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Almost minimal embeddings of quotient singular points into rational surfaces

By

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0. Introduction

Let k be an algebraically closed field of characteristic zero. Let $\bar{X}$ be a normal algebraic surface with only one quotient singular point P. Let $f: X \to \bar{X}$ be a minimal resolution of $\bar{X}$ and let $D = \sum_{i=1}^{n} D_i$ be the reduced exceptional divisor with respect to f, where the D_i are irreducible components. Define a $\mathbf{Q}$-divisor $D^{\#} = \sum_{i=1}^{n} \alpha_i D_i$ such that $(D^{\#} + K_X \cdot D_i) = 0$ for every $1 \leq i \leq n$. Since the intersection matrix of D is negative definite and $(D_i^2) \leq -2$, $D^{\#}$ is uniquely defined and $0 \leq \alpha_i < 1$. We say that a pair (X, D) is *almost minimal* if, for every irreducible curve C, $(D^{\#} + K_X \cdot C) \geq 0$ or the intersection matrix of $C + \mathrm{Bk}(D)$ is not negative definite, where $\mathrm{Bk}(D) = D - D^{\#}$ (see Miyanishi-Tsunoda [11, p. 226]). We say that the singular point P is *almost minimal* in $\bar{X}$ if the pair (X, D) is almost minimal. Then $D^{\#} + K_X \equiv f^*(K_{\bar{X}})$, and $\bar{X}$ is log relatively minimal (cf. Gurjar-Miyanishi [3]). By virtue of [11, 1.11], we can construct the almost minimal singular points from any quotient singular points which might be changed from the original singularities.

In the present article, we study such singularities. In the section 1, we study the case of the logarithmic Kodaira dimension $\bar{\kappa}(X-D) = -\infty$ using the Mori theory [13], and classify such singular points when $\mathrm{Supp}(D)$ is contained in a fiber of a certain $\mathbf{P}^1$-fibration (Theorem 1.1). In [17, Proposition 2.2], Tsunoda classified all the almost minimal quotient singular points on rational surfaces for which $\bar{\kappa}(X-D) = 0$. In the section 2, we study and give a classification in the case where $\bar{\kappa}(X-D) = 1$ and X is a rational surface (Theorem 2.5). In the section 3, we classify the almost minimal pair (X, D) where D is irreducible by using some results in Mohan Kumar-Murthy [14] and Iitaka [5]. Finally, in the section 4, we study the structure of (X, D) and give examples when $\bar{\kappa}(X-D) = 0$ and X is a rational surface (Theorem 4.1).

The terminology is the same as the one in [11]. By a $(-n)$-curve we mean a nonsingular rational curve with self-intersection number $-n$. A reduced effective divisor D is called an SNC divisor (an NC divisor, resp.) if

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D has only simple normal crossings (normal crossings, resp.).

We employ the following notation.

K_X (or simply, K): the canonical divisor on X.
$\kappa(X)$: the Kodaira dimension of a complete surface X.
$\bar{\kappa}(X-D)$: the logarithmic Kodaira dimension of an open surface $X-D$.
$\rho(X)$: the Picard number of X.
$\mathbf{F}_n$ $(n\geq 0)$: a Hirzebruch surface of degree n.
M_n $(n\geq 0)$: a minimal section of $\mathbf{F}_n$.
$D^{\#}$: $=D-\mathrm{Bk}(D)$.
$h^i(D)$: $=\dim_k H^i(X, \mathcal{O}_X(D))$.

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1. The case $\bar{\kappa}(X-D)=-\infty$

Let (X, D) be the same pair as in the introduction. In this section, we assume that $\bar{\kappa}(X-D)=-\infty$ and (X, D) is almost minimal. Since $K_{\bar{X}}$ is not numerically effective, there exists an extremal rational curve $\bar{l}$ on $\bar{X}$. Let l be the proper transform of $\bar{l}$ on X. Then, by [11, Lemma 2.7], one of the following two cases takes place:

(A) The intersection matrix of $l+\mathrm{Bk}(D)$ is negative semidefinite, but not negative definite. Furthermore, $(\bar{l}^2)=0$.

(B) The Picard number $\rho(\bar{X})$ is equal to one, and $-K_{\bar{X}}$ is ample.

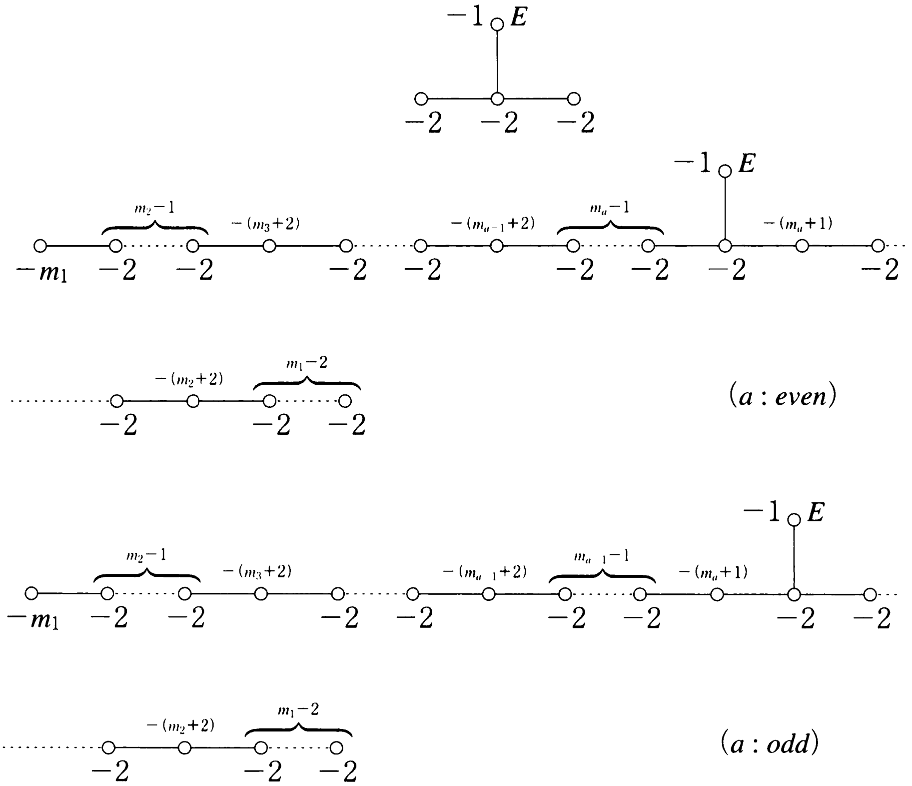
In the subsequent arguments, we consider only the case (A), leaving the case (B) to a forthcoming paper (cf. [7]).

Theorem 1.1. *Let $\bar{X}$ be a normal projective surface with only one quotient singular point P such that $\bar{\kappa}(\bar{X}-P)=-\infty$ and let (X, D) be the minimal resolution of $\bar{X}$. Suppose that (X, D) is almost mimimal and that the case (A) takes place. Then the following assertions hold*:

(1) *There exists a $\mathbf{P}^1$-fibration h: $X\to C$ over a curve C such that* Supp(D) *is contained in* Supp F_0, *where F_0 is a unique reducible fiber of h. Furthermore, there exists a unique (-1)-curve E on X such that* Supp$(E+D)$ *coincides with* Supp(F_0).

(2) *The weighted dual graph of $E+D$ is one of the following*:

(i) *Case*: Supp D *is a linear chain. Then the dual graph of $E+D$ is given as in Figure* 1, *where $m_1\geq 2$, $a\geq 2$ and $m_i\geq 1$ for $2\leq i\leq a$.*



$(a : even)$

$(a : odd)$

Figure 1

(ii) *Case*: Supp D *is not a linear chain. Then the dual graph of* $E+D$ *is given as in Figures* 2 *and* 3. *In Figure* 2, $n \geq 0$ *and the subgraph denoted by the encircled* $S_a (a \geq 0)$ *stands for one of the linear chains plus the* (-1) *-curve* E *given as in Figure* 4, *where* $m_1 \geq 3$ *and* $m_i \geq 1$ *for* $2 \leq i \leq a$.

(Type D)

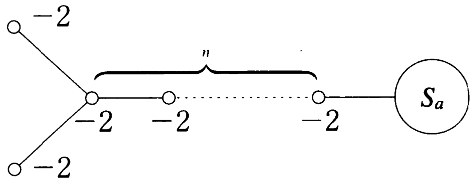


Figure 2

(Type E_6)

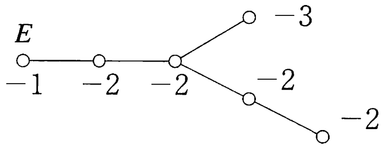


Figure 3

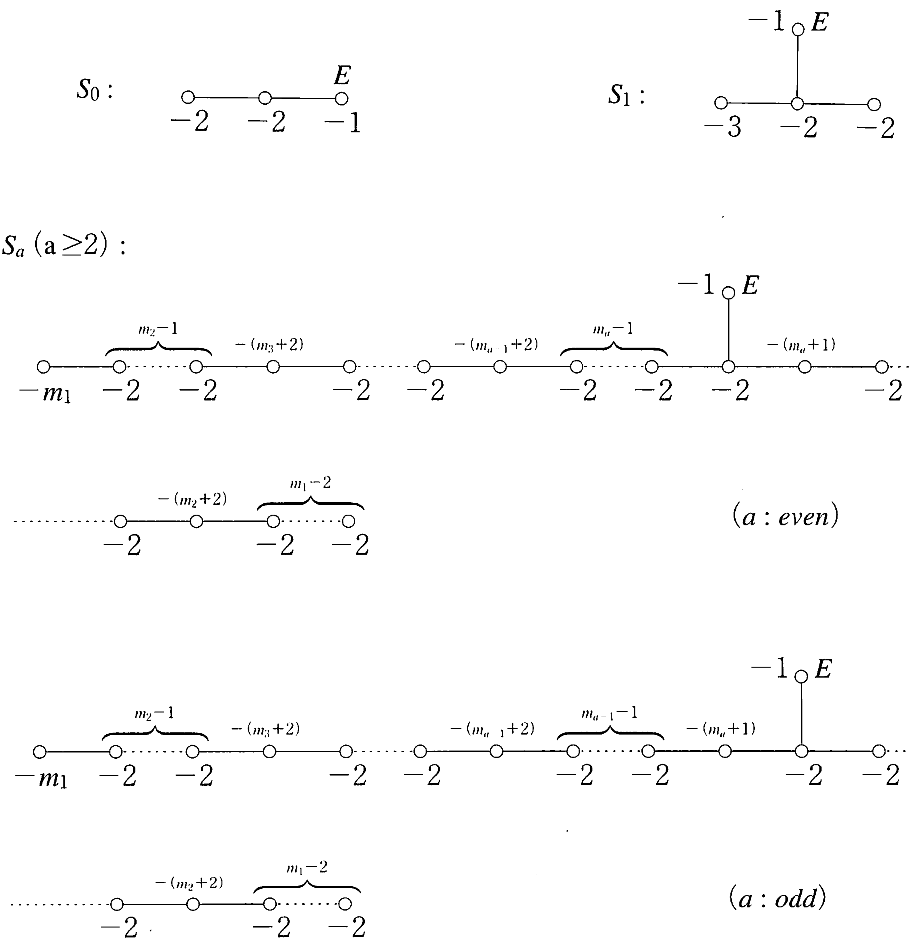


Figure 4

Proof. (1) Let f: $X \rightarrow \bar{X}$ be the minimal resolution of $\bar{X}$ and let $D = \sum_{i=1}^{r} D_i$ be the decomposition of D into irreducible components. Since $\bar{X}$ is a normal projective surface with only one quotient singular point, there exists an integer $N>0$ such that, for every Weil divisor $\bar{G}$ on $\bar{X}$, $N\bar{G}$ is a Cartier divisor on $\bar{X}$ (cf. [1, Theorem 2.3] and [11, 2.5]). By [11, Lemma 2.8], for a sufficiently large n, the linear system $|nNf^*(\bar{l})|$ is composed of an irreducible pencil, free from base points, whose general members are isomorphic to $\mathbf{P}^1$.

Let h: $X \to C$ be the $\mathbf{P}^1$-fibration of X over a curve C defined by the linear system $|nNf^*(\bar{l})|$. Since $(f^*(\bar{l}) \cdot D_i) = 0$ for all i, $1 \leq i \leq r$ and $\mathrm{Supp}(D)$ is connected, it follows that $\mathrm{Supp}(D)$ is contained in $\mathrm{Supp}(F_0)$, where F_0 is a fiber $h^{-1}(a)$ for some $a \in C$. Note that F_0 is a unique reducible fiber of h. Hence there exists a (-1)-curve E in $\mathrm{Supp}(F_0)$. If $(D \cdot E) = 0$, then $(E \cdot D^{\#} + K_X) = -1 < 0$ and the intersection matrix of $E + \mathrm{Bk}(D)$ is negative definite. Therefore, $(E \cdot D) = 1$ because $\mathrm{Supp}(D)$ is connected and $\mathrm{Supp}(F_0)$ contains no loops of curves. Since $\mathrm{Supp}(E+D) \subseteq \mathrm{Supp}(F_0)$, the intersection matrix of $E + \mathrm{Bk}(D)$ is negative semidefinite. Furthermore, since $(E \cdot D^{\#} + K) < (E \cdot D) - 1 = 0$ and (X, D) is almost minimal, it follows that $E+D$ is not negative definite. Hence, $\mathrm{Supp}(E+D) = \mathrm{Supp}(F_0)$. The uniqueness of such a (-1)-curve E is clear.

(2) By virtue of (1), we know that E meets only one irreducible component of $\mathrm{Supp}(D)$. Let f: $X \to Y$ be the contraction of all possible contractible components of $E + \mathrm{Bk}(D)$ including E, i.e., irreducible components of $E + \mathrm{Bk}(D)$ which become exceptional curves of the first kind after a succession of several contractions. Let $F = f_*(F_0)$. By (1), F is an irreducible rational curve with self-intersection number zero. We shall consider the following two cases separately.

Case 1: $\mathrm{Supp}(D)$ is a linear chain. By the above remark, we can easily see that the weighted dual graph of $E+D$ is given as in Figure 1.

Case 2: $\mathrm{Supp}(D)$ is not a linear chain. Then there exists only one irreducible component D_0 such that $(D - D_0 \cdot D_0) = 3$. The dual graph of $\mathrm{Supp}(D)$ is as follows:

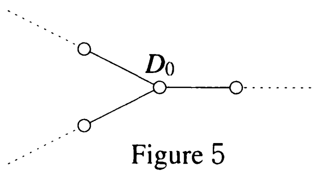


Figure 5

where each connected component of $\mathrm{Supp}(D - D_0)$ is a linear chain. Let $B = D_i$ be a unique irreducible component of $\mathrm{Supp}(D)$ which meets E. Then we have the following:

Lemma 1.2. *With the notation and assumptions as above, we have*:

(1) *B is a (-2)-curve.*
(2) *$B \neq D_0$.*
(3) *Let $\bar{g}: X \to \bar{X}$ be the contraction of* Supp$(E+D)$ *such that $\bar{g}_*(D_0)$ is a (-1)-curve. Then* Supp $\bar{g}_*(E+D)$ *is a linear chain.*

Proof. (1) Let g': $X \to X'$ be the contraction of E. Since $g'_*(F_0)$ is a reducible fiber of a $\mathbf{P}^1$-fibration, there exits a (-1)-curve in Supp $g'_*(F_0)=$ Supp $g'_*(D)$. Since E intersects only $B=D_i$ and each irreducible component of Supp(D) has self-intersection number ≤ -2, $g'_*(B)$ is then a (-1)-curve. Hence $(B^2)=-2$.

(2) Suppose that $B=D_0$. Then D_0 is a (-2)-curve by (1). Let g'': $X \to X''$ be the contraction of E and D_0. Then Supp $g''_*(D)=$ Supp $g''_*(F_0)$ is a reducible fiber of a $\mathbf{P}^1$-fibration which contains three irreducible components meeting in one point. This contradicts [8, Lemma 2.2, p. 115].

(3) By virtue of (2), it is clear.

Lemma 1.3. *Let $D=\sum_{i=0}^{r} D_i$ be the decomposition of D into irreducible components. Suppose that the weighted dual graph is as follows:*

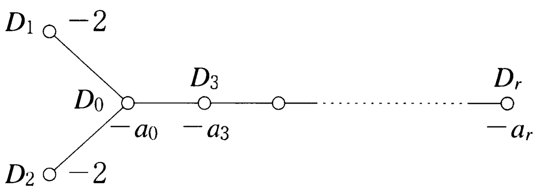


Figure 6

where $a_j=-(D_j^2)\geq 2, j=0, 3, \cdots, r$. Then the dual graph is given as in Figure 2.

Proof. By Lemma 1.2, (1), $B \neq D_0$. We first consider the case $B=D_1$ (similarly, $B=D_2$). Then, $a_0=2$. Let g: $X \to X_0$ be the contraction of E, D_1, and D_0. Let D'_i: $=g_*(D_i)$. Then $g_*(E+D)$ has the following configuration:

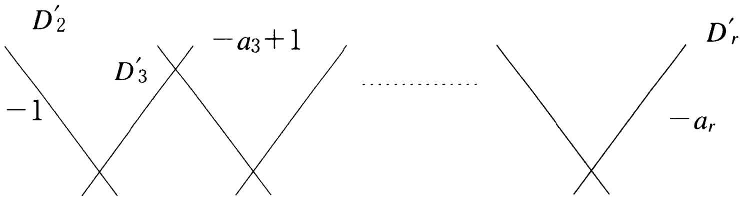


Figure 7

In this case, we know that P is a rational double point of Type D_4. Similarly, if $B=D_r$, then P is a rational double point of Type D_{r+1}.

Next we consider the case $B=D_i$, $3\leq i<r$. Let μ: $X\rightarrow Y$ be a sequence of contractions of (-1)-curve E and subsequently contractible curves in Supp (F_0) such that $\mu_*(D_r)$ is a (-1)-curve. Then Supp $\mu_*(E+D-D_r)$ can be contracted to a rational double point of Type A_3 or $D_n (n\geq 4)$. Hence we know that the weighted dual graph of $E+D$ is one of Figure 2.

By Lemma 1.3, we may assume that the configuration of D has one of the graphes of Types E_6, E_7 and E_8 in the following arguments.

Suppose that the configuration of $E+D$ is as follows:

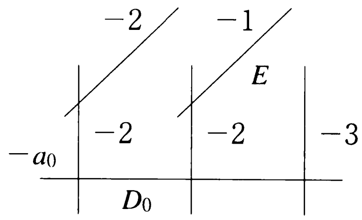


Figure 8

Then, we can easily see that $a_0=-(D^2{}_0)=2$ and this case can occur. It suffices to show that with the above assumption, the cases except for the above one cannot occur. We assume, for example, that the configuration of D is as follows:

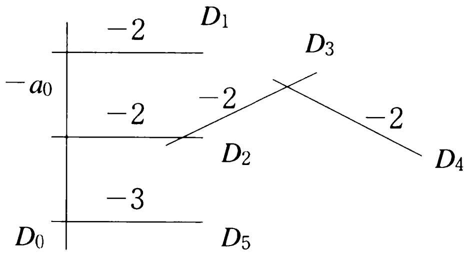


Figure 9

By Lemma 1.2, E meets one of $D_1, \cdots, D_4$. We first consider the case $(E\cdot D_1)=1$. Suppose that $a_0\geq 3$. Let u_1: $X\rightarrow Y$ be the contraction of E and D_1. Then Supp $u_{1*}(D+E)=$ Supp $u_{1*}(F_0)$ contains no (-1)-curves. Since $u_{1*}(F_0)$ is a reducible fiber of a $\mathbf{P}^1$-fibration, this is a contradiction. Hence $a_0=2$. Let u: $X\rightarrow Y$ be the contraction of E, D_1, D_2, D_3 and D_4. Then, $u_*(F_0)=u_*(D_5)$ is a nonsingular rational curve with self-intersection number 1. This is also a contradiction. Next, we consider the case $(E\cdot D_2)=1$. Contracting E, D_2, D_3 and D_4, we know that $a_0=4$ by Lemma 1.2, (3). Then it is clear that $E+D$ can be contracted to a nonsingular point. This is a contradiction. In the case $(E \cdot D_3)=1$, it is clear that $E+D$ can not be a fiber of a $\mathbf{P}$-fibration. In the same way as in the case $(E \cdot D_2)=1$, it follows that the case $(E \cdot D_4)=1$ cannot occur. The other cases with configurations different from the above ones can be treated in similar fashions.

This completes the proof of Theorem 1.1.

Remark 1.4. Let $(\bar{X}, P)$ (or (X, D)) be the same pair as in the introduction. Suppose that the case (B) occurs. If P is a rational double or triple singular point, then such pairs have been classified completely. See Miyanishi-Zhang [12] and Zhang [19].

2. The case $\bar{\kappa}(X-D)\geq 0$

Let (X, D) be the same pair as in the introduction.

Lemma 2.1. *Suppose that $\bar{\kappa}(X-D)\geq 0$ and every irreducible component of D is a (-2)-curve. Then $\bar{\kappa}(X)\geq 0$.*

Proof. Since every irreducible component of D is a (-2)-curve, we have

$$D^{\#}+K_X=K_X.$$

By the assumption $\bar{\kappa}(X-D)\geq 0$, we have $\bar{\kappa}(X)\geq 0$.

In the subsequent arguments, we assume that X is a nonsingular rational surface and $\bar{\kappa}(X-D)\geq 0$. Furthermore, we assume that (X, D) is an almost minimal pair. Then, by [11, Theorem 2.11], it follows that $D^{\#}+K_X$ is numerically effective. Lemma 2.1 implies that there exists an irreducible component D_i of D with $(D_i^2)\leq -3$.

Now we state the following lemma.

Lemma 2.2. *Suppose that $\bar{\kappa}(X-D)\geq 0$. Then we have*

$$(K_X^2)\leq -1.$$

Proof. Suppose to the contrary that $(K^2)\geq 0$. By the Riemann-Roch theorem, we have

$$h^0(X, \mathcal{O}_X(-K))\geq (K^2)+1.$$

Hence $|-K|\neq 0$. On the other hand, since D is an SNC divisor on a rational surface, we have $|D+K|=\emptyset$ (cf. Miyanishi [9, Lemma 2.1.3]). Then, by [9, Lemma 2.1.1], $p_a(D)=0$, where $p_a(D)$ is the arithmetic genus of the divisor D. Now the Riemann-Roch theorem yields

$$\begin{aligned} h^0(D+2K)+h^0(-D-K) &\geq \frac{1}{2}(D+2K \cdot D+K)+1 \\ &= (K \cdot D+K) \end{aligned}$$

$$\geq 1.$$

CLAIM. $|-D-K|=\emptyset$.

In fact, suppose to the contrary that $|-D-K|\neq\emptyset$. Since $\bar{\kappa}(X-D)\geq 0$, it follows that $|n(D+K)|\neq\emptyset$ for some $n>0$. Hence $D+K$ is linearly equivalent to zero. This is a contradiction because $|D+K|=\emptyset$ as remarked above.

By the above inequality and the claim,

$$h^0(D+2K)\geq 1,$$

i.e., $|D+2K|\neq\emptyset$. Combining this with $|-K|\neq\emptyset$, we have

$$|D+K|\neq\emptyset.$$

This contradicts [9, Lemma 2.1.3].

In the subsequent arguments, we asssume that $\bar{\kappa}(X-D)=0$ or 1. We will give the configuration of the divisor D in such cases.

Since X is a rational surface and the dual graph of D is a tree, we have $|D+K|=\emptyset$ (cf. [9, Lemma 2.1.3]). Hence, by [17, Proposition 2.2], we have the following result:

Theorem 2.3. *Let $\bar{X}$ be a normal projective rational surface with only one quotient singular point P and let (X, D) be the mininal resolution of $\bar{X}$. Suppose that (X, D) is almost minimal and $\bar{\kappa}(X-D)=0$. Then $D+2K\sim 0$ and $h^0(2(D+K))=1$. Furthermore, the configuration of D is given in Figure* 10, *where* $0\leq n\leq 8$.

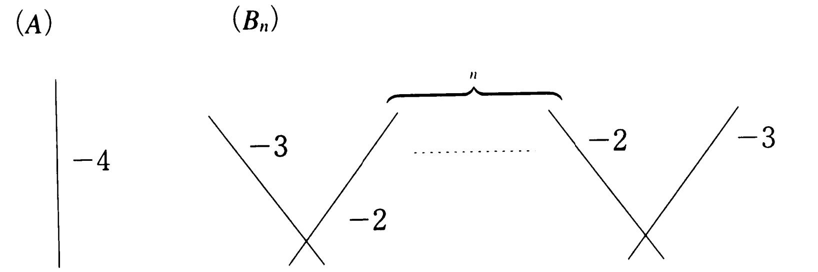


Figure 10

The case (A) was studied in [14]. For an example in the case (B_8), see [20, Example 3.2].

We consider the case $\bar{\kappa}(X-D)=1$. The linear system $|j(D^{\#}+K)|$ then gives rise to an irreducible pencil of elliptic curves or rational curves h: $X\rightarrow\mathbf{P}^1$ for a sufficiently large j by taking, if necessary, the Stein factorization of $\Phi_{|j(D^{\#}+K)|}$ (cf. Kawamata [6, Theorem 2.3]). More precisely, the following assertion holds:

Lemma 2.4. *h is an elliptic fibration. Furthermore,* Supp(D) *is* *contained in a fiber F_0 of h.*

Proof. If h is a $\mathbf{P}^1$-fibration, D contains a component, say D_1, which is a section or 2-section of h. Since $(D^{\#}+K_X \cdot F)=0$ for a general fiber F of h, the coefficient of D_1 in $D^{\#}$ must be 1. However, the coefficients of all components of $D^{\#}$ are less than 1. This is a contradiction. Hence h is an elliptic fibration. Since $(D^{\#}+K_X \cdot D)=0$ and $\mathrm{Supp}(D)$ is connected, $\mathrm{Supp}(D)$ is contained in a fiber of h.

In order to state the following theorem, we define linear chains A_m and $A_{a,m}$ $(a \geq 1, m \geq 0)$ as in Figure 11, where $m_i \geq 1$ for $1 \leq i \leq a$:

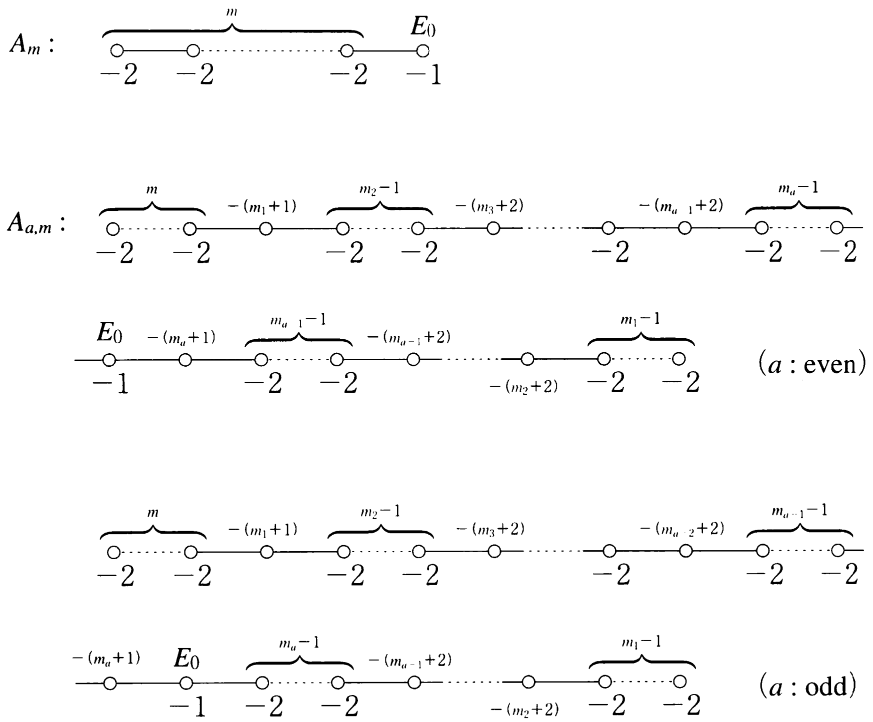


Figure 11

Theorem 2.5. *Let $\bar{X}$ be a normal projective rational surface with only one quotient singular point P and let (X, D) be a minimal resolution of $\bar{X}$. Suppose (X, D) is almost minimal and $\bar{\kappa}(X-D)=1$. Then the following assertions hold:*

(1) *Let h be as in Lemma 2.4 and let F_0 be the fiber of h which contains D. Then there exists a unique (-1)-curve E_0 such that* $\mathrm{Supp}(D+E_0)=\mathrm{Supp}(F_0)$. *Furthermore, all the fibers of h except for F_0 contain no (-1)-curves.*

(2) *The configuration or the weighted dual graph of $D+E_0$ is given as in Figures* 12, 13 *and* 14:

(a) *Cases B_2 and B_3.*

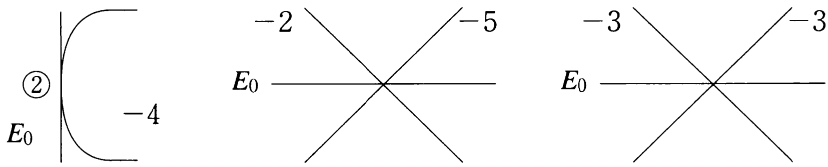


Figure 12

(b) *Case A_1'.*

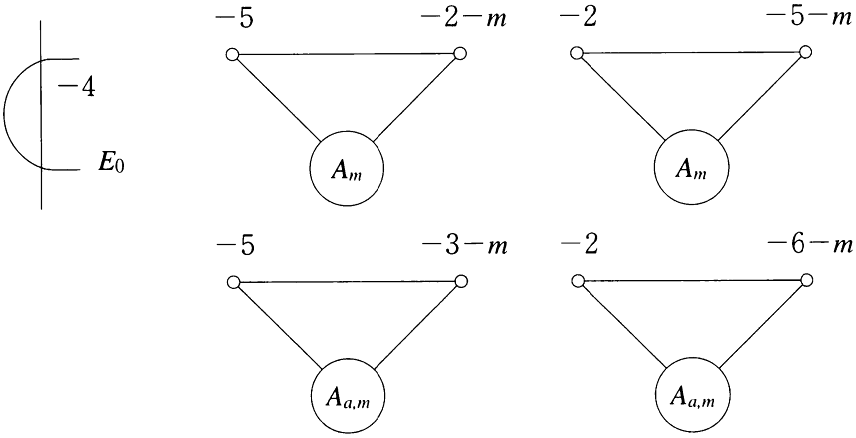


Figure 13

(c) *Case $A_n'(n \geq 2)$.*

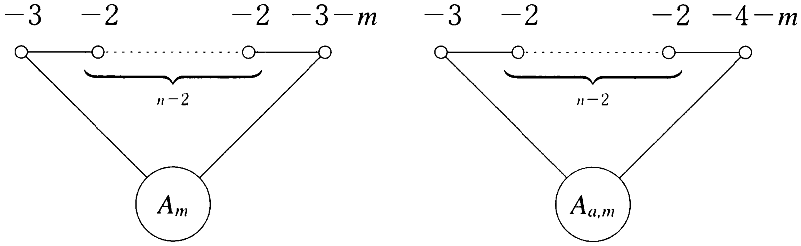


Figure 14

Proof. Let h: $X \to \mathbf{P}^1$ be the elliptic fibration as above. By Lemma 2.2, h is not a minimal elliptic fibration. Hence there exists a (-1)-curve E_0 such that E_0 is contained in a fiber of h. By the almost minimality of (X, D) and by Lemma 2.4, E_0 is contained in the same fiber F_0 as $\mathrm{Supp}(D)$ is connected.

Let $D = \sum_{i=1}^{n} D_i$ be the irreducible decomposition of D and write $D^{\#} = \sum_{i=1}^{n}$

$\alpha_i D_i$ where $\alpha_i \in \mathbf{Q}$. Then, by Lemma 2.1 and by the fact that $\alpha_i = 0$ for some i if and only if D consists of (-2)-curves, we have $0 < \alpha_i < 1$ for any i, $1 \leq i \leq n$. Since $(D^{\#} + K \cdot E_0) = 0$, it follows that $(D^{\#} \cdot E_0) = 1$.

We first consider the case where D is irreducible. Then,

$$D^{\#} = \frac{a-2}{a} D \qquad \text{with} \quad a = -(D^2).$$

In this case, the theorem follows from the claim below.

CLAIM 1. We have $a = 4$. Furthermore, F_0 is a multiple of $D + 2E_0$

Proof. Since $(D^{\#} + K)^2 = 0$, we have

$$(K^2) = 4 - a - \frac{4}{a}.$$

Since (K^2) is an integer and $a \geq 2$, it follows that $a = 2$ or 4. By Lemma 2.1, we have $a = 4$. Then, since $D^{\#} + K = 1/2D + K$, it follows that $(D \cdot E_0) = 2$ and $(D + 2E_0)^2 = 0$. Hence F_0 is a multiple of $D + 2E_0$.

We may hereafter assume that D is reducible. Then F_0 is obtained from G, which is a fiber of a minimal elliptic fibration on a nonsingular surface X_0, by blowing up some points or infinitely near points on $\mathrm{Supp}(G)$. Let $\mu: X \to X_0$ be the converse which is the contraction of all possibly contractible components of $\mathrm{Supp}(F_0)$ including E_0, i.e., irreducible components of $\mathrm{Supp}(F_0)$ which become exceptional curves of the first kind after a succession of several contractions. We put $X = \mathrm{dil}_{P_l} \circ \cdots \circ \mathrm{dil}_{P_0}(X_0)$ and $\mu_i = \mathrm{dil}_{P_i} \circ \cdots \circ \mathrm{dil}_{P_0}$, where dil_{P_i} is the blowing-up with center P_i and $\mu_{-1} = \mathrm{id}_{X_0}$.

CLAIM 2. P_i is a singular point on $\mu_{i-1}^*(G)$ for every $i \geq 0$. In particular, G is a singular fiber of the minimal elliptic fibration.

Proof. Suppose to the contrary that there exists i $(0 \leq i \leq l)$ such that P_i is a smooth point on $\mu_{i-1}^*(G)$. Let F_0' be the total transform of $P_i \in \mu_{i-1}(X_0)$ on X. Let E be a (-1)-curve which is contained in $\mathrm{Supp}\,(F_0')$. It is then clear that $((F_0')_{\mathrm{red}} - E \cdot E) \leq 2$. In particular, if $((F_0')_{\mathrm{red}} - E \cdot E) = 2$, then we have $E \cap \mathrm{Supp}((F_0)_{\mathrm{red}} - (F_0')_{\mathrm{red}}) = \emptyset$. Since $(F_0')_{\mathrm{red}}$ is an SNC divisor and $\mathrm{Supp}(F_0')$ contains no loops of curves, we have one of the following cases:

(a) $\mathrm{Supp}((F_0)_{\mathrm{red}} - E)$ has two connected components, say A and B.

(b) $\mathrm{Supp}((F_0)_{\mathrm{red}} - E)$ is connected and E is an end component of F_0'.

Suppose that the case (b) occurs. Then, since $((F_0)_{\mathrm{red}} - (F_0')_{\mathrm{red}} \cdot (F_0')_{\mathrm{red}}) = 1$, E meets at most one irreducible component of D. This contradicts the above remark $(D^{\#} \cdot E) = 1$. Hence the case (a) occurs. Then D is contained in A or B because D is connected. On the other hand, since $(E \cdot (F_0)_{\mathrm{red}} - E) = 2$, E meets at most one irreducible component of D. This is also a contradiction.

CLAIM 3. E_0 meets just two irreducible components of D.

Proof. First, suppose that G is a singular fiber of type B_4, where our naming of the fiber types accords with the one in Shafarevich [16].

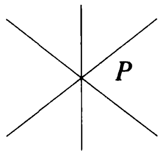


Figure 15

Then, by Claim 2 and the assumption that D is reducible, $\mathrm{Supp}(F_0)-E_0$ has two connected components A and B. Since $\mathrm{Supp}(D)$ is connected, D is contained in A or B. This is a contradiction because E_0 then meets at most one irreducible component of D and $(F_0)_{\mathrm{red}}$ is an SNC divisor.

Next, we consider the case G is of type B_2 or B_3. By the above argument and the assumption that D is reducible, the configuration of F_0 is one of the following:

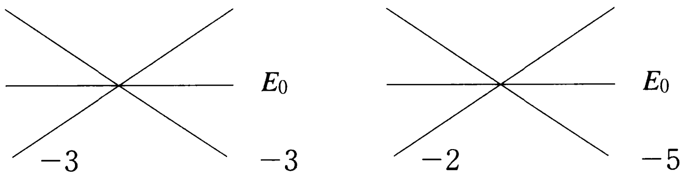


Figure 16

Hence, in this case, the assertion is verified.

We may assume that G is not of the type B_2, B_3 or B_4. Then G is an NC or SNC sivisor. Hence, by Claim 2, the assertion is verified.

Using the arguments in the proof of Claims 2 and 3, we have the following:

CLAIM 4. $\mathrm{Supp}((F_0)_{\mathrm{red}}-E_0)$ *is connected.*

CLAIM 5. *G is of type A'_n $(n\geq 1)$, B_2 or B_3. Hence the dual graph of D is a linear chain.*

Proof. By the proof of Claim 3, we know that G is not of type B_4. Suppose that G is of type $A''_n\,(n\geq 0)$, B_6, B_7, B_8, or B_{10}. Then, by Claim 2, $\mathrm{Supp}((F_0)_{\mathrm{red}}-E_0)$ is disconnected. This contradicts Claim 4.

By Claims 1, 2, 4 and 5, we can easily verify the following claim:

CLAIM 6. *F_0 contains only one (-1)-curve. Furthermore, for all $i\,(0\leq i$*

$\leq l$), $\mu_i^*(G)$ has only one (-1)-curve.

Using the above claims, we can easily verify the theorem.

3. Irreducible curves with negative self intersection on rational surfaces

Let X be a nonsingular projective rational surface and let D be a nonsingular rational curve on X with $n:=-(D^2)\geq 2$.

Definition 3.1. (cf. Iitaka [5]). A pair (X, D) is *relatively minimal* if $(E\cdot D)\geq 2$ for every (-1)-curve E on X.

Then we have the following lemma.

Lemma 3.2. *A pair (X, D) is almost minimal if and only if it is relatively minimal.*

Proof. Assume that (X, D) is almost minimal. Suppose to the contrary that there exists a (-1)-curve E such that $(E\cdot D)\leq 1$. It is then clear that $(E\cdot D)=1$. Then we have

$$(D^{\#}+K_X\cdot E)=\frac{n-2}{n}-1=-\frac{2}{n}<0,$$

because $n\geq 2$. Furthermore, the intersection matrix of $E+\mathrm{Bk}(D)$ is then negative definite. This contradicts the hypothesis that (X, D) is almost minimal. Hence (X, D) is relatively minimal.

Conversely, assume that (X, D) is relatively minimal. Suppose to the contrary that (X, D) is not almost minimal. Then there exists an irreducible curve C such that $(C\cdot D^{\#}+K)<0$ and the intersection matrix of $C+\mathrm{Bk}(D)$ is negative definite. We note that $C\neq D$. Then we have clearly $(C^2)<0$ and $(C\cdot K)<0$, i.e., C is a (-1)-curve. We have

$$(C\cdot D^{\#})=\frac{n-2}{n}(C\cdot D)<-(C\cdot K)=1.$$

Since (X, D) is relatively minimal, we have $(C\cdot D)\geq 2$. Hence, by the above inequality, we have $n=2$ or 3. On the other hand, we have

$$(C+D)=(C^2)+2(C\cdot D)+(D^2)\geq -1+2(C\cdot D)-3\geq 0.$$

This contradicts the hypothesis that the intersection matrix of $C+\mathrm{Bk}(D)$ is negative definite. Hence (X, D) is almost minimal.

Using Lemma 3.2 and the results in [5] and [14], we have the following:

Theorem 3.3. *Let X be a nonsingular projective rational surface and let D be a $(-n)$-curve $(n\geq 2)$ on X. Suppose that (X, D) is an almost minimal pair. Then the following assertions hold:*

(1) $\bar{\kappa}(X-D)=-\infty$ *if and only if* X *is a Hirzebruch surface* $\mathbf{F}_n$ *of degree* n *and* D *is a minimal section* M_n *of* X.

(2) $\bar{\kappa}(X-D)=0$ *if and only if* $n=4$ *and* $D+2K_X$ *is linearly equivalent to zero. Furthermore, if* E *is any* (-1)*-curve, the linear system* $|D+2E|$ *is an irreducible pencil of elliptic curves. We also have a birational morphism* f: $X \to \mathbf{P}^2$ *such that* $f(D)$ *is a sextic with ten double points (possibly including infinitely near points).*

(3) $\bar{\kappa}(X-D)=1$ *if and only if* $n=4$ *and* $|D+3K| \neq \emptyset$. *There exists a unique* (-1)*-curve* E_0 *such that* $(E_0 \cdot D)=2$. *Furthermore, the linear system* $|D+2E_0|$ *is an irreducible pencil of elliptic curves. There also exists a birational morphism* f: $X \to \mathbf{P}^2$ *such that* $f(D)$ *is a curve of degree* $3m$, $m \geq 3$ *with nine* m*-tuple points and one double point (possibly including infinitely near points).*

(4) *If* $\bar{\kappa}(X-D)=2$, *then we have* $n \geq 5$.

Proof. (2) and (3) If $\bar{\kappa}(X-D)=0$ or 1, then by Theorems 2.3 and 2.5, it follows that $n=4$ and $(K^2)=-1$. Hence, by Lemma 3.2, (X, D) satisfies the hypothesis in [14, Theorem 3.3]. Hence follow our assertions.

(1) Let f: $X \to \bar{X}$ be the contraction of D to a quotient singular point P. If $\bar{\kappa}(X-D)=-\infty$, then $K_{\bar{X}}$ is not numerically effective. If there exists an extremal rational curve $\bar{l}$ with $(\bar{l}^2)=0$, D must be reducible by virtue of Theorem 1.1. Hence $\bar{X}$ is a log del Pezzo surface of rank one with contractible boundary (for the definition, see [18]). Then we have $\rho(X)=2$. Hence X is a Hirzebruch surface $\mathbf{F}_n$ of degree n and D is a minimal section of X.

(4) Since (X, D) is almost minimal and $\bar{\kappa}(X-D)=2$, we have

$$(D^{\#}+K)^2=(D^{\#} \cdot K)+(K^2)=\frac{(n-2)^2}{n}+(K^2)>0.$$

By Lemma 2.2, we have

$$1<\frac{(n-2)^2}{n}.$$

Since $n \geq 2$ it follows that $n \geq 5$.

Let D be an irreducible complete curve and let X and X' be nonsingular projective surfaces. Let i: $D \hookrightarrow X$ and i': $D \hookrightarrow X'$ be two closed immersions of the curve D. We say that (X, D) and (X', D) are *equivalent* if there exist Zariski-open neighbourhoods $U \subset X$ and $U' \subset X'$ of $i(D)$ and $i(D')$ respectively, and an isomorphism g: $U \to U'$ such that $g \circ i=i'$.

We state the following theorem due to Mohan Kummar and Murthy [14, Theorem 2.1]. They proved the theorem by using Fujita-Miyanishi-Sugie theory (cf. [2] and [10]). Our proof depends on the Mori theory.

Theorem 3.4. *Let* X *be a nonsingular projective rational surface and* D *a*

$(-n)$-curve $(n\geq 1)$ on X. Then the following conditions are equivalent:

(i) *(X, D) is equivalent to $(\mathbf{F}_n, M_n)$.*

(ii) $\bar{\kappa}(X-D)=-\infty$.

Proof. It suffices to show that (i) follows from (ii). If $n=1$ then (X, D) is equivalent to $(\mathbf{F}_1, M_1)$ by Nagata [15, Theorem 3]. Hence we may assume that $n\geq 2$. Let $X'=\mathrm{dil}_P(X)$ with $P\in D$ and D' the proper transform of D in X'. We note that (X, D) is equivalent to $(\mathbf{F}_n, M_n)$ if and only if (X', D') is equivalent to $(\mathbf{F}_{n+1}, M_{n+1})$. If there exists a (-1)-curve E with $(E\cdot D)=1$, then (X, D) is equivalent to $(\mathbf{F}_n, M_n)$ by the above remark and the induction hypothesis. So, we may assume that $(E\cdot D)\geq 2$ or $(E\cdot D)=0$ for any (-1)-curve E on X. By contracting all (-1)-curves with $(E\cdot D)=0$, we may assume that

$$(E\cdot D)\geq 2$$

for any (-1)-curve E on X. Then, by Lemma 3.2, (X, D) is almost minimal. Hence $X=\mathbf{F}_n$ and $D=M_n$ by Theorem 3.3, (1). This proves the theorem.

By Theorem 3.4, we have the following:

Corollary 3.5. (cf. [14]). *Let (X, D) be the same as in Theorem 3.4. Let f: $X\rightarrow\bar{X}$ be the contraction of D to a quotient singular point P and A the local ring of $\bar{X}$ at P. If $\bar{\kappa}(X-D)=-\infty$, then we have*

$$A\cong k[X^n, X^{n-1}Y, \cdots, X^{n-i}Y^i, \cdots, Y^n]_m,$$

where m is the maximal ideal corresponding to the origin.

4. The structure of surfaces $X-D$ for the case $\bar{\kappa}(X-D)=0$

Let (X, D) be as in the introduction. We assume that (X, D) is almost minimal and $\bar{\kappa}(X-D)=0$. By Theorem 2.3, we know the configuration of such a divisor D. In this section, we shall study the structure of such a pair (X, D) and give some examples.

If (X, D) is of Type (A) in Theorem 2.3, then $|D+2E|$ is an irreducible pencil of elliptic curves for any (-1)-curve E (cf. Theorem 3.3, (2)). This is the case also in Type (B_n) with additional conditions. More precisely, we have the following structure therem on such a pair (X, D).

Theorem 4.1. *Let (X, D) be the same as in Theorem 2.3. If (X, D) is of Type (A) or (B_n) $(n\neq 8)$, then there exists a (-1)-curve E on X such that $|D+2E|$ is an irreducible pencil of elliptic curves. Namely, X has an elliptic fibration over $\mathbf{P}^1$ which contains D in a fiber.*

In the case (B_n), let $D=\sum_{i=0}^{n+1}D_i$ be the irreducible decomposition of the above D with $(D^2{}_0)=(D_{n+1}^2)=-3$ and $(D_i^2)=-2$, $1\leq i\leq n$. To prove Theorem 4.1, we need the following lemma.

Lemma 4.2. *Suppose that (X, D) is of Type (B_n) and that there exists a (-1)-curve E such that $(E \cdot D_0) = (E \cdot D_{n+1}) = 1$ and $(E \cdot D_i) = 0$ for all i $(1 \leq i \leq n)$. Then $|D+2E|$ gives an irreducible pencil of elliptic curves.*

Proof. First, we shall prove that $h^0(D+2E) \geq 2$. By the Riemann-Roch theorem,

$$h^0(E-K) \geq \frac{1}{2}(E-K \cdot E-2K)+1=1,$$

where we note that the hypothesis $D+2K \sim 0$ implies $(K^2) = -1$. Hence $|E-K| \neq \emptyset$. In the same way, we have $h^0(D+E+K) \geq 1$, i.e., $|D+E+K| \neq \emptyset$. Meanwhile, by [9, Lemma 2.1.3], we have $|D+K| = \emptyset$. We also note that $|-K| = \emptyset$. Indeed, suppose to the contrary that $|-K| \neq \emptyset$. We have

$$D+K=(D+2K)+(-K).$$

Since $D+2K \sim 0$ and $|-K| \neq \emptyset$, we have $|D+K| \neq \emptyset$. This is a contradiction. Since $|D+K| = \emptyset$, E is not a fixed component of $|E+D+K|$. Furthermore, since $|-K| = \emptyset$, E is not a fixed component of $|E-K|$. Since

$$D+2E=(D+E+K)+(E-K),$$

the above remark implies that E is not a fixed component of $|D+2E|$. Since $D+2E$ is effective, this implies that $h^0(D+2E) \geq 2$.

Next, we shall prove that $h^0(D+2E) \leq 2$, hence $h^0(D+2E) = 2$. The following exact sequence

$$0 \to \mathcal{O}_X(D_0+\cdots+D_n+2E) \to \mathcal{O}_X(D+2E) \to \mathcal{O}_{D_{n+1}} \to 0$$

implies that $h^0(D+2E) \leq h^0(D_0+\cdots+D_n+2E)+1$. From the following exact sequence

$$\begin{aligned} 0 \to &\mathcal{O}_X(D_0+\cdots+D_{n-1}+2E) \\ &\to \mathcal{O}_X(D_0+\cdots+D_n+2E) \to \mathcal{O}_{D_n}(-1) \to 0, \end{aligned}$$

it follows that $h^0(D_0+\cdots+D_n+2E) = h^0(D_0+\cdots+D_{n-1}+2E)$. Hence we know that

$$h^0(D+2E) \leq h^0(D_0+2E)+1.$$

Furthermore, from the following exact sequence

$$0 \to \mathcal{O}_X(2E) \to \mathcal{O}_X(D_0+2E) \to \mathcal{O}_{D_0}(-1) \to 0,$$

we have

$$h^0(D_0+2E) = h^0(2E) = 1.$$

Hence $h^0(D+2E) \leq 2$, i.e., $h^0(D+2E) = 2$.

From the above argument, it follows that none of D_0, D_{n+1} and E are fixed components of $|D+2E|$. Suppose that there exists an integer i $(1 \leq i \leq n)$ such that D_i is contained in the fixed part of $|D+2E|$. Then we can easily verify that $D_0, \cdots, D_n$ and D_{n+1} are also contained in the fixed part of $|D+2E|$. This is a contradiction. Hence $|D+2E|$ contains no fixed conponents. Since $(D+2E)^2=0$, it follows that $|D+2E|$ is a pencil of curves without base points. Hence $|D+2E|$ is an irreducible pencil of elliptic curves because $p_a(D+2E)=1$ and $D+2E$ is a connected member of $|D+2E|$.

Proof of Theorem 4.1. We shall prove the case $n=4$ only. The other cases can be proved in the same way. The configuration of D is given as follows:

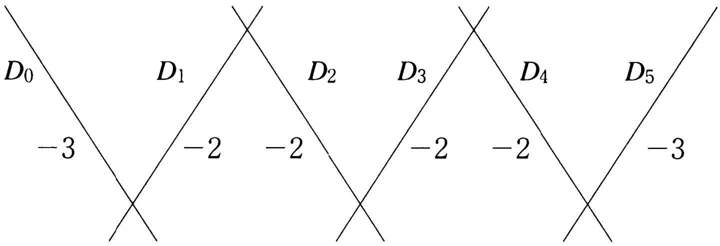


Figure 17

By Lemma 4.2 it suffices to show that there exists a (-1)-curve E such that

$$(E \cdot D_0)=(E \cdot D_5)=1.$$

We prove our assertion by the *reductio absurdum*. Namely, suppose that there exist no (-1)-curves which meet D_0 and D_5.

Since $(K_X^2)=-1$, there exist (-1)-curves on X. Now $D+2K_X \sim 0$ implies that $(E \cdot D)=2$ for any (-1)-curve E. Then we have one of the following two cases:

(I) There exists a (-1)-curve E which intersects two distinct irreducible components of D.

(II) For any (-1)-curve E, there exists an integer i $(0 \leq i \leq 5)$ such that

$$(D \cdot E)=(D_i \cdot E)=2.$$

We consider the above two cases separately.

Case (I). Let E be a (-1)-curve as above. By the above hypothesis, we may assume that one of the following nine cases takes place:

(I-1) $(E \cdot D_5)=(E \cdot D_4)=1,$
(I-2) $(E \cdot D_5)=(E \cdot D_1)=1,$
(I-3) $(E \cdot D_5)=(E \cdot D_2)=1,$
(I-4) $(E \cdot D_5)=(E \cdot D_3)=1,$
(I-5) $(E \cdot D_4)=(E \cdot D_1)=1,$
(I-6) $(E \cdot D_4)=(E \cdot D_2)=1,$

(I-7) $(E \cdot D_4) = (E \cdot D_3) = 1$,
(I-8) $(E \cdot D_3) = (E \cdot D_1) = 1$,
(I-9) $(E \cdot D_3) = (E \cdot D_2) = 1$.

We shall consider separately each of the above cases.

Case (I-1). Let μ: $X \rightarrow Y$ be the contraction of E, D_4, D_3, D_2 and D_1. Then we have

$$(\mu_*(D_0)^2) = -2,\ (\mu_*(D_5)^2) = 14,\ (\mu_*(D_0) \cdot (\mu_*(D_5))) = 2.$$

Let f: $Y \rightarrow \mathbf{F}_n$ be a birational morphism. Put $\overline{D_0} = (f \circ \mu)_*(D_0)$ and $\overline{D_5} = (f \circ \mu)_*(D_5)$.

Case (I-1-a): $\overline{D_0} = M_n$. Since $\overline{D_0} + \overline{D_5} \sim -2K_{\mathbf{F}_n}$, $\overline{D_5} \sim 3M_n + 2(n+2)l$, where l is a fiber of the ruling on $\mathbf{F}_n$. By making use of the hypothesis that there are no (-1)-curves meeting D_0 and D_5, we can readily show that $n=2$. Hence Y dominates a surface obtained by blowing up a point P on $\mathbf{F}_2$ lying outside of M_2. Let l_P be the fiber of the ruling on $\mathbf{F}_2$ which passes through P. Since $(l_P \cdot \overline{D_5}) = 3$, the proper transform of l_P on X is a (-1)-curve which intersects D_0 and D_5. This contradicts the hypothesis of the reductio absurdum.

Case (I-1-b): $\overline{D_0} \neq M_n$. We prove the following claims 1 and 2.

CLAIM 1. $n \leq 2$.

Proof. Since $\overline{D_0}$, $\overline{D_5} \neq M_n$, we have

$$(M_n \cdot \overline{D_0} + \overline{D_5}) = (M_n \cdot -2K_{\mathbf{F}_n}) = 2(2-n) \geq 0.$$

Hence it follows that $n \leq 2$.

CLAIM 2. *We have* $(\overline{D_0} \cdot \overline{D_5}) = (\mu_*(D_0) \cdot \mu_*(D_5)) = 2$.

Proof. It is clear that $(\overline{D_0} \cdot \overline{D_5}) \geq 2$. Suppose that $(\overline{D_0} \cdot \overline{D_5}) \geq 3$. Since $\mu_*(D_0) + \mu_*(D_5) + 2K_Y \sim 0$, there exists a (-1)-curve C on Y such that $(C \cdot \mu_*(D_0)) = (C \cdot \mu_*(D_5)) = 1$. The proper transform $\widetilde{C}$ of C on X is then a (-1)-curve with $(C \cdot D_0) = (C \cdot D_5) = 1$. This contradicts the hypothesis of the reductio absurdum.

If $n=2$ then $\mathrm{Supp}(\overline{D_0} + \overline{D_5}) \cap M_2 = \emptyset$. Hence Y dominates a surface obtained by blowing up a point on $\mathbf{F}_2$ lying outside of M_2 and hence dominates $\mathbf{F}_1$. If $n=0$ then Y dominates a surface obtained by blowing up a point on $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$ and hence dominates $\mathbf{F}_1$. Hence by the hypothesis of the reductio absurdum, there exists a birational morphism g: $X \rightarrow \mathbf{P}^2$ such that $g_*(D) = g_*(D_0) + g_*(D_1) \sim -2K_{\mathbf{P}^2}$ and $(g_*(D_0) \cdot g_*(D_5)) = 2$. This is a contradiction.

Case (I-2). Let μ: $X \rightarrow Y$ be the contraction of E, D_1, D_2, D_3 and D_4.

Then we have

$$(\mu_*(D_0)^2)=1,\ (\mu_*(D_5)^2)=5,\ (\mu_*(D_0)\cdot\mu_*(D_5))=5.$$

Let f: $Y \to \mathbf{F}_n$ be a birational morphism. Put $\overline{D_i}=(f\circ\mu)_*(D_i)$ $(i=0, 5)$. As in Claims 1 and 2, it follows that $n\leq 2$ and $(\overline{D_0}\cdot\overline{D_5})=5$. Hence, as in Case (I-1-b), there exists a birational morphism g: $Y\to\mathbf{P}^2$, such that $(g\circ\mu)_*(D)=(g\circ\mu)_*(D_0)+(g\circ\mu)_*(D_5)\sim -2K_{\mathbf{P}^2}$, $\#\,(g\circ\mu)_*(D_0)\cap(g\circ\mu)_*(D_5)=1$ and $((g\circ\mu)_*(D)\cdot(g\circ\mu)_*(D_5))=5$.

Put D_0': $=(g\circ\mu)_*(D_0)$ and D_5': $=(g\circ\mu)_*(D_5)$. Then $P=D_0'\cap D_5'$ is a double point of D_5'. Since D_5' is singular and $(D_0'\cdot D_5')=5$, it follows that D_0' is a line and D_5' an irreducible rational curve of degree 5. Then D_5' has another double point, say Q. Let H be a line which passes through P and Q. Then we have one of the following three cases, where $i(D_5'\cdot H;P)$ is a local intersection number of D_5' and H at P.

(i) $i(D_5'\cdot H;P)=i(D_5'\cdot H;Q)=2$.

(ii) $i(D_5'\cdot H;P)=3$ and $i(D_5'\cdot H;Q)=2$.

(iii) $i(D_5'\cdot H;P)=2$ and $i(D_5'\cdot H;Q)=3$.

We consider the above three cases separately.

Case (i). There exists a point $R\in D_5'\cap H$ other than P and Q. Then D_5 is smooth at R. Let $\widetilde{H}$ be the proper transform of H on X. Since R is not a fundamental point of g, $\widetilde{H}$ is a (-1)-curve with $(\widetilde{H}\cdot D_4)=(\widetilde{H}\cdot D_5)=1$. By arguing as in Case (I-1), we have a contradiction.

Case (ii). Let ν: $\mathbf{F}_1\to\mathbf{P}^2$ be the inverse of a blowing up with center P. Then we have

$$(\nu'(D_5')\cdot\nu'(H))=3,\ (\nu'(D_0')\cdot\nu'(H))=0,\ i(\nu'(D_5')\cdot\nu'(H);\ \nu^{-1}(Q))=2.$$

Hence $P'=\nu'(D_5')\cap\nu'(H)\cap\nu^{-1}(P)$ is a smooth point of $\nu'(D_5)$. Let $\widetilde{H}$ be the proper transform of H on X. Then $\widetilde{H}$ is a (-1)-curve with $(\widetilde{H}\cdot D_5)=(\widetilde{H}\cdot D_4)=1$. By arguing as in Case (I-1), we have a contradiction.

Case (iii). By an argument similar to Case (ii), we have a contradiction.

Case (I-3). Let μ: $X\to Y$ be the contractions of E, D_2, D_3 and D_4. Then we have

$$(\mu_*(D_0)^2)=-3,\ (\mu_*(D_1)^2)=1,\ (\mu_*(D_5)^2)=4,$$
$$(\mu_*(D_0)\cdot\mu_*(D_5))=0,\ (\mu_*(D_1)\cdot\mu_*(D_5))=4.$$

Let f: $X\to\mathbf{F}_n$ be a birational morphism. Put $(\overline{D_i})=(f\circ\mu)_*(D_i)$ $(i=0, 1, 5)$.

Case (I-3-*a*): $\overline{D_0}=M_n$. As in Case (I-1-a), we know that there exists a (-1)-curve $\widetilde{E}$ on X such that $\widetilde{E}$ intersects two of the three components D_0, D_1 and D_5. By returning to Case (I-1) or (I-2), we have a contradiction.

Case (I-3-*b*): $\overline{D_0} \neq M_n$. As in Claim 2, it follows that

$$(\overline{D_0} \cdot \overline{D_1}) = 1, \ (\overline{D_0} \cdot \overline{D_5}) = 0, \ (\overline{D_1} \cdot \overline{D_5}) = 4.$$

Furthermore, as in Claim 1, we know that $n \leq 2$. By arguing as in Case (I-1-b), we have a contradiction.

By an argument similar to Case (I-3), we can show that Cases (I-4) ~ (I-9) do not occur.

Case (II). Let $f: X \to \mathbf{F}_n$ be a birational morphism. Note that $f_*(D) + 2K_{\mathbf{F}_n} \sim 0$ and $f_*(D_i) \neq 0$ for any i $(0 \leq i \leq 5)$. We consider the following two cases (II-1) and (II-2) separately.

Case (II-1) : $f_*(D_i) = M_n$ for some i. Then by arguing as in Case (I-1-a), there exists a (-1)-curve E' on X which meets two irreducible components of D. This contradicts the hypothesis in Case (II).

Case (II-2) : $f_*(D_i) \neq M_n$ for any i. Then, by the hypothesis in Case (II), we have

$$(f_*D_i) \cdot f_*(D_j) = \begin{cases} 1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$$

whenever $i \neq j$. Hence by arguing as in Case (I-1-b), we have a contradiction.

This completes the proof of Therem 4.1.

Remark 4.3. Let (X, D) be the same as above. Gurjar and Zhang proved that there exists a (-1)-curve C on X such that C meets a terminal component of D (cf. [4, Proposition 3.1]).

Corollary 4.4. *Let (X, D) be the same pair as in Theorem* 2.3. *If (X, D) is of Type (B_n), there is a one-to-one correspondence between the following sets* (X) *and* (Y):

(X) *consists of (-1)-curves E on X such that $(E \cdot D_0) = (E \cdot D_{n+1}) = 1$ and $(E \cdot D_i) = 0$ for any $i\,(1 \leq i \leq n)$.*

(Y) *consists of irreducible elliptic pencils Λ such that* $\mathrm{Supp}(D)$ *is contained in a member of Λ.*

If (X, D) is of Type (A), there exists a one-to-one correspondence between the set of (-1)-curves on X and the above set (Y).

Proof. Let Λ be an irreducible elliptic pencil which contains $\mathrm{Supp}(D)$ in a member C_0. Then C_0 is a reducible member whose components have all self-intersection numbers ≤ -1. Since $(C_0 - D \cdot K_X) = -2$, there exists a component E of C_0 such that $(E \cdot K_X) < 0$. Then E is a (-1)-curve. Since $(D+2E)^2 = 0$, we have that $C_0 = n(D+2E)$, where $n > 0$. If D is irreducible then, by Theorem 3.3, (2), the assertion is clear. If D is reducible then $(E \cdot D_0) = (E \cdot D_{n+1}) = 1$ and $(E \cdot D_i) = 0$ for any $i\,(1 \leq i \leq n)$. Hence by Lemma 4.2 $|D+2E|$ is an irreducible elliptic pencil and hence $\Lambda = |D+2E|$.

Now we shall give several examples.

Example 4.5. (cf. [14]). Let C be a rational sextic plane curve with ten double points. Let μ: $X \to \mathbf{P}^2$ be the minimal resolution of singularities of C and let D be the proper transform of C on X. Then we have that $(D^2) = -4$ and $D+2K_X \sim 0$. Hence the pair (X, D) is of Type (A).

Concerning the above example, we shall explain a construction due to Miyanishi of the pair of Type(A).

Example 4.6. Let k be an algebraically closed field whose transcendence degree over the prime field Π is infinite. Let $P_1, \cdots, P_8$ be independent generic points of $\mathbf{P}^2$ over a field k_0 (not necessarily an algebraic closure of Π) and let λ: $=|3l - \sum_{i=1}^{8} P_i|$ be the linear pencil of cubic curves passing through $P_1, \cdots, P_8$, where l is a line on $\mathbf{P}^2$. Let C be a generic member of λ over k_0 $(P_1, \cdots, P_8)$ and let P_9 be a point of C such that $2(P_1+\cdots+P_9)=0$ is a unique relation among $P_1, \cdots, P_9$ In particular, P_9 is not a base point B of λ. In fact, the tangent line l_B of C at B meets C at a point Q, and there are three other lines through Q which are tangent to C. The point P_9 is one of the three points whose tangent line l_{P_9} passes through Q. Then there exists an irreducible sextic curve S_0 such that $S_0 \cdot C = \sum_{i=1}^{9} 2P_i$. Let L be a linear pencil spanned by S_0 and $2C$.

Let ρ: $Y \to \mathbf{P}^2$ be the blowing-up with centers $P_1, \cdots, P_9$, and let L': $=\rho' L$ the proper transform of L. Then L' is a linear elliptic pencil without base points, in which $2\rho'(C)$ is a unique multiple member. Since $c_2(Y)=12$, L' has singular members. In fact, all fibers are irreducible. Let α be the number of members which have nodes, and let β be the number of members which have cusps. Then we have $\alpha + 2\beta = 12$. Namely, L' has at most 12 singular members (except $2\rho'(C)$) and at least 6 singular members (except $2\rho'(C)$). Let S' be one of the singular members of $\rho' L$ and let P_{10} be the singular point of S'. Let σ: $X \to Y$ be the blowing-up with center P_{10} and let D: $=\sigma'(S')$ be the proper transform of S' on X. Then it is clear that $(D^2) = -4$ and $D+2K_X \sim 0$. Hence the pair (X, D) is of Type (A).

We shall give some examples of the pairs (X, D) of Type(B_n).

Example 4.7. Let C_1, C_2 be two cuspidal cubic curves on $\mathbf{P}^2$ such that C_1 has a cusp P_1 and C_2 has a cusp P_2, where $P_1 \neq P_2$. Furthermore, assume that C_1 and C_2 meet each other in nine distinct points $\{P_3, \cdots, P_{11}\}$. Let μ: $X \to \mathbf{P}^2$ be the blowing-up with centers $P_1, \cdots, P_{10}$ and let D_i: $=\mu'(C_i)$, $i=1, 2$. Then the pair (X, D_1+D_2) is of Type(B_0).

Example 4.8. Let C_1, C_2 and C_3 be three nonsingular conics on $\mathbf{P}^2$. Put $C_1 \cap C_2 = \{P_1, \cdots, P_4\}$, $C_2 \cap C_3 = \{P_5, \cdots, P_8\}$ and $C_3 \cap C_1 = \{P_9, \cdots, P_{12}\}$. Assume that $P_i \neq P_j$ whenever $i \neq j$. Let μ: $X \to \mathbf{P}^2$ be the blowing up with centers $P_1, \cdots, P_{12}$ except for P_4 and P_5 and let $D_i = \mu'(C_i)$ $(i=1, 2, 3)$. Then $(X, D_1+D_2+D_3)$ is of Type(B_1).

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