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A Bifurcation phenomenon for the periodic solutions of the Duffing equation

By

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1. Introduction and Result

In this paper, we study a bifurcation phenomenon for the periodic solutions of the following Duffing equation which describes a nonlinear forced oscillation:

$$u''(t)+\mu u'(t)+\kappa u(t)+\alpha u^3(t)=f(t),\ t\in\mathbf{R}, \tag{1.1}$$

where μ and α are positive constants, κ is a nonnegative constant, and $f(t)$ is a given periodic external force. It is known that for any periodic external force there exists at least one periodic solution of (1.1) with same period as the external force. Furthermore, if the external force is suitably small, then the periodic solution is proved to be unique and asymptotically stable. On the other hand, in the case of the relatively large external force, numerical computations show a possibility of not only the non-uniqueness of the periodic solution but also the existence of various bifurcation phenomena. In particular, a strange attractor discovered by Ueda[6], so called Japanese attractor, is well known. However, it is surprising that there have been no mathematical proofs of the existence of bifurcation for the periodic solutions of (1.1). The aim of this paper is to give a mathematical proof of the existence of bifurcation for a special family of external forces. To do that, we define the one-parameter families of periodic functions $\{u_\lambda(t)\}_{\lambda>0}$ and $\{f_\lambda(t)\}_{\lambda>0}$ with period one by

$$\begin{cases} u_\lambda(t):=\lambda\sin 2\pi t,\ \lambda>0, \\ f_\lambda(t):=u''_\lambda(t)+\mu u'_\lambda(t)+\kappa u_\lambda(t)+\alpha u^3_\lambda(t), \end{cases} \tag{1.2}$$

so that the equation (1.1) has the trivial periodic solution $u(t)=u_\lambda(t)$ to the external force $f(t)=f_\lambda(t)$ for any $\lambda>0$. Then our main Theorem is

Theorem 1. *Suppose μ and κ satisfy*

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$$0\leq\kappa<4\pi^2,\quad \mu\leq\min\left(\frac{3(4\pi^2-\kappa)}{20\pi},\ \frac{(16\pi^2-\kappa)^2}{384\pi^3}\right)$$

and the external force $f(t)=f_\lambda(t)$ is given by (1.2). Then there exist at least three positive constants $\Lambda_i(i=1,2,3;\Lambda_1<\Lambda_2<\Lambda_3)$, which depend only on μ and κ such that a nontrivial periodic solution of (1.1) with period one bifurcates from $\{u_\lambda(t)\}_{\lambda>0}$ at $\lambda_i=\sqrt{\Lambda_i/\alpha}(i=1,2,3)$.

To prove Theorem 1, we first reformulate the problem on the periodic solution of (1.1) to an integral equation in Section 2, and apply the Krasnosel'skii's Theorem [3] on bifurcation to the integral equation in Section 3. A crucial part in this process is to show the eigenvalue problem of the linealized equation at $u(t)=u_\lambda(t)$ has at least three algebraicly simple eigenvalues. We investigate the eigenvalue problem in Section 4 and 5 by making use of the arguments on the continued fraction along the same line as in the paper Meshalkin and Sinai [4], where they studied the stability of stationary solutions of Navier-Stokes equation. Finally we show some results of numerical computations in Section 6.

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2. Reformulation of the problem

We shall seek the periodic solution of (1.1) with period one in the form

$$u(t)=u_\lambda(t)+\lambda v(t). \tag{2.1}$$

Substituting (2.1) to (1.1), we obtain the following problem:

$$\begin{cases} v''(t)+\mu v'(t)+\kappa v(t)+\alpha\lambda^2(L(v)(t)+N(v)(t))=0, \\ v(t+1)=v(t),\ t\in\mathbf{R}, \end{cases} \tag{2.2}$$

where $L(v)$ and $N(v)$ are defined by

$$\begin{cases} L(v)(t):=3v(t)\sin^2 2\pi t, \\ N(v)(t):=3v^2(t)\sin 2\pi t+v^3(t). \end{cases} \tag{2.3}$$

We reformulate the problem (2.2) into an integral equation in the space E defined by

$$E=\{u(t)\in C(\mathbf{R});\mathrm{u(t+1)=u(t),\ t}\in\mathbf{R}\} \tag{2.4}$$

with the norm $\|u\|:=\sup_{t\in[0,1]}|u(t)|$.

We first consider the case $\kappa\neq0$. It is easy to see that the problem

$$\begin{cases} v''(t)+\mu v'(t)+\kappa v(t)=f(t), \\ v(t+1)=v(t),\ t\in\mathbf{R} \end{cases} \tag{2.5}$$

has a unique solution $v\in E\cap C^2(\mathbf{R})$ for any $f\in E$. Let us denote this solution by $G(f)$. The problem (2.2) is reduced to the integral equation in E:

$$v=-\alpha\lambda^2 G(L(v)+N(v)). \tag{2.6}$$

In the case of $\kappa=0$, since G is not well defined, we need a further consideration. Noting that the solution of (2.2) satisfies

$$\int_0^1 L(v)(t)+N(v)(t)dt=0, \tag{2.7}$$

we rewrite the problem (2.2) as

$$\begin{cases} v''+\mu v'=-\alpha\lambda^2(L(v)+N(v))+\alpha\lambda^2\int_0^1(L(v)(t)+N(v)(t))dt, \\ \int_0^1 v(t)\sin^2 2\pi t\,dt=-\frac{1}{3}\int_0^1 N(v)(t)dt, \\ v(t+1)=v(t),\ t\in\mathbf{R}. \end{cases} \tag{2.8}$$

To solve (2.8), we consider the following two linear equations for $f\in E$ and $\beta\in\mathbf{R}$,

$$\begin{cases} v''(t)+\mu v'(t)=f(t)-\int_0^1 f(t)dt, \\ \int_0^1 v(t)dt=0,\ v(t+1)=v(t),\ t\in\mathbf{R}, \end{cases} \tag{2.9}$$

$$\begin{cases} w''(t)+\mu w'(t)=0, \\ \int_0^1 w(t)\sin^2 2\pi t\,dt=\beta,\ w(t+1)=w(t),\ t\in\mathbf{R}. \end{cases} \tag{2.10}$$

It is standard to see that the problem (2.9) has a unique solution $v\in E\cap C^2(\mathbf{R})$, denoting it by $\tilde{G}(f)$, and the solution of (2.10) is explicitly given by a constant

$$w=\frac{\beta}{\int_0^1 \sin^2 2\pi t\,dt}=2\beta. \tag{2.11}$$

Thus, the problem (2.2) with $\kappa=0$ is reduced to the integral equation in E:

$$\begin{aligned} v=&-\alpha\lambda^2\tilde{G}(L(v)+N(v)) \\ &+2\alpha\lambda^2\int_0^1(\sin^2 2\pi t)\tilde{G}(L(v)+N(v))(t)dt-\frac{2}{3}\int_0^1 N(v)(t)dt. \end{aligned} \tag{2.12}$$

3. Proof of Theorem 1

To show Theorem 1, we apply the Krasnosel'skii's Theorem [3] to the integral equation (2.6)(*resp.*(2.12)) for $\kappa>0$(*resp.* $\kappa=0$).

Theorem A (Krasnosel'skii's Theorem). *Let E be a Banach space and $f(x, \lambda)$ be an operator with domain $D\subset E\times\mathbf{R}$ into E of the form*

$$f(x, \lambda)=x-\lambda Tx+g(x, \lambda).$$

Suppose the followings:

(1) $\lambda_0\neq 0$, $(0, \lambda_0)\in D$.

(2) *T is a linear compact operator $E\to E$.*

(3) *$g(x, \lambda)$ is a nonlinear compact operator $D\to E$, which satisfies $g(0, \lambda)\equiv 0$, $g(x, \lambda)=o(\|x\|)$ uniformly in the neighborhood $\lambda=\lambda_0$.*

(4) *$1/\lambda_0$ is an eigenvalue of T with odd algebraic multiplicity.*

Then $(0, \lambda_0)$ is a bifurcation point for $f(x, \lambda)=0$.

Now, let E be a Banach space defined by (2.4) and T be an operator $E\to E$ defined by

$$Tv=\begin{cases} G(-L(v)) & \text{if } \kappa\neq 0, \\ \tilde{G}(-L(v))+2\int_0^1(\sin^2 2\pi t)\tilde{G}(L(v)(t)dt & \text{if } \kappa=0, \end{cases} \tag{3.1}$$

and $g(v, \lambda)$ be an operator with the domain $D=E\times\mathbf{R}_+$ into E defined by

$$g(v, \lambda)=\begin{cases} G(\alpha\lambda^2 N(v)) & \text{if } \kappa\neq 0, \\ \tilde{G}(\alpha\lambda^2 N(v))-2\alpha\lambda^2\int_0^1(\sin^2 2\pi t)\tilde{G}(N(v))(t)dt & \\ +\frac{2}{3}\int_0^1 N(v)(t)dt & \text{if } \kappa=0, \end{cases} \tag{3.2}$$

where $\mathbf{R}_+=\{\lambda\in\mathbf{R}\,;\,\lambda>0\}$. Then the both integral equations (2.6) and (2.12) are equivalent to

$$f(v, \lambda):=v-\alpha\lambda^2 Tv+g(v, \lambda)=0. \tag{3.3}$$

Therefore, we must show the corresponding assumptions (1)~(4) in Theorem A for the equation (3.3). These are verified by the following two Propositions.

Proposition 3.1. (i) *$G(f)$, $\tilde{G}(f)\in E\cap C^2(\mathbf{R})$ for any $f\in E$.*

(ii) *There exists a positive constant C such that for any $f\in E$*

$$\|G(f)\|,\ \|\tilde{G}(f)\|\leq C\|f\|,$$

$$\left\| \frac{d}{dt} G(f) \right\|, \left\| \frac{d}{dt} \tilde{G}(f) \right\| \leq C \|f\|.$$

(iii) *G and $\tilde{G}$ are compact oparators in E.*

Proposition 3. 2. *Suppose μ and κ are positive constants satisfying the assumption of Theorem 1. Then there exist at least three positive constants Λ_i ($i=1, 2, 3$, $\Lambda_1<\Lambda_2<\Lambda_3$) which depend only on μ and κ such that Λ_i^{-1} are algebraicly simple eigenvalues of T.*

The proof of Proposition 3.1 is given by a quite standard argument on the ordinary differential equation, so omitted. We shall give the proof of Proposition 3.2 in the next section. Then applying Theorem A to the equation (3.3), we can prove that a nontrivial periodic solution of (3.3) bifurcates at $\lambda_i=\sqrt{\Lambda_i/\alpha}$ $(i=1,2,3)$.

4. Eigenvalue problem of linearized equation

In this section, we give the proof of Proposition 3.2. First we note that the eigenvalue problem for T is again equivalent to the problem:

$$(4.1) \qquad \begin{cases} w''(t)+\mu w'(t)+\kappa w(t)+3\Lambda w(t) \sin^2 2\pi t=0, \\ w(t+1)=w(t),\ t\in\mathbf{R}, \end{cases}$$

where we set $\Lambda=\alpha\lambda^2$. We expand the solution by Fourier series as

$$(4.2) \qquad w(t)=\sum_{n=-\infty}^{\infty} a_n e^{2n\pi i t}, \quad \{a_n\}_{n\in\mathbf{Z}}\in \ell^2.$$

Substituting (4.2) to (4.1), we obtain

$$\sum_{n=-\infty}^{\infty} (-4\pi^2 n^2+2\pi\mu n i+\kappa+3\Lambda \sin^2 2\pi t)a_n e^{2n\pi i t}=0,$$

which implies that $\{a_n\}_{n\in\mathbf{Z}}$ satisfies the following recurrence formula:

$$(4.3) \qquad A_n(\Lambda)a_n+a_{n-2}+a_{n+2}=0,\ n\in\mathbf{Z},$$

where

$$A_n(\Lambda)=-2+\frac{16\pi^2 n^2-4\kappa}{3\Lambda}-\frac{8\pi\mu n i}{3\Lambda}.$$

We study this recurrence formula according as n is odd or even. In the case $n=2m+1$ $(m\in\mathbf{Z})$, setting $b_m=a_{2m+1}$ and $B_m(\Lambda)=A_{2m+1}(\Lambda)$, we rewrite (4.3) for $\{b_m\}_{m\in\mathbf{Z}}$ as

$$(4.4) \qquad B_m(\Lambda)b_m+b_{m-1}+b_{m+1}=0,\ m\in\mathbf{Z}.$$

In the cace $n=2m(m\in\mathbf{Z})$, setting $d_m=a_{2m}$ and $D_m(\Lambda)=A_{2m}(\Lambda)$, we rewrite for $\{d_m\}_{m\in\mathbf{Z}}$ as

$$D_m(\Lambda)d_m+d_{m-1}+d_{m+1}=0,\ m\in\mathbf{Z}. \tag{4.5}$$

For the solvability of these recurrence formulas (4.4) and (4.5), the following lemma holds.

Lemma 4.1. (I) *The recurrence equation (4.4) with $\Lambda=\Lambda_0$ has a nontrivial solution $\{b_m(\Lambda_0)\}_{m\in\mathbf{Z}}\in \ell^2$, if and only if $\{B_m(\Lambda_0)\}_{m\in\mathbf{Z}}$ satisfies the condition,*

$$|B_0(\Lambda_0)-\mathfrak{B}(\Lambda_0)|=1, \tag{4.6}$$

where

$$\mathfrak{B}(\Lambda)=\cfrac{1}{B_1(\Lambda)-\cfrac{1}{B_2(\Lambda)-\cfrac{1}{B_3(\Lambda)-\ddots}}}$$

(II) *The recurrence equation (4.4) with $\Lambda=\Lambda_0$ has a nontrivial solution $\{d_m(\Lambda_0)\}_{m\in\mathbf{Z}}\in \ell^2$, if and only if $\{D_m(\Lambda_0)\}_{m\in\mathbf{Z}}$ satisfies condition,*

$$D_0(\Lambda_0)=2\mathrm{Re}\,\mathfrak{D}(\Lambda_0), \tag{4.7}$$

where

$$\mathfrak{D}(\Lambda)=\cfrac{1}{D_1(\Lambda)-\cfrac{1}{D_2(\Lambda)-\cfrac{1}{D_3(\Lambda)-\ddots}}}$$

We shall give a proof of Lemma 4.1 in Section 5. Let us admit Lemma 4.1 for the moment. Then, to prove Proposition 3.2, we have only to show that there exist $\Lambda_i\in\mathbf{R}_+(i=1,2,3)$ which satisfy the equality (4.6) or (4.7) and which correspond to the algebraicly simple eigenvalues of T. To do that, we make use of the following Worpitzky's Theorem [1] concerning a family of continued fractions.

Theorem B (Worpitzky's Theorem). *Let $\mathfrak{F}$ be a family of the formal continued fractions :*

$$\mathfrak{F}=\left\{ C=\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\ddots}}}\ ;\ a_k\in\mathbf{C},\ \ |a_k|\leq\frac{1}{4}\ \textit{for any}\ k\in\mathbf{N} \right\}$$

Let $w_n(C)$ and $w(C)$ respectively denote the n-th approximant and the value of a convergent continued fraction C. Then a family $\mathfrak{F}$ is uniformly convergent, that is,

$$\lim_{n\to\infty}\sup_{C\in\mathfrak{F}}|w_n(C)-w(C)|=0.$$

Furthermore, it holds that $|w(C)|\leq\frac{1}{2}$ *for any* $C\in\mathfrak{F}$.

First we consider the existence of $\Lambda_i(i=1,2)$ satisfying the equality (4.6). We introduce some notations and definitions. Let $[h\,;r]$ denote the ball which has the center $h\in\mathbf{C}$ and the radius $r\in\mathbf{R}_+$. When the ball $[h\,;r]$ does not include the origin, we define the reciprocal ball by $[h\,;r]^{-1}=\left[\frac{\bar{h}}{|h|^2-r^2}\,;\frac{r}{|h|^2-r^2}\right]$. We next define the constants $\{\tilde{\Lambda}_i\}_{i=0}^{5}(0<\tilde{\Lambda}_0<\tilde{\Lambda}_1<\tilde{\Lambda}_2<\tilde{\Lambda}_3<\tilde{\Lambda}_4<\tilde{\Lambda}_5)$ by

$$\tilde{\Lambda}_0=\frac{8(4\pi^2-\kappa)}{21},\quad \tilde{\Lambda}_1=\frac{2(4\pi^2-\kappa)}{3},\quad \tilde{\Lambda}_2=\frac{4(4\pi^2-\kappa)}{3},$$
$$\tilde{\Lambda}_3=\frac{4(16\pi^2-\kappa)}{9},\quad \tilde{\Lambda}_4=\frac{2(16\pi^2-\kappa)}{3},\quad \tilde{\Lambda}_5=\frac{4(36\pi^2-\kappa)}{9},$$

so that they satisfy the following relation:

Λ	$\tilde{\Lambda}_0$	$\tilde{\Lambda}_1$	$\tilde{\Lambda}_2$	$\tilde{\Lambda}_5$
$\mathrm{Re}\,B_0(\Lambda)$	$\frac{3}{2}$	0	-1	
$\mathrm{Re}\,B_1(\Lambda)$	$\frac{59}{2}+\frac{28\kappa}{4\pi^2-\kappa}$	$16+\frac{16\kappa}{4\pi^2-\kappa}$	$7+\frac{8\kappa}{4\pi^2-\kappa}$	1
$\mathrm{Re}\,B_2(\Lambda)$		$48+\frac{48\kappa}{4\pi^2-\kappa}$	$23+\frac{24\kappa}{4\pi^2-\kappa}$	$\frac{19}{3}+\frac{16\kappa}{3(36\pi^2-\kappa)}$

Λ	$\tilde{\Lambda}_3$	$\tilde{\Lambda}_4$
$\mathrm{Re}\,D_1(\Lambda)$	1	0
$\mathrm{Re}\,D_2(\Lambda)$	$10+\frac{9\kappa}{16\pi^2-\kappa}$	$6+\frac{6\kappa}{16\pi^2-\kappa}$

In the following, we note that the continued fraction $\mathfrak{B}(\Lambda)$ can be rewritten in the form

$$(4.8)\qquad \mathfrak{B}(\Lambda)=\cfrac{\frac{1}{B_1(\Lambda)}}{1+\cfrac{-\frac{1}{B_1(\Lambda)B_2(\Lambda)}}{1+\cfrac{-\frac{1}{B_2(\Lambda)B_3(\Lambda)}}{1+\ddots}}}$$

In the case $0<\Lambda\leq\tilde{\Lambda}_0$, it holds that $\mathrm{Re}B_0(\Lambda)\geq 3/2$, $\mathrm{Im}B_0(\Lambda)<0$, and $|B_i(\Lambda)|>4$ for $i\geq 1$. Then Theorem B imples $|\mathfrak{B}(\Lambda)|\leq 1/2$, and it easily follows

$$(4.9)\qquad |B_0(\Lambda)-\mathfrak{B}(\Lambda)|>1 \text{ for } 0<\Lambda\leq\tilde{\Lambda}_0.$$

In the case $\Lambda=\tilde{\Lambda}_1$, it holds $|B_i(\tilde{\Lambda}_1)|\geq 16$ for $i\geq 1$. So, Theorem B gives $|\mathfrak{B}(\tilde{\Lambda}_1)|\leq\frac{2}{|B_1(\Lambda_1)|}$ and $|B_0(\tilde{\Lambda}_1)-\mathfrak{B}(\tilde{\Lambda}_1)|\leq|B_0(\tilde{\Lambda}_1)|+\frac{1}{8}$. From the assumption of Theorem 1, $|B_0(\tilde{\Lambda}_1)|=|\mathrm{Im}B_0(\tilde{\Lambda}_1)|\leq 3/5$. Therefore we have

$$(4.10)\qquad |B_0(\tilde{\Lambda}_1)-\mathfrak{B}(\tilde{\Lambda}_1)|<1.$$

By (4.9) and (4.10), the Intermediate Value Theorem proves the existence of a constant $\Lambda_1\in(\tilde{\Lambda}_0,\tilde{\Lambda}_1)$ such that

$$(4.11)\qquad |B_0(\Lambda_1)-\mathfrak{B}(\Lambda_1)|=1.$$

In the case $\tilde{\Lambda}_2\leq\Lambda\leq\tilde{\Lambda}_5$, it holds that $\mathrm{Re}B_0(\Lambda)\leq -1$ and $|B_i(\Lambda)B_{i+1}(\Lambda)|>4$ for $i\geq 1$. Then, Theorem B imples $\mathfrak{B}(\Lambda)\in\frac{1}{B_1(\Lambda)}\left[1;\frac{1}{2}\right]^{-1}$. From the assumption of Theorem 1, we can see $|\arg B_1(\Lambda)|<\pi/6$, so that $|\arg\mathfrak{B}(\Lambda)|<\pi/3$. Therefore $\mathrm{Re}\mathfrak{B}(\Lambda)>0$, and we easily have

$$(4.12)\qquad |B_0(\Lambda)-\mathfrak{B}(\Lambda)|>1 \text{ for } \tilde{\Lambda}_2\leq\Lambda\leq\tilde{\Lambda}_5.$$

By (4.10) and (4.12), the Intermediate Value Theorem again proves the existence of a constant $\Lambda_2\in(\tilde{\Lambda}_1,\tilde{\Lambda}_2)$ such that

$$(4.13)\qquad |B_0(\Lambda_2)-\mathfrak{B}(\Lambda_2)|=1.$$

Next, we consider the existence of $\Lambda_3\in\mathbf{R}_+$ such that $\{D_m(\Lambda_3)\}_{m\in\mathbf{Z}}$ satisfies (4.7). In the case $0<\Lambda\leq\tilde{\Lambda}_3$, we have $|D_i(\Lambda)D_{i+1}(\Lambda)|>4$ for $i\geq 1$. Similarly as in the previous case, we obtain $\mathrm{Re}\mathfrak{D}(\Lambda)>0$ which implies that

$$D_0(\Lambda)<2Re\mathfrak{D}(\Lambda) \quad \text{for } 0<\Lambda\leq\tilde{\Lambda}_3. \tag{4.14}$$

In the case $\Lambda=\tilde{\Lambda}_4$, it holds $|D_i(\tilde{\Lambda}_4)|>4$ for $i\geq 3$. Noting that $\mathfrak{D}(\Lambda)$ can be rewritten in the form

$$\mathfrak{D}(\Lambda)=\cfrac{1}{D_1(\Lambda)+\cfrac{-\cfrac{1}{D_2(\Lambda)}}{1+\cfrac{\cfrac{-1}{D_2(\Lambda)}\cdot\cfrac{3}{D_3(\Lambda)}}{1+\ddots}}}$$

we have $\mathfrak{D}(\Lambda)\in[D_1(\Lambda)-h(\Lambda);r(\Lambda)]^{-1}$ where $h(\Lambda)$ and $r(\Lambda)$ are the center and the radius of the ball $\dfrac{1}{D_2(\Lambda)}\left[1;\dfrac{1}{2|D_2(\Lambda)|}\right]^{-1}$ respectively, which are explicitly given by

$$h(\Lambda)=\frac{\overline{D_2(\Lambda)}}{|D_2(\Lambda)|^2-\dfrac{1}{4}},\ r(\Lambda)=\frac{\dfrac{1}{2}}{|D_2(\Lambda)|^2-\dfrac{1}{4}}.$$

Therefore we obtain the following estimate

$$\mathrm{Re}\mathfrak{D}(\tilde{\Lambda}_4)\leq\frac{\mathrm{Re}(D_1(\tilde{\Lambda}_4)-h(\tilde{\Lambda}_4))+r(\tilde{\Lambda}_4)}{|D_1(\tilde{\Lambda}_4)-h(\tilde{\Lambda}_4)|^2-r^2(\tilde{\Lambda}_4)} \tag{4.15}$$

Set $\xi=\mathrm{Re}D_2(\tilde{\Lambda}_4)$ and $\eta=\mathrm{Im}\,D_2(\tilde{\Lambda}_4)$, then $D_0(\tilde{\Lambda}_4)=-\xi/3$, $D_1(\tilde{\Lambda}_4)=\eta i/2$, and $\eta=-\mu\xi/(6\pi)$. Substituting ξ and η into (4.15),

$$\begin{aligned}\mathrm{Re}\mathfrak{D}(\tilde{\Lambda}_4)&\leq\frac{\dfrac{-\xi+\dfrac{1}{2}}{\xi^2+\eta^2-\dfrac{1}{4}}}{\left(\dfrac{\xi}{\xi^2+\eta^2-\dfrac{1}{4}}\right)^2+\eta^2\left(\dfrac{1}{2}+\dfrac{1}{\xi^2+\eta^2-\dfrac{1}{4}}\right)^2-\dfrac{1}{4\left(\xi^2+\eta^2-\dfrac{1}{4}\right)^2}}\\ &=\frac{\left(-\xi+\dfrac{1}{2}\right)}{\dfrac{\eta^2}{4}\left(\xi^2+\eta^2-\dfrac{1}{4}\right)+\eta^2+1}.\end{aligned} \tag{4.16}$$

In order to see $2\mathrm{Re}\mathfrak{D}(\tilde{\Lambda}_4)<D_0(\tilde{\Lambda}_4)$, we must show the inequality

$$\frac{1}{6}\left(\frac{\eta^2}{4}\left(\xi^2+\eta^2-\frac{1}{4}\right)+\eta^2+1\right)+\frac{1}{2\xi}-1<0. \tag{4.17}$$

To do that, we define $f(x):=x^2+9\cdot159x-24\cdot27\cdot36$. Using $\xi\geq6$, we have

$$\begin{aligned}
&\frac{1}{6}\left(\frac{\eta^2}{4}\left(\xi^2+\eta^2-\frac{1}{4}\right)+\eta^2+1\right)+\frac{1}{2\xi}-1\\
&=\frac{1}{6}\left\{\frac{1}{4}\left(\frac{\mu\xi}{6\pi}\right)^2\left\{\xi^2+\left(\frac{\mu\xi}{6\pi}\right)^2-\frac{1}{4}\right\}+\left(\frac{\mu\xi}{6\pi}\right)^2+1\right\}+\frac{1}{2\xi}-1\\
&=\frac{1}{24\xi^4}\left(\frac{\mu^2\xi^4}{36\pi^2}\right)^2+\frac{1}{24}\left(\frac{\mu^2\xi^4}{36\pi^2}\right)+\frac{15}{96\xi^2}\left(\frac{\mu^2\xi^4}{36\pi^2}\right)+\frac{1}{2\xi}-\frac{5}{6}\\
&\leq\frac{1}{24\cdot36^2}\left(\frac{\mu^2\xi^4}{36\pi^2}\right)^2+\frac{1}{24}\left(\frac{\mu^2\xi^4}{36\pi^2}\right)+\frac{15}{48\cdot72}\left(\frac{\mu^2\xi^4}{36\pi^2}\right)-\frac{3}{4}\\
&=\frac{1}{24\cdot36^2}f\left(\frac{\mu^2\xi^4}{36\pi^2}\right).
\end{aligned}$$

From the assumption of Theorem 1, $0<\mu^2\xi^4/(36\pi^2)\leq16$, so that $f\left(\frac{\mu^2\xi^4}{36\pi^2}\right)<0$.

Therefore we obtain (4.17), which implies

$$2\mathrm{Re}\mathfrak{D}(\tilde{\Lambda}_4)<D_0(\tilde{\Lambda}_4). \tag{4.18}$$

By (4.14) and (4.18), the Intermediate Value Theorem proves the existence of a constant $\Lambda_3\in(\tilde{\Lambda}_3,\tilde{\Lambda}_4)$ such that

$$D_0(\tilde{\Lambda}_3)=2\mathrm{Re}\mathfrak{D}(\Lambda_3). \tag{4.19}$$

Thus we have proved that there exist at least three eigenvalues of T. Finally we show these are algebraicly simple eigenvalues, which means $\dim\bigcup_{n=1}^{\infty}\mathrm{Ker}(\Lambda_i^{-1}I-T)^n=1$. Since (4.12) and (4.14) shows that (4.6) does not hold for $\Lambda=\Lambda_3$, and neither does (4.7) for $\Lambda=\Lambda_1$ and $\Lambda=\Lambda_2$, we can see $\dim\mathrm{Ker}(\Lambda_i^{-1}I-T)=1$. So we need to show $\mathrm{Ker}(\Lambda_i^{-1}I-T)^2=\mathrm{Ker}(\Lambda_i^{-1}I-T)$. If it does not hold, there exists a nontrivial solution $u\in E$ of $(\Lambda_i^{-1}I-T)u=w$ for any $w\in\mathrm{Ker}(\Lambda_i^{-1}I-T)$, which is equivalent to the solvability of the following equation:

$$u''(t)+\mu u'(t)+\kappa u(t)+3\Lambda_i u(t)\sin^2 2\pi t=-3\Lambda_i w\sin^2 2\pi t. \tag{4.20}$$

From the standard argument, the solvability of (4.20) is equivalent to the condition

$$\int_0^1 w(t)w_*(t)\sin^2 2\pi t dt=0, \tag{4.21}$$

where w_* is a solution of the adjoint equation to (4.1):

(4.22) $$w''_*(t)-\mu w'_*(t)+\kappa w_*(t)+3\Lambda w_*(t)\sin^2 2\pi t=0.$$

Noting that $w_*(t)=w(-t)=\sum_{n=-\infty}^{\infty}\overline{a_n}e^{2n\pi i t}$, (4.21) is rewritten as $\sum_{n=-\infty}^{\infty}(a_n-a_{n+2})^2=0$. Therefore we must show

(4.23) $$\sum_{n=-\infty}^{\infty}(a_n-a_{n+2})^2\neq 0 \quad at\ \Lambda=\Lambda_i(i=1,2,3).$$

In the case $\Lambda=\Lambda_1$, from the definition of $\{b_m\}_{m\in Z}$ given by (5.4) in the next section,

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(a_n-a_{n+2})^2&=\sum_{m=-\infty}^{\infty}(b_m-b_{m+1})^2\\
&=2\{-2(\mathrm{Im}\ b_0)^2+\mathrm{Re}(b_0^2(1-\rho_1)^2)+\sum_{m\geq 1}\mathrm{Re}(b_m-b_{m+1})^2\}\\
&=2\{-3(\mathrm{Im}\ b_0)^2+(\mathrm{Re} b_0)^2+\mathrm{Re}(b_0^2\rho_1(\rho_1-2))+\sum_{m\geq 1}\mathrm{Re}(b_m-b_{m+1})^2\}\\
&=-2\{3-4(\mathrm{Re}\ b_0)^2-\mathrm{Re}(b_0^2\rho_1(\rho_1-2))-\sum_{m\geq 1}\mathrm{Re}(b_m-b_{m+1})^2\}\end{aligned}$$

Taking notice that $(\mathrm{Re}\ b_0)^2=\frac{1}{2}(1+\mathrm{Re}(b_0)^2)$, we have

(4.24) $$\begin{aligned}&3-4(\mathrm{Re}\ b_0)^2-\mathrm{Re}(b_0^2\rho_1(\rho_1-2))-\sum_{m\geq 1}\mathrm{Re}(b_m-b_{m+1})^2\\
&=1-2\mathrm{Re}(b_0^2)-\mathrm{Re}(b_0^2\rho_1^2-2b_0^2\rho_1)-\sum_{m\geq 1}\mathrm{Re}(b_m-b_{m+1})^2\\
&\geq 1+2\mathrm{Re}(B_0+\rho_1)-|\rho_1|^2-2|\rho_1|-\sum_{m\geq 1}|b_m-b_{m+1}|^2\\
&\geq 1+2\mathrm{Re}(B_0+\rho_1)-|\rho_1|^2-2|\rho_1|\\
&\quad-\sum_{m\geq 0}|b_1|^2(1+|\rho_2|)^2|\rho_2|^{2m}\\
&\geq 1-4|\rho_1|-|\rho_1|^2-\frac{(1+|\rho_2|)^2|\rho_1|^2}{1-|\rho_2|^2}.\end{aligned}$$

However, Theorem B implies $|\rho_m|\leq\frac{2}{|B_m|}$ so that $|\rho_1|<\frac{1}{8}$ and $|\rho_2|<\frac{1}{24}$. Using these estimates, we obtain (4.24) >0. Therefore we can show the inequality (4.23) at $\Lambda=\Lambda_1$.

In the case $\Lambda=\Lambda_2$, we can rewrite (4.24)as

(4.25) $$\begin{aligned}&3-4(\mathrm{Re}\ b_0)^2-\mathrm{Re}(b_0^2\rho_1(\rho_1-2))-\sum_{m\geq 1}\mathrm{Re}(b_m-b_{m+1})^2\\
&=1+2\mathrm{Re}(B_0+\rho_1)+2\mathrm{Re}(b_0^2\rho_1)-\mathrm{Re}(b_0^2\rho_1^2)-\sum_{m\geq 1}\mathrm{Re}(b_m-b_{m+1})^2\\
&=1+2\mathrm{Re}(B_0+\rho_1)+2\mathrm{Re}(-(\overline{B_0}+\overline{\rho_1})\rho_1)-\mathrm{Re}(b_0^2\rho_1^2)-\sum_{m\geq 1}\mathrm{Re}(b_m-b_{m+1})^2\\
&=-1+2\mathrm{Re}(B_0+\rho_1)+2|B_0|^2+2\mathrm{Re}\ B_0\overline{\rho_1}+\mathrm{Re}((\overline{B_0}+\overline{\rho_1})\rho_1^2)\end{aligned}$$

$$-\sum_{m\geq 1}\mathrm{Re}(b_m-b_{m+1})^2.$$

Since it follows $-1<\mathrm{Re}B_0<0$ and $|\mathrm{Im}\,B_0|\leq\frac{3}{5}$ from the assumption of Theorem 1, and Theorem B implies $|\rho_1|<\dfrac{1}{7-\frac{2}{23}}$ and $|\rho_2|\frac{2}{23}$, we have

$$\begin{aligned}(4.25)\qquad &\leq -1+2\mathrm{Re}\,B_0(1+\mathrm{Re}\,B_0)+2|\mathrm{Im}\,B_0|^2+2\mathrm{Re}\,\rho_1(1+\mathrm{Re}B_0)\\ &\quad+2\mathrm{Im}\,B_0\,\mathrm{Im}\,\rho_1+|\rho_1|^2-\sum_{m\geq 1}\mathrm{Re}(b_m-b_{m+1})^2\\ &< -1+2|\mathrm{Im}\,B_0|^2+2\mathrm{Im}\,B_0\,\mathrm{Im}\,\rho_1+|\rho_1|^2+\sum_{m\geq 1}|b_m-b_{m+1}|^2\\ &\leq -1+2|\mathrm{Im}\,B_0|^2+2\mathrm{Im}\,B_0\,\mathrm{Im}\,\rho_1+|\rho_1|^2+\frac{(1+|\rho_2|)^2|\rho_1|^2}{1-|\rho_2|^2}\\ &\leq -1+2|\mathrm{Im}\,B_0|^2+2|\mathrm{Im}\,B_0|\,|\rho_1|+\frac{2(1+|\rho_2|)|\rho_1|^2}{1-|\rho_2|^2}\\ &<0.\end{aligned}$$

Therefore we can show the inequality (4.23) at $\Lambda=\Lambda_2$.

In the case $\Lambda=\Lambda_3$, from the definition of $\{d_m\}_{m\in\mathbf{Z}}$ given by (5.5) in the next section,

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(a_n-a_{n+2})^2&=\sum_{m=-\infty}^{\infty}(d_m-d_{m+1})^2\\ &=2\{\mathrm{Re}((1-d_1)^2)+Re((d_1-d_2)^2)+\sum_{m\geq 2}\mathrm{Re}((d_m-d_{m+1})^2)\}\\ &=-2\{2|d_1|^2-3\Big(1+\frac{2\kappa}{3\Lambda_3}\Big)^2-\Big(\frac{2\kappa}{3\Lambda_3}\Big)^2\\ &\quad-\mathrm{Re}\{d_1^2\sigma_2(\sigma_2-2)\}-\sum_{m\geq 2}\mathrm{Re}((d_m-d_{m+1})^2)\}.\end{aligned}$$

In order to see (4.23) at $\Lambda=\Lambda_3$, we must show

$$\begin{aligned}(4.26)\qquad &2|d_1|^2-3\Big(1+\frac{2\kappa}{3\Lambda_3}\Big)^2-\Big(\frac{2\kappa}{3\Lambda_3}\Big)^2-\mathrm{Re}\{d_1^2\sigma_2(\sigma_2-2)\}\\ &-\sum_{m\geq 2}\mathrm{Re}((d_m-d_{m+1})^2)\neq 0\end{aligned}$$

Noting that $|\sigma_2|<\frac{2}{11}$ and $|\sigma_3|<\frac{2}{31}$, we have

$$2|d_1|^2-3\Big(1+\frac{2\kappa}{3\Lambda_3}\Big)^2-\Big(\frac{2\kappa}{3\Lambda_3}\Big)^2-\mathrm{Re}\{d_1^2\sigma_2(\sigma_2-2)\}-\sum_{m\geq 2}\mathrm{Re}((d_m-d_{m+1})^2)$$

$$\geq 2|d_1|^2-3\Big(1+\frac{2\kappa}{3\Lambda_3}\Big)^2-\Big(\frac{2\kappa}{3\Lambda_3}\Big)^2-2|d_1|^2|\sigma_2|-|d_1|^2|\sigma_2|^2$$

$$-\frac{(1+|\sigma_3|)^2|\sigma_2|^2|d_1|^2}{1-|\sigma_3|^2}$$

$$\geq\Big(2-\frac{4}{11}-\frac{4}{121}-\frac{12}{319}\Big)|d_1|^2-4\Big(1+\frac{2\kappa}{3\Lambda_3}\Big)^2$$

$$>\frac{38}{25}|d_1|^2-6\Big(1+\frac{2\kappa}{3\Lambda_3}\Big).$$

Since Re $d_1=-\mathrm{Re}(D_1+\sigma_2)/|D_1+\sigma_2|^2$ and Re $D_1(\Lambda_3)>0$, we get

$$|d_1|^2=-\frac{1}{\mathrm{Re}(D_1+\sigma_2)}\Big(1+\frac{2\kappa}{3\Lambda_3}\Big)>\frac{11}{2}\Big(1+\frac{2\kappa}{3\Lambda_3}\Big). \tag{4.28}$$

By (4.27) and (4.28), we can show the inequality (4.23) at $\Lambda=\Lambda_3$. This completes the proof of Proposition 3.2.

5. Proof of Lemma 4.1

In this section, we give a proof of Lemma 4.1, by the similar argument in Meshalkin and Sinai [3]. Throughout this section, $\{b_m\}_{m\in\mathbf{Z}}$, $\{d_m\}_{m\in\mathbf{Z}}$, $\mathfrak{B}$, and $\mathfrak{D}$ denote $\{b_m(\Lambda_0)\}_{m\in\mathbf{Z}}$, $\{d_m(\Lambda_0)\}_{m\in\mathbf{Z}}$, $\mathfrak{B}(\Lambda_0)$, and $\mathfrak{D}(\Lambda_0)$ respectively.

(I) The case $n=2m+1$

First we show the proof of necessity. We assume that $\{b_m\}_{m\in\mathbf{Z}}\in\ell^2$ satisfies (4.4). Then $b_m\neq0$ for any $m\in\mathbf{Z}$. In fact, suppose that there exists k such that $b_k=0$. *Then* $b_{k+l}=C_lb_{k+1}$ for any $l\in\mathbf{Z}$, where C_l is a constant which satisfies $|C_l|\to\infty$ as $|l|\to\infty$. Let us separate the cases accoding as b_{k+1} is zero or not. If $b_{k+1}=0$, then $b_m=0$ for any $m\in\mathbf{Z}$, which contradicts that $\{b_m\}_{m\in\mathbf{Z}}$ is a nontrivial sequence. If $b_{k+1}\neq0$, then $|b_{k+l}|\to\infty$ as $|l|\to\infty$ which contradicts that $\{b_m\}_{m\in\mathbf{Z}}\in\ell^2$. Therefore we can well define $\rho_m=\frac{b_m}{b_{m-1}}(m\geq1)$ and $\dot\rho_m=\frac{b_{m-1}}{b_m}(m\leq0)$. So we can rewrite

(4.4) *as*
$$\begin{cases} B_m+\dfrac{1}{\rho_m}+\rho_{m+1}=0, & (m\geq1)\\ B_m+\dot\rho_m+\dfrac{1}{\dot\rho_{m+1}}=0, & (m\leq-1)\end{cases}$$

which implies respectively,

$$(5.1)\qquad \begin{cases} \rho_m = -\cfrac{1}{B_m - \cfrac{1}{B_{m+1} - \cfrac{1}{B_{m+2} - \ddots}}} & (m \geq 1) \\[2ex] \dot{\rho}_m = -\cfrac{1}{B_{m-1} - \cfrac{1}{B_{m-2} - \cfrac{1}{B_{m-3} - \ddots}}} & (m \leq 0) \end{cases}$$

On the other hand, it follows from (4.4) with $m=0$ that

$$(5.2)\qquad B_0 = \cfrac{1}{B_1 - \cfrac{1}{B_2 - \cfrac{1}{B_3 - \ddots}}} + \cfrac{1}{B_{-1} - \cfrac{1}{B_{-2} - \cfrac{1}{B_3 - \ddots}}}$$

Using the relation $\overline{B_m} = B_{-(m+1)}$, *we have*

$$(5.3)\qquad |B_0 - \mathfrak{B}| = 1,$$

where

$$\mathfrak{B} = \cfrac{1}{B_1 - \cfrac{1}{B_2 - \cfrac{1}{B_3 - \ddots}}}$$

This completes the proof of necessity. Next, we show the proof of sufficiency. We assume that $\{B_m\}_{m\in\mathbf{Z}}$ satisfies (4.6). Let us define $\{b_m\}_{m\in\mathbf{Z}}$ as follows,

$$(5.4)\qquad \begin{cases} b_0 = \overline{\dot{\rho}}_0^{\frac{1}{2}}, & 0 \leq \arg \overline{\dot{\rho}}_0^{\frac{1}{2}} < \pi \\ b_m = b_0 \rho_1 \rho_2 \cdots \rho_m, & (m \geq 1) \\ b_{-1} = \dot{\rho}_0^{\frac{1}{2}}, & \pi \leq \arg \dot{\rho}_0^{\frac{1}{2}} < 2\pi \\ b_m = b_{-1} \dot{\rho}_{-1} \dot{\rho}_{-2} \cdots \dot{\rho}_{m+1}, & (m \leq -2) \end{cases}$$

where ρ_m and $\dot{\rho}_m$ are given by (5.1). Then it is easy to see that $\{b_m\}_{m\in\mathbf{Z}}$ satisfies the recurrence equation (4.4) and $\{b_m\}_{m\in\mathbf{Z}} \in \ell^2$. This completes the proof of sufficiency.

(II) The case $n=2m$

Let us define $\{d_m\}_{m\in\mathbf{Z}}$ as follows,

$$
(5.5)\qquad \begin{cases} d_0 = 1, \\ d_m = \sigma_1 \sigma_2 \cdots \sigma_m, & (m \geq 1) \\ d_m = \dot{\sigma}_0 \dot{\sigma}_{-1} \cdots \dot{\sigma}_{m+1}, & (m \leq -1) \end{cases}
$$

where

$$
(5.6)\qquad \begin{cases} \sigma_m = -\cfrac{1}{D_m - \cfrac{1}{D_{m+1} - \cfrac{1}{D_{m+2} - \ddots}}} & (m \geq 1) \\[2em] \dot{\sigma}_m = -\cfrac{1}{D_{m-1} - \cfrac{1}{D_{m-2} - \cfrac{1}{D_{m-3} - \ddots}}} & (m \leq 0) \end{cases}
$$

Noting that $\{D_m\}_{m \in Z}$ satisfies $\overline{D_m} = D_{-m}$, we can give a proof similarly as in the case $n = 2m + 1$. This completes the proof of Lemma 4.1.

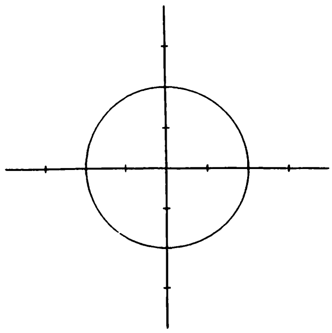
6. Numerical Computations

In this section, we show some results of the numerical computations. First, we made the numerical computations by the typical Runge-kutta scheme of four steps and forth order for the equation (1.1), (1.2) with the coefficients $\mu = \kappa = \alpha = 1$ and the initial data $(x(0), x'(0))$ which are suitably taken in a neighborhood of $(0, 2\pi\lambda)$:

$$
(6.1)\qquad \begin{cases} x''(t) + x' + x(t) + x^3(t) = P(\lambda \sin 2\pi t) \\ P(\lambda \sin 2\pi t) := (\lambda \sin 2\pi t)'' + (\lambda \sin 2\pi t)' + \lambda \sin 2\pi t + (\lambda \sin 2\pi t)^3. \end{cases}
$$

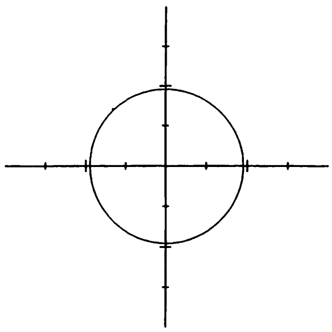
The pictures of the trajectories (x, x') are given in Figure 1$\sim$ Figure 7. "Range" in the left upper side of the figure indicates the range of x, while the range of x' is taken to be 2π times of it so that the trivial solution u_λ depicts a fixed circle. Figure 1 shows the trajectory at $\lambda = 3$. We observe that the trajectory of the trivial solution u_λ is stable. As we continue to increase λ, the trivial solution u_λ loses stability at $\lambda = \lambda_1 \doteqdot 4.19\cdots$. Figure 2 shows the trajectory at $\lambda = 4.3$. We observe that the trajectory is a slightly different circle from the trivial solution. Tracing this branch forward, we have a typical trajectory of this branch at $\lambda = 8$ in Figure 3. The trajectories of this branch have a symmetry with respect to the origin. As we continue to increase λ, the trivial solution becomes stable again at $\lambda = \lambda_2 \doteqdot 6.71\cdots$. The trajectory at $\lambda = 6.5$, which is a little behind λ_2, is given in Figure 4. Tracing this branch back, we have a typical trajectory at $\lambda = 2$ in

Duffing Equations.
X''+1.0X+1.0X'+1.0X^3=P(3.000*sin(2Pi*t))
RANGE=6.0000



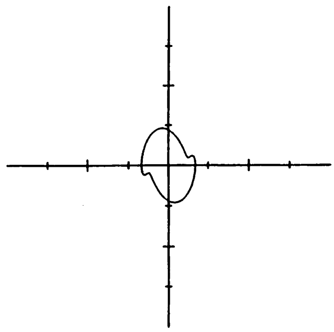
LAMBDA= 3.0000
Fig. 1

Duffing Equations.
X''+1.0X+1.0X'+1.0X^3=P(4.300*sin(2Pi*t))
RANGE=8.6000



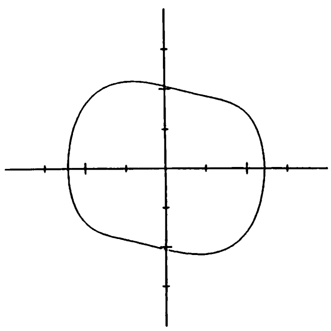
LAMBDA= 4.3000
Fig. 2

Duffing Equations.
X''+1.0X+1.0X'+1.0X^3=P(8.000*sin(2Pi*t))
RANGE=16.0000



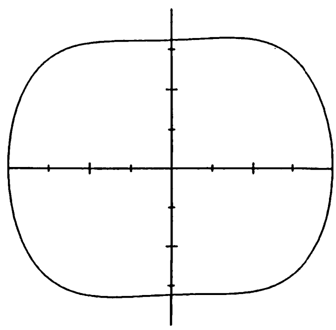
LAMBDA= 8.0000
Fig. 3

Duffing Equations.
X''+1.0X+1.0X'+1.0X^3=P(6.500*sin(2Pi*t))
RANGE=13.0000



LAMBDA= 6.5000
Fig. 4

Duffing Equations.
X''+1.0X+1.0X'+1.0X^3=P(2.000*sin(2Pi*t))
RANGE=8.0000



LAMBDA= 2.0000
Fig. 5

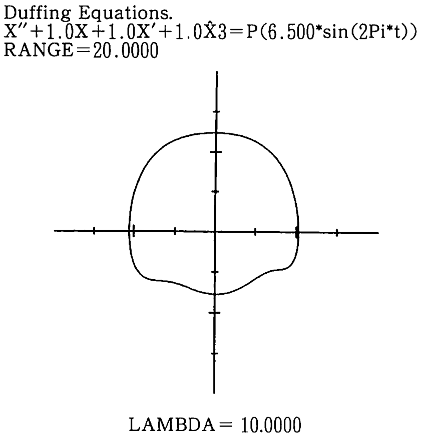


Fig. 6

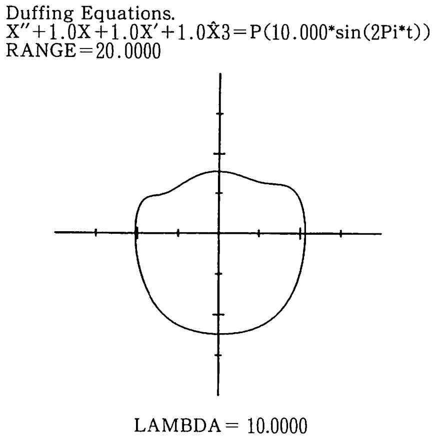


Fig. 7

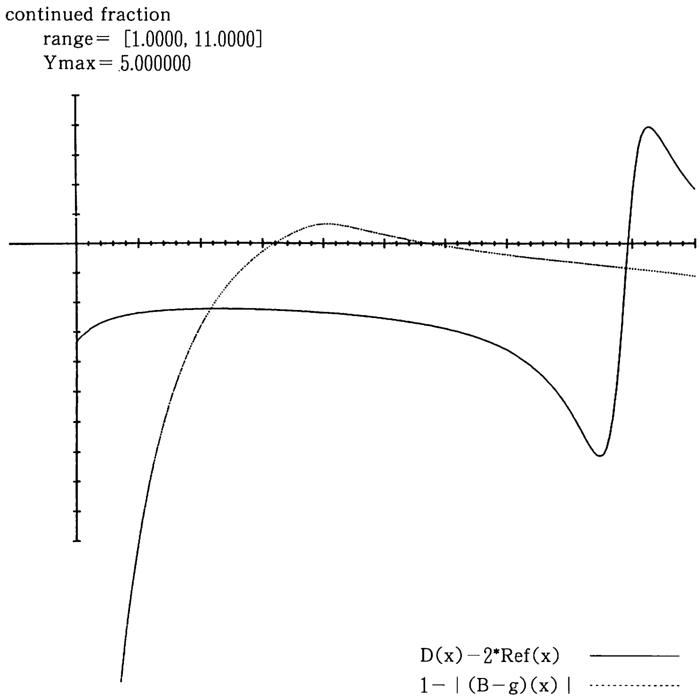


Fig. 8

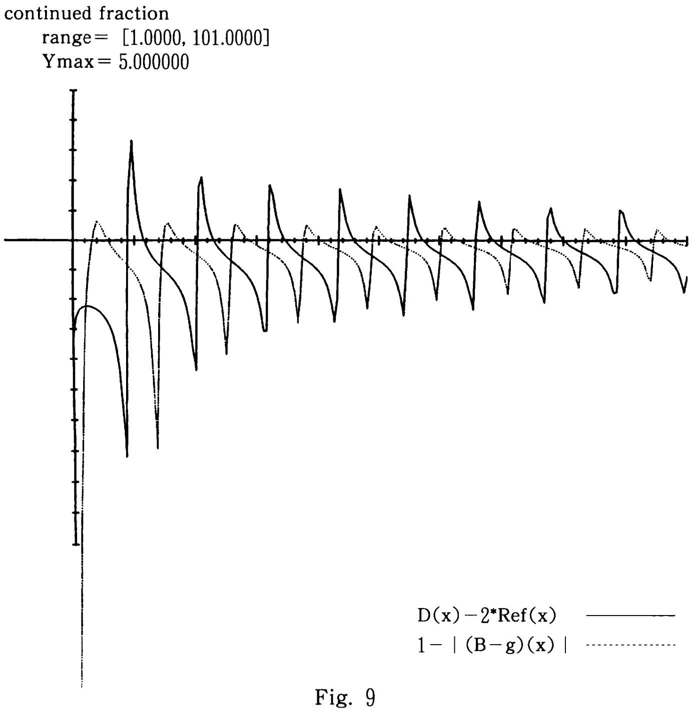


Fig. 9

Figure 5. The trajectories also have a symmetry with respect to the origin. With a further increase of λ, the trivial solution loses stability again at $\lambda=\lambda_3 \fallingdotseq 9.93\cdots$, and we have a pair of stable trajectories. The trajectories at $\lambda=10$ are given in Figure 6 and 7. In this case, each trajectory has not a symmetry, different from the previous cases.

On the other hand, we mathematically studied the eigenvalue problem for the linearized equation in Section 4, and we had the necessary and sufficient condition (4.6) and (4.7). Therefore, we next tried the numerical computations for the continued fractions in (4.6) and (4.7) until the error become less than 10^{-10}. The graphs of the functions $1-|B_0(\lambda)-\mathfrak{B}(\lambda)|$ and $D_0(\lambda)-2\,\mathrm{Re}\,\mathfrak{D}(\lambda)$ for $1\leq\lambda\leq 21$ are given in Figure 8. We have that $1-|B_0(\lambda)-\mathfrak{B}(\lambda)|$ intersects λ axis at $\lambda\fallingdotseq 4.19\cdots$ and $\lambda\fallingdotseq 6.71\cdots$, $D_0(\lambda)-2\mathrm{Re}\,\mathfrak{D}(\lambda)$ intersects λ axis at $\lambda\fallingdotseq 9.93\cdots$. These results are well consistent with both the statement in Theorem 1 and the above mentioned results of numerical computations for the equation (6.1). Finally we should point out that the number of bifurcation points is expected to be not only three but infinity, as suggested from computations for $1\leq\lambda\leq 101$ in Figure 9.

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