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On maps from BS^1 to classifying spaces of certain gauge groups

By

SHUICHI TSUKUDA

1. Introduction

Let G be a compact connected Lie group, $\pi: P \to X$ a principal G bundle over a compact connected manifold X, and $\mathscr{G}$ its gauge group. $\mathscr{G}$ is identified with $\Gamma(AdP)$, all continuous sections of the adjoint bundle of P, and we give the compact open topology on it.

Assume that the structure group of P reduces to $Z(S^1)$, the centralizer of a closed subgroup S^1 of G, then

$$AdP = P \times_{Ad} G = P_{Z(S^1)} \times_{Ad} G,$$

therefore $\mathscr{G}$ naturally contains S^1. Conversely if $\mathscr{G}$ contains S^1 as a subgroup, one can show that the structure group of P reduces to $Z(S^1)$ (see Appendix). We can show similar results in the level of classifying spaces in some cases. The homotopy theory of classifying spaces of compact Lie groups has been developed since 80's ([9] is a good survey) and using the results of [8], [6] we have following results.

Theorem 1.1. *Let P be a principal* $\mathrm{SU}(m)$ *or* $\mathrm{Sp}(m)$ *bundle over an n dimensional sphere S^n. Then the following three conditions are equivalent.*

1. *There exists a homotopically non trivial map from BS^1 to $B\mathscr{G}$.*
2. *There exists a non trivial homomorphism from S^1 to $\mathscr{G}$.*
3. *There exists a non trivial homomorphism $\rho: S^1 \to G$ $(G = \mathrm{SU}(m), \mathrm{Sp}(m))$ and the structure group of P reduces to $Z(\rho(S^1))$.*

Theorem 1.2. *Let P be a principal* $\mathrm{SU}(2)$ *bundle over a smooth simply connected spin 4 manifold X or CP^2. Then the following three conditions are equivalent.*

1. *There exists a homotopically non trivial map from BS^1 to $B\mathscr{G}$.*
2. *There exists a non trivial homomorphism from S^1 to $\mathscr{G}$.*
3. *The structure group of P reduces to S^1.*

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In the case of $X=\boldsymbol{C}P^2$, we can show a similar result in classical way using the ring structure of the cohomology of $B\mathscr{G}$. In section 3 we prove the following result. See section 3 for details.

Proposition 1.3 (weaker version of 1.2). *If $M_{l,\, CP^2}\simeq M_{-k^2,\, CP^2}$, then there exists an integer m and $l=-m^2$.*

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2. Proof of Theorem 1.1, 1.2

By [2], we have a homotopy equivalence

$$B\mathscr{G}\simeq \mathrm{Map}_P(X,\ BG),$$

where $\mathrm{Map}_P(X,\ BG)$ denotes the connected component of $\mathrm{Map}(X,\ BG)$ containing the map inducing P and a fibration

$$\mathrm{Map}_P^*(X,\ BG)\to \mathrm{Map}_P(X,\ BG)\xrightarrow{ev} BG,$$

where $\mathrm{Map}_P^*(X,\ BG)$ is the space of based maps. Consider a map $f: BS^1\to \mathrm{Map}_P(S^n,\ BG)$. The following holds.

Lemma 2.1. *Assume that n is even or $\pi_j(G)\otimes\boldsymbol{Q}=0$ for $j>n$. If $ev\circ f$ is homotopically trivial then so is f.*

Proof. If $ev\circ f$ is homotopically trivial then we have a lifting $\tilde{f}: BS^1\to \mathrm{Map}_P^*(S^n,\ BG)\simeq\Omega^{n-1}G$. By [6], if Y is a finite dimensional connected complex and $\pi_i(Y)$ is finitely generated for each $i>1$, for $j\geq 1$

$$\pi_j(\mathrm{Map}^*(BG,\ Y))\cong\prod_{k>j}H^{k-j}(BG\ ;\ \pi_{k+1}(Y)\otimes\hat{\boldsymbol{Z}}/\boldsymbol{Z}),$$

where $\hat{\boldsymbol{Z}}=\prod\boldsymbol{Z}_p$ is the product over all p-adic integers.

Thus we have

$$\begin{aligned}[BS^1,\ \Omega^{n-1}G]&=[\Sigma^{n-1}BS^1,\ G]=\pi_{n-1}(\mathrm{Map}^*(BS^1,\ G))\\&\cong\prod_{k>n-1}H^{k-n+1}(BS^1,\ \pi_{k+1}(G)\otimes\hat{\boldsymbol{Z}}/\boldsymbol{Z})=0,\end{aligned}$$

where [ ] denotes based homotopy classes. Therefore $\tilde{f}\simeq *$ and so is f.

We can prove a similar result in the case of principal SU(2) bundles over simply connected 4 manifolds. Let X be a simply connected 4 manifold with 2nd betti number b. Then we have a cofibering

$$\bigvee_b S^2 \xrightarrow{i} X \xrightarrow{q} S^4$$

and obtain a fibering

$$\Omega^3 S^3 \rightarrow \mathrm{Map}_k^*(X, B\mathrm{SU}(2)) \rightarrow \prod_b \Omega S^3. \tag{1}$$

Note that principal SU(2) bundles over X are classified by their 2nd Chern classes and $\mathrm{Map}_k(X, B\mathrm{SU}(2))$ denotes the component corresponds to the bundle P_k with $c_2(P_k)=k$.

Lemma 2.2. *Consider a map* $f : BS^1 \rightarrow \mathrm{Map}_k(X, B\mathrm{SU}(2))$. *If* $ev \circ f$ *is homotopically trivial, then so is* f.

Proof. If $ev \circ f$ is homotopically trivial, we have a lifting $\tilde{f} : BS^1 \rightarrow \mathrm{Map}_k^*(X, B\mathrm{SU}(2))$. Since

$$[BS^1, \prod_b \Omega S^3] = \prod_b [BS^1, \Omega S^3] = \prod_b [\Sigma BS^1, S^3]$$

$$\cong \bigoplus_b \pi_1(\mathrm{Map}^*(BS^1, S^3)) \cong \prod_{\substack{k>1 \\ b}} H^{k-1}(BS^1 ; \pi_{k+1}(S^3) \otimes \hat{\boldsymbol{Z}}/\boldsymbol{Z}) = 0,$$

$i^\# \circ \tilde{f}$ is trivial and $\tilde{f}$ lifts to $\Omega^3 S^3$.

$$[BS^1, \Omega^3 S^3] \cong \prod_{k>3} H^{k-3}(BS^1 ; \pi_{k+1}(S^3) \otimes \hat{\boldsymbol{Z}}/\boldsymbol{Z}) = 0$$

hence f is homotopically trivial.

Note that this lemma also holds if X is a finite complex with only even cells.

We recall some results from [8]. Let $\rho : S^1 \rightarrow G$ be a homomorphism. Denote by $Z(\rho)$ the centralizer of this homomorphism and by $\mathrm{Map}_\rho(BS^1, BG)$ the component which contains the map $B\rho$. The obvious homomorphism

$$Z(\rho) \times S^1 \rightarrow G$$

induces a map

$$BZ(\rho) \times BS^1 \rightarrow BG,$$

which has as adjoint

$$ad_\rho : BZ(\rho) \rightarrow \mathrm{Map}_\rho(BS^1, BG).$$

Denote by $Rep(S^1, G)$ the set of conjugation classes of homomorphisms.

Theorem 2.3 ([8]). *The map*

$$Rep(S^1, G) \to [BS^1, BG]$$

is a bijection.

Theorem 2.4 ([8]). *ad_ρ induces an isomorphism in the mod p homology.*

Moreover one can describe the homotopy fiber X_ρ of ad_ρ. Suppose that X is a space with an action of a topological group H. We define $X^H = \mathrm{Map}_H(pt, X)$ to be the fixed point set and $X^{hH} = \mathrm{Map}_H(EH, X)$ to be the homotopy fixed point set where $\mathrm{Map}_H(\,,\,)$ denote the space of all equivariant maps. $\hat{X}_p$ denote the p-adic completion in the sense of Bousfield and Kan and $\hat{X} = \Pi \hat{X}_p$ is the product over all p-adic completions. Let S^1 act on G via ρ and conjugation. By choosing a fixed point as base point of $(\hat{G})^{hS^1}$, S^1 acts on the homotopy fiber F of $G \to \hat{G}$. This induces a homotopy fibration (see [8] for details)

$$F^{hS^1} \to G^{hS^1} \to (\hat{G})^{hS^1}$$

and one can compute the homotopy groups $\pi_j(F^{hS^1})$.

Proposition 2.5 ([8]).

$$\pi_j(F^{hS^1}) \cong \prod_{i \geq j} H^{i-j}(BS^1\,;\, \pi_{i+1}(G) \otimes \hat{\boldsymbol{Z}}/\boldsymbol{Z}).$$

One can also compute the homotopy groups of the homotopy fiber F_{fix} of $G^{S^1} \to \hat{G}^{S^1}$.

Proposition 2.6 ([8]).

$$\pi_j(F_{fix}) \cong \pi_{j+1}(G^{S^1}) \otimes \hat{\boldsymbol{Z}}/\boldsymbol{Z} \cong H^2(BS^1\,;\, \pi_{j+1}(G^{S^1}) \otimes \hat{\boldsymbol{Z}}/\boldsymbol{Z})$$

and the map $\pi_j(F_{fix}) \to \pi_j(F^{hS^1})$ is given by the canonical homomorphism between the coefficients of the homology groups.

Note that [8] contains more general results.

Proof of Theorem 1.1. We must show that 1 implies 3. We consider the case of $G = \mathrm{SU}(m)$.

Let $\rho_m : S^1 \to \mathrm{SU}(m)$ be a homomorphism given by

$$\rho_m(z) = \begin{pmatrix} z^{m-1} & & & \\ & z^{-1} & & \\ & & \ddots & \\ & & & z^{-1} \end{pmatrix},$$

then $Z(\rho_m)=\mathrm{SU}(m)\cap(S^1\times\mathrm{U}(m-1))$. If $n\leq 2m-1$, since $\pi_{n-1}(\mathrm{SU}(m)\cap(S^1\times\mathrm{U}(m-1)))\to\pi_{n-1}(\mathrm{SU}(m))$ is surjective, the structure group of any principal $\mathrm{SU}(m)$ bundle over S^n reduces to $Z(\rho_m)$ hence 1, 2 and 3 always hold.

Assume that $n\geq 2m$. Suppose there exists a non trivial map $f: BS^1\to \mathrm{Map}_P(S^n, BG)$. By 2.1, $ev\circ f$ is homotopically nontrivial and by 2.3 there exists a non trivial homomorphism $\rho: S^1\to G$ such that $ev\circ f\simeq B_\rho$ hence taking adjoint of f we obtain a map g

$$\begin{array}{c} BZ(\rho) \\ \downarrow{\scriptstyle ad_\rho} \\ S^n \xrightarrow{g} \mathrm{Map}_\rho(BS^1, BG) \\ \downarrow{\scriptstyle ev} \\ BG. \end{array}$$

Note that $ev\circ g$ induces P and $ev\circ ad_\rho$ is homotopic to the map induced by the inclusion $Z(\rho)\hookrightarrow G$.

By [8] there is a fibration

$$F_{fix}\to F^{hS^1}\to X_\rho.$$

Since $\pi_j(G)\otimes \boldsymbol{Q}=0$ for $j>2m-1$ and $\pi_j(G^{S^1})\otimes\boldsymbol{Q}=0$ for $j>2m-2$, $\pi_j(F^{hS^1})=0$ for $j>2m-2$ and $\pi_j(F_{fix})=0$ for $j>2m-3$ hence by the homotopy exact sequence for the fibration we have $\pi_j(X_\rho)=0$ for $j>2m-2$. Therefore $(ad_\rho)_*: \pi_n(BZ(\rho))\to\pi_n(\mathrm{Map}_\rho(BS^1, BG))$ is surjective hence we have a lift of g, $\tilde{g}: S^n\to BZ(\rho)$ and the structure group of P reduces to $Z(\rho)$.

The proof in the case of $\mathrm{Sp}(m)$ is similar.

Proof of Theorem 1.2. As above, if there exists a non trivial map f: $BS^1\to\mathrm{Map}_P(X, B\mathrm{SU}(2))$ we obtain a map

$$g: X\to\mathrm{Map}_\rho(BS^1, B\mathrm{SU}(2)),$$

where $\rho: S^1\to\mathrm{SU}(2)$ is a non trivial homomorphism. Note that $Z(\rho)=S^1$ and $ev\circ ad_\rho\simeq Bi: BS^1\to B\mathrm{SU}(2)$ where $i: S^1\hookrightarrow\mathrm{SU}(2)$ is an inclusion. $Bi^*c_2=-c_1^2$ where $c_2\in H^4(B\mathrm{SU}(2);\boldsymbol{Z})$ is the universal 2nd Chern class and $c_1\in H^2(BS^1;\boldsymbol{Z})$ is the universal 1st Chern class. By 2.4 there exists an element $\alpha\in H^2(\mathrm{Map}_\rho(BS^1, B\mathrm{SU}(2));\boldsymbol{Z}/p)$ and $ad_\rho^*\alpha=c_1$. We have

$$c_2(P)=g^*ev^*c_2=g^*(-\alpha^2)=-(g^*(\alpha))^2\in H^4(X;\boldsymbol{Z}/p).$$

If X is a spin manifold with the 2nd betti number $b_2=0$, we have $c_2(P)\equiv 0 \pmod p$ for any prime p hence $c_2(P)=0$.

Note that the intersection form Q of a simply connected spin 4 manifold is even. If X is smooth, by a result of Donaldson [4], Q is indefinite hence of the form $Q=mH\oplus nE_8$ where

$$H=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

and $m, n\in \boldsymbol{Z}_{\geq 0}$. By [5] if $n>0$, $m\geq 3$ therefore if $b_2>0$, Q has at least one H part hence the structure group of P reduces to S^1 if and only if $c_2(P)$ is even. Since we have an element $v\in H^2(X;\boldsymbol{Z}/2)$ such that $c_2(P)=-v^2=0\in H^4(X;\boldsymbol{Z}/2)$, the result follows.

If X is $\boldsymbol{C}P^2$, we have an integer m_p for each prime p such that $c_2(P)\equiv -m_p^2 \pmod p$ therefore $c_2(P)=-m^2$ for some integer m.

Remark 2.7. The proof above breaks for general simply connected 4 manifolds because of algebraic reason. For example, in the case of $X=\boldsymbol{C}P^2\#\boldsymbol{C}P^2$, we have integers m_p, n_p for each prime p such that $-c_2(P)\equiv m_p^2+n_p^2 \pmod p$ but this does not imply that $-c_2(P)$ is a sum of square numbers. In fact $6=6+0=2+4=3+3$ and $\left(\frac{6}{p}\right)=\left(\frac{2}{p}\right)\left(\frac{3}{p}\right)$ where $\left(\frac{a}{p}\right)$ is the Legendre's symbol.

3. Cohomology of Map($\boldsymbol{C}P^2$, BSU(2))

In this section we determine the cohomology of Map($\boldsymbol{C}P^2$, BSU(2)) in low degree. Of course the calculation is based on the Serre spectral sequences for the fibrations

$$\mathrm{Map}_k^*(\boldsymbol{C}P^2, B\mathrm{SU}(2))\to \mathrm{Map}_k(\boldsymbol{C}P^2, B\mathrm{SU}(2))\xrightarrow{ev_k} B\mathrm{SU}(2), \tag{2}$$

$$\Omega_0^3S^3\xrightarrow{q}\mathrm{Map}_k^*(\boldsymbol{C}P^2, B\mathrm{SU}(2))\xrightarrow{i}\Omega S^3. \tag{3}$$

Denote $\mathrm{Map}_k(X, B\mathrm{SU}(2))$ (resp. $\mathrm{Map}_k^*(X, B\mathrm{SU}(2))$) by $M_{k,X}$ (resp. $M_{k,X}^*$).

It is well known that $\mathrm{Map}_k^*(\boldsymbol{C}P^2, B\mathrm{SU}(2))\to\Omega S^3$ is a rational equivalence.

Proposition 3.1.

$$H^*(\mathrm{Map}_k^*(\boldsymbol{C}P^2, B\mathrm{SU}(2));\boldsymbol{Q})\cong\boldsymbol{Q}[x],$$

where $\deg x=2$.

Let p be a prime. Note that for $*\leq 2p$

$$H^*(\Omega S^3;\boldsymbol{Z}_{(p)})\cong\boldsymbol{Z}_{(p)}[u_1, u_2]/(u_1^p-pu_2)$$

as algebras where $\deg u_1=2$, $\deg u_2=2p$,

$$H^j(\Omega S^3\,;\,\boldsymbol{Z}/p)=\begin{cases}0 & \text{if } j \text{ is odd}\\ \boldsymbol{Z}/p & \text{if } j \text{ is even,}\end{cases}$$

if $p\geq 3$ the p component of the homotopy groups of S^3 is given by

$$\pi^{(p)}_{3+k}(S^3)=\begin{cases}0 & 0<k<2p-3,\ 2p-3<k<4p-6\\ \boldsymbol{Z}/p & k=2p-3\end{cases}$$

and

$$H^j(\Omega^3_0 S^3\,;\,\boldsymbol{Z}/p)=\begin{cases}0 & 0<j<2p-3,\ 2p-2<j<4p-6\\ \boldsymbol{Z}/p & j=2p-3,\ 2p-2.\end{cases}$$

From the homotopy exact sequence for the fibrations (2), (3), using results of [10, ChV] we have $\pi_1(M^*_{k,CP^2})=\pi_1(M_{k,CP^2})=0$ and

$$\begin{array}{ccc}\boldsymbol{Z} & \xrightarrow{2\times} & \boldsymbol{Z}\\ \| & & \|\\ \pi_2(M^*_{k,CP^2}) & \longrightarrow & \pi_2(\Omega S^3)\\ \cong\downarrow & & \\ \pi_2(M_{k,CP^2}) & & \end{array}$$

hence $H^2(M^*_{k,CP^2}\,;\,\boldsymbol{Z})\cong H^2(M_{k,CP^2}\,;\,\boldsymbol{Z})\cong\boldsymbol{Z}$. Let $\tilde{a}$ be a generator of $H^2(M_{k,CP^2}\,;\,\boldsymbol{Z})$ and a its image in $H^2(M^*_{k,CP^2}\,;\,\boldsymbol{Z})$. Note that we can choose u_1 to satisfy $i^*(u_1)=a\in H^2(M^*_{k,CP^2}\,;\,\boldsymbol{Z}_{(p)})$.

We show the following results.

Theorem 3.2. *Let p be an odd prime. For $*\leq 2p-2$*

$$H^*(M^*_{k,CP^2}\,;\,\boldsymbol{Z}_{(p)})\cong\boldsymbol{Z}_{(p)}[a,\,b]/(a^{p-1}-pb)$$

as rings where $\deg a=2$, $\deg b=2p-2$.

$$H^j(M^*_{k,CP^2}\,;\,\boldsymbol{Z}/p)\cong\begin{cases}\boldsymbol{Z}/p & 2p-1\leq j\leq 4p-7,\ odd\\ \boldsymbol{Z}/p\oplus\boldsymbol{Z}/p & 2p\leq j\leq 4p-8,\ even\end{cases}$$

as vector spaces.

Corollary 3.3. *For $*\leq 2p-2$*

$$H^*(M_{k,CP^2}\,;\,\boldsymbol{Z}/p)\cong H^*(M^*_{k,CP^2}\,;\,\boldsymbol{Z}/p)\otimes H^*(B\mathrm{SU}(2)\,;\,\boldsymbol{Z}/p)$$

as vector spaces and $\tilde{a}^{p-1}\equiv 0$ mod $(ev_k^*c_2)\in H^{2p-2}(M_{k,CP^2}\,;\,\boldsymbol{Z}/p)$.

Theorem 3.4. *For $*\leq 4$*

$$H^*(M^*_{k,CP^2}\,;\,\boldsymbol{Z})\cong\boldsymbol{Z}[a,\,b]/(a^2-6b)$$

as algebras where deg $a=2$, deg $b=4$.

Proof of Theorem 3.2. *We give some remarks on the fibration*

$$\Omega_0^3 S^3 \to \mathrm{Map}_k(S^4, B\mathrm{SU}(2)) \xrightarrow{ev_k} B\mathrm{SU}(2).$$

Consider the transgression $\tau_k : H^{2p-3}(\Omega^3 S^3 ; \boldsymbol{Z}/p) \to H^{2p-2}(B\mathrm{SU}(2) ; \boldsymbol{Z}/p)$. It is easy to see that $\tau_k = k\tau_1$.

Lemma 3.5. $\tau_1 \neq 0$ *if* $p \geq 3$.

Proof. If $p \geq 5$, this is deduced from Lemma 2.2, 2.3, 2.4 of [11]. For $p=3$, see [7].

The first possibly nontrivial differential for the $\boldsymbol{Z}/p$ coefficient Serre spectral sequence for the principal fibration (3) in total degree $\leq 2p-3$ is

$$d_{2p-2} : E_{2p-2}^{0,\,2p-3} = H^{2p-3}(\Omega^3 S^3 ; \boldsymbol{Z}/p) \to H^{2p-2}(\Omega S^3 ; \boldsymbol{Z}/p) = E_{2p-2}^{2p-2,\,0}.$$

Lemma 3.6. *If* $p \geq 3$, $d_{2p-2} \neq 0$.

Proof. Note that the fibration (3) is independent of k. Consider the following commutative diagram

$$\begin{array}{ccccc}
\Omega^3 S^3 & \xrightarrow{q} & M^*_{-1,\,CP^2} & \xrightarrow{i} & \Omega S^3 \\
\downarrow & & \downarrow & & \\
M_{-1,\,S^4} & \longrightarrow & M_{-1,\,CP^2} & & \\
\downarrow & & \downarrow & & \\
B\mathrm{SU}(2) & = & B\mathrm{SU}(2) & &
\end{array}$$

Assume that $d_{2p-2}=0$ then for a generator $x \in H^{2p-3}(\Omega^3 S^3 ; \boldsymbol{Z}/p)$, there exists an element $y \in H^{2p-3}(M^*_{-1,\,CP^2} ; \boldsymbol{Z}/p)$ and $q^*(y)=x$. Then $\tau(x)=\tau q^*(y)=\tau(y) \neq 0$. There exists a map $f : BS^1 \to M_{-1,\,CP^2}$ such that $(ev \circ f)^* \neq 0 : H^{2p-2}(B\mathrm{SU}(2) ; \boldsymbol{Z}/p) \to H^{2p-2}(BS^1 ; \boldsymbol{Z}/p)$ therefore

$$0 \neq (ev \circ f)^* \tau(y) = f^*(ev^* \tau(y)) = 0,$$

which is a contradiction.

By proposition 3.1 $H^{2p-2}(M^*_{k,\,CP^2} ; \boldsymbol{Z})$ has a free part hence $d_2=0 : E_2^{0,\,2p-2} \to E_2^{2,\,2p-3}$ therefore $H^{2p-2}(M^*_{k,\,CP^2} ; \boldsymbol{Z}/p) \cong \boldsymbol{Z}/p$. Consider the $\boldsymbol{Z}_{(p)}$ coefficient Serre spectral sequence for the fibration (3). Since $E_2^{s,\,t}=0$ for $0<t<2p-2$, we have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_\infty^{0,2p-2} & \longrightarrow & H^{2p-2}(M^*_{k,CP^2}\,;\,\boldsymbol{Z}_{(p)}) & \longrightarrow & E_\infty^{2p-2,0} & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & \boldsymbol{Z}_{(p)} & & & & \boldsymbol{Z}/p. & & \end{array}$$

This sequence does not split because $H^{2p-2}(M^*_{k,CP^2}\,;\,\boldsymbol{Z}/p)\cong\boldsymbol{Z}/p$, therefore $a^{p-1}=i^*(u_1^{p-1})=pb$ where b is a generator of $H^{2p-2}(M^*_{k,CP^2}\,;\,\boldsymbol{Z}_{(p)})\cong\boldsymbol{Z}_{(p)}$ and we complete the proof of the first part of theorem 3.2.

Again we consider the $\boldsymbol{Z}/p$ coefficient spectral sequence. Since $d_2=0$: $E_2^{0,2p-2}\to E_2^{2,2p-3}$, we have $d_2=0$: $E_2^{s,2p-2}\to E_2^{s,2p-3}$ for any s.

Lemma 3.7. *If* $0<s\leq 2p-2$,

$$d_{2p-2}=0 : E_{2p-2}^{s,2p-3}\to E_{2p-2}^{s+2p-2,0}.$$

Proof. We may assume s is even. Note that $E_{2p-2}^{s,2p-3}=E_2^{s,2p-3}=E_2^{s,0}\otimes E_2^{0,2p-3}$ and if $j\leq 2p-2$, the cup product

$$H^j(\Omega S^3\,;\,\boldsymbol{Z}/p)\otimes H^{2p-2}(\Omega S^3\,;\,\boldsymbol{Z}/p)\to H^{j+2p-2}(\Omega S^3\,;\,\boldsymbol{Z}/p)$$

is zero. Let v be a generator of $H^{2p-3}(\Omega_0^3S^3\,;\,\boldsymbol{Z}/p)=\boldsymbol{Z}/p$. Then $d(u_1^{s/2}\otimes v)=u_1^{s/2}\cdot d(v)=0$.

Therefore we have

$$\sum_{s+t=j} E_\infty^{s,t}=\begin{cases} E_{2p-2}^{j-2p+3,\,2p-3}=\boldsymbol{Z}/p & 2p-1\leq j\leq 4p-7,\ \text{odd} \\ E_2^{j-2p+2,\,2p-2}\oplus E_{2p-2}^{j,0}=\boldsymbol{Z}/p\oplus\boldsymbol{Z}/p & 2p\leq j\leq 4p-8,\ \text{even} \end{cases}$$

as vector spaces which completes the proof.

Proof of Corollary 3.3. Consider the $\boldsymbol{Z}/p$ coefficient Serre spectral sequence for the fibration (2). By theorem 3.2 $E_2^{s,t}$, $t\leq 2p-2$ are concentrated in even dimensions. Therefore

$$H^*(M_{k,CP^2}\,;\,\boldsymbol{Z}/p)\cong H^*(M^*_{k,CP^2}\,;\,\boldsymbol{Z}/p)\otimes H^*(B\mathrm{SU}(2)\,;\,\boldsymbol{Z}/p)$$

as vector spaces for $*\leq 2p-2$.

Since $\tilde{a}^{p-1}$ is in the kernel of $H^{2p-2}(M_{k,CP^2}\,;\,\boldsymbol{Z}/p)\to H^{2p-2}(M^*_{k,CP^2}\,;\,\boldsymbol{Z}/p)$, we have $\tilde{a}^{p-1}\equiv 0 \bmod (ev_k^*c_2)$.

At this stage we can prove proposition 1.3.

Proof of Proposition 1.3. Note that we have a canonical map $f : BS^1\to M_{-k^2,CP^2}$ (see the proof of the following lemma).

Lemma 3.8.

$$f^*(\tilde{a})=\varepsilon k c_1 \in H^2(BS^1;\boldsymbol{Z}),$$

where $\varepsilon=1$ *or* -1.

Proof. The map f decomposes as follows.

$$BS^1 \hookrightarrow \mathrm{Map}_0(\boldsymbol{CP}^2, BS^1) \xrightarrow{\cong} \mathrm{Map}_k(\boldsymbol{CP}^2, BS^1) \xrightarrow{j_*} \mathrm{Map}_{-k^2}(\boldsymbol{CP}^2, B\mathrm{SU}(2)),$$

where $j: BS^1 \to B\mathrm{SU}(2)$ is an inclusion. Consider the following commutative diagram

$$\begin{tikzcd}
\boldsymbol{Z} \arrow[r, equal] & \pi_2(\mathrm{Map}(S^2, B\mathrm{SU}(2))) & \pi_2(M_{-k^2,\, CP^2}) \arrow[l, "(i^\#)_*"'] \arrow[r, equal] & \boldsymbol{Z} \\
\boldsymbol{Z} \arrow[r, equal] \arrow[d, equal] & \pi_2(\mathrm{Map}_k(S^2, BS^1)) \arrow[u, "(j_*)_*"] & \pi_2(\mathrm{Map}_k(\boldsymbol{CP}^2, BS^1)) \arrow[l] \arrow[u, "(j_*)_*"] & \\
\boldsymbol{Z} \arrow[r, equal] \arrow[d, equal] & \pi_2(\mathrm{Map}_0(S^2, BS^1)) \arrow[u, "\cong"] & \pi_2(\mathrm{Map}_0(\boldsymbol{CP}^2, BS^1)) \arrow[l] \arrow[u, "\cong"] & \\
\boldsymbol{Z} \arrow[r, equal] & \pi_2(BS^1) \arrow[u, "\cong"] \arrow[r, equal] & \pi_2(BS^1) \arrow[u] &
\end{tikzcd}$$

From the homotopy exact sequence for fibrations (2), (3)

$$(i^\#)_*=2\times : \pi_2(M_{-k^2,\, CP^2})=\boldsymbol{Z}\to\boldsymbol{Z}=\pi_2(\mathrm{Map}(S^2, B\mathrm{SU}(2))).$$

Let $k: S^2\to S^2$ be a map of degree k then we have a commutative diagram

$$\begin{tikzcd}
\boldsymbol{Z} \arrow[r, equal] & \pi_2(\mathrm{Map}_k(S^2, BS^1)) \arrow[r, "(j_*)_*"] & \pi_2(\mathrm{Map}(S^2, B\mathrm{SU}(2))) \arrow[r, equal] & \boldsymbol{Z} \\
\boldsymbol{Z} \arrow[r, equal] \arrow[u, equal] & \pi_2(\mathrm{Map}_1(S^2, BS^1)) \arrow[r, "(j_*)_*"] \arrow[u, "(k^\#)_*"] & \pi_2(\mathrm{Map}(S^2, B\mathrm{SU}(2))) \arrow[r, equal] \arrow[u, "(k^\#)_*"] & \boldsymbol{Z}. \arrow[u, "k\times"']
\end{tikzcd}$$

A generator of $\pi_2(\mathrm{Map}(S^2, B\mathrm{SU}(2)))$ is given by the adjoint of the degree 1 map $S^2\wedge S^2=S^4\to B\mathrm{SU}(2)$ and that of $\pi_2(\mathrm{Map}_1(S^2, BS^1))$ is given by the adjoint of the map $h: S^2\times S^2\to BS^1$ which represents the line bundle with $c_1=\alpha\otimes 1+1\otimes\alpha$. Then we have a commutative diagram

$$\begin{tikzcd}
S^2\times S^2 \arrow[r, "h"] \arrow[d] & BS^1 \arrow[d, "j"] \\
S^4 \arrow[r, "-2"] & B\mathrm{SU}(2).
\end{tikzcd}$$

This shows that

$$\begin{array}{ccc} \pi_2(\mathrm{Map}_1(S^2, BS^1)) & \xrightarrow{(j_*)_*} & \pi_2(\mathrm{Map}(S^2, B\mathrm{SU}(2))) \\ \| & & \| \\ \boldsymbol{Z} & \xrightarrow{2\times} & \boldsymbol{Z}. \end{array}$$

Thus we have

$$f_* = k\times : \pi_2(BS^1) = \boldsymbol{Z} \to \boldsymbol{Z} = \pi_2(M_{-k^2, CP^2}).$$

Recall that $\pi_1(BS^1) = \pi_1(M_{-k^2, CP^2}) = 0$ hence

$$f^* = k\times : H^2(M_{-k^2, CP^2}; \boldsymbol{Z}) \to H^2(BS^1; \boldsymbol{Z}).$$

Let $g : M_{-k^2, CP^2} \to M_{l, CP^2}$ be a homotopy equivalence, p prime to k. By the above lemma we have $(g\circ f)^*((a\otimes 1)^{p-1}) \neq 0 \in H^{2p-2}(BS^1; \boldsymbol{Z}/p)$ hence by 3.3 $(ev_l \circ g \circ f)^* c_2 \neq 0$.

Lemma 3.9. *For any continuous map $f : BS^1 \to B\mathrm{SU}(2)$, $f^*c_2 = 2m^2c_1^2 \in H^4(BS^1; \boldsymbol{Z})$ where m is an integer.*

Proof. Put $u = -c_2$. Let $f^*u = lc_1^2$. We must show that l is a square number. We have

$$\mathscr{P}^1(f^*u) = l\mathscr{P}^1(c_1^2) = 2lc_1^{p+1},$$

on the other hand

$$f^*\mathscr{P}^1(u) = f^*(2u^{p+1/2}) = 2l^{p+1/2}c_1^{p+1}$$

therefore if $(l, p) = 1$, $l^{p-1/2} \equiv 1 \pmod p$ and by Euler's criterion, $(\frac{l}{p}) = 1$ hence l is a square number.

By this lemma we can put $(ev_l \circ g \circ f)^*(c_2) = -m^2c_1^2$. Taking the adjoint of $g\circ f : BS^1 \to M_{l, CP^2}$, we obtain a map $\Phi : BS^1 \times \boldsymbol{C}P^2 \to B\mathrm{SU}(2)$ and we have

$$\Phi^*(u) = m^2c_1^2 \otimes 1 + nc_1 \otimes c_1 - l1 \otimes c_1^2.$$

Let p be a prime satisfying $(l, p) = (m, p) = 1$ and consider the mod p cohomology.

$$\begin{aligned} \mathscr{P}^1(\Phi^*u) &= \mathscr{P}^1(m^2c_1^2 \otimes 1 + nc_1 \otimes c_1 - l1 \otimes c_1^2) \\ &= 2m^2c_1^{p+1} \otimes 1 + nc_1^p \otimes c_1. \end{aligned}$$

On the other hand

$$\Phi^*\mathscr{P}^1(u) = 2\Phi^*(u^{p+1/2})$$

$$=2m^{p+1}c_1^{p+1}\otimes 1+(p+1)m^{p-1}nc_1^p\otimes c_1$$
$$+\left\{\frac{1}{4}(p+1)(p-1)m^{p-3}n^2-(p+1)lm^{p-1}\right\}c_1^{p-1}\otimes c_1^2$$

hence we have

$$lm^2\equiv\frac{1}{4}(p+1)(p-1)n^2\equiv-\left(\frac{p+1}{2}n\right)^2 \pmod p,$$

therefore

$$\left(\frac{-l}{p}\right)=\left(\frac{-lm^2}{p}\right)=1$$

and $-l$ is a square number.

Proof of Theorem 3.4. For $p=2$, since we cannot use 3.5, we consider the Postonikov decomposition of $\mathrm{Map}^*(\boldsymbol{CP}^2, B\mathrm{SU}(2))$. For a space Y and a non negative integer q, let $Y<q>=Y\cup e_\alpha^{q+1}\cup\cdots$ be a space obtained from Y by killing the homotopy groups in dimension $\geq q$. From the homotopy exact sequence of the fibering (3), using results of [10, ChV] we have

$$M_{0,CP^2}^*<4>=M_{0,CP^2}^*<5>$$

and a fibration

$$K(\boldsymbol{Z}/6,3)\to M_{0,CP^2}^*<4>\xrightarrow{p}K(\boldsymbol{Z},2)=BS^1. \tag{4}$$

Let $\boldsymbol{k}\in H^4(BS^1;\boldsymbol{Z}/6)$ be the Postnikov invariant.

If $2\boldsymbol{k}=0$, the fibration (4) localized at 3 is trivial hence $H^3(M_{0,CP^2}^*;\boldsymbol{Z}/3)\cong H^3(M_{0,CP^2}^*<4>;\boldsymbol{Z}/3)\neq 0$ which contradicts to theorem 3.2. Therefore $2\boldsymbol{k}\neq 0$.

Lemma 3.10. $3\boldsymbol{k}\neq 0$.

Proof. If $3\boldsymbol{k}=0$, there is a map $s: BS^1\to M_{0,CP^2}^*<4>$ such that

$$(p\circ s)_*=3\times:\pi_2(BS^1)\to\pi_2(BS^1).$$

Restricting to $\boldsymbol{CP}^2\subset BS^1$, we obtain a lift $\tilde{s}:\boldsymbol{CP}^2\to M_{0,CP^2}^*$ and its adjoint $\Phi:\boldsymbol{CP}^2\times\boldsymbol{CP}^2\to B\mathrm{SU}(2)$. Then we obtain a principal SU(2) bundle over $\boldsymbol{CP}^2\times\boldsymbol{CP}^2$ with 2nd Chern class is $6\alpha\otimes\alpha\in H^4(\boldsymbol{CP}^2\times\boldsymbol{CP}^2;\boldsymbol{Z})$ where $\alpha\in H^2(\boldsymbol{CP}^2)$ is a generator. Note that $K(B\mathrm{SU}(2))\cong\boldsymbol{Z}[u]$, $K(\boldsymbol{CP}^2\times\boldsymbol{CP}^2)\cong\boldsymbol{Z}[a,b]/(a^3,b^3)$ and $ch(u)=c_2-\frac{1}{12}c_2^2$, $ch(a)=\alpha\otimes 1+\frac{1}{2}\alpha^2\otimes 1$, $ch(b)=1\otimes\alpha+\frac{1}{2}(1\otimes\alpha^2)$. Put $\Phi^*(u)=6ab+\lambda_1a^2b+\lambda_2ab^2+\lambda_3a^2b^2$ where $\lambda_i\in\boldsymbol{Z}$.

$$\Phi^* ch(u) = \Phi^*\left(c_2 - \frac{1}{12}c_2^2\right)$$
$$= 6\alpha\otimes\alpha - 3\alpha^2\otimes\alpha^2.$$

On the other hand

$$ch(\Phi^*(u)) = ch(6ab + \lambda_1 a^2 b + \lambda_2 ab^2 + \lambda_3 a^2 b^2)$$
$$= 6\alpha\otimes\alpha + (3+\lambda_1)\alpha^2\otimes\alpha + (3+\lambda_2)\alpha\otimes\alpha^2 + \left\{\frac{3}{2} + \frac{1}{2}(\lambda_1+\lambda_2) + \lambda_3\right\}\alpha^2\otimes\alpha^2$$

and we have equations

$$3+\lambda_1 = 0$$
$$3+\lambda_2 = 0$$
$$\frac{3}{2} + \frac{1}{2}(\lambda_1+\lambda_2) + \lambda_3 = -3,$$

which is a contradiction.

Thus $\boldsymbol{k}\in H^4(BS^1\,;\,\boldsymbol{Z}/6)$ is a generator. Therefore

$$d_4 : E_4^{0,3} = H^3(K(\boldsymbol{Z}/6, 3)\,;\,\boldsymbol{Z}/6) \to H^4(BS^1\,;\,\boldsymbol{Z}/6) = E_4^{4,0}$$

is an isomorphism in the $\boldsymbol{Z}/6$ coefficient Serre spectral sequence for the fibration (4). Then we can prove theorem 3.4 quite similarly to theorem 3.2.

It is known that all differentials in the $\boldsymbol{Z}/2$ coefficient Serre spectral sequence for the fibration (1) vanishes if X is spin ([3]).

Appendix

In this appendix we study compact subgroups of gauge groups. Fix a base point $p_0\in P$ and $\pi(p_0)=x_0$. Then we can naturally identify AdP_{x_0}, the fiber over x_0 of AdP, with G by $G\ni g \mapsto [p_0, g]\in AdP_{x_0}$. In this appendix we always identify AdP_{x_0} with G by this identification.

Define an evaluation map

$$ev : X\times\mathcal{G}\to AdP$$

by $ev(x, u) = u(x)$, and a restriction map

$$r_{x_0} : \mathcal{G}\to AdP_{x_0} = G$$

by $r_{x_0}(u) = ev(x_0, u)$. Note that the evaluation map is a fiberwise homomorphism and the restriction map is a group homomorphism. Then we will show the following.

Theorem. *For any compact subgroup $\mathscr{K}$ of $\mathscr{G}$, the evaluation map restricted to $X\times\mathscr{K}$*

$$ev: X\times\mathscr{K}\to AdP$$

is injective. In particular, $r_{x_0}: \mathscr{K}\to G$ is injective.

Compact subgroups of a gauge group is related to the reduction of the bundle.

Let H be a closed subgroup of G. A sub H bundle of P is a subset $P_H\subset P$ which is a principal H bundle over X with respect to the natural H action. Note that if P contains a sub H bundle, the structure group of P naturally reduces to H.

Assume that the structure group of P reduces to $Z(K)$, the centralizer of a closed subgroup K of G, then

$$AdP=P\times_{Ad}G=P_{Z(K)\times Ad}G,$$

therefore $\mathscr{G}$ naturally contains K. If K is a tori then any compact subgroup of $\mathscr{G}$ such that $r_{x_0}(\mathscr{K})=K$ is obtained in this way. More precisely, we have the following.

Theorem. *Let K be a closed torus subgroup of G. Then there exists a natural one to one correspondence,*

$$\{\mathscr{K}\subset\mathscr{G}\mid \textit{compact subgroup of } \mathscr{G} \textit{ such that } r_{x_0}(\mathscr{K})=K\}$$

$$\updownarrow 1 \textit{ to } 1$$

$$\{Z(K) \textit{ sub bundles of } P \textit{ which contains } p_0\}.$$

Let $\mathscr{G}_f\subset\mathscr{G}$ denotes all the elements of finite order of $\mathscr{G}$, $G_f\subset G$ all the elements of finite order of G. Note that for any $u\in\mathscr{G}_f$, $ev(x, u)$ is of finite order for all $x\in X$, hence $u\in\Gamma(P\times_{Ad}(\bigcup_{g\in G} gr_{x_0}(u)g^{-1}))$. Since there is an isomorphism

$$P/Z(r_{x_0}(u))\cong P\times_{Ad}\left(\bigcup_{g\in G} gr_{x_0}(u)g^{-1}\right)$$

sending $[p]$ to $[p, r_{x_0}(u)]$, we can consider u as a section of $P/Z(r_{x_0}(u))$. Let $p: P\to P/Z(r_{x_0}(u))$ be the natural projection. We define a subspace of P, $P_{(u)}:=p^{-1}(u(X))$, then $P_{(u)}$ is a sub $Z(r_{x_0}(u))$ bundle of P and $p_0\in P_{(u)}$.

Proposition. *For any $g\in G_f$, $P_{(\cdot)}$ gives a one to one correspondence,*

$$\{u\in\mathscr{G}_f r_{x_0}(u)=g\}$$

$$\downarrow 1 \text{ to } 1$$

$$\{\textit{sub } Z(g) \textit{ bundles of } P \textit{ which contains } p_0\}.$$

Proof. We construct the inverse to $P_{(\cdot)}$. Let a sub $Z(g)$ bundle $p_0 \in P_Z \subset P$ be given. The inclusion $P_Z \hookrightarrow P$ induces an element of $\mathscr{G}_f$

$$u : X = P_Z/Z(g) \hookrightarrow P/Z(g) \cong P \times_{Ad}\Big(\bigcup_{h \in G} hgh^{-1}\Big),$$

where the last isomorphism is given by sending $[p]$ to $[p, g]$ and $r_{x_0}(u) = g$. In a sense this u is a constant section i. e.

$$u : X \ni x \mapsto [p_x, g] \in P_Z \times_{Ad} G.$$

It can be easily shown that this construction gives the inverse to $P_{(\cdot)}$.

Proof of the first Theorem. Note that for any $u \in \mathscr{G}_f$ of order n, $ev(x, u)$ is of order n for all $x \in X$. Since $\mathscr{K}$ is compact, $\ker[ev(x, \cdot) : \mathscr{K} \to G]$ should contain elements of finite order, hence $ev(x, \cdot) \mid_{\mathscr{K}}$ is injective.

Let $\mathrm{Inj}^0(K, G)$ denote the component of all the injective homomorphisms from K to G including the natural inclusion $K \hookrightarrow G$.

Corollary. *For any compact subgroup K of G, there is a natural one to one correspondence,*

$$\{\mathscr{K} \subset \mathscr{G} \mid \textit{compact subgroup of } \mathscr{G} \textit{ such that } r_{x_0}(\mathscr{K}) = K\}$$

$$\updownarrow 1 \text{ to } 1$$

$$\{s \in \Gamma(P \times_{Ad} \mathrm{Inj}^0(K, G)) \mid s(x_0) = [p_0, i]\},$$

where $i : K \hookrightarrow G$ is the natural inclusion.

Proof. For $\mathscr{K} \subset \mathscr{G}$, taking the adjoint of

$$X \times K \xrightarrow[1 \times r_{x_0}^{-1}]{} X \times \mathscr{K} \xrightarrow[ev]{} P \times_{Ad} G,$$

we obtain a section of $P \times_{Ad} \mathrm{Inj}^0(K, G)$. This gives the desired correspondence.

If K is a tori, G acts on $\mathrm{Inj}^0(K, G)$ transitively, we have $\mathrm{Inj}^0(K, G) \cong G/Z(K)$ and

$$P \times_{Ad} \mathrm{Inj}^0(K, G) \cong P \times_G G/Z(K) \cong P/Z(K).$$

Then define a sub $Z(K)$ bundle $P_{(\mathscr{K})}$ for each compact subgroup $\mathscr{K} \subset \mathscr{G}$ such that $r_{x_0}(\mathscr{K})=K$ as before. Then just as the proposition before, $P_{(\cdot)}$ gives a desired one to one correspondence of the second theorem.

Remark. In fact $P_{(\mathscr{K})} := \bigcap_{u\in\mathscr{K}'} P_{(u)}$. Since G and $\mathscr{K}$ are compact, any $u\in \mathscr{K}$ is a section of $P\times_{Ad}\left(\bigcup_{g\in G} gr_{x_0}(u)g^{-1}\right)$, hence $P_{(u)}$ can be defined and $P_{(\mathscr{K})}= \bigcap_{u\in\mathscr{K}} P_{(u)}$.

We can similarly describe conjugacy classes of subgroups.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

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