi))d\theta=ct\{1+h_2(y, \xi)\}^{1/2}$$

is valid. Hence, we can see from Theorem 1.2 that if and only if the Cauchy problem (0.1) is L^2 well posed on $[0, T]$ for a $T>0$, $\mathrm{Im}c$ is non-positive.

Example 1.2. (c.f. Example 1.2 in [6]). Let $h_2(t, x, \xi)=\frac{1}{2m}|\xi|^2+\frac{m\omega^2}{2}\times|x|^2$ and $h_1(t, x, \xi)=\sum_{j=1}^n c_j(t)\xi_j$, where m and ω are positive constants, and $c_j(t)$ $(j=1, 2, \cdots, n)$ are continuous functions on $[0, T]$ for $T>0$. Then, the solution of (1.2) is given by

$$(q, p)(t, 0; y, \xi)=\left(y\cos\omega t+\frac{\xi}{m\omega}\sin\omega t,\ -my\omega\sin\omega t+\xi\cos\omega t\right).$$

Assume that the Cauchy problem (0.1) is L^2 well posed on $[0, T]$. Then, it follows from Theorem 1.1 that there must be a $T_1(0<T_1\leq T)$ such that

$$\sup_{0\leq t\leq T_1,(y,\xi)\in R^{2n}}\sum_{j=1}^n\int_0^t(-my_j\omega\sin\omega\theta+\xi_j\cos\omega\theta)\mathrm{Im}c_j(\theta)\mathrm{d}\theta<\infty$$

is valid. Hence, we get

$$\text{(1.9)} \qquad \mathrm{Im}c_j(t)=0 \text{ on } [0, T_1]\ (j=1, 2, \cdots, n)\,.$$

Conversely, if (1.9) is valid, we can easily see that (0.1) is L^2 well posed on [0,

T_1].

Example 1.3. Let $h_2(t, x, \xi)=\frac{1}{2m}|\xi-x|^2+\frac{m\omega^2}{2}|x|^2$ and $h_1(t, x, \xi)=\sum_{j=1}^{n} c_j(t)(\xi_j-x_j)$, where m, ω and $c_j(t)$ ($j=1, , \cdots, n$) are the same ones in Example 1.2. It is easy to see that the transformation from $R^{2n}_{x',\xi'}$ onto $R^{2n}_{x,\xi}$ defined by $(x, \xi)=(x', \xi'+x')$ is canonical. So, it follows from Remark 1.1 and Example 1.2 that if (0.1) is L^2 well posed on $[0, T]$, the inequality (1.9) must be valid for a $T_1>0$ ($0<T_1\leq T$).

Example 1.4. We take $\frac{1}{2}|x|^2$ and $\sum_{j=1}^{n} c_j\xi_j$ as $h_2(t, x, \xi)$ and $h_1(t, x, \xi)$, where c_j ($j=1, 2, \cdots, n$) are constants. Then, the solution of (1.2) is given by

$$(q(t, 0; y, \xi), p(t, 0; y, \xi))=(y, -ty+\xi).$$

So, it follows from Theorem 1.2 that if and only if (0.1) is L^2 well posed on $[0, T]$ for a $T>0$,

$$\sup_{0\leq t\leq T_1,(y,\xi)\in R^{2n}} \sum_{j=1}^{n} \mathrm{Im}\, c_j \int_0^t (-\theta y_j+\xi_j)d\theta<\infty$$

is valid for a T_1 ($0<T_1\leq T$). Hence, we get

$$\mathrm{Im}\, c_j=0 \ (j=1, 2, \cdots, n) \tag{1.10}$$

as a necessary and sufficient condition for L^2 well-posedness of (0.1) on $[0, T]$. This result is not new, because we have known it in [9] as a result for kowalewskian type of equations to be L^2 well posed.

Instead of the above $h_j(t, x, \xi)$ ($j=1, 2$), let $h_2(t, x, \xi)=\frac{1}{2}|\xi+x|^2$ and $h_1(t, x, \xi)=\sum_{j=1}^{n} c_j\xi_j$. Then, we also get (1.10) as a necessary and sufficient condition for (0.1) to be L^2 well posed on $[0, T]$. For using the canonical transformations defined by $(x', \xi')=(\xi+x, \xi)$, $h_2(t, x, \xi)$ and $h_1(t, x, \xi)$ are expressed as $\frac{1}{2}|x'|^2$ and $\sum_{j=1}^{n} c_j\xi'_j$ respectively. So, applying Corollary 1.3, we get the above result.

2. Proof of Theorem 1.1

We first prove the following lemma and proposition. Though we already had the similar results in [4] and [10], we need more detailed results than theirs.

Lemma 2.1. *Let $p(x, \xi)\in T^m$ for an m, $S(x)\in \mathcal{B}^{k,\infty}(R^{2n})$ for a $k\geq 0$ be a real valued function and $\lambda\geq 1$ a parameter. We set*

(2.1) $$\tilde{\nabla} S(x, y)=\int_0^1 \frac{\partial S}{\partial x}(y+\theta(x-y))d\theta\ .$$

Then, we get for $u(x)\in \mathcal{S}$

(2.2) $$e^{-i\lambda S(x)}p(x, \lambda^{-1}D_x)e^{i\lambda S(x)}u(x)$$
$$=\sum_{|\alpha|<N}\frac{1}{\alpha!}\lambda^{-|\alpha|}D_y^{\alpha}\{p^{(\alpha)}(x, \tilde{\nabla} S(x, x+y))u(x+y)\}|_{y=0}$$
$$+N\lambda^{n-N}\sum_{|\gamma|=N}\frac{1}{\gamma!}\int_0^1(1-\theta)^{N-1}d\theta Os-\iint e^{-i\lambda y\cdot\eta}$$
$$\times D_y^{\gamma}\{p^{(\gamma)}(x, \theta\eta+\tilde{\nabla} S(x\ x+y))u(x+y)\}dy đ\eta\ ,$$

where $N=1, 2, \cdots$ *and* $Os-\iint(\cdots)dyđ\eta$ *denotes the oscillatory integral in* [8].

Proof. It follows from the definition of pseudo-differential operators and the change of variables that

$$Q(x)\equiv e^{-i\lambda S(x)}p(x, \lambda^{-1}D_x)e^{i\lambda S(x)}u(x)$$
$$=\iint e^{i(x-y)\cdot\xi+i\lambda S(y)-i\lambda S(x)}p(x, \lambda^{-1}\xi)u(y)\mathrm{d}yđ\xi$$
$$=\lambda^n\iint e^{i\lambda(x-y)\cdot\xi-i\lambda(x-y)\cdot\tilde{\nabla} S(x,y)}p(x, \xi)u(y)\mathrm{d}yđ\xi$$

is valid. So, changing variables (y, ξ) to $(y', \eta')=(y-x, \xi-\tilde{\nabla} S(x, y))$, we get

(2.3) $$Q(x)=\lambda^n\iint e^{-i\lambda y'\cdot\eta'}p(x, \eta'+\tilde{\nabla} S(x, x+y'))u(x+y')\mathrm{d}y'đ\eta'\ .$$

The term $p(x, \eta'+\tilde{\nabla} S(x, x+y'))$ is expansioned as

$$\sum_{|\alpha|<N}\frac{1}{\alpha!}\eta'^{\alpha}p^{(\alpha)}(x, \tilde{\nabla} S(x, x+y'))$$
$$+N\sum_{|\gamma|=N}\frac{1}{\gamma!}\eta'^{\gamma}\int_0^1(1-\theta)^{N-1}p^{(\gamma)}(x, \theta\eta'+\tilde{\nabla} S(x, x+y'))\mathrm{d}\theta\ .$$

Hence, using

$$\lambda^n\iint e^{-i\lambda y\cdot\eta}g(y)\mathrm{d}yđ\eta=g(0) \qquad \text{for } g(x)\in \mathcal{S}$$

and the integral by parts, we obtain

(2.4) $$Q(x)=\sum_{|\alpha|<N}\frac{1}{\alpha!}\lambda^n\iint e^{-i\lambda y'\cdot\eta'}\eta'^{\alpha}p^{(\alpha)}(x, \tilde{\nabla} S(x, x+y'))$$

$$\times u(x+y')dy'đ\eta'+N\sum_{|\gamma|=N}\frac{1}{\gamma!}\int_0^1(1-\theta)^{N-1}d\theta\lambda^n$$

$$\times Os-\iint e^{-i\lambda y'\cdot\eta'}\eta'^{\gamma}p^{(\gamma)}(x,\theta\eta'+\tilde{\nabla}S(x,x+y'))u(x+y')dy'đ\eta'$$

$$=\sum_{|\alpha|<N}\frac{1}{\alpha!}\lambda^{-|\alpha|}D_{y'}^{\alpha}\{p^{(\alpha)}(x,\tilde{\nabla}S(x,x+y'))u(x+y')\}|_{y'=0}$$

$$+N\lambda^{n-N}\sum_{|\gamma|=N}\frac{1}{\gamma!}\int_0^1(1-\theta)^{N-1}d\theta Os-\iint e^{-i\lambda y'\cdot\eta'}$$

$$\times D_{y'}^{\gamma}\{p^{(\gamma)}(x,\theta\eta'+\tilde{\nabla}S(x,x+y'))u(x+y')\}dy'đ\eta'.$$

This completes the proof. Q.E.D.

Proposition 2.2. *Let $S(x)\in\mathcal{B}^{2,\infty}(R^{2n})$ be a real valued function and $\lambda\geq 1$ a parameter. We assume that $p(x,\xi)\in T^m$ satisfies*

$$\text{if}\quad |\alpha|=N,\quad p^{(\alpha)}(x,\xi)\in\mathcal{B}^{\infty}(R^{2n})$$

for an integer $N\geq 1$, where $\mathcal{B}^{\infty}(R^{2n})$ denotes the space of all C^{∞} functions on R^{2n} whose derivatives of any order are bounded. Then, we get for $u(x)\in\mathcal{S}$

$$e^{-i\lambda S(x)}p(x,\lambda^{-1}D_x)e^{i\lambda S(x)}u(x) \tag{2.5}$$

$$=\sum_{|\alpha|<N}\frac{1}{\alpha!}\lambda^{-|\alpha|}D_y{}^{\alpha}\{p^{(\alpha)}(x,\tilde{\nabla}S(x,x+y))u(x+y)\}|_{y=0}$$

$$+\lambda^{-N}R_Nu(x)$$

and

$$\|R_Nu(\cdot)\|\leq C_N\{\sum_{2\leq|\alpha|\leq l_N+1}\sup_{x\in R^n}|D_x{}^{\alpha}S(x)|\}\{\sum_{|\alpha|\leq l_N}\|D_x{}^{\alpha}u(\cdot)\|\} \tag{2.6}$$

with a constant C_N independent of $\lambda\geq 1$, $S(x)$ and $u(x)$, where $l_N=2[n/2+1]+N$ and $[\cdot]$ denotes the Gauss symbol.

Proof. Since we have had (2.2), we have only to prove (2.6). $R_Nu(x)$ is expressed as

$$N\sum_{|\gamma|=N}\frac{1}{\gamma!}\int_0^1(1-\theta)^{N-1}d\theta\lambda^n Os-\iint e^{-i\lambda y\cdot\eta}$$

$$\times D_y{}^{\gamma}\{p^{(\gamma)}(x,\theta\eta+\tilde{\nabla}S(x,x+y))u(x+y)\}dyđ\eta.$$

Integrating the above each term by parts, we get from the assumptions on $S(x)$ and $p(x,\xi)$

$$(2.7)\qquad \left|\lambda^n Os-\iint e^{-i\lambda y\cdot\eta}D_y^{\gamma}\{p^{(\gamma)}(x,\theta\eta+\tilde{\nabla}S(x,x+y))u(x+y)\}dy đ\eta\right|$$

$$=\left|Os-\iint e^{-iy\cdot\eta}D_y^{\gamma}\{p^{(\gamma)}(x,\theta\lambda^{-1}\eta+\tilde{\nabla}S(x,x+y))u(x+y)\}dy đ\eta\right|$$

$$=\Big|Os-\iint e^{-iy\cdot\eta}\langle y\rangle^{-l_0}\langle D_\eta\rangle^{l_0}\langle\eta\rangle^{-l_0}\langle D_y\rangle^{l_0}$$

$$\times D_y^{\gamma}\{p^{(\gamma)}(x,\theta\lambda^{-1}\eta+\tilde{\nabla}S(x,x+y))u(x+y)\}dy đ\eta\Big|$$

$$\le C_N'\{\textstyle\sum_{2\le|\alpha|\le l_N+1}\sup_{x\in R^n}|D_x^{\alpha}S(x)|\}\sum_{|\alpha|\le l_N}\displaystyle\iint\langle y\rangle^{-l_0}$$

$$\times\langle\eta\rangle^{-l_0}|D_x^{\alpha}u(x+y)|dy đ\eta\,,$$

where $l_0=2[n/2+1]$ and $\langle x\rangle=(1+|x|^2)^{1/2}$. Hence, we can complete the proof from the Hausdorff-Young inequality. Q.E.D.

The following lemma is fundamental in the proof of Theorem 1.1.

Lemma 2.3. *Suppose the same assumptions as in Theorem 1.1 and let* $(q,p)(t,s;y,\xi)$ *be the solution of* (1.2). *We set*

$$(2.8)\qquad \Omega(t,y,\xi)=-i\int_0^t h_1(\theta,q(\theta,0;y,\xi),p(\theta,0;y,\xi))d\theta\,.$$

Then,

$$(2.9)\qquad \text{if } |\alpha|\ge1\,,\quad |D_y^{\alpha}\Omega(t,y,\xi)|\le C_\alpha t \quad\text{on } [0,T]\times R^{2n}_{y,\xi}$$

are valid, where C_α *are constants independent of* (t,y,ξ).

Proof. We have known in Proposition 3.1 of [7] or Lemma 2.1 of [6] that

$$(2.10)\qquad \begin{cases}\text{if } |\alpha|\ge1,\ |D_y^{\alpha}\{q_j(t,s;y,\xi)-y_j\}|\le C_\alpha'(t-s) \text{ and}\\ \qquad |D_y^{\alpha}p_j(t,s;y,\xi)|\le C_\alpha'(t-s) \text{ on } [0,T]\times[0,T]\times R^{2n}_{y,\xi}\end{cases}$$

with constants C_α' independent of (t,s,y,ξ) are valid for $j=1,2,\cdots,n$. Hence, we can easily prove (2.9) from (2.10) and the assumption on $h_1(t,x,\xi)$. Q.E.D.

Proof of Theorem 1.1. Assume that (1.3) is not valid for any $T_1(0<T_1\le T)$. Then, we will show that we can construct the solution of (0.1) which contradicts the energy inequality (1.1). Thus, we will prove this theorem.

Since (1.3) is assumed not to be valid for any $T_1(0<T_1\le T)$, we can take points $(t_m,y^{(m)},\xi^{(m)})\in[0,T]\times R^{2n}_{y,\xi}(m=1,2,\cdots)$ such that

$$(2.11)\qquad \begin{cases} t_m \to 0 \text{ as } m\to\infty\,, \\ \mathrm{Re}\,\Omega(t_m, y^{(m)}, \xi^{(m)}) \to \infty \text{ as } m\to\infty\,, \end{cases}$$

where $\Omega(t, y, \xi)$ was determined by (2.8). $\mathrm{Re}\,c$ denotes the real part of c. Then, we can suppose that these points satisfy

$$(2.12)\qquad \mathrm{Re}\,\Omega(t, y^{(m)}, \xi^{(m)}) \le \mathrm{Re}\,\Omega(t_m, y^{(m)}, \xi^{(m)}) \ (0\le t\le t_m)\,.$$

Let $S_m(t, x)$ $(m=1, 2, \cdots)$ be the solution of the eiconal equation

$$(2.13)\qquad \partial_t S_m(t, x) + h_2\Big(t, x, \frac{\partial S_m}{\partial x}(t, x)\Big) = 0\,, \quad S_m|_{t=0} = x\cdot\xi^{(m)}\,.$$

Then, we know from Lemmas 2.1, 2.3 and 2.4 in [6] (or see [7]) that $S_m(t, x)$ for all $m=1, 2, \cdots$ exist on commom region $[0, T_0]\times R^n$ for a $T_0 (0 < T_0 \le T)$ and that

$$(2.14)\qquad \text{if } |\alpha|\ge 2\,, \quad |D_x^\alpha S_m(t, x)| \le C_\alpha \text{ on } [0, T_0]\times R^n$$

are valid, where C_α are constants independent of (t, x) and $m=1, 2, \cdots$. Let $v(t, x)$ be a C^∞ function on $[0, T_0]\times R^n$ with compact support. Then, we can apply Proposition 2.2 as $\lambda=1$ to each term $e^{-iS_m(t,x)}H_j(t, x, D_x)\{e^{iS_m(t,x)}v(t, x)\}(j=1, 2)$ because of the assumption (0.2) and (2.14). Hence, we get together with (2.13)

$$(2.15)\qquad e^{-iS_m(t,x)}L\{e^{iS_m(t,x)}v(t, x)\}$$
$$= \frac{1}{i}\Big[\partial_t v(t, x) + \textstyle\sum_{j=1}^n \dfrac{\partial h_2}{\partial \xi_j}\Big(t, x, \dfrac{\partial S_m}{\partial x}(t, x)\Big)\dfrac{\partial v}{\partial x_j}(t, x)$$
$$+ \frac{1}{2}\Big\{\mathrm{Tr}\frac{\partial^2 h_2}{\partial \xi^2}\Big(t, x, \frac{\partial S_m}{\partial x}(t, x)\Big)\frac{\partial^2 S_m}{\partial x^2}(t, x)\Big\}v(t, x)$$
$$+ ih_1\Big(t, x, \frac{\partial S_m}{\partial x}(t, x)\Big)v(t, x)\Big] + \mathrm{Rv}(t, x) \text{ on } [0, T_0]\times R^n$$

and

$$(2.16)\qquad \|\mathrm{Rv}(t, \cdot)\| \le M_1 \textstyle\sum_{|\alpha|\le l_2} \|D_x^\alpha v(t, \cdot)\|\,,$$

where $\mathrm{Tr}(\cdot)$ denotes the trace of matrix and M_1 is a constant independent of $m=1, 2, \cdots$ and $v(t, x)$.

Take a C^∞ function $\psi(x)$ such that

$$(2.17)\qquad \psi(0)\ne 0\,, \quad \mathrm{supp}\,\psi(\cdot)\subset\{x; |x|\le 1\} \text{ and } \int|\psi(x)|^2 dx = 1\,.$$

$\mathrm{supp}\,\psi(\cdot)$ denotes the support of $\psi(x)$. We define $v_m(t, x) (m=1, 2, \cdots)$ as the solution of

$$
(2.18)\qquad \begin{cases} \partial_t v_m(t,x)+\sum_{j=1}^n \dfrac{\partial h_2}{\partial \xi_j}\Big(t,x,\dfrac{\partial S_m}{\partial x}(t,x)\Big)\dfrac{\partial v_m}{\partial x_j}(t,x) \\ \qquad +\dfrac{1}{2}\Big\{\mathrm{Tr}\dfrac{\partial^2 h_2}{\partial \xi^2}\Big(t,x,\dfrac{\partial S_m}{\partial x}(t,x)\Big)\dfrac{\partial^2 S_m}{\partial x^2}(t,x)\Big\}v_m(t,x) \\ \qquad +ih_1\Big(t,x,\dfrac{\partial S_m}{\partial x}(t,x)\Big)v_m(t,x)=0\,, \\ v_m|_{t=0}=\psi(x-y^{(m)})\,. \end{cases}
$$

We can assume $\det\dfrac{\partial q}{\partial y}(t,0;y,\xi^{(m)})>0$ on $[0,T_0]\times R_y^n$ $(m=1,2,\cdots)$ from (2.10). Then, we know well (c.f. [2] or [10]) that these solution $v_m(t,x)$ are given by

$$
(2.19)\qquad \begin{aligned} & v_m(t,q(t,0;y,\xi^{(m)})) \\ &= \Big\{\det\frac{\partial q}{\partial y}(t,0;y,\xi^{(m)})\Big\}^{-1/2}\psi(y-y^{(m)}) \\ &\quad \times\exp\Big[-i\int_0^t h_1(\theta,q(\theta,0;y,\xi^{(m)}),p(\theta,0;y,\xi^{(m)}))\mathrm{d}\theta \\ &\quad +\frac{1}{2}\int_0^t \mathrm{Tr}\frac{\partial^2 h_2}{\partial x\partial \xi}(\theta,q(\theta,0;y,\xi^{(m)}),p(\theta,0;y,\xi^{(m)}))\mathrm{d}\theta\Big] \\ &\quad \text{on } [0,T_0]\times R_x^n \qquad (y=y(t,0;x,\xi^{(m)}))\,. \end{aligned}
$$

Here, $y=y(t,s;x,\xi^{(m)})=(y_1(t,s;x,\xi^{(m)}),\cdots,y_n(t,s;x,\xi^{(m)}))$ on $[0,T_0]\times[0,T_0]\times R_x^n$ is defined as the inverse of the mapping: $R^n\ni y\to x=q(t,s;y,\xi^{(m)})\in R^n$, whose well-definedness on $[0,T_0]\times[0,T_0]\times R_x^n$ and properties were studied in Lemma 2.3 in [6] (or see [7]). We shall prove (2.19), because the situation in [2] and [10] is some different from ours. We know well that

$$
\frac{\partial S_m}{\partial x}(t,q(t,0;y,\xi^{(m)}))=p(t,0;y,\xi^{(m)}) \text{ on } [0,T_0]\times R_y^n
$$

are valid. For example, see Lemma 2.4 in [6]. So, we can see from (1.2) that (2.18) can be written as

$$
(2.18)'\qquad \begin{cases} \dfrac{d}{dt}v_m(t,q(t,0;y,\xi^{(m)})) \\ \qquad +\dfrac{1}{2}\Big\{\mathrm{Tr}\dfrac{\partial^2 h_2}{\partial \xi^2}(t,q,p)\dfrac{\partial^2 S_m}{\partial x^2}(t,q)\Big\}v_m(t,q) \\ \qquad +ih_1(t,q,p)v_m(t,q)=0 \qquad (p=p(t,0;y,\xi^{(m)}))\,, \\ v_m(t,q(t,0;y,\xi^{(m)}))|_{t=0}=\psi(y-y^{(m)})\,. \end{cases}
$$

Also, since

$$
\frac{dq}{dt}(t,0;y,\xi^{(m)})=\frac{\partial h_2}{\partial \xi}\Big(t,q,\frac{\partial S_m}{\partial x}(t,q)\Big)
$$

is valid, we have

$$\frac{d}{dt}\frac{\partial q}{\partial y}(t, 0; y, \xi^{(m)})=\left\{\frac{\partial^2 h_2}{\partial x\partial\xi}(t, q, p)+\frac{\partial^2 h_2}{\partial\xi^2}(t, q, p)\frac{\partial^2 S_m}{\partial x^2}(t, q)\right\}\frac{\partial q}{\partial y}.$$

Consequently, we obtain by the Liouville formula

$$(2.20)\qquad \frac{d}{dt}\log\det\frac{\partial q}{\partial y}(t, 0; y, \xi^{(m)})$$

$$=\mathrm{Tr}\left\{\frac{\partial^2 h_2}{\partial x\partial\xi}(t, q, p)+\frac{\partial^2 h_2}{\partial\xi^2}(t, q, p)\frac{\partial^2 S_m}{\partial x^2}(t, q)\right\}$$

$$\text{on } [0, T_0]\times R^n_y.$$

Hence, we obtain (2.19) from (2.18)′ and (2.20).

We define $u_m(t, x)\in\mathcal{E}^0_t([0, T]; L^2)$ $(m=1, 2, \cdots)$ by

$$u_m(t, x)=\begin{cases}\{\exp iS_m(t, x)\}v_m(t, x) & (0\leq t\leq T_0)\\ \{\exp iS_m(T_0, x)\}v_m(T_0, x) & (T_0\leq t\leq T).\end{cases}$$

Recall from Lemmma 2.3 in [6] that

$$(2.21)\qquad \text{if } |\alpha|\geq 1, \qquad |D_x^{\alpha}\{y_j(t, s; x, \xi)-x_j\}|\leq C''_{\alpha}(t-s)$$

$$\text{on}\quad [0, T_0]\times[0, T_0]\times R^{2n}_{x,\xi}$$

are valid with constants C''_{α} independent of (t, s, x, ξ). Let's insert $u_m(t, x)$ into the energy inequality (1.1). Then, it follows from (2.15) and (2.16) that we have

$$(2.22)\qquad \|v_m(t, \cdot)\|\leq C(T)(\|\psi(\cdot)\|+M_1 t\textstyle\sum_{|\alpha|\leq l_2}\max\limits_{\theta\in[0,t]}\|D_x^{\alpha}v_m(\theta, \cdot)\|)$$

$$(0\leq t\leq T_0).$$

M_1 is the constant in (2.16). We shall first estimate $\|v_m(t, \cdot)\|$ from below. We can easily see from Lemma 2.3 in the present paper that

$$(2.23)\qquad |\mathrm{Re}\Omega(t, y', \xi)-\mathrm{Re}\Omega(t, y, \xi)|\leq Kt|y'-y|$$

are valid for all (t, y, ξ), $(t, y', \xi)\in[0, T_0]\times R^{2n}$, where K is a constant independent of (t, y, ξ) and (t, y', ξ). Hence, noting (0.2) and (2.17), we obtain from (2.19)

$$(2.24)\qquad \|v_m(t, \cdot)\|^2$$

$$=\int|\psi(y-y^{(m)})|^2\Big[\exp 2\mathrm{Re}\Big\{-i\int_0^t h_1(\theta, q(\theta, 0; y, \xi^{(m)}),$$

$$p(\theta, 0; y, \xi^{(m)}))\mathrm{d}\theta+\frac{1}{2}\int_0^t \mathrm{Tr}\frac{\partial^2 h_2}{\partial x \partial \xi}(\theta, q, p)\mathrm{d}\theta\Big\}\Big]\mathrm{d}y$$

$$\geq \delta^2 \int |\psi(y-y^{(m)})|^2 \exp\{2\mathrm{Re}\Omega(t, y^{(m)}, \xi^{(m)})-2Kt|y-y^{(m)}|\}\mathrm{d}y$$

$$\geq \delta^2 \exp\{2\mathrm{Re}\Omega(t, y^{(m)}, \xi^{(m)})-2Kt\} \qquad (0\leq t\leq T_0)$$

for a constant $\delta>0$ independent of t and $m=1, 2, \cdots$. The above first equality was derived by changing x variables to $y=y(t, 0; x, \xi^{(m)})$. Similarly, taking account of (0.2), (2.10) and (2.21), we can easily see from (2.19) and (2.23)

$$\sum_{|\alpha|\leq l_2} \max_{\theta\in[0,t]} \|D_x{}^\alpha v_m(\theta, \cdot)\| \tag{2.25}$$

$$\leq M_2 \max_{\theta\in[0,t]} \exp\{\mathrm{Re}\Omega(\theta, y^{(m)}, \xi^{(m)})+Kt\} \qquad (0\leq t\leq T_0)$$

for a constant M_2 independent of t and $m=1, 2, \cdots$, where we also used (2.17). Insert (2.24) and (2.25) into (2.22) and set $t=t_m$ $(m=1, 2, \cdots)$, which can be assumed to be less than T_0 because of (2.11). Then, we get together with (2.12)

$$(\delta-C(T)M_1M_2t_me^{2Kt_m})e^{\mathrm{Re}\Omega(t_m,y^{(m)},\xi^{(m)})}\leq C(T)e^{Kt_m} \tag{2.26}$$

for $m=1, 2, \cdots$. It follows from the choice (2.11) of $(t_m, y^{(m)}, \xi^{(m)})$ that the above (2.26) is not valid, when m is much large. Thus, we can complete the proof. Q.E.D.

3. On a sufficient condition

In this section we will prove the following theorem, whose similar result has been obtained generally in [6]. Our method is easier than in [6], because we study only the limited equations.

Theorem 3.1. *Suppose the same assumptions as in Theorem 1.1. Then if* (1.3) *is valid for a* $T_1(0<T_1\leq T)$, *there exists a* T_1' $(0<T_1'\leq T_1)$ *in order that the Cauchy problem* (0.1) *is* L^2 *well posed on* $[0, T_1']$.

Proof. We can prove this theorem in the similar way as in [11]. We determine a symbol $k(t, x, \xi)$ on $[0, T_1]\times R^{2n}$ as the solution of

$$\begin{cases} \dfrac{\partial k}{\partial t}(t, x, \xi)+\{h_2(t, x, \xi), k(t, x, \xi)\}+ih_1(t, x, \xi)k(t, x, \xi)=0\,, \\ k(0, x, \xi)=1\,, \end{cases} \tag{3.1}$$

where $\{h_2, k\}$ denotes the Poisson bracket $\sum_{j=1}^n\Big\{\frac{\partial h_2}{\partial \xi_j}(t, x, \xi)\frac{\partial k}{\partial x_j}(t, x, \xi)-\frac{\partial h_2}{\partial x_j}(t, x, \xi)\frac{\partial k}{\partial \xi_j}(t, x, \xi)\Big\}$. We can easily see from (1.2) that (3.1) is written as

$$\begin{cases} \dfrac{d}{dt}k(t, q(t, 0; y, \eta), p(t, 0; y, \eta)) + ih_1(t, q, p)k(t, q, p) = 0\,, \\ k(t, q(t, 0; y, \eta), p(t, 0; y, \eta))|_{t=0} = 1\,. \end{cases}$$

So, we obtain

$$k(t, x, \xi) = \exp\left\{-i\int_0^t h_1(\theta, q(\theta, 0; y, \eta), p(\theta, 0; y, \eta))d\theta\right\} \tag{3.2}$$

$$\text{on } [0, T_1]\times R^{2n}_{x,\xi} \qquad ((y, \eta) = (q(0, t; x, \xi), p(0, t; x, \xi))\,,$$

because $(q, p)(t, 0; y, \eta) = (x, \xi)$ is equivalent to $(y, \eta) = (q, p)(0, t; x, \xi)$. Hence, noting (0.2) with $j=1$ and (2.10), we can see from the assumption (1.3) that

$$|k^{(\alpha)}_{(\beta)}(t, x, \xi)| \le C_{\alpha,\beta} \text{ on } [0, T_1]\times R^{2n} \tag{3.3}$$

are valid for all α and β with constants $C_{\alpha,\beta}$ independent of (t, x, ξ). Also, noting the assumption (0.2), we can prove from (3.1) and (3.3)

$$\frac{\partial k}{\partial t}(t, x, \xi) \in T^1(R^{2n}) \ (0 \le t \le T_1)\,.$$

Now, we will find a solution $u(t, x)$ of (0.1) in the form

$$u(t, x) = K(t, x, D_x)v(t, x) \equiv Kv(t, x)\,. \tag{3.4}$$

Then, we have

$$L\{K(t, x, D_x)v(t, x)\} = K\circ\left[\frac{1}{i}\partial_t v(t, x) + H_2(t, x, D_x)v\right]$$

$$+\left\{\frac{1}{i}\frac{\partial K}{\partial t}(t, x, D_x) + (H_2\circ K - K\circ H_2) + H_1\circ K\right\}v\,,$$

where $\dfrac{\partial K}{\partial t}(t, x, D_x)$ denotes the pseudo-differential operator with symbol $\dfrac{\partial k}{\partial t}(t, x, \xi)$ and '$\circ$' the product of operators. Applying the expansion formula of pseudo-differential operators in section 3 of chapter 2 in [8], it follows from the assumption (0.2) and (3.3) that

$$\begin{cases} \sigma(H_2\circ K - K\circ H_2)(t, x, \xi) = \dfrac{1}{i}\{h_2(t, x, \xi), k(t, x, \xi)\} + r_1(t, x, \xi)\,, \\ \sigma(H_1\circ K)(t, x, \xi) = h_1(t, x, \xi)k(t, x, \xi) + r_2(t, x, \xi) \end{cases} \tag{3.5}$$

and estimates

$$|r^{(\alpha)}_{j(\beta)}(t, x, \xi)| \le C'_{\alpha,\beta} \qquad \text{on } [0, T_1]\times R^{2n} \ (j=1, 2) \tag{3.6}$$

with constants $C'_{\alpha,\beta}$ independent of (t, x, ξ) are obtained. Consequently,

$$L\{K(t, x, D_x)v(t, x)\}=K\circ\left[\frac{1}{i}\partial_t v(t, x)+H_2 v\right]+(R_1+R_2)v$$

is valid, because $k(t, x, \xi)$ is the solution of (3.1). Hence, setting $R=R_1(t, x, D_x)+R_2(t, x, D_x)$, the Cauchy problem (0.1) can be expressed as

$$\begin{cases} K\circ\left[\frac{1}{i}\partial_t v(t, x)+H_2(t, x, D_x)v\right]+Rv=f(t, x), \\ v(0, x)=u_0(x). \end{cases} \tag{3.7}$$

The existence of the inverse operator $K^{-1}=K(t, x, D_x)^{-1}$ of $K(t, x, D_x)$ as the mapping from L^2 space onto L^2 space can be proven as follows. Set

$$\tilde{k}(t, x, \xi)=1/k(t, x, \xi) \tag{3.8}$$

and consider the product $\tilde{K}(t, x, D_x)\circ K(t, x, D_x)$. Then, applying the expansion formula of pseudo-differential operators again, we can prove from (3.2) and (2.10) in the same way as in the proof of (3.3) that

$$\sigma(\tilde{K}\circ K)(t, x, \xi)=1+ts(t, x, \xi) \tag{3.9}$$

is valid, where $s(t, x, \xi)$ has the same estimates with another constants as (3.6). We see from the Calderón-Vaillancourt theorem in [3]

$$\sup_{0\leq t\leq T_1}\|S(t, x, D_x)\|<\infty,$$

where $\|S(t, x, D_x)\|$ denotes the operator norm as the mapping from L^2 space into L^2 space. If T_1' $(0<T_1'\leq T_1)$ is small, we can construct the inverse as the mapping from L^2 space onto L^2 space of $I+tS(t, x, D_x)$ by the Neumann series for each $t\in[0, T_1']$. Thus, we can see the existence of K^{-1} for each $t\in[0, T_1']$, because $\tilde{K}(t, x, D_x)$ also becomes an L^2 bounded operator from the Calderón-Vaillancourt theorem.

If we operate $K(t, x, D_x)^{-1}$ on both sides of equations in (3.7), we obtain the Cauchy problem

$$\frac{1}{i}\partial_t v(t, x)+H_2 v+K^{-1}\circ Rv=K^{-1}f(t, x), \quad v(0, x)=u_0(x). \tag{3.10}$$

Applying the Calderón-Vaillancourt theorem to $R=R_1(t, x, D_x)+R_2(t, x, D_x)$ again, we can see that (3.10) is L^2 well posed on $[0, T_1']$ in a sense of Definition 1.1. Also, we can prove from this fact that (0.1) is also L^2 well posed on $[0, T_1']$, because the solution $u(t, x)$ of (0.1) is determined by (3.4). Q.E.D.

Remark 3.1. We note that we can also complete the proof of Theorem 1.2 from this Theorem 3.1 in place of Corollary 1.2 in [6].

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References

[1] V. I. Arnold, Mathematical method of classical mechanics, Springer-Verlag, New York, 1978.
[2] G. D. Birkhoff, Quantum mechanics and asymptotic series, Bull. Amer. Math. Soc., **39** (1933), 681-700.
[3] A. P. Calderón and R. Vaillancourt, On the boundedness of pseudo-differential operators, J. Math. Soc. Japan, **23** (1971), 374-378.
[4] W. Ichinose, The Cauchy problem for Schrödinger type equations with variable coefficients, Osaka J. Math., **24** (1987), 853-886.
[5] W. Ichinose, On L^2 well posedness of the Cauchy problem for Schrödinger type equations on the Riemannian manifold and the Maslov theory, Duke Math. J., **56** (1988), 549-588.
[6] W. Ichinose, On the Cauchy problem for Schrödinger type equations and Fourier integral operators, J. Math. Kyoto Univ., **33** (1993), 583-620.
[7] H. Kitada, On a construction of the fundamental solution for Schrödinger equations, J. Fac. Sci. Univ. Tokyo, Ser. IA, **27** (1980), 193-226.
[8] H. Kumano-go, Pseudo-differential operators, M. I. T. Press, Cambridge, 1981.
[9] P. D. Lax, Asymptotic solutions of oscillatory initial value problems, Duke Math. J., **24** (1957), 627-646.
[10] V. P. Maslov and M. V. Fedoriuk, Semi-classical approximation in quantum mechanics, D. Reidel, Dordrecht, 1981.
[11] S. Mizohata, Sur quelques équations du type Schrödinger, Jounées "Equations aux dérivées partielles", Saint-Jean de Monts, 1981.